A Scheme of Notation and Nomenclature for Aircraft Dynamics and Associated Aerodynamics

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Part 4 – Aerodynamic Data for Dynamics

LONDON: HER MAJESTY'S STATIONERY OFFICE
1970
PRICE £1 0s 0d (£1) NET
PREFACE

For many years the notation and nomenclature used in the UK for aircraft dynamics has consisted of a basic scheme introduced by Bryant and Gates (R. & M. 1801, 1937) together with various additions and amendments due to Neumark, Mitchell, and others. Modifications were not co-ordinated and resulted in a complex mixture having at least two serious drawbacks. First, further rational extensions would be extremely difficult to make and probably confusing. Secondly, some parts of the scheme appeared to possess a pattern which in fact they did not possess, and this hidden ambiguity sometimes led to mistakes.

The present Report is the fourth in a series of five separate parts of R. & M. 3562 in which a unified system of notation and nomenclature is described. The system will present few difficulties to those familiar with the scheme of Bryant and Gates and its variants, and has the great advantage that it has a built-in potential for extension. At the same time, reliable patterns are incorporated and furthermore a great deal of freedom is available to an author who wishes in special cases to simplify the notation without ambiguity – for example, by omitting suffixes.

The new system is the outcome of many years of discussion at the Royal Aircraft Establishment, in co-operation with the National Physical Laboratory. It has been adopted by the Royal Aeronautical Society for its Data Items on Dynamics. Moreover, agreements reached by the International Organisation for Standardisation on terms and symbols used in flight dynamics have so far been completely consistent with the principles of the system.

The Aeronautical Research Council hopes that publication in the R. & M. Series will encourage the acceptance of the new notation and nomenclature and its use in the general field of aircraft dynamics by workers in research establishments and universities and in industry.
Summary.

A scheme of notation and nomenclature applicable to the dynamics and associated aerodynamics of both aeroplanes and missiles is proposed. The proposals are intended to supersede the attempts made, notably by Bryant and Gates, to revise and extend the existing standard reference in this field, namely R. & M. 1801.

Part 1 contains an extensive introduction describing the main objectives and summarising a considerable amount of historical background. It also lists the symbols, references, and most of the tables for the whole report, and provides an index. All illustrations are appended to Part 1, and copies included in the remaining parts where required.

Parts 2, 3, and 4 deal with basic notation and nomenclature, aircraft dynamics, and associated aerodynamic data respectively, and they can be read almost independently of Part 1. A great deal of reference information is included in the main text and in the twelve appendices which comprise Part 5.

Parts 2 to 5 do not contain separate lists of symbols, references or indexes.

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15. Aerodynamic Coefficients.

For some purposes the aerodynamic forces and moments may be represented indirectly, and even an
analysis based on particle dynamics can yield useful results, but in general aerodynamic data must be
available in direct form in order to determine the force and moment components acting on the aircraft.
The usefulness of data is extended by introducing non-dimensional quantities. Forces and moments are
usually divided by factors proportional to \( \rho V^2 \), and it is traditional to call the resulting quantities force
coefficients and moment coefficients: all these are aerodynamic coefficients. Such quantities may also be
formed from non-aerodynamic forces or moments, and much of the notation introduced below is applicable
to these as well as the aerodynamic coefficients. The variables upon which the forces and moments
depend may also be non-dimensionalised, and the resulting quantities are called normalised variables.


Aerodynamic forces and moments depend on the following quantities.

(a) The overall shape of the aircraft, and the roughness of its skin.
(b) The size of the aircraft, typified by some length (l).
(c) The properties of the air, specified by the pressure (P), density (\( \rho \)), and viscosity (\( \mu \)).
(d) The motion of the aircraft relative to the air, defined by the components of linear velocity (\( u, v, w \))
and of angular velocity (\( \rho, q, r \)).
(e) The motivator deflections (e.g. \( \xi, \eta, \zeta, \nu, \delta, \kappa \)).
(f) The nearness and the orientation of the aircraft with respect to another object, usually the earth.
Displacement relative to the earth may be defined by the components \( x_0, y_0, z_0 \), and by the attitude
angles \( \Phi, \Theta, \Psi \). When 'ground effects' are assumed to be due to an idealised level surface, the variables
\( z_0, \Phi, \Theta, \Theta \), or equivalent, should suffice.

In many problems the factors above are taken to be constant with the exception of (d) and (e), which
in a linearized treatment can be expressed in terms of Taylor expansions involving derivatives of the
quantities concerned. If the properties of the air vary along the flight-path, derivatives of \( P, \rho, \mu \) are also
required. In some cases other variables may have to be allowed for, an example being the temperature
of the aircraft skin, which can affect the surface friction; or again the flexibility of the aircraft structure,
which will affect the shape. In particular conditions such as take-off and landing, the effects of (f) may be
important; and of course they play a predominant role in the behaviour of hovercraft.

If we assume that (a) is constant and consider a typical force component \( Z \), we can write

\[ Z = f_1 (l, P, \rho, \mu, x_0, y_0, z_0, \Phi, \Theta, \Psi, u, v, w, \rho, q, r, \xi, \eta, \zeta, \nu, \delta, \kappa, \hat{u}, \hat{v}, \hat{w}, \text{ etc.}), \]

where the dependence of \( Z \) on the shape and skin of the aircraft is absorbed in the function \( f_1 \). Equation
(15.1) is rather long, and in the arguments that follow it will be abbreviated by retaining only a typical
selection of variables: we shall write

\[ Z = f_2 (l, P, \rho, \mu, x_0, z_0, \Phi, \Theta, V, u, w, q, \eta, \hat{V}, \hat{w}). \]

We can reduce equation (15.2) to a relation between eleven non-dimensional combinations of the
fourteen quantities considered (see Duncan\(^{44}\)). Before this operation it is convenient to replace \( u \) by the
resultant speed \( V \), where (in this case) \( V^2 = u^2 + w^2 \). Substituting for \( u \) in equation (15.2), we obtain

\[ Z = f_3 (l, P, \rho, \mu, x_0, z_0, \Theta, \Phi, \rho, q, \eta, V, \hat{V}, \hat{w}). \]

Then, choosing \( l, \rho, V \) as the variables to be absorbed, we obtain

\[ \frac{Z}{\rho V^2 F^2} = f_4 \left( \frac{P}{\rho V^2}, \frac{\mu}{\rho V^2}, \frac{x_0}{l}, \Theta, \frac{w}{V}, \frac{q}{V}, \frac{\eta}{V}, \frac{\hat{V}}{V}, \frac{\hat{w}}{V} \right). \]
Although the same length $l$ has been used throughout in equation (15.4), we may replace $l$ in any term by some other length of the aircraft if more convenient. It is usual to replace $l$ on the left-hand side by $\frac{1}{2}S$, where $S$ represents some area relevant to the aircraft. With the further substitutions for Mach number ($M$), Reynolds number ($R$), and angle of downslip* ($\alpha$), namely

$$M = \frac{\rho V}{\sqrt{\gamma P}},$$
$$R = \rho V l / \mu,$$
$$\alpha = \sin^{-1}(w/V),$$

where $\gamma$ is the ratio of the specific heats (assumed constant), we can write equation (15.4) in the more familiar form

$$Z = f_5 \left( M, R, x_0, z_0, \Theta, \alpha, q_l, \eta, V_l, \delta_l \right).$$

The ratio of the force $Z$ and the force $\frac{1}{2}\rho V^2 S$ is called the $Z$-force coefficient and written $C_Z$. Similarly the ratio of the pitching moment $M$ and the moment $\frac{1}{2}\rho V^2 l$ is called the pitching-moment coefficient and written** $C_m$. The symbols $C_n, C_m, C_n$ employ lower-case suffixes in order to avoid confusion with the lift coefficient $C_L$ and normal-force coefficient $C_N$. These two, and also the drag coefficient $C_D$ and the longitudinal-force coefficient $C_D$, are formed by means of the divisor $\frac{1}{2}\rho V^2 S$ from the components of aerodynamic force (excluding thrust contributions) along the $x$- and $z$-axes of the transposed airpath and transposed configuration type (see Sections 3, 4.1, 4.3).

Force and moment coefficients are introduced essentially for aerodynamic contributions, because very often the variation with speed and air density can be largely embodied in the term $\frac{1}{2}\rho V^2$. Other forces and some aerodynamic ones (for example the thrust) may not depend mainly on $\frac{1}{2}\rho V^2$, and for one of these an aerodynamic coefficient is less useful, and can even be misleading, particularly when derivatives of the coefficients are introduced. On the other hand, if most of the forces in a system yield useful coefficients, it is probably convenient in some applications to form coefficients for the other forces as well.

Symbols are required for the normalised variables on the right-hand side of equation (15.5). It is proposed that the whole set of variables be called the $\gamma$ set, and $\gamma$ used as a general suffix. We thus define, for example, a normalised distance $z_{o\gamma} = z_0/l$, a normalised angular velocity $q_{\gamma} = q_l/V$, and so on. The compact form of equation (15.5) is then

$$C_Z = f_5 \left( M, R, x_{o\gamma}, z_{o\gamma}, \Theta, \alpha, q_{\gamma}, \eta, V_{\gamma}, \delta_{\gamma} \right).$$

*The incidence angle chosen can be either the angle of downslip or the angle of attack, which are distinguishable by the symbols $\alpha_d$ and $\alpha_t$ when necessary. If, as here, no suffix is used, it should be made clear which incidence angle is meant. Comments on the use of, for example, $w/V$ instead of $z$ are to be found in Section 19.

**$C_m$ is preferred to $C_{\alpha}$ mainly to agree with old notation. When an aircraft is able to hover or fly at very low speed the divisor should contain some other speed. For example, a rotorcraft having a rotor with radius $R$ and angular velocity $\Omega$ would, when $V$ is small, be subject to aerodynamic forces that are functions of the approximate rotor tip speed $\Omega R$. Divisors for force and moment coefficients would then be based on $\frac{1}{2}\rho(\Omega R)^2 \pi R^2$ and $\frac{1}{2}\rho(\Omega R)^2 \pi R^2$, and the symbol $G$ would replace $C$ (see Appendix J).
Section 15.1

It may be useful on occasions to add the suffix $\gamma$ even to variables such as $M, R, \Theta, \alpha, \eta$. For example, $\partial C_{w}/\partial \Theta$, can be uniquely interpreted as the partial derivative with respect to $\Theta$ when all other variables belonging to the $\gamma$ set are constant*. Such derivatives can be symbolised as $(C_{\gamma})$, or $C_{\omega \beta}$, when necessary, and examples of similar usage in relation to $\omega$ and $\Omega$ sets of variables arise in Section 22.

The choice of $l, \rho, V$, as the absorbed variables is arbitrary and a similar process could be carried out with any three dimensionally independent variables. However, for the special case of steady straight motion in an inviscid incompressible fluid, equation (15.3) reduces to

$$Z = f_3 (l, \rho, x_0, z_0, \Theta, V, w, \eta),$$

and if this case is not to be excluded $l, \rho, V$ must be chosen – $l$ is to be preferred to $x_0$ or $z_0$, which may vary during the motion, and $w$ cannot be used as it may vanish.

The aerodynamicist will wish to investigate the forces and moments acting on portions of an aircraft, such as the body, wing, tail, or fin, and will form non-dimensional coefficients for these portions. Appropriate representative areas and lengths will be chosen for the separate parts, and examples are given in Appendix H. Additional comments on choice of dressings are given in Appendices H and J. For the complete aircraft it will often be convenient to choose two values of the representative length ($l$), one for longitudinal coefficients ($l_1$) and another for lateral ones ($l_2$). The choice of areas and lengths is considered in more detail in Section 16.

15.2. Aerodynamic Coefficients and Variables Commonly Used.

**Force coefficients.**

$$C_R = \frac{R}{1/2 \rho V^2 S}$$

$$(C_X, C_Y, C_Z) = \frac{1}{1/2 \rho V^2 S} (X, Y, Z)$$

$$(C_L, C_D) = \frac{1}{1/2 \rho V^2 S} (L, D)$$

$$(C_A, C_N) = \frac{1}{1/2 \rho V^2 S} (A, N)$$

$X, Y, Z$ represent the components of the resultant aerodynamic force ($R$ or $R$). $L, D$ represent the lift and drag, and $A, N$ the longitudinal and normal aerodynamic forces attributed to the airframe (see Section 3 and 4.3).

**Moment coefficients.**

$$(C_B, C_M) = \frac{1}{1/2 \rho V^2 S L_2} (B, M)$$

*The choice of $\gamma$ stems from earlier thinking that $\gamma$, seemed an appropriate symbol for the quantity $qI/V$, which can be interpreted as an angle. Another symbol may be found to be more attractive, and one proposal (International Standards Organisation) is that the asterisk should be adopted, at least for the normalised $p, q, r$. It might seem tempting to adopt a common dressing for the normalised forms of $p, q, r$ and $u, v, w$, namely $qI/V$ and $w/V$, etc., but it should be noted that to exploit the dressing for clarifying partial derivative expressions would then not be possible for the set of variables considered here. See Sections 17.3, 19, and Appendix L.
Section 15.2

\[ C_m = \frac{M}{\frac{1}{2} \rho V^2 S l_1} \]

Other force and moment coefficients that are sometimes used are given in Appendix J.

**Normalised variables.**

(a) Mach number and Reynolds number:

\[ M = V(\rho/\gamma P)^{\frac{1}{4}}, \quad R = \rho V l/\mu. \]

(b) Normalised distance:

\[ z_{0\gamma} = z_0 / l_1. \]

(c) Angles of incidence, for example the angle of attack or downslip, i.e.

\[ \alpha = \tan^{-1}(w/u) \quad \text{or} \quad \alpha = \sin^{-1}(w/V), \]

together with the angle of sideslip,

\[ \beta = \sin^{-1}(v/V). \]

(d) Normalised angular velocities:

\[ q_1 = \frac{q l_1}{V}, \quad p_1 = \frac{p l_2}{V}, \]

\[ \dot{q}_1 = \frac{\dot{q} l_1}{V}, \quad r_1 = \frac{r l_2}{V}, \]

\[ \beta_1 = \frac{\beta l_2}{V}. \]

(e) Normalised acceleration:

\[ \dot{V}_1 = \frac{\dot{V} l_1}{V^2}. \]

Other normalised variables will sometimes be needed, and they can be formed in a similar way as follows.

\[ \dot{\eta}_1 = \frac{\dot{\eta} l_1}{V}, \quad \dot{q}_1 = \frac{\dot{q} l_1^2}{V^2}, \quad \beta_1 = \frac{\beta l_2^2}{V^2}, \]

\[ \dot{V}_1 = \frac{\dot{V} l_1^2}{V^2}, \]

and likewise a normalised frequency

\[ \omega_1 = \frac{\omega l}{V}, \]

where \( \omega \) represents an angular frequency such as is introduced in Appendix K.
Care must be taken to distinguish between the symbol \( l \), which denotes any one of the various representative lengths used in forming aerodynamic coefficients, and the symbol \( l_0 \), the representative length (usually equal to one of the \( l \)'s) which is the unit of length in the aero-normalised system. It is especially important, of course, that when aerodynamic coefficients and normalised variables are used it should be known without any doubt which representative areas and lengths they are based upon.

16. The Choice of Representative Area and Representative Length.

In practice the representation of aerodynamic forces and moments in terms of non-dimensional coefficients still leaves room for considerable variations with shape and size. Thus, in order to have coefficients that can usefully be compared, aerodynamic objects must be grouped, and within a group a standard choice of representative area and length should be made unless there is some overwhelming objection. One convenient group is that for which the main interest is in an aerofoil, for example the wing. Another group is that for which the body is the main object of interest, and a third is that for which airscrews are particularly important. Examples of aircraft in these respective categories are aeroplanes, most missiles, and helicopters.

We discuss below the choice for the wing group and the body group, and the airscrew group is considered in Appendix J. Notation for subsidiary aerofoils such as a fin is given in Appendix H.

(a) The wing group.

The representative area for a wing is generally taken as a projected area. When the wing is combined with a body, etc. a projected gross wing area should be taken. The exact definition is arbitrary, since the way in which the leading and trailing edges of the exposed wing are continued until they meet on the plane of symmetry must be chosen to suit the configuration.

The choice of representative length is more difficult, especially in the longitudinal case. Here, there is an additional advantage in locating a definite line (of length \( l_1 \)) along the aircraft longitudinal axis in such a way that a certain point on it can be taken as a natural datum to which various aerodynamic quantities like pitching moment are referred. Although it is impossible to find such a datum that is convenient for all aspect ratios and all speeds, it is recommended that the second mean chord be used for \( l_1 \) unless its aerodynamic significance is exceedingly small. Its applicability as a datum will be discussed later. In magnitude it is defined as

\[
\bar{c} = \frac{I_2}{I_1},
\]

where

\[
I_2 = \int_{-b}^{b} c^2 \, dy,
\]

\[
I_1 = \int_{-b}^{b} c \, dy = S,
\]

and \( b \) denotes the span, \( c \) being the length of the chord at spanwise station \( y \) from the longitudinal axis. When the second mean chord is inappropriate it is recommended that the centreline chord be used. This is the distance between the points at which the leading and trailing edges meet on the plane of symmetry when they are continued for the purpose of defining the gross wing area. The centreline chord is likely to be adopted for wings with low aspect ratio or wings of complicated shape. Apart from this the second mean chord is preferred to the first mean chord as explained below (see also Ref. 45).
Section 16

The first mean chord is defined as

$$\tilde{c} = I_1/I_0,$$

where

$$I_0 = \int_{-\frac{b}{2}}^{\frac{b}{2}} dy = b,$$

i.e.

$$\tilde{c} = S/b.$$

It was at one time widely used in U.K. but it cannot be assigned a longitudinal position in such a direct way as the second mean chord. The practice was to aline its own quarter point (i.e. $\frac{1}{4}\tilde{c}$) with the mean quarter-chord point given by

$$\bar{x}_k = \frac{1}{S} \int_{-\frac{b}{2}}^{\frac{b}{2}} c x_k dy,$$

where $x_k$ is the distance of the quarter-chord point (at station $y$) aft of an arbitrary datum. This procedure stemmed from its relevance to two-dimensional flow at subsonic speeds, but for supersonic speeds a different position based on $\frac{1}{2}\tilde{c}$ and $\bar{x}_k$ would be the sensible choice. By contrast we find that

$$\bar{x}_k = \bar{x}_0 + \frac{1}{2} \tilde{c},$$

and indeed

$$\bar{x}_k = \bar{x}_0 + k \tilde{c},$$

where $\bar{x}_0$ defines the mean leading edge, and $k$ is any number from zero to unity. It follows that the natural location for the second mean chord is at $c = \tilde{c}$, where its own $k$-chord point coincides with the mean $k$-chord point of the wing. Furthermore a natural datum to take is the aerodynamic centre of the wing, and for high aspect ratio wings (which tend to have geometric loading) this is on the longitudinal axis and more or less in line with the quarter point and mid point of the second mean chord for subsonic and supersonic speeds respectively. Other examples of specifying the aerodynamic centre in terms of a mean chord have been given by Yates.46

Other arguments in favour of the second mean chord are that it is widely used in U.S.A. and other countries, and that using $S\tilde{c}$ to form the pitching moment coefficient is consistent with the use of $c^2$ for the two-dimensional pitching-moment coefficient. In wing-body combinations the second mean chord of the gross wing should be taken.

It is recommended that the span be used for lateral aerodynamic coefficients and normalised variables (i.e. for $I_2$). Up to now the span has been used for the former and the semi-span for the latter.

(b) The body group.

For the representative area the maximum cross-sectional area is recommended. It is widely used for this purpose in the missile field, and seems superior to the area of the enclosing square, which is used by ballisticians – their interest is restricted to axially-symmetric bodies.
The choice of representative length is again more difficult, and although in some respects some fore-
and-aft length would seem desirable, the best choice is considered to be the maximum diameter. It is
already widely used, and has the advantage of being more definite, and a more fundamental dimension
of a missile than, say, its length. If the body is not axially-symmetric an equivalent maximum diameter
given by $2\sqrt{S_B/\pi}$, where $S_B$ is the maximum cross-sectional area, should be taken.

17. Expansions of Forces and Moments.

17.1. General.

For application to dynamics the forces and moments should be known at all times, but they are rarely
known as direct functions of time. Aerodynamic forces, for example, are functions of many variables
which are themselves functions of time. It may be sufficient to expand functional forms, such as $f_1,
f_2, \ldots$ of Section 15.1, in terms of the variables and their time derivatives. This is profitable provided
that the expansion yields a fairly small number of terms, and that the importance of the time derivatives
diminishes rapidly as we proceed to first derivatives, second derivatives, and so on. A further restriction
may be imposed whereby an expansion is carried out in terms of only some of the variables: the co-
efficients in the expansion are then functions of the remaining variables.

Taylor series are used mostly. If only small perturbations from a reference condition are considered,
so that a linearized treatment is justified, the Taylor series can be terminated with the first derivatives
of the forces and moments. In general the forces and moments are non-linear functions of the variables,
the series are infinite, and any analytical treatment depends on the possibility of making assumptions
that justify the omission of higher derivatives. For some purposes, however, it is convenient to represent
groups of terms by compact symbols, and schemes are given in Appendix K for oscillatory derivatives,
mean derivatives, and other shorthand notations. These are unspecified functions of the variables and
may have to be expanded again for further analysis.

17.2. Derivatives of Forces and Moments.

As an example we consider an aerodynamic force as given by (15.1), but for simplicity we take a selection
of the variables. These are sufficient to indicate the development of the notation without any loss in
generality, for the treatment of a non-aerodynamic force and also additional variables would be similar.
We thus take the aerodynamic force component $Z$ as

$$Z = Z(l; P, \rho, \mu, z_0, \Theta, u, q, \eta). \tag{17.1}$$

In principle we could expand the right-hand side of this equation by a Taylor series based on the
variables $P, \rho, \mu, z_0, \Theta, u, q, \eta$. To do this we would choose datum values $P_e, \rho_e, \ldots$ of these variables, and
would express $Z$ as a series in terms of the increments $P', \rho', \ldots$, which are given by $P' = P - P_e$, etc.
When, however, we are concerned with the motion of an aircraft through air which is in equilibrium, we
can embody the variations with $P, \rho, \mu$ as a variation with altitude ($h$, usually above sea-level), because
they are related along with the temperature by the equation of state, the equation of static equilibrium,
and the viscosity-temperature relationship (see, for example, Ref. 47).

Some of the variables may be better excluded from the Taylor expansion. For example, $z_0$ and $\Theta$ are
significant variables for ground effects, and these are likely to be too complicated to be described in
terms of derivatives. In the expressions below it is implied that we are dealing with a problem in which
the dependence of $Z$ on $z_0$ is too complicated, but not the dependence on $\Theta$. In other problems it may be
possible to represent $z_0$ effects also in terms of derivatives: on the other hand it may be necessary to
exclude derivatives based on $\Theta, h$, or some other variable, but the treatment is analogous to that
described below for $z_0$. Since ground effect is usually investigated for a horizontal plane earth, the variables
$x_0, y_0$ will very seldom if ever be introduced, and $z_0$ could be replaced by $-h$, provided that a clear
distinction were made between ground effects and those due to variations in $P, \rho, \mu$.

In order to express a Taylor expansion compactly it is convenient to define a linear differential operator
Section 17.2

\[
T = h \frac{\partial}{\partial h} + \left( \phi \frac{\partial}{\partial \phi} + \theta \frac{\partial}{\partial \theta} + \psi \frac{\partial}{\partial \psi} \right) + \left( u' \frac{\partial}{\partial u} + v' \frac{\partial}{\partial v} + w' \frac{\partial}{\partial w} \right) + \\
+ \left( p' \frac{\partial}{\partial p} + q' \frac{\partial}{\partial q} + r' \frac{\partial}{\partial r} \right) + \ldots, \tag{17.2}
\]

where \( h', u', v', w', p', q', r', \ldots \) are increments in \( h, u, \ldots \), and \( \phi, \theta, \psi \) are attitude-deviation angles. We may then write in general, and for a reference value \( Z_d \),

\[
Z = Z_d + \left( T + \frac{T^2}{2!} + \frac{T^3}{3!} + \ldots \right) Z, \tag{17.3}
\]

and if we retain only some of the variables as in (17.1) the full expression is

\[
Z = Z_d + Z_{hh'} + Z_{uu'} + Z_{qq'} + Z_{rr'} + Z_{\theta\theta'} + \\
+ \frac{1}{2} \left( Z_{hh'h'^2} + Z_{uu'u'^2} + Z_{qq'q'^2} + Z_{rr'r'^2} + Z_{\theta\theta'^2} \right) + \\
+ Z_{hh'} u' + Z_{uu'} q' + Z_{qq'} r' + \ldots + \\
+ \frac{1}{6} Z_{hh'h'^3} + \ldots + \\
+ \ldots,
\tag{17.4}
\]

where the expansion is taken relative to an arbitrary value of \( z_0 \), so that

\[
Z_d = Z(l, z_0; \Theta, h, u, q, \eta),
\]

and, for example,

\[
Z_u = \frac{\partial Z}{\partial h},
\]

\[
Z_{uu} = \frac{\partial^2 Z}{\partial h \partial u} = Z_{uh}.
\]

\( Z_d \) and the derivatives are functions of \( z_0 \), and are evaluated at the datum values of \( \Theta, h, u, q, \eta \).

The functions of \( z_0 \) may be very complicated and in applications are likely to be replaced by simpler functions which fit well enough over the range of interest. If no particular form is suggested by theory or experiment a power series approximation may be acceptable.

In certain applications such as aeroelasticity, it is sometimes preferred to expand the expression for aerodynamic force or moment in terms of 'displacement' variables rather than 'velocity' variables. In place of equation (15.2) we then write

\[
Z = f_0 (l, P, \mu, x_0, z_0, \Theta, x_0, z_0, \Theta, \eta, x_0, z_0), \tag{17.5}
\]

where \( x_0, z_0 \) have replaced \( u, w, \) and \( \Theta \) has replaced \( q, \) and so on.

As in the treatment of equation (17.1), we do not represent ground effect in terms of derivatives with respect to \( x_0, z_0 \), but in forming the Taylor series in terms of the other variables it is found necessary to define derivatives with respect to perturbations in \( x_0, z_0 \). This is because, in addition to the dependence
Section 17.2

of $Z$ on ground effect, there is an implicit dependence on $x_0, z_0$ which could almost be called spurious – an explanation is given in Section 22. It is also convenient as before to work in terms of the attitude deviation $\theta$ rather than the increment $\Theta'$, and in terms of vector increments $x', z'$ rather than the scalar increments $x, z$ (see Section 7 and Appendix M). The expansion in terms of displacement variables is therefore written*   

$$Z = Z_d + Z_h x' + Z_z z' + Z_\theta \theta + Z_x x' + Z_z z' + Z_h h' + Z_\theta \theta' + \ldots,$$

(17.6)

where $Z_d$ and the derivatives are functions of $z_0$, and are evaluated at datum values of $h, \Theta, x_0, z_0, \Theta, \eta$. The relations between derivatives with respect to displacement and velocity variables (these might be labelled the $\Omega$ and $\omega$ sets respectively) are established in Section 22.

17.3. Aerodynamic-Coefficient Derivatives.

As an example we consider a force coefficient as represented by (15.6). As in the previous section, we take a selection of the variables and write

$$C_z = C_z (M, R, z_0'; \Theta, \alpha, q_y, \eta).$$

(17.7)

If $\Theta_e, \alpha_e, q_{y e}, \eta_e$ are datum values of the variables $\Theta, \alpha, q_y, \eta$, then for arbitrary values of $M, R, z_0'$ a Taylor expansion in terms of increments $x' = \alpha - \alpha_e$ etc. and of the pitch-deviation angle $\theta$ gives

$$C_z = C_{zd} + C_{z\alpha} x' + C_{zq_y} q_y' + C_{z\theta} \theta' +$$

$$+ \frac{1}{2} (C_{zzx} x'^2 + C_{zzq_y} q_y'^2 + C_{z\theta\theta} \theta'^2) +$$

$$+ C_{z\alpha} \alpha' q_y' + \ldots +$$

$$+ \frac{1}{6} C_{zzzz} x'^3 + \ldots +$$

$$+ \ldots,$$

(17.8)

where the reference value is

$$C_{zd} = C_z (M, R, z_0'; \Theta_e, \alpha_e, q_{y e}, \eta_e),$$

and, for example,

$$C_{z\alpha} = \frac{\partial C_z}{\partial \alpha}, \quad C_{zq_y} = \frac{\partial C_z}{\partial q_y},$$

$$C_{zzq_y} = \frac{\partial^2 C_z}{\partial \alpha \partial q_y} = C_{zzq_y}.$$

*The symbols $Z_\alpha, Z_q$, etc. are not likely to be ambiguous. It might very occasionally be necessary to distinguish between (say) $\partial Z/\partial z_0$ and $\partial Z/\partial z_0'$, but derivatives with respect to $x_0, z_0, x_0, z_0, \Theta, \eta$ are not usually employed. If they were, the additional terms would be $Z_{z_0} z_0'$, etc., where $Z_{z_0} \equiv \partial Z/\partial z_0$, and $z_0 = z_0 - z_{0 e}, z_{0 e}$ being a constant.
Section 17.3

$C_{z_d}$ and the derivatives are functions of $M$, $R$, and $z_{0r}$ and are evaluated at datum values of $\Theta$, $\alpha$, $q_{r}$, $\eta$. When, in addition, the datum values of $M$, $R$, and $z_{0r}$ are taken, $C_{z_d}$ becomes $C_{z_0}$ and the derivatives are written $(C_{z_0})_e$ etc.

As for the force and moment derivatives in Section 17.2, the Taylor expansion is not extended to the variable $z_{0r}$ nor for the same reason is it extended here to the variables $M$, $R$. As mentioned in that same section, approximate functions may be introduced, and unless theory or experiment suggests particular functions then power series may be assumed. In many applications a function of only one of the three variables will suffice: during a landing manoeuvre, for example, the effects of variations in $M$ and $R$ could be taken as negligible.

It can be argued that $\partial C_{z_d}/\partial q_{r}$ ought to be represented as $C_{z_d}$, rather than $C_{z_0}$, and indeed there is no objection to this course other than its clumsiness. It is thought, however, that there will be no serious disadvantage in adopting the convention that $C_{z_0}$ always stands for $C_{z_d}$ unless this is specifically denied. Care must be taken not to confuse these new British symbols $C_{z_0}$, $C_{z_0}$, etc. with similar ones used in American reports, namely $C_{z_0}$, $C_{z_0}$, etc., which usually correspond to the normalised derivatives $Z_{w}$, $Z'_{w}$, etc. that are introduced in Section 18. Writing $C_{z_d}$ instead of $\partial C_{z_d}/\partial q_{r}$ is also acceptable, but the danger of confusion with American symbols is then enhanced. This is discussed at the end of Section 2.

18. Normalised Derivatives of Forces and Moments.

As explained in Section 2 and 11 it is convenient to normalise aerodynamic forces and moments and their derivatives according to the aero-system, which is based on divisors or units of magnitude $l_0$ for lengths, $V_e$ for velocities, and $\frac{1}{2} \rho V_e^2 S$ for forces, other divisors being formed in a consistent way as in systems of units. Table 11 gives a list of most divisors likely to be required, and it will be remembered that aero-normalised quantities are denoted by the overscript -. For example, $\dot{w} = w/V_e$ and $\tilde{I} = tV_e/l_0$.

In the case of derivatives such as $Z_u = \partial Z/\partial u$, the aero-normalised quantity will be distinguished by one overscript on the main symbol. Thus $Z_u$ will represent $\partial Z/\partial u$. If the need should arise to denote $\partial Z/\partial u$ by $Z_u$ – and this is extremely unlikely – the departure from the usual notation would have to be emphasised.

It is not usually helpful to express aerodynamic quantities in the dynamic-normalised system, but should this be desired the divisors are easily obtained from the aero-normalising divisors by substituting $\mu l_0$ for $l_0$.

Since $\mu = m_e/\frac{1}{2} \rho S l_0$ and $\tau = m_e/\frac{1}{2} \rho V_e S = \mu l_0/V_e$, the divisors for obtaining dynamic-normalised quantities sometimes have simpler alternative forms. For example, if $\tilde{q} = q l_0/V_e$, then $\tilde{q} = q \mu l_0/V_e = q \tau$. Similarly, if $M_w = M_w/\frac{1}{2} \rho V_e S l_0$, then

$$M_w = M_w/\frac{1}{2} \rho V_e S l_0 = M_w/m_e V_e.$$


Since the aero-normalised forces are based on the divisor $\frac{1}{2} \rho V_e^2 S$, and the force coefficients on the divisor $\frac{1}{2} \rho V_e^2 S$, there is an affinity between corresponding quantities and between their derivatives. The same is true for aero-normalised moments and the corresponding moment coefficients, and for their derivatives. To establish the relations between the first derivatives of $\bar{Z}$ and $C_{z_0}$, for example, we merely write

$$Z = \frac{1}{2} \rho V_e^2 S C_{z_0} \quad (19.1)$$

and differentiate each side (partially) with respect to each one of the variables concerned on the left-hand side, in turn. Finally we divide each derivative of $Z$ by the appropriate divisor as given in Table 11.

Some variables appear on both sides of the equation, but they belong to different sets, and must sometimes be given a distinguishing mark so that it is clear which variables are considered constant.
### TABLE 11

**Divisors for Normalised Systems of Units.**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Example</th>
<th>Divisor for obtaining normalised quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>in the aero-system</td>
</tr>
<tr>
<td>Length</td>
<td></td>
<td>( l_0 )</td>
</tr>
<tr>
<td>Mass</td>
<td></td>
<td>( \frac{1}{2} \rho_c S l_0 )</td>
</tr>
<tr>
<td>Time</td>
<td></td>
<td>( l_0 / V_e )</td>
</tr>
<tr>
<td>Linear velocity</td>
<td>( w )</td>
<td>( V_e )</td>
</tr>
<tr>
<td>Linear acceleration</td>
<td>( \dot{w} )</td>
<td>( V_e^2 / l_0 )</td>
</tr>
<tr>
<td>Angular displacement</td>
<td>( \eta )</td>
<td>1</td>
</tr>
<tr>
<td>Angular velocity</td>
<td>( q )</td>
<td>( V_e l_0 )</td>
</tr>
<tr>
<td>Angular acceleration</td>
<td>( \dot{q} )</td>
<td>( V_e^2 / l_0^2 )</td>
</tr>
<tr>
<td>Force</td>
<td>( Z )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S )</td>
</tr>
<tr>
<td>Moment</td>
<td>( \mathcal{M} )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S l_0 )</td>
</tr>
<tr>
<td>Pressure</td>
<td>( P )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S l_0 )</td>
</tr>
<tr>
<td>Density</td>
<td>( \rho )</td>
<td>( \frac{1}{2} \rho_c S / l_0^2 )</td>
</tr>
<tr>
<td>Viscosity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Force derivatives</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with respect to</td>
<td></td>
<td></td>
</tr>
<tr>
<td>linear displacement</td>
<td>( Z_h )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S / l_0 )</td>
</tr>
<tr>
<td>linear velocity</td>
<td>( Z_w )</td>
<td>( \frac{1}{2} \rho_c S / l_0 )</td>
</tr>
<tr>
<td>linear acceleration</td>
<td>( Z_\omega )</td>
<td>( \frac{1}{2} \rho_c S / l_0 )</td>
</tr>
<tr>
<td>angular displacement</td>
<td>( Z_\eta )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S )</td>
</tr>
<tr>
<td>angular velocity</td>
<td>( Z_q )</td>
<td>( \frac{1}{2} \rho_c V_e S l_0 )</td>
</tr>
<tr>
<td>Moment derivatives</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with respect to</td>
<td></td>
<td></td>
</tr>
<tr>
<td>linear displacement</td>
<td>( M_h )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S )</td>
</tr>
<tr>
<td>linear velocity</td>
<td>( M_w )</td>
<td>( \frac{1}{2} \rho_c V_e S l_0 )</td>
</tr>
<tr>
<td>linear acceleration</td>
<td>( M_\omega )</td>
<td>( \frac{1}{2} \rho_c S l_0^2 )</td>
</tr>
<tr>
<td>angular displacement</td>
<td>( M_\eta )</td>
<td>( \frac{1}{2} \rho_c V_e^2 S l_0 )</td>
</tr>
<tr>
<td>angular velocity</td>
<td>( M_q )</td>
<td>( \frac{1}{2} \rho_c V_e S l_0 )</td>
</tr>
</tbody>
</table>
when a partial derivative is taken. We have already mentioned that the force coefficient is expressed in terms of the \( \gamma \) set of variables: \( M, R, x_0, y_0, z_0, \Phi, \Theta, \Psi, x, \beta, p, q, r, \xi, \eta, \zeta, \nu, \delta, \kappa, \dot{V}, \dot{x}, \dot{\beta}, \dot{\gamma}, \) etc. Suppose that the force is expressed in terms of an \( \omega \) set of variables: \( h, x_0, y_0, z_0, \Phi, \Theta, \Psi, u, v, w, p, q, r, \xi, \eta, \zeta, \nu, \delta, \kappa, \dot{h}, \dot{v}, \dot{w}, \) etc. Differentiating both sides of equation (19.1) with respect to any \( \omega \) variable we obtain

\[
Z_{\omega} = \left( \frac{\partial Z}{\partial \omega} \right)_e = \left( \rho VSC_Z \frac{\partial V}{\partial \omega} \right)_e + \frac{1}{2} \rho_e V_e^2 S \sum \left( \frac{\partial C_z}{\partial \gamma} \frac{\partial \gamma}{\partial \omega} \right)_e,
\]

where \( \gamma \) is any variable in the \( \gamma \) set, and the suffix \( e \) after a bracket implies that all quantities inside the bracket are to be given their datum values. We thus have

\[
\frac{Z_{\omega}}{\frac{1}{2} \rho_e V_e S} = \left( 2C_Z \frac{\partial V}{\partial \omega} \right)_e + V_e \sum \left( \frac{\partial C_z}{\partial \gamma} \frac{\partial \gamma}{\partial \omega} \right)_e. \tag{19.2}
\]

To substitute in the right hand side of equation (19.2) we need the basic definitions of the variables, such as*

\[
V^2 = u^2 + v^2 + w^2,
\]

\[
M = \frac{V}{\alpha},
\]

\[
R = \frac{\rho Vl}{\mu} = \frac{\rho Vl_0}{\mu},
\]

\[
(\cos \sigma, \sin \beta, \sin \alpha) = \left( \frac{u}{V}, \frac{v}{V}, \frac{w}{V} \right),
\]

\[
(x_0, y_0, z_0) = \left( \frac{x_0}{l_1}, \frac{y_0}{l_2}, \frac{z_0}{l_1} \right),
\]

\[
(p_r, q_r, r_r) = \left( \frac{pl_2}{V}, \frac{ql_1}{V}, \frac{rl_2}{V} \right),
\]

\[
(\dot{V}, \dot{\beta}, \dot{\alpha}) = \frac{1}{V} \left( \frac{\dot{V}_l}{V}, \frac{l_2 \dot{\beta}}{l_1}, \frac{l_1 \dot{\alpha}}{\cos \alpha} \right)
\]

\[
= \frac{1}{V^2} \left( l_0 \dot{V}, \frac{l_2 (\dot{\beta} - \dot{V} \sin \beta)}{\cos \beta}, \frac{l_1 (\dot{\alpha} - \dot{V} \sin \alpha)}{\cos \alpha} \right).
\]

It should be noted that the representative lengths \( l \) and \( l_0 \) are taken to be the same and equal to either the longitudinal representative length \( (l_1) \) or the lateral one \( (l_2) \).

*The variable \( \sigma \) is redundant when \( \alpha \) and \( \beta \) are present, but it is a convenient auxiliary quantity for writing some expressions compactly.
Section 19

The partial derivatives $\partial \gamma / \partial \omega$ are simple expressions except when the $\omega$ variable is a function of $V$, that is when $\omega$ is one of the variables $u, v, w, \hat{u}, \hat{v}, \hat{w}$, etc. Some of the derivatives $\partial \gamma / \partial u, \partial \gamma / \partial v$, etc. are quoted in Appendix L. Applying equation (19.2) to each of the $\omega$ variables in turn we obtain the following relations.

The local symbols $\Pi_\omega$ etc. are used to denote the product terms $C_{zq} q, \rho$, etc.

\[
\left( \frac{\partial Z}{\partial x_0}, \frac{\partial Z}{\partial y_0}, \frac{\partial Z}{\partial z_0} \right) = \left( \frac{l_0 \partial C}{\partial x_0}, \frac{l_0 \partial C}{\partial y_0}, \frac{l_0 \partial C}{\partial z_0} \right).
\]

\[
(\hat{Z}_x, \hat{Z}_y, \hat{Z}_z) = (C_{z\phi}, C_{z\theta}, C_{z\psi})_e.
\]

\[
\hat{Z}_u \sec \gamma = 2C_{ze} + M_e \left( \frac{\partial C}{\partial M} \right)_e + R_e \left( \frac{\partial C}{\partial R} \right)_e - (C_{z\alpha} \tan \alpha + C_{z\beta} \tan \beta)_e - \]

\[
-(\Pi_\rho + \Pi_\eta + \Pi_\beta + 2\Pi_\phi)_e - \]

\[
-(\Pi_\phi (2 + \tan^2 \alpha - \sin^2 \alpha \sec^2 \gamma) + \Pi_\beta (2 + \tan^2 \beta - \sin^2 \beta \sec^2 \gamma))_e - \]

\[
-(C_{z\phi}) \sec^2 \gamma \left( \frac{l_0}{l_1} \hat{x}_y \sin \alpha \cos \alpha + \frac{l_0}{l_2} \hat{\beta}_x \sin \beta \cos \beta \right)_e + \]

\[
+(C_{z\alpha}) \tan \alpha \left( \frac{l_0}{l_1} \hat{y}_y + \frac{l_1}{l_2} \hat{\gamma}_x \sin \alpha \cos \alpha \sec \gamma \right)_e + \]

\[
+(C_{z\beta}) \cos \beta = C_{z\alpha} \tan x \sin \beta)_e - \]

\[
-\sin \beta_e \left[ \Pi_\rho + \Pi_\eta + \Pi_\beta + 2\Pi_\phi + 2\Pi_\beta + \Pi_\phi (2 + \tan^2 \alpha) \right)_e + \]

\[
+ \cos \beta_e \left( \frac{l_0}{l_2} C_{z\phi} \hat{\beta}_x - \frac{l_2}{l_0} C_{z\phi} \hat{\gamma}_y \right)_e + \]

\[
+(C_{z\phi}) \tan \alpha \left( \frac{l_0}{l_1} \hat{y}_y \sin \beta = \frac{l_0}{l_2} \hat{\gamma}_x \cos \beta \right)_e .
\]

\[
\hat{Z}_w = \sin \beta_e \left[ 2C_{ze} + M_e \left( \frac{\partial C}{\partial M} \right)_e + R_e \left( \frac{\partial C}{\partial R} \right)_e \right] + \]

\[
+(C_{z\phi} \cos \beta - C_{z\alpha} \tan x \sin \beta)_e - \]

\[
-\sin \beta_e \left[ \Pi_\rho + \Pi_\eta + \Pi_\beta + 2\Pi_\phi + 2\Pi_\beta + \Pi_\phi (2 + \tan^2 \alpha) \right)_e + \]

\[
+ \cos \beta_e \left( \frac{l_0}{l_2} C_{z\phi} \hat{\beta}_x - \frac{l_2}{l_0} C_{z\phi} \hat{\gamma}_y \right)_e + \]

\[
+(C_{z\phi}) \tan \alpha \left( \frac{l_0}{l_1} \hat{y}_y \sin \beta - \frac{l_0}{l_2} \hat{\gamma}_x \cos \beta \right)_e .
\]

\[
\hat{Z}_w = \text{as for } \hat{Z}_v \text{ but with } \alpha \text{ and } \beta \text{ interchanged, and also } l_1 \text{ and } l_2 .
\]

\[
(\hat{Z}_p, \hat{Z}_q, \hat{Z}_r) = \frac{1}{l_0} (l_2 C_{zp}, l_1 C_{zq}, l_2 C_{zr})_e.
\]

\[
(\hat{Z}_\phi, \hat{Z}_\eta, \hat{Z}_\psi, \text{ etc.}) = (C_{z\phi}, C_{z\eta}, C_{z\psi}, \text{ etc.})_e .
\]
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\[ \dot{Z}_u = \left( C_{Z\phi} \cos \sigma - \frac{l_2}{l_0} C_{Z \alpha} \cos \alpha \tan \alpha - \frac{l_2}{l_0} C_{Z \beta} \cos \sigma \tan \beta \right) e. \]

\[ \dot{Z}_v = \left( C_{Z \beta} \cos \beta - \frac{l_1}{l_0} C_{Z \alpha} \sin \alpha \tan \alpha + \frac{l_2}{l_0} C_{Z \phi} \cos \beta \right) e. \]

\[ \dot{Z}_w = \left( C_{Z \phi} \sin \alpha + \frac{l_1}{l_0} C_{Z \alpha} \cos \alpha - \frac{l_2}{l_0} C_{Z \beta} \sin \alpha \tan \beta \right) e. \]

All the coefficient derivatives are evaluated at the datum values of the variables retained in the Taylor expansion of \( Z \) in terms of \( Z_u, Z_v, Z_w \), etc. This implies datum values of \( M \) and \( R \) as well as \( \Phi, \Theta, \Psi, \alpha, \beta, \psi, r, \psi, r, \) \( \cdots \), but excludes \( \psi_0, \rho_0, \zeta_0 \), if we follow the example given in Section 17.2. The derivatives of \( Z \) and of \( C_Z \) are then evaluated at the relevant values of \( \psi_0, \rho_0, \zeta_0 \), and the suffix \( e \) is not attached to the first set of relations above for this reason.

It is seen that the derivatives \( \partial C_Z / \partial M \) and \( \partial C_Z / \partial R \) are introduced in the relations for \( u, v, w \) derivatives, and there seems no objection to denoting these by \( C_{Z M} \) and \( C_{Z R} \).

The relations between moment derivatives and moment-coefficient derivatives follow exactly the same pattern except that all terms on the right hand sides are multiplied by an additional factor \( l_2/l_0 \) for pitching-moment derivatives, and \( l_2/l_0 \) for rolling- or yawing-moment derivatives. For example,

\[ M_\alpha = \left( C_{\alpha M} l_1/l_0 \right), \]

\[ L_\alpha = \left( \frac{l_2}{l_0} C_{Z \phi} \sin \phi - \frac{l_2}{l_0} l_1 \Phi \sin \phi - \frac{l_2}{l_0} C_{Z \beta} \cos \beta \right) e. \]

The incidence magnitude appears in the relations for \( u \) and \( \dot{u} \) derivatives, and should not be confused with the relative air density \((\rho/\rho_0)\), which is also represented by \( \sigma, \rho_0 \) being the air density at a datum sea-level.

If the coefficients and their derivatives are given for a particular body-axes system, and it is desired to derive the normalised force and moment derivatives for another body-axes system, it is convenient to obtain the corresponding derivatives for the original system of axes first, and then to convert to the desired system as described in Section 20. If, on the other hand, the coefficients and their derivatives are known for the air-path axes associated with a particular body-axes system, the corresponding quantities for the body-axes system can be obtained as shown in Section 21, and the normalised force and moment derivatives for these body axes then become available as before by application of the conversion formulae of this Section.

The exact relations given above will simplify when the datum values of some of the variables are zero. It is usual for the datum values of \( \psi_0, \rho_0, \zeta_0 \), \( \psi_0 \), \( \zeta_0 \) to be zero, and only the first two lines of the expressions for \( Z_u, Z_v, Z_w \) are then required. Further simplification is achieved when aerodynamic-body axes are chosen, since \( \dot{x}_e \) is then zero.

If the angle of attack \( \alpha \) is used instead of the angle of downslip \( \phi \), the relations between \( Z_u, Z_v, Z_w, \) \( \dot{Z}_u, \dot{Z}_v, \dot{Z}_w \) \( \cdots \) and \( C_{Z \phi}, C_{Z \alpha}, \cdots \) or \( C_{Z \alpha}, C_{Z \phi}, \cdots \) are different, but can be found in a similar way (useful formulae are set out in Appendix I). Examples are given below for the special conditions in which the datum values of \( \psi_0, \rho_0, r, \psi, \zeta, \dot{\psi}, \dot{\zeta} \) are zero.

\[ \dot{Z}_u = \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] e \cos \alpha \cos \beta_e - (C_{Z \alpha} \sin \alpha \sec \beta + C_{Z \beta} \cos \alpha \sin \beta) e. \]

\[ \dot{Z}_v = \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] e \sin \beta_e + (C_{Z \beta} \cos \beta) e. \]
\[ \dot{Z}_w = \left[ 2C_Z + M\frac{\partial C_Z}{\partial M} + R\frac{\partial C_Z}{\partial R} \right] \sin \alpha \cos \beta \ \sin \beta - C_{Z\beta} \sin \alpha \sin \beta \ e. \]

\[ \dot{Z}_a = \left( C_{Z\beta} \cos \alpha \cos \beta - \frac{l_1}{l_0} C_{Za} \sin \alpha \sec \beta - \frac{l_2}{l_0} C_{Z\beta} \cos \alpha \sin \beta \right) e. \]

\[ \dot{Z}_\phi = \left( C_{Z\phi} \sin \beta + \frac{l_2}{l_0} C_{Z\phi} \cos \beta \right) e. \]

\[ \dot{Z}_w = \left( C_{Z\phi} \sin \alpha \cos \beta + \frac{l_1}{l_0} C_{Zw} \cos \alpha \sec \beta - \frac{l_2}{l_0} C_{Z\beta} \sin \alpha \sin \beta \right) e. \]

Only in this paragraph have we written \( \alpha, \beta \) for \( \alpha_t, \beta_t \).

It is possible that the \( \gamma \) set of normalised variables might in some circumstances be rejected in favour of a closely related set, here called the \( \lambda \) set, which is identically the same except that \( \alpha, \beta, \alpha_t, \beta_t, \) etc. are replaced by \( \omega, \psi, \omega_t, \psi_t, \) and so on*. These variables are defined as follows, together with the auxiliary quantity \( u_\lambda \) which is analogous to the incidence magnitude.

\[
(u_\lambda, \ v_\lambda, \ w_\lambda) = \left( \begin{array}{c} u \\ \psi \\ w \end{array} \right).
\]

\[
(\dot{u}_\lambda, \ \dot{w}_\lambda) = \frac{1}{V^2} (l_2 \dot{\psi}_t, \ l_1 \dot{w}_t) + \frac{d}{dt} (u_\lambda, \ w_\lambda).
\]

As before \( l \) and \( l_0 \) are taken to be the same and equal to either \( l_1 \) or \( l_2 \). Also, just as we write \( C_{Z\alpha} = \partial C_Z/\partial \alpha \), we write \( C_{Zw} = \partial C_Z/\partial \omega, \ C_{Z\psi} = \partial C_Z/\partial \psi \), and so on. By applying the formula

\[ Z_\alpha = \left( \rho V^2 C_Z \frac{\partial V}{\partial \omega} \right) e + \frac{1}{2} \beta \ V^2 S \sum_{\lambda} \left( \frac{\partial C_Z}{\partial \lambda} \right) \frac{\partial \lambda}{\partial \omega} e, \]

we obtain the following general relations, where \( \Pi_q \) stands for \( C_{Zq} q_\lambda \) and so on. The variables \( q_\lambda, \ V_\lambda, \) etc. are identical with \( q_r, \dot{V}_r, \) etc.

\[
\dot{Z}_u = u_{\lambda\lambda} \left[ 2C_Z + M\frac{\partial C_Z}{\partial M} + R\frac{\partial C_Z}{\partial R} \right] e - u_{\lambda\lambda} \left[ \Pi_w + \Pi_\psi + \Pi_{\psi^2} + 2\Pi_{\omega} + 2\Pi_{\omega^2} \right] e - \left[ \Pi_{\psi} \left( 3u_\lambda - \frac{1}{u_\lambda} \right) \right] e - \left[ \frac{C_{Z\psi}}{u_\lambda} \left( \frac{l_0}{l_2} \dot{\psi}_r v_\lambda + \frac{l_0}{l_1} \dot{\psi}_r w_\lambda \right) \right] e.
\]

*There is no need for two distinguishing signs (\( \gamma, \lambda \)) unless both the \( \alpha, \beta \) and the \( w/V, \nu/V \) sets are being discussed, and we might well write \( w_t = w/V \), and so on.
Section 19

\[
\dot{Z}_v = v_{e\lambda} \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] e - \frac{l_0}{l_2} \left[ C_{Zv} \dot{\psi}_e \right] e.
\]

\[
\dot{Z}_w = \text{as for } \dot{Z}_v \text{ but with } v \text{ and } w \text{ interchanged, and also } l_2 \text{ and } l_1.
\]

\[
\dot{Z}_d = \left[ C_{Z\psi} u_e \right] e.
\]

\[
\dot{Z}_\phi = \left[ C_{Z\theta} v_e + l_2 C_{Z\theta} \right] e.
\]

\[
\dot{Z}_\omega = \left[ C_{Z\psi} w_e + l_1 C_{Z\omega} \right] e.
\]

Other derivatives are related as for the \( \gamma \) variables, and moment derivatives are obtained as explained earlier in this section. It is seen that the general expressions for the derivatives \( \dot{Z}_m \) etc. are much simpler in terms of \( w/V, v/V, \) etc. than in terms of \( \alpha, \beta, \) etc. When the datum values of \( p, q, r, \dot{V}, \dot{v}, \dot{w} \) are zero, however, the expressions are about as simple for the \( \gamma \) variables as for the \( \lambda \) variables. A preference for the latter would then be justified only if aerodynamic forces and moments were significantly more nearly linear in terms of these variables.

It is interesting to give a few examples of the relations between force derivatives and lift, drag, thrust, and cross-stream force derivatives. The formulae developed in Section 21 are applied, but for the purpose of illustration the simple datum conditions mentioned above are assumed \( (p_{\text{der}}, \text{etc.} \text{ zero}), \) and in addition we take \( \beta = 0, \) so that \( \alpha_e = \alpha. \) The expressions for \( \ddot{X}_w, \ddot{X}_w, \ddot{Z}_w, \ddot{Z}_w, \ddot{Y}_e \) are then as follows:

\[
\ddot{X}_u = \left[ 2C_x^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right] \cos^2 \alpha_e e.
\]

\[
- \left[ C_x^u + C_{x\alpha}^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right] \sin \alpha_e \cos \alpha_e e + (C_{x\alpha}^u + C_{x\alpha}^u) \sin^2 \alpha_e e,
\]

\[
\ddot{X}_w = (C_{xa}^u - C_{xw}^u) \cos^2 \alpha_e e + \left[ C_x^u - C_{xz}^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right] \sin \alpha_e \cos \alpha_e e - \left[ 2C_x^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right] \sin^2 \alpha_e e,
\]

\[
Z_u = \left( 2C_x^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right) \cos^2 \alpha_e e + \left[ C_x^u - C_{xz}^u + M \frac{\partial C_x^u}{\partial M} + R \frac{\partial C_x^u}{\partial R} \right] \sin \alpha_e \cos \alpha_e e + (C_x^u - C_{xw}^u) \sin^2 \alpha_e e.
\]
To introduce lift and drag coefficients we usually have to separate the thrust contributions to the force components, since, as mentioned in Section 15, the introduction of a thrust coefficient is likely to be inconvenient when derivatives are in use. In the notation suggested in Section 3 we write a total force component as

\[ X = X^g + X^c + X, \]

where the superfixes \( G, C, \) refer to gravitational and earth contact contributions, the remaining part being expressed as

\[ X = X^T + X^A, \]

where \( X^T \) represents the thrust contribution, and \( X^A \) the aerodynamic force unaccounted for in the former. The lift and drag are defined along air-path axes, and hence

\[ X^A = -D, \quad Z^A = -L, \]

while \( C^A_{xa} \) (otherwise written \( C^A_{ya} \)) can be replaced by \( -C_D \), and \( C^A_{px} \) by \( -C_{px} \), and so on. The cross-stream force \( Y^A \) is usually taken to be equal to \( Y_a \) or \( Y \), since \( Y^T \) is negligible.

Any derivative such as \( \dot{X}_a \) can be expressed as the sum of contributions \( \dot{X}^T_a \) and \( \dot{X}^A_a \), and, unless a thrust coefficient has been adopted, only the second one should be found from the relations given above. We write, for example,

\[
\dot{X}_a^A = -2C_D + M \frac{\partial C_D}{\partial M} + R \frac{\partial C_D}{\partial R} \cos^2 \alpha_a +
\]

\[
+ \left[ C_L + C_{pa} + M \frac{\partial C_L}{\partial M} + R \frac{\partial C_L}{\partial R} \right] \sin \alpha_a \cos \alpha_a -
\]

\[-(C_{la} + C_{pa}) \sin^2 \alpha_a.\]

The thrust contribution can be expressed as

\[ \dot{X}_a^T = \frac{1}{\frac{1}{2} \rho_a V_e S} \frac{\partial X^T}{\partial u}, \]
and is evidently zero when \( X^T \) is independent of speed. In contrast, the same quantity but in terms of a thrust coefficient and derivatives can be written as

\[
X^T_u = 2CT_X + M \frac{\partial CT_X}{\partial M} + R \frac{\partial CT_X}{\partial R},
\]

and it is not immediately obvious that this is zero when the thrust is independent of speed. The thrust of course is probably specified in relation to body axes and not to air-path axes, so there is little point in bringing in \( C^T_X \) and its derivatives, and the relations would be interpreted with \( \alpha_x \) zero.

Checks for consistency can often be devised: for the examples given we have

\[
\begin{align*}
X^T_u + Z_w &= \left[ C^2_{Zu} + 3C^2_X + M \frac{\partial C^2_X}{\partial M} + R \frac{\partial C^2_X}{\partial R} \right], \\
X^T_w - Z_u &= \left[ C^2_{Xw} - 3C^2_z - M \frac{\partial C^2_z}{\partial M} - R \frac{\partial C^2_z}{\partial R} \right].
\end{align*}
\]

Derivatives with respect to height must be considered separately. We write, for example,

\[
Z_h = \frac{\partial Z}{\partial P} \frac{dP}{dh} + \frac{\partial Z}{\partial \rho} \frac{d\rho}{dh} + \frac{\partial Z}{\partial \mu} \frac{d\mu}{dh} = \frac{1}{2} \rho_v V^2_s S \tilde{Z}_h/l_0.
\]

Hence, using the definitions \( Z = \frac{1}{2} \rho V^2 SC_z, M = V(\rho/\gamma P)^{1/2}, R = \rho Vl/\mu, \) we obtain

\[
Z_h = \left[ -Ml_0 \frac{\partial C_z}{\partial M} \frac{dP}{dh} + l_0 \left( C_z + \frac{M}{2} \frac{\partial C_z}{\partial M} + R \frac{\partial C_z}{\partial R} \right) \frac{d\rho}{dh} - \frac{Rl_0}{\mu} \frac{\partial C_z}{\partial R} \frac{d\mu}{dh} \right].
\]

and similarly, since the pitching moment is given by \( \mathcal{M} = \frac{1}{2} \rho V^2 S l_1 C_{\alpha w} \)

\[
\mathcal{M}_h = \left[ -Ml_1 \frac{\partial C_{\alpha w}}{\partial M} \frac{dP}{dh} + l_1 \left( C_{\alpha w} + \frac{M}{2} \frac{\partial C_{\alpha w}}{\partial M} + R \frac{\partial C_{\alpha w}}{\partial R} \right) \frac{d\rho}{dh} - \frac{Rl_1}{\mu} \frac{\partial C_{\alpha w}}{\partial R} \frac{d\mu}{dh} \right].
\]

If \( \mathcal{R} \) denotes the gas constant, and \( k \) the temperature lapse rate (equal to \( -dT/dh \), where \( T \) denotes the temperature), then we may substitute further:

\[
1 \frac{dP}{dh} = -\frac{g}{\mathcal{R}T^4} \quad \text{and} \quad 1 \frac{d\rho}{dh} = -\frac{g}{\mathcal{R}T^4} \frac{k}{T^3},
\]

where \( g \) denotes the acceleration due to gravity.

Higher derivatives such as \( Z_{\alpha w} \) are formed in a similar way. The general relations are very complicated and so rarely used that they are not given here, although a few examples applicable to simplified conditions are considered in Appendix E.
Section 20


It is frequently necessary to convert the values of derivatives to correspond to a new choice of axes. Examples are the conversion from one set of body axes to another, and from body axes to earth axes. The two systems of axes will have a relative orientation which may be constant or varying with time, and they may not have the same origin. Conversion formulae are most concise when written in matrix notation, and this is elaborated later in this Section. There is much less advantage, however, when the two systems have a common xz-plane and both origins lie in this plane. This is frequently met and simplified conversion formulae are therefore developed below independently of the matrix treatment.

Section 20.1 gives the simple analysis for two sets of body axes, and Section 20.2 deals similarly with body axes and earth axes. Sections 20.3 and 20.4 give the general matrix formulae. When considering body-axes and some other type of axes system we find it convenient to assume a common origin, and to introduce an origin shift only in the conversion formulae for two body systems. Any axes transformation can then be tackled as a combination of an origin shift for body axes followed by a conversion from body axes to the desired axes system.

In general the orientation of body \( x_1, y_1, z_1 \)-axes relative to body \( x, y, z \)-axes is expressed in terms of Euler angles \( \phi, \theta, \psi \). It will be found that symbols of the form \( Z_2 \) are introduced to denote derivatives relevant to a second body-axes system. Quantities represented as \( Z_1 \) also exist, but they do not seem to be required except incidentally in determining the conversion formulae, as in this Section. It would accordingly often be reasonable for an author to use \( Z_1 \) to stand for \( Z_2 \), provided that he made this clear. The reader would otherwise expect \( \frac{\partial Z_2}{\partial u} \) instead of \( \frac{\partial Z_1}{\partial u} \).

The conversion formulae for aerodynamic coefficients are obtained merely by replacing the force and moment symbols by the corresponding coefficient ones.


Consider two sets of body axes \( 0xyz \) and \( 0_{1x_1y_1z_1} \), whose y-axes are parallel, and which are displaced relative to each other as shown in Fig. 5. The origin shift \( 0x \) has components \( h, k \) in the \( x, z \) directions respectively, and \( h_1, k_1 \) in the \( x_1, z_1 \) directions. The angle \( \epsilon \) between \( 0x_1 \) and \( 0x \) is positive if a clockwise rotation of \( 0x \) about the y-axis and equal to \( \epsilon \) would make it point in the same direction as \( 0x_1 \). If we represent \( \sin \epsilon \) by \( s \) and \( \cos \epsilon \) by \( c \) we can write:

\[
\begin{align*}
h_1 &= ch-sk \quad \text{and} \quad h = ch_1 + sk_1 \\
k_1 &= ck+sh \quad \text{and} \quad k = ck_1 - sh_1 \\
p_1 &= cp-sr \quad \text{and} \quad p = cp_1 + sr_1 \\
r_1 &= cr+sp \quad \text{and} \quad r = cr_1 - sp_1
\end{align*}
\]

and \( q_1 = q \). Expressions for linear velocity components include additional terms if we define \( u, v, w \) as the components along \( 0xyz \) of the velocity of \( 0 \), and \( u_1, v_1, w_1 \) as the components along \( 0_1x_1y_1z_1 \) of the velocity of \( 0_1 \). Thus:

\[
\begin{align*}
u_1 &= cu-sw+k_1q \\
v_1 &= v-kp+hr \\
w_1 &= cw+su-h_1q
\end{align*}
\]

\[
\begin{align*}
u &= cu_1+sw_1-kq \\
v &= v_1+k_1p_1-h_1r_1 \\
w &= cw_1-su_1+hq
\end{align*}
\]

A system represented by forces \( X, Y, Z \) acting at \( 0 \) and by moments \( L, M, N \) acting about \( 0 \) is assumed to be also represented by forces \( X_1, Y_1, Z_1 \) acting at \( 0_1 \) together with moments \( L_1, M_1, N_1 \) about \( 0_1 \).
We then have:

\[
\begin{align*}
X_t &= cX - sZ \\
Z_t &= cZ + sX \\
L_t &= cL - sN + k_1 Y \\
N_t &= cN + sL - h_1 Y
\end{align*}
\]

and also \( Y_t = Y, M_1 = M + hZ - kX \). From now on we write \( X^1 \) instead of \( X_1 \) in order to simplify the derivative notation. This means that

\[
\frac{\partial X^1}{\partial u_1} = X^1_{u_1},
\]

and so on.

By direct application of the relations given above we obtain the following conversion equations.

\[
\begin{align*}
X^1_{u_1} &= c^2 X_u - s(c X_w + Z_u) + s^2 Z_w \\
Z^1_{u_1} &= c^2 Z_u + s(c X_w + Z_u) + s^2 X_u \\
X^1_{w_1} &= c^2 X_w + s(c X_u - Z_w) - s^2 Z_w \\
Z^1_{w_1} &= c^2 Z_w + s(c X_u - Z_w) - s^2 X_u \\
X^1_{q_1} &= cX_q - sZ_q + h(cX_w - sZ_w) + k(sZ_u - cX_u) \\
Z^1_{q_1} &= cZ_q + sX_q + h(cZ_w + sX_u) - k(sX_u + cZ_u) \\
M^1_{u_1} &= cM_u - sM_w + h(cZ_u - sZ_w) + k(sX_w - cX_u) \\
M^1_{w_1} &= cM_w + sM_u + h(cZ_u + sZ_w) - k(sX_u + cX_w) \\
M^1_{q_1} &= M_q + h(Z_q + M_u) - k(X_q + M_u) + h^2 Z_w - hk(Z_u + X_w) + k^2 X_u \\
Y^1_{v_1} &= Y_v \\
L^1_{v_1} &= cL_v - sN_v + k_1 Y_v \\
N^1_{v_1} &= cN_v + sL_v - h_1 Y_v \\
Y^1_{p_1} &= cY_p - sY_v + k_1 Y_v \\
Y^1_{r_1} &= cY_r + sY_p - h_1 Y_v \\
L^1_{p_1} &= c^2 L_p + k_1 [k_1 Y_v + c(L_v + Y_p)] + s [sN_v - c(L_v + N_p) - k_1 (Y_r + N_p)]
\end{align*}
\]

*This notation for the linear velocities and moments clashes with the general proposal in this Report that the suffix 1 merely distinguishes components in the \( x_1, y_1, z_1 \) directions, so that, for example, \( u \) refers to the \( x \)-component of the velocity of \( 0 \) and \( u_1 \) to the \( x_1 \)-component of the velocity of \( 0 \) (not of \( 0_1 \) as in this Section). A more complete notation such as \( u^1 \) for the \( x_1 \)-component of the velocity of \( 0_1 \), however, would become unwieldy for derivatives, and is therefore scorned.
Section 20.1

\[ N_{1r} = c^2 N_r + h_1 \left[ h_1 Y_r - c (Y_r + N_r) \right] + s \left[ sL_p + c (L_r + N_p) - h_1 (L_r + Y_p) \right] \]

\[ L_{1r} = c^2 L_r + c (k_1 Y_r - h_1 L_o) - h_1 k_1 Y_v + s \left[ h_1 N_r + k_1 Y_p - s N_p + c (L_p - N_r) \right] \]

\[ N_{1p} = c^2 N_p + c (k_1 N_r - h_1 Y_p) - h_1 k_1 Y_v + s \left[ h_1 Y_r + k_1 L_o - s L_r + c (L_p - N_r) \right] \]

Useful checks can be devised by combining certain pairs of relations. For example,

\[ X_{w1}^1 + Z_{w1}^1 = X_u + Z_w, \]

and

\[ L_{p1}^1 + N_{r1}^1 = L_p + N_r - h (Y_r + N_r) + k (L_r + Y_p) + (h^2 + k^2) Y_v. \]

The formulae above are also valid if \( u, v, w, p, q, r \) are replaced by their rates of change \( \dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r} \). When a variable is independent of the axes system, the conversion of a derivative accords with that of the force or moment. Thus \( X_{u1}^1 \) bears the same relation to \( X_u \) as \( X^1 \) does to \( X \).

Attitude derivatives are sometimes introduced and are best expressed in terms of attitude-deviation angles: \( \phi, \theta, \psi \) for \( 0xyz \), and \( \phi_1, \theta_1, \psi_1 \) for \( 0_{1x_1y_1z_1} \). These variables satisfy the relations

\[ \phi_1 = c \phi - s \psi \]

\[ \psi_1 = c \psi + s \phi \]

and \( \theta_1 = \theta \). The force and moment derivatives are then related as follows.

\[ X_{\theta1}^1 = c X_\theta - s Z_\theta, \]

\[ Z_{\theta1}^1 = c Z_\theta + s X_\theta, \]

\[ M_{\theta1}^1 = M_\theta + h Z_\theta - k X_\theta; \]

\[ Y_{\phi1}^1 = c Y_\phi - s Y_\phi, \]

\[ Y_{\psi1}^1 = c Y_\psi + s Y_\phi, \]

\[ L_{\phi1}^1 = c^2 L_\phi - sc (L_\phi + N_\phi) + s^2 N_\phi + k_1 Y_\phi, \]

\[ N_{\phi1}^1 = c^2 N_\phi + sc (L_\phi + N_\phi) + s^2 L_\phi - h_1 Y_\psi, \]

\[ L_{\psi1}^1 = c^2 L_\psi + sc (L_\psi - N_\psi) + s^2 L_\psi + k_1 Y_\phi, \]

\[ N_{\psi1}^1 = c^2 N_\psi + sc (L_\psi - N_\psi) - s^2 L_\psi - h_1 Y_\psi. \]

It follows that

\[ L_{\phi1}^1 + N_{\phi1}^1 = L_\phi + N_\phi + k Y_\phi - h Y_\psi, \]

\[ L_{\psi1}^1 - N_{\psi1}^1 = L_\psi - N_\psi + h Y_\phi + k Y_\psi. \]
Section 20.2

20.2. Conversion between Body Axes and Earth Axes.

We first consider systems of axes with a common origin and a common y-axis: the earth axes \(0x_0y_0z_0\) and the body axes \(Oxyz\), which are at an orientation \(\Theta\) to the earth system* (see Fig. 6). This implies that motion is restricted to the vertical plane through 0, and that only derivatives of \(X, Z, M\) with respect to \(u, w, q, etc.\) are required. As in the previous Section we have relations between the variables, and between forces and moments referred to the two systems. Thus

\[
\begin{align*}
  u_0 &= cu + sw, \quad \dot{u}_0 = c\dot{u} + s\dot{w} + w_0q, \\
  w_0 &= cw - su, \quad \dot{w}_0 = c\dot{w} - s\dot{u} - u_0q, \\
  q_0 &= q = \Theta; \\
  X &= cX^0 - sZ^0, \\
  Z &= cZ^0 + sX^0, \\
  M &= M^0,
\end{align*}
\]

where \(c, s\) stand for \(\cos \Theta, \sin \Theta\) respectively.

It is convenient to carry out the conversion in two stages, the first consisting of the application of the formulae

\[
Z_\omega = \sum_{\omega_0} \frac{\partial Z}{\partial \omega_0} \frac{\partial \omega_0}{\partial \omega},
\]

where \(\omega\) is any one of the variables \(\Theta, u, w, q, \dot{u}, . . .\), and \(\omega_0\) runs through the variables \(\Theta, u_0, w_0, q_0, \dot{u}_0, . . .\). The variable \(\Theta\) appears in each set of variables, and we shall distinguish the attitude derivatives belonging to the \(\omega_0\) set by means of a suffix \(\omega\). Also, as it is generally preferred to express perturbations in attitude in terms of attitude-deviation angles, attitude terms in the Taylor expansion will be written as \(Z_\theta\), for example, rather than \(Z_\Theta\), although for the simple case considered here \(Z_\theta\) is equal to \(Z_\Theta\), and the pitch-deviation angle \(\theta\) is equal to \(\Theta\), the increment in \(\Theta\).

Accordingly, \(Z_\theta\) stands for \(\partial Z/\partial \theta\) when \(u, w, . . .\) are kept constant, and \(Z_{\omega_0}\) stands for \(\partial Z/\partial \omega\) when \(u_0, w_0, . . .\) are constant. Application of the formula above yields the following expressions:

\[
\begin{align*}
  Z_\theta &= Z_{\theta_0} + Z_{u_0} (-su + cw) + Z_{w_0} (-sw - cu) + \\
           &\quad + Z_{\dot{u}_0} (-s\dot{u} + cw) + Z_{\dot{w}_0} (-s\dot{w} - c\dot{u}) + \\
           &\quad + qZ_{\dot{u}_0} (-sw - cu) + qZ_{\dot{w}_0} (su - cw), \\
  Z_u &= cZ_{u_0} - sZ_{w_0} - qsz_{u_0} - qcz_{w_0}, \\
  Z_w &= cZ_{w_0} + sZ_{u_0} + qcz_{u_0} - qsz_{w_0},
\end{align*}
\]

and similarly

\[
\begin{align*}
  Z_{\dot{u}} &= cZ_{\dot{u}_0} - sZ_{\dot{w}_0} - qsz_{\dot{u}_0} - qcz_{\dot{w}_0},
\end{align*}
\]

and so on, where the terms involving \(\dot{u}_0, \dot{w}_0\) are usually neglected.

The second stage consists of substituting for \(Z_{\theta_0}, Z_{\omega_0}, \text{ etc.}\) by means of relationships such as

\[
X_{\theta_0} = cX^0_{\theta_0} - sZ^0_{\theta_0} - sX^0 - cZ^0,
\]

*The results of this Section are equally valid when \(0z_0\) is not vertical.
Section 20.2

\[ Z_{\theta_0} = cZ^0_{\theta_0} + sX^0_{\theta_0} - sZ^0 + cX^0, \]

and

\[ X_{u_0} = cX^0_{u_0} - sZ^0_{u_0}, \]

\[ Z_{u_0} = cZ^0_{u_0} + sX^0_{u_0}, \]

the simpler relationships being valid for any of the variables \( u_0, w_0, \dot{u}_0, \dot{w}_0, \) etc. We thus obtain the required conversion equations in the following form, where for convenience some terms on the right hand sides are expressed as body-axes quantities.

\[ X_{\theta} = cX^2_{\theta_0} - sZ^0_{\theta_0} - Z + \]
\[ + w (c^2 X^2_{\theta_0} - scZ^0_{\theta_0} + Z^0_{\theta_0} + s^2 Z^0_{w_0}) - u (c^2 X^2_{w_0} + sX^0_{w_0} - s^2 Z^0_{w_0} + s^2 Z^0_{w_0}) + \]
\[ + (\ddot{w} - qu)(c^2 X^2_{\theta_0} - scZ^0_{\theta_0} - scX^0_{w_0} + s^2 Z^0_{w_0}) - \]
\[ - (\ddot{u} + qw)(c^2 X^2_{\theta_0} + scX^0_{\theta_0} - scZ^0_{w_0} + s^2 Z^0_{w_0}) \]
\[ = cX^2_{\theta_0} - sZ^0_{\theta_0} - Z + (wX_u - uX_w) + (\ddot{w}X_a - \ddot{u}X_w) \]

\[ Z_{\theta} = cZ^2_{\theta_0} + sX^0_{\theta_0} + X + (wZ_u - uZ_w) + (\ddot{w}Z_a - \ddot{u}Z_w) \]

\[ X_u = c^2 X^2_{\theta_0} - sc (X^0_{\theta_0} + Z^0_{\theta_0}) + s^2 Z^0_{w_0} - qX_w \]

\[ Z_w = c^2 Z^2_{\theta_0} + sc (X^0_{\theta_0} + Z^0_{\theta_0}) + s^2 Z^0_{w_0} + qZ_d \]

\[ X_w = c^2 X^2_{\theta_0} + sc (X^0_{\theta_0} - Z^0_{\theta_0}) - s^2 Z^0_{\theta_0} + gX_d \]

\[ Z_d = c^2 Z^2_{\theta_0} + sc (X^0_{\theta_0} - Z^0_{\theta_0}) - s^2 Z^0_{\theta_0} + qZ_w \]

\[ X_a = c^2 X^2_{\theta_0} - sc (X^0_{\theta_0} + Z^0_{\theta_0}) + s^2 Z^0_{w_0} \]

\[ Z_d = c^2 Z^2_{\theta_0} + sc (X^0_{\theta_0} - Z^0_{\theta_0}) - s^2 Z^0_{\theta_0} + qZ_w \]

\[ X_q = c (X^0_{\theta_0} - u_0X^0_{\theta_0} + w_0X_{\theta_0}) - s (Z^0_{\theta_0} - u_0Z^0_{\theta_0} + w_0Z_{\theta_0}) \]
\[ = cX^0_{\theta_0} - sZ^0_{\theta_0} + wX_a - uX_w \]

\[ Z_q = c (Z^0_{\theta_0} - u_0Z^0_{\theta_0} + w_0Z_{\theta_0}) + s (X^0_{\theta_0} - u_0X^0_{\theta_0} + w_0X_{\theta_0}) \]
\[ = cZ^0_{\theta_0} + sX^0_{\theta_0} + wZ_a - uZ_w \]

\[ M_{\theta} = M^0_{\theta} + (wM_u - uM_w) + (\ddot{w}M_a - \ddot{u}M_w) \]

\[ M_u = cM^0_{u_0} - sM^0_{w_0} - qM_w \]

\[ M_w = cM^0_{w_0} + sM^0_{u_0} + qM_d \]
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\[ M_a = cM_{a0}^0 - sM_{a0}^0 \]
\[ M_w = cM_{w0}^0 + sM_{w0}^0 \]
\[ M_q = M_{q0}^0 - u_0 M_{w0}^0 + w_0 M_{a0}^0 = M_{q0}^0 - uM_w + wM_a \]

All quantities on the right hand sides should be evaluated for the datum conditions. We therefore usually have \( \dot{u}, \dot{w}, \dot{q} \) equal to zero. If also \( \Theta = 0 \) in the datum condition the relationships simplify as follows.

\[ X_\theta = X_{\theta 0}^0 - Z_c + w_\theta X_u - u_\theta X_w, \]
\[ Z_\theta = Z_{\theta 0}^0 + X_c + w_\theta Z_u - u_\theta Z_w, \]
\[ M_\theta = M_{\theta 0}^0 + w_\theta M_u - u_\theta M_w, \]
\[ X_q = X_{q 0}^0 + w_\theta X_u - u_\theta X_w, \]
\[ Z_q = Z_{q 0}^0 + w_\theta Z_u - u_\theta Z_w, \]
\[ M_q = M_{q 0}^0 + w_\theta M_u - u_\theta M_w, \]

\[ (X_w, X_{w'}, X_{\theta}, X_{\theta'}) = (X_{w 0}^0, X_{w' 0}^0, X_{\theta 0}^0, X_{\theta' 0}^0), \]

and similarly for \((Z_w, Z_{w'}, Z_{\theta}, Z_{\theta'})\) and \((M_w, M_{w'}, M_{\theta}, M_{\theta'})\).

As already noted, \( \Theta = 0 \) implies that in the datum condition the body axes are coincident with the earth axes but the z-axes need not be vertical; furthermore the x-axes may not be aligned with the air-path or flight-path direction. For example, in the wind-tunnel experiments of Thompson and Fail\(^{22}\) both body axes and sting axes are inclined to the air stream. In real flight problems it is usual to align the x-axes with a datum straight path and to have \( w_\theta \) zero.

The inverse forms of the simple relationships are obvious, but in general they can be obtained as analogous ones in which terms containing the variables explicitly have their signs changed and \( c, s \) are replaced by \( c, -s \). For example,

\[ M_{w0}^0 = cM_w - sM_u - q_0 M_{a0}^0, \]
\[ M_{a0}^0 = cM_a + sM_w, \]
\[ M_{q0}^0 = M_\theta + (u_0 M_{w0}^0 - w_0 M_{a0}^0) + (u_0 M_{w0}^0 - w_0 M_{a0}^0). \]

It should be noted that an attitude derivative such as \( X_{\theta 0}^0 \) is the sum of two contributions, one being \( X_\theta \), which represents the 'genuine' dependence of the resultant aerodynamic force on attitude. When there is no other object in the neighbourhood the resultant aerodynamic force is of course constant when there is a change in attitude but no change in the body-axes variables \( u, w, q, \) etc., and \( X_\theta \) is then zero. The other contribution to \( X_{\theta 0}^0 \) arises because components of the force and the variables are taken along earth-fixed axes, and it must always be present. It is often taken for granted that \( X_\theta \) is zero except for landing or take-off, and the term \( X_\theta \) accordingly left out in the Taylor expansion relevant to body axes. There is then a danger that the term in \( \theta \) would be omitted when earth axes are used, and such an expansion would not be correct. Even when \( \theta \) terms are included in the expansion relevant to earth axes, there is a danger of regarding \( \theta \) as representing an increment in the angle of attack, and in addition contributions due to the datum aerodynamic forces might be overlooked.

The remarks made above are similar to those given in Section 22, where choice of 'displacement' variables is shown to give rise to additional terms in the expressions for attitude derivatives, and to
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additional linear-displacement derivatives.

So far we have allowed the body axes to possess merely a small pitch deviation (represented by \( \theta \)) relative to the datum attitude (represented by \( \Theta \)). Now consider the more general relative orientation represented by \( \phi, \theta, \psi \), all small. It is found from the analysis given in Section 20.4 that the relationships established earlier in the present section are still valid provided that the datum values of \( v, p, r, \psi, \) etc. are all zero, as the additional terms are then of second order magnitude. With these restrictions we can establish the following relationships between lateral derivatives. Again some terms on the right hand sides are not expressed in earth-axes.

\[
\begin{align*}
Y_v &= Y_v^0 \\
L_v &= cL_v^0 - sN_v^0 \\
N_v &= cN_v^0 + sL_v^0 \\
Y_\phi &= Y_\phi^0 \\
L_\phi &= cL_\phi^0 - sN_\phi^0 \\
N_\phi &= cN_\phi^0 + sL_\phi^0 \\
Y_p &= cY_p^0 - sY_p^0 - wY_\phi \\
L_p &= c^2L_p^0 - sc(L_p^0 + N_p^0) + s^2N_p^0 - wL_\phi \\
N_p &= c^2N_p^0 + sc(L_p^0 + N_p^0) + s^2L_p^0 + uN_\phi \\
L_\theta &= c^2L_\theta^0 + sc(L_\theta^0 - N_\theta^0) - s^2N_\theta^0 + uL_\phi \\
N_\theta &= c^2N_\theta^0 + sc(L_\theta^0 + N_\theta^0) + s^2L_\theta^0 + uN_\phi \\
L_\psi &= c^2L_\psi^0 + sc(L_\psi^0 - N_\psi^0) - s^2N_\psi^0 + M + uL_\phi - qL_\psi \\
N_\psi &= c^2N_\psi^0 + sc(L_\psi^0 + N_\psi^0) - s^2L_\psi^0 - M - wN_\phi + qZ - wN_\theta \\
N_\phi^0 &= c^2N_\phi^0 - sc(L_\phi^0 - N_\phi^0) - s^2L_\phi^0 + M^0 + w_0N_\phi^0 + q_0Z_\phi^0 + \dot{w}_0N_\phi^0.
\end{align*}
\]

Inverse relationships can be deduced by replacing \( c, s \) with \( c, -s \) and changing the signs of other corresponding terms in the variables. For example,

\[
N_\phi^0 = c^2N_\phi - sc(L_\phi - N_\phi) - s^2L_\phi + M^0 + w_0N_\phi^0 + q_0Z_\phi^0 + \dot{w}_0N_\phi^0.
\]

When the datum values of \( v, p, q, r, \dot{u}, \dot{v}, \dot{w} \) are zero, and also \( \Theta = 0 \), the lateral derivatives are related as follows.
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\[(Y_{\psi}, Y_{\phi}, Y_{\theta}, Y_{\eta}) = (Y_{\psi}^0, Y_{\phi}^0, Y_{\theta}^0, Y_{\eta}^0),\]

and similarly for \((L_{\psi}, L_{\phi}, L_{\theta}, L_{\eta})\) and \((N_{\psi}, N_{\phi}, N_{\theta}, N_{\eta})\).

\[Y_{\psi} = Y_{\psi}^0 + Z_\psi - w_\psi Y_{\eta}\]
\[Y_{\phi} = Y_{\phi}^0 - X_\phi + u_\phi Y_{\eta}\]
\[L_{\phi} = L_{\phi}^0 - w_\phi L_{\eta}\]
\[N_{\phi} = N_{\phi}^0 + u_\phi N_{\eta}\]
\[L_{\eta} = L_{\eta}^0 + M_\eta + u_\eta L_{\phi}\]
\[N_{\eta} = N_{\eta}^0 - M_\eta - w_\eta N_{\phi}\]

As previously mentioned under longitudinal derivatives, axes are usually chosen such that \(w_\psi\) is zero.

In principle, cross derivatives \(X_{\psi}, X_{\phi}, X_{\theta}, X_{\eta}, X_{\phi}, X_{\theta}, M_{\theta}, Z_{\theta}, \ldots\) exist, and the conversion from earth axes to body axes is exactly according to the pattern for \(L_{\psi}, L_{\phi}, L_{\theta}, L_{\eta}, N_{\psi}, N_{\phi}, N_{\theta}, N_{\eta}, \ldots\). Similarly the pattern for \(L_{\psi}, L_{\phi}, L_{\theta}, L_{\eta}, N_{\psi}, N_{\phi}, N_{\theta}, N_{\eta}, \ldots\) is the same as that for \(X_{\psi}, X_{\phi}, X_{\theta}, X_{\eta}, X_{\phi}, X_{\theta}, M_{\theta}, Z_{\theta}, \ldots\).


Consider two systems of body axes 0xyz and 01x1y1z1. Components of the origin shift 001 may be taken either along 0xyz and denoted by \(h, j, k\), or along 01x1y1z1 and denoted by \(h_1, j_1, k_1\). In the notation of Appendix M, the corresponding column and row matrices are denoted by \([h] \equiv [h \ j \ k]\), \([h] \equiv [h \ j \ k], \) respectively, and likewise \([h_1]\) and \([h_1]\). The direction cosine matrix representing the orientation of 0xyz relative to 01x1y1z1 is written as \(S_1\), so that

\[\{h\} = S_1 \{h_1\}. \quad (20.1)\]

In terms of the Euler angles \(d, e, f\) this matrix is given by

\[S_1 = Y_{-f} P_{-e} R_{-d},\]

since successive rotations through \(-d, -e, -f\) would bring 01x1y1z1 into alignment with 0xyz. In the simple example of Section 20.1 the matrix is simply \(P_{-e}\) that is

\[
\begin{bmatrix}
\cos e & 0 & \sin e \\
0 & 1 & 0 \\
-\sin e & 0 & \cos e
\end{bmatrix}
\]

The components of angular velocity \(\{\rho\}\) and their time derivatives, and also components of force \(\{X\}\) are converted as in equation (20.1), that is

\[\{\rho\} = S_1 \{\rho_1\}, \quad (20.2)\]
\[\{X\} = S_1 \{X_1\}. \quad (20.3)\]

The relationship between the components \(p, q, r\) and \(p_1, q_1, r_1\), and between the others, can also be expressed in the forms
where $S_1^T$ is the transpose of $S_1$. Such simple relationships are not obtained for linear velocity components and moments because of the effects of origin shift.

Let $\{u\}$ represent the components along $Oxyz$ of the velocity of $0$, and let $\{u_1\}$ represent* the components along $0_1x_1y_1z_1$ of the velocity of $0_1$. Similarly let $\{L\}$ refer to moments about $0$ and $\{L^1\}$ to moments about $0_1$. We then have

\[
\{u\} = S_1 \{u_1\} + A_h S_1 \{p_1\} = S_1 \{u_1\} + S_1 A_h \{p_1\},
\]

\[
\{L\} = S_1 \{L^1\} + A_h S_1 \{X^1\} = S_1 \{L^1\} + S_1 A_h \{X^1\}.
\]

The inverse relations are obtained by multiplying by $S_1^T$, for example

\[
\{u_1\} = S_1^T \{u\} - A_h \{p_1\} = S_1^T \{u\} - A_h S_1^T \{p\} = S_1^T \{u\} - S_1^T A_h \{p\}.
\]

Alternative forms equivalent to (20.5) and (20.6) also exist, such as

\[
[L] = [L^1] S_1^T - [X^1] A_h S_1^T.
\]

The matrix $A_h$ is the anti-symmetric one based on $\{h\}$ as defined in Appendix M, and it is useful to remember that for any matrix $\{x\} = \{x_1 x_2 x_3\}$ we have

\[
A_h S_1 = S_1 A_h, \quad S_1^T A_h = A_h S_1^T, \quad S_1^T A_x = A_x S_1^T,
\]

provided that $\{x\} = S_1 \{x_1\}$.

We define differential operator matrices of the form

\[
\{D\} = \{D_u \quad D_v \quad D_w\},
\]

\[
= \left\{ \frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial}{\partial w} \right\},
\]

and it follows from the relations between $\{u\}$ and $\{u_1\}$, etc. that

*Here we omit a raised suffix. According to Appendix M we should write $u_1$, because $u_1$ usually denotes the component of the vector (i) along $0x_1$, so that $\{u\} = S_1 \{u_1\}$. See remarks in Section 20.1.
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\[
\{D_u\} = S^T \{D_u\}, \quad (20.9)
\]
\[
\{D_p\} = S^T \{D_p\} - S^T A_h \{D_u\}. \quad (20.10)
\]

Alternative forms are

\[
[D_u] = [D_u] S_1, \quad (20.11)
\]
\[
[D_p] = [D_p] S_1 + [D_u] S_1 A_h, \quad (20.12)
\]

First derivatives.

Conversion formulae for derivatives are obtained by applying equations (20.9), (20.10), (20.4), and (20.8). For example,

\[
\{D_u\} [X'] = S^T \{D_u\} [X] S_1, \quad (20.13)
\]

and this relation may be transformed into one that contains the derivatives \(X_u, \ Y_v, \text{ etc.}\) in direct fashion. Since

\[
\{D_u\} [X] \equiv [D_u X]
\]

\[
= \begin{bmatrix}
D_u X & D_u Y & D_u Z \\
D_v X & D_v Y & D_v Z \\
D_w X & D_w Y & D_w Z
\end{bmatrix},
\]

it is also identical with \([X_u]^T\) if we take \([X_u]\) to be the Jacobian matrix as defined in Appendix M, namely

\[
[X_u] = \begin{bmatrix}
X_u & X_v & X_w \\
Y_u & Y_v & Y_w \\
Z_u & Z_v & Z_w
\end{bmatrix}.
\]

We may therefore replace (20.13) by

\[
[X_u']^T = S^T [X_u]^T S_1,
\]

and, since for any matrices \(P = BCD \ldots W\), we have \(P^T = W^T \ldots D^T C^T B^T\), we can write

\[
[X_u'] = S^T [X_u] S_1. \quad (20.14)
\]

Similarly we can establish that

\[
[X_p'] = S^T [X_p] S_1 + S^T [X_u] S_1 A_h,
\]

\[
= S^T [X_u] S_1 + [X_u'] A_h, \quad (20.15)
\]
\[
[L_u'] = S^T [L_u] S_1 - S^T A_h [X_u] S_1
\]

\[
= S^T [L_u] S_1 - A_h \{X_u\}', \quad (20.16)
\]

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\[
[L_p^1] = S_1^T[L_p] S_1 + S_1^T[L_u] S_1 A_{h_i} - A_{h_i} [X_{p_i}]
= S_1^T[L_p] S_1 + [L_u^1] A_{h_i} - A_{h_i} S_1^T[L_x] S_1 .
\] (20.17)

The relations (20.14) to (20.17) are also valid if the variables \(u_1, u, p_1, p, \) etc. are replaced by \(\hat{u}_1, \hat{u}, \hat{p}_1, \hat{p}, \) etc.

Attitude derivatives might be expected to be related in complicated fashion, owing to the dependence of the transformation matrix \(S_1\) itself on the attitude angles. Since, however, these derivatives are expressed in terms of attitude-deviation angles rather than increments, and it can be shown* that for small deviations

\[
\{\phi\} = S_1 \{\phi_1\},
\]

the relations between \([X_0]\) and \([X_{\phi_1}^1]\) follow the same pattern as those between \([X_u]\) and \([X_{\phi_1}^1]\). We thus have

\[
[X_{\phi_1}^1] = S_1^T[X_0] S_1 ,
\] (20.18)

and similarly

\[
[L_{\phi_1}^1] = S_1^T[L_0] S_1 - A_{h_i} [X_{\phi_1}^1].
\] (20.19)

Derivatives with respect to other variables, such as \(h\) (height), \(\xi, \eta, \zeta,\) are more easily handled, as they require just a simple axes transformation as given in equations (20.3) and (20.6). For example,

\[
\{X_{\eta}\} \equiv \{X_{\eta} \ Y_{\eta} \ Z_{\eta}\}
= S_1 \{X_{\phi_1}^1\},
\]

and of course

\[
\{X_{\eta}^1\} = S_1^T \{X_{\eta}\}.
\]

Second derivatives.

It is not convenient to deal simultaneously with the derivatives of \(X, Y,\) and \(Z\) except when they are first-order ones. It is easier to make use of matrices such as \([D_{pu}]\), where

\[
[D_{pu}] \equiv [D_p D_u] \equiv \{D_p\} [D_u] \equiv
\begin{bmatrix}
D_{pu} & D_{pu} & D_{pu} \\
D_{qu} & D_{qu} & D_{qu} \\
D_{ru} & D_{ru} & D_{ru}
\end{bmatrix}
\]

*Datum and perturbed attitudes of the two axes systems are defined by \(\{\Phi_0\}, \{\Phi_{\phi_1}\},\) and \(\{\Phi\}, \{\Phi_1\},\) and deviation angles \(\{\phi\}, \{\phi_1\}\) by the equations

\[
S_0 = S_0 S_{\phi_1}, \quad S_{\phi_1} = S_{\phi_1} S_{\phi_1},
\]

where

\[
S_{\phi} = S_1 S_{\phi_1}, \quad \text{and} \quad S_{\phi_1} = S_1 S_{\phi_1}.
\]
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From the equations (20.9) to (20.12) we obtain the following conversion formulae.

\[ [D_{u1u1}] = S_1^T [D_{uu}] S_1 . \]  
(20.20)

\[ [D_{p1u1}] = S_1^T [D_{pu}] S_1 - S_1^T A_h [D_{uu}] S_1 , \]  
(20.21)

\[ [D_{p1p1}] = S_1^T [D_{pp}] S_1 + S_1^T [D_{pu}] A_h S_1 - 
-S_1^T A_h [D_{uu}] S_1 - S_1^T A_h [D_{uu}] A_h S_1 . \]  
(20.22)

Since \([D_{up}] = [D_{pu}]\), the last expression may also be written as

\[ S_1^T [D_{pp}] S_1 + S_1^T (B + B^T) S_1 - S_1^T A_h [D_{uu}] A_h S_1 , \]

or

\[ S_1^T [D_{pp}] S_1 + S_1^T (B + B^T) S_1 - A_{h1} [D_{u1u1}] A_{h1} , \]

where

\[ B = [D_{pu}] A_h . \]

Matrices such as \([Z_{u1u1}]\), \([Z_{p1u1}]\), \([M_{u1u1}]\), etc., where

\[ [Z_{pu}] = \begin{bmatrix} Z_{pu} & Z_{pc} & Z_{pw} \\ Z_{qu} & Z_{qv} & Z_{qw} \\ Z_{ru} & Z_{rv} & Z_{rw} \end{bmatrix} , \]

are obtained from (20.20) to (20.22) merely by replacing \(D\) with \(Z\) or \(M\), etc.

Since second derivatives are not often used, and it is easy to express \(Z^1\), \(M^1\), etc. in terms of \(X\), \(Y\), \(Z\), \(L\), \(M\), \(N\) by means of equations (20.3) and (20.6), it does not seem necessary to erect a fabric of tensor notation in order to achieve symbolical simplicity in relating the whole set \(X_{u1u1} , Y_{u1u1} , Z_{u1u1} , X_{u1u1} , \ldots, X_{u1u1} , \ldots, \) and so on.

For second derivatives involving \(\phi, \theta, \psi\) we can establish the following relations.

\[ [D_{\phi\psi\phi}] = S_1^T [D_{\phi\phi}] S_1 \] and \([D_{u1\phi}] = S_1^T [D_{u\phi}] S_1 , \]

but

\[ [D_{p1\phi}] = S_1^T [D_{p\phi}] S_1 - S_1^T A_h [D_{u\phi}] S_1 . \]

Second derivatives involving other variables such as \(h\) (height), \(\eta\), can be handled like first derivatives. Thus, if we write

\[ [X_{u\eta}] = \begin{bmatrix} X_{u\eta} & X_{v\eta} & X_{w\eta} \\ Y_{u\eta} & Y_{v\eta} & Y_{w\eta} \\ Z_{u\eta} & Z_{v\eta} & Z_{w\eta} \end{bmatrix} , \text{ etc.,} \]
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the relation between \([X_{x1}]\) and \([-X_{j}]\), between \([X_{p1}]\) and \([X_{m}]\), and so on, will have exactly the same pattern as the relations (20.14) to (20.17), which are valid when \(\eta\) is omitted. Derivatives like \(X_{n}\) are treated as \(X\) itself; and moment derivatives like \(M_{n}\) are treated as \(M\).


Consider a system of body axes \(Oxyz\) and a system of earth axes \(Ox_0y_0z_0\) having a common origin. As before we write

\[ \{u\} = S\{u_0\}, \quad (20.23) \]

where \(S\) is the attitude matrix introduced in Appendix M, and usually expressed in the form \(S_\theta\) as given in Section 5.6. Other quantities such as \(\{p\}, \{X\}, \{L\}\) are transformed in the same way as \(\{u\}\) because the origin is common. When, however, we consider rates of change such as \(\{\dot{u}\}\), we must remember that \(S\) is not zero, being given by

\[ \dot{S} = -A_pS = -SA_{p_0}. \]

We thus have

\[ \{\dot{u}\} = S\{\dot{u}_0\} - A_pS\{u_0\} \]
\[ = S\{\dot{u}_0\} - SA_{p_0}\{u_0\} \quad (20.24) \]
\[ = S\{\dot{u}_0\} + SA_{m}\{p_0\}. \quad (20.24a) \]

Since \(A_{p_0}\{p_0\}\) is identically zero the form of \(\{\dot{p}\}\) is still simple:

\[ \{\dot{p}\} = S\{\dot{p}_0\}. \quad (20.25) \]

Moreover, as explained in Section 20.2, attitude derivatives must be taken into account even when the aerodynamic forces and moments do not physically depend on attitude. As in Section 20.3 we consider these derivatives in terms of the appropriate deviation angles \(\phi, \theta, \psi, \phi_0, \theta_0, \psi_0\), which are taken to be small and therefore related by the equation

\[ \phi = S\{\phi_0\}. \]

For a perturbation in attitude the increment in \(S\) relative to the datum-attitude condition given by \(S_e\) is equal to

\[ S' = -A_pS_e. \]

From equations (20.23) to (20.25) we then derive the following relations between increments.

\[ \{u'\} = S\{u'_0\} + A_pS\{\phi_0\} \]
\[ \{p'\} = S\{p'_0\} + A_pS\{\phi_0\} \]
\[ \{\dot{u}'\} = S\{\dot{u}'_0\} - A_pS\{u'_0\} + A_pS\{p'_0\} + A_pS\{\phi_0\} \]

The associated differential operators therefore satisfy the relations
Section 20.4

\[
\begin{align*}
\{D_{m}\} &= S^T\{D_{u}\} + A_p\{D_{u}\}, \\
\{D_{p}\} &= S^T\{D_{p}\} - A_u\{D_{u}\}, \\
\{D_{a}\} &= S^T\{D_{a}\}, \\
\{D_{p0}\} &= S^T\{D_{p}\}, \\
\{D_{\phi0}\} &= S^T\{D_{\phi}\} - A_u\{D_{u}\} - A_p\{D_{p}\} - A_a\{D_{a}\} - \ldots. 
\end{align*}
\]

(20.26)

Alternative row matrix forms are

\[
\begin{align*}
[D_{\phi0}] &= ([D_{\phi}] + [D_{a}] A_u \ldots) S, \\
[D_{a0}] &= ([D_{a}] - [D_{a}] A_p) S,
\end{align*}
\]

(20.27)

For all derivatives other than attitude derivatives a procedure analogous to that of the previous Section can be followed, and the resulting conversion formulae for first derivatives are given below.

\[
\begin{align*}
[X_{m0}] &= S^T[X_{a}] S - S^T[X_{a}] A_p S \\
&= S^T[X_{a}] S - S^T[X_{a}] S A_{p0}, \\
[X_{p0}] &= S^T[X_{p}] S + S^T[X_{a}] A_u S \\
&= S^T[X_{p}] S + S^T[X_{a}] S A_{u0}, \\
[X_{a0}] &= S^T[X_{a}] S \\
[X_{p0}] &= S^T[X_{p}] S \\
\end{align*}
\]

(20.28)

and so on.

All quantities on the right hand sides should be evaluated for the datum conditions, and the conversion formulae for moment derivatives are obtained by replacing \(X\) with \(L\). Premultiplying by \(S\) and post-multiplying by \(S^T\) will yield inverse relationships. Thus

\[
S[X_{m0}] S^T = [X_{a}] - [X_{a}] A_p = [X_{a}] - [X_{a}] S A_{p0} S^T,
\]

and since

\[
S[X_{a0}] S^T = [X_{a}],
\]

then

\[
[X_{a}] = S[X_{a0}] S^T + S[X_{m0}] A_{p0} S^T.
\]

For attitude derivatives we have
Section 20.4

\[
[X^0_\phi]^T = \{D_\phi\} [X^0] = \{D_\phi\} [X] S
\]
\[
= ST \left( \{D_\phi\} [X] \right) S + ST \left[ \left[ [l_{i\phi}]^T \{X\} \right]_{i=1,2,3} \right] - \sum_{u,p} ST \{D_u\} [X] S,
\]
(20.29)

where

\[
[l_{i\phi}] \equiv \begin{bmatrix} l_\phi & l_\theta & l_\psi \\ m_{i\phi} & m_\theta & m_\psi \\ n_{i\phi} & n_\theta & n_\psi \end{bmatrix},
\]

and \(l_{i\phi} \equiv D_\phi l_i\). To express \([l_{i\phi}]\) in the simplest form, we consider the \(i\)th column of \(S\'), namely \(-A_\phi \{l_i\}\), which is equal to \(A_{ii} \{\phi\}\). Since

\[
S' = \frac{\partial S}{\partial \phi} \phi + \frac{\partial S}{\partial \theta} \theta + \frac{\partial S}{\partial \psi} \psi
\]

\[
= \left[ \left[ [l_{i\phi}] \{\phi\} \right]_{i=1,2,3} \right],
\]

the \(i\)th column is also given by \([l_{i\phi}] \{\phi\}\), and since the expressions are valid for any values of \(\phi, \theta, \psi\), then

\[
[l_{i\phi}] = A_{ii}.
\]

Taking the transpose of both sides of (20.29), and using the identities

\[
A_{ii} \{l_2\} \equiv \{l_3\}, \quad A_{ij} \{l_3\} \equiv \{l_1\}, \quad A_{ij} \{l_1\} \equiv \{l_2\},
\]

we finally have

\[
[X^0_\phi]^T = ST[X_\phi] S - ST A_\phi S + \sum_{u,p} ST[X_\phi] A_u S.
\]
(20.30)

Since \(\{X\} = S\{X^0\}\), the second term on the right hand side is equal to \(-A_{\chi_0}\). The relation between derivatives of \(L, M, N\) has the same form, and \(X\) in equation (20.30) can be replaced with \(L\).

The inverse of (20.30) is obtained as outlined earlier, but care should be taken in substituting for \(A_\phi\) as

\[
\{\dot{u}\} = S\{\dot{u}_0\} - A_p \{u\}
\]

and hence

\[
A_\phi = SA_{\phi 0} S^T - A_p A_u + A_u A_p.
\]

It is found that
Section 20.4

\[ [X_{\phi}] = S[X_{\phi_0}] S^T + S A_{\chi_0} S^T - \sum_{u_0, \rho_0, \ldots} S[X_{w_u}] A_{w_0} S^T. \]  

(20.31)

For the simple case treated in Section 20.2 the attitude matrix \( S \) becomes

\[
\begin{pmatrix}
\cos \Theta & 0 & -\sin \Theta \\
0 & 1 & 0 \\
\sin \Theta & 0 & \cos \Theta
\end{pmatrix},
\]

and \( A_{\chi_0} \) is given by

\[
\begin{pmatrix}
0 & X \sin \Theta - Z \cos \Theta & Y \\
Z \cos \Theta - X \sin \Theta & 0 & -(X \cos \Theta + Z \sin \Theta) \\
-Y & X \cos \Theta + Z \sin \Theta & 0
\end{pmatrix}.
\]

As always, quantities in equations (20.30) and (20.31) should be given values for the datum conditions.

First derivatives with respect to variables like \( \eta \) are treated as the force components themselves, so that for example

\[
\{X^0_\eta\} = \{X^0_\eta\} = S^T \{X_\eta\}.
\]

Second derivatives may also be developed as in the previous section by means of the relations (20.26) and (20.27). For example

\[
[D_{w_0 w_0}] = [D_{w_0}] [D_{w_0}]
\]

\[
= S^T [D_{w_0}] S + S^T A_p [D_{w_0}] S - S^T [D_{w_0}] A_p S - S^T A_p [D_{w_0}] A_p S,
\]

whereas \([X^0_{w_0}]\) is treated like \([X^0_{w_0}]\).

The first-order relationships (20.26) and (20.27) are easily inverted by premultiplying with \( S \) and postmultiplying with \( S^T \), respectively, and inverted second-order relationships can then be determined by direct substitution in expressions such as \([D_{w_0}][D_{w_0}]\).


Expressions are given in Section 19 for aero-normalised force- and moment-derivatives \((\dot{X}_w, \dot{M}_w, \ldots)\) in terms of corresponding aerodynamic-coefficient derivatives \((\partial C_{xw}/\partial M, C_{mz}, \ldots)\). If data are available in the form of coefficient derivatives for air-path axes (for example, \(C_{x\alpha}, C_{\beta\alpha}, C_{\beta\beta}, \ldots\)), we need the relations between the two sets of coefficient derivatives. It will be shown that it is enough to find the relations between \{\(X_\gamma\)\} and \{\(X^0_\eta\)\}, where \(\gamma\) represents any one of the normalised variables \(M, R, \alpha, \beta, V, \gamma, \dot{\alpha}, \dot{\beta}, \ldots\), introduced in Section 15.

The force and moment components along body axes and air-path axes are related through the incidence matrix:
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\[ \{X\} = S_a\{X_a\} = S_a\{X^\alpha\}, \]

and

\[ \{\mathcal{L}\} = S_a\{\mathcal{L}_a\} = S_a\{\mathcal{L}_a^\alpha\}. \]  

(21.1)

Since the incidence matrix is a function of \( \alpha \) and \( \beta \) only, partial derivatives with respect to any \( \gamma \) variable other than \( \alpha \) or \( \beta \) are related just like the forces and moments themselves:

\[ \{X_\gamma\} = S_a\{X^\alpha_\gamma\}, \]  

(21.2)

and so on. But for derivatives with respect to \( \alpha \) we must write

\[ \{X_a\} = S_a\{X^\alpha_a\} + \frac{\partial S_a}{\partial \alpha}\{X^\alpha_a\}, \]

(21.3)

and similarly for \( \beta \) derivatives.

It is clear that force and moment coefficients for body axes and air-path axes are also related by equations of the form (21.1):

\[ \{C_X\} = S_a\{C_X^\alpha\}, \]

\[ \{C_\gamma\} = S_a\{C_\gamma^\alpha\}. \]

It follows that coefficient derivatives are related in the manner of (21.2) and (21.3), namely that

\[ \{C_{X_\gamma}\} = S_a\{C_{X_\gamma}^\alpha\}, \]

when \( \gamma \) does not stand for \( \alpha \) or \( \beta \), and so on. We thus have the necessary machinery for relating all coefficient derivatives (for forces and moments) embodied in equations (21.2) and (21.3) with its counterpart for \( \beta \).

When \( \alpha, \beta_s \) are used, we have

\[
S_a = \begin{bmatrix}
\cos \sigma & -\cos \sigma \tan \beta_s & -\sin \alpha \sec \beta_s \\
\sin \beta_s & \cos \beta_s & 0 \\
\sin \alpha_s & -\sin \alpha_s \tan \beta_s & \cos \sigma \sec \beta_s
\end{bmatrix},
\]

where \( \sin^2 \sigma = \sin^2 \alpha_s + \sin^2 \beta_s \), and hence

\[
\frac{\partial S_a}{\partial \alpha_s} = \cos \alpha_s \begin{bmatrix}
-\sec \sigma \sin \alpha_s & \sec \sigma \sin \alpha_s \tan \beta_s & -\sec \beta_s \\
0 & 0 & 0 \\
1 & -\tan \beta_s & -\sec \sigma \sin \alpha \sec \beta_s
\end{bmatrix},
\]

\[
\frac{\partial S_a}{\partial \beta_s} = \begin{bmatrix}
-\sec \sigma \sin \beta_s \cos \beta_s & -(\sec \sigma \sin^2 \beta_s) & -\sin \alpha_s \sec \beta_s \tan \beta_s \\
\cos \beta_s & -\sin \beta_s & 0 \\
0 & -\sin \alpha_s \sec \beta_s & (\cos \sigma \sec \beta_s \tan \beta_s)
\end{bmatrix}.
\]

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When, on the other hand, \( t \) and \( \theta \) are used, we have

\[
S_a = \begin{bmatrix}
\cos \alpha \cos \beta_a & -\cos \alpha \sin \beta_a & -\sin \alpha \\
\sin \beta_a & \cos \beta_a & 0 \\
\sin \alpha \cos \beta_a & -\sin \alpha \sin \beta_a & \cos \alpha 
\end{bmatrix}
\]

and hence

\[
\frac{\partial S_a}{\partial \alpha} = \begin{bmatrix}
-\sin \alpha \cos \beta_a & \sin \alpha \sin \beta_a & -\cos \alpha \\
0 & 0 & 0 \\
\cos \alpha \cos \beta_a & -\cos \alpha \sin \beta_a & -\sin \alpha 
\end{bmatrix}
\]

\[
\frac{\partial S_a}{\partial \beta_a} = \begin{bmatrix}
-\cos \alpha \sin \beta_a & -\cos \alpha \cos \beta_a & 0 \\
\cos \beta_a & -\sin \beta_a & 0 \\
-\sin \alpha \sin \beta_a & -\sin \alpha \cos \beta_a & 0 
\end{bmatrix}
\]

When \( \beta_a = 0 \), \( \alpha \) and \( \alpha \) are the same and the matrices reduce to

\[
S_a = \begin{bmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha 
\end{bmatrix}
\]

\[
\frac{\partial S_a}{\partial \alpha} = \begin{bmatrix}
-\sin \alpha & 0 & -\cos \alpha \\
0 & 0 & 0 \\
\cos \alpha & 0 & -\sin \alpha 
\end{bmatrix}
\]

\[
\frac{\partial S_a}{\partial \beta} = \begin{bmatrix}
0 & -\cos \alpha & 0 \\
1 & 0 & 0 \\
0 & -\sin \alpha & 0 
\end{bmatrix}
\]

and hence

\[
X_a = (X_a - Z^a) \cos \alpha - (Z_a + X^a) \sin \alpha , \\
Y_a = Y_a^a , \\
Z_a = (Z_a + X^a) \cos \alpha + (X_a - Z^a) \sin \alpha ,
\]

\[
(21.4)
\]
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\[
\begin{align*}
X_\beta &= (X^a_\beta - Y^a) \cos \alpha - Z^a_\beta \sin \alpha, \\
Y_\beta &= Y^a_\beta + X^a, \\
Z_\beta &= Z^a_\beta \cos \alpha + (X^a - Y^a) \sin \alpha.
\end{align*}
\]

(21.5)

If derivatives of the lift, drag, and thrust are employed, we substitute

\[
\begin{align*}
X^a &= X_a = X_a^T + X_a^A = X_a^T - D, \\
Y^a &= Y_a = Y_a^T + Y_a^T, \\
Z^a &= Z_a = Z_a^T + Z_a^A = Z_a^T - L,
\end{align*}
\]

where \(X_a^T, Y_a^T, Z_a^T\) are the components of thrust along air-path axes. The thrust, however, is usually known with reference to body axes, and \(Y_a^T\) is usually negligible. Equation (21.3) is then more conveniently written as

\[
\begin{bmatrix}
X_a \\
Y_a \\
Z_a
\end{bmatrix} =
\begin{bmatrix}
X_a^T \\
0 \\
Z_a^T
\end{bmatrix} -
S_a
\begin{bmatrix}
D_a \\
-Y_a^a \\
L_a
\end{bmatrix} -
\frac{\partial S_a}{\partial \alpha}
\begin{bmatrix}
D \\
-Y^a \\
L
\end{bmatrix}
\]

22. Relations between Derivatives of Forces and Moments with Respect to Different Sets of Variables.

In Section 17.2 it was explained that a force or moment may be expressed as a function of an \(\omega\) set of variables \((x_0, y_0, z_0; \Phi, \Theta, \Psi, h, u, v, w, p, q, r, \zeta, \xi, \ldots)\) or as a function of an \(\Omega\) set \((x_0, y_0, z_0; \Phi, \Theta, \Psi, h, x_0, y_0, z_0, \Phi, \Theta, \Psi, \xi, \zeta, \ldots)\). The Taylor expansion for the \(\omega\) set was in terms of the perturbations \(\phi, \theta, \Psi, h, u', v', w', p', q', r', \zeta', \xi', \ldots\) but the expansion for the \(\Omega\) set was in terms of perturbations \(x^+, y^+, z^+, \phi, \theta, \Psi, h', x', y', z', \phi', \theta', \Psi', h', x', y', z', \ldots\). The relations between the derivatives \(Z_\omega, Z_\Theta, Z_\Phi, \ldots\) and \(Z_x, Z_\phi, Z_\theta, Z_\Psi, Z_h, Z_x, Z_\phi, \ldots\) are obtained by applying the formula

\[
\frac{\partial Z}{\partial \Omega} = \sum \frac{\partial Z}{\partial \omega_i} \frac{\partial \omega_i}{\partial \Omega},
\]

where \(\omega\) and \(\Omega\) denote any variables in the respective sets. We therefore have to establish the relations between the two sets of perturbations. The variables \(h, \xi, \ldots\) are basically independent of the variables describing the kinematics of the aircraft as a whole, and the derivatives \(Z_\omega, Z_\Theta, \ldots\), etc. therefore the same in the two sets. It should be remembered that \(\xi\) is included in order to cover the effects of a non-uniform atmosphere (see Section 17.2).

For the other variables it can be shown (see Appendix M) that to the first order of small perturbations

\[
\begin{align*}
u' &= \dot{x}^+ + v_e \dot{\psi} - w_e \dot{\phi} + q_e z^+ - r_e y^+, \\
v' &= \dot{y}^+ + w_e \dot{\phi} - u_e \dot{\psi} - p_e z^+ - r_e x^+, \\
w' &= \dot{z}^+ + u_e \dot{\phi} + p_e y^+ - q_e x^+,
\end{align*}
\]

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\[ p' = \dot{p} + q_\phi \psi - r \theta, \]
\[ q' = \dot{q} + r \phi - p \psi, \]
\[ r' = \dot{r} + p \theta - q \phi. \]

From these we can obtain the relations given below. It should be noted that, purely for exposing the technique involved, the assumption is made that the derivatives are to be used in a problem where the dependence of the forces and moments on \( x_0, y_0, z_0 \) is too complicated for these to be represented by terms in the Taylor expansion. On the other hand, the variables \( \Phi, \Theta, \Psi \) are taken care of in the expansions, and the treatment is thus completely in accord with that given in Section 17.2, where it is made clear that these different assumptions about the linear and angular displacements are arbitrary, and introduced only to clarify the interpretation of the notation.

\[ Z_x = r_e Z_v - q_e Z_w, \]
\[ Z_y = p_e Z_w - r_e Z_u, \]
\[ Z_z = q_e Z_u - p_e Z_v, \]
\[ (Z_\phi)_\Omega = (Z_\phi)_\omega + w_e Z_v - v_e Z_w + r_e Z_q - q_e Z_r, \]
\[ (Z_\theta)_\Omega = (Z_\theta)_\omega + u_e Z_w - w_e Z_u + p_e Z_r - r_e Z_p, \]
\[ (Z_\psi)_\Omega = (Z_\psi)_\omega + v_e Z_u - u_e Z_v + q_e Z_p - p_e Z_q, \]
\[ Z_\phi = Z_\phi + r_e Z_v - q_e Z_w, \]
\[ Z_\theta = Z_\theta + p_e Z_w - r_e Z_u, \]
\[ Z_\psi = Z_\psi + q_e Z_u - p_e Z_v, \]
\[ Z_\phi = (Z_\phi)_\omega + w_e Z_v - v_e Z_w + r_e Z_q - q_e Z_r, \]
\[ Z_\theta = (Z_\theta)_\omega + u_e Z_w - w_e Z_u + p_e Z_r - r_e Z_p, \]
\[ Z_\psi = (Z_\psi)_\omega + v_e Z_u - u_e Z_v + q_e Z_p - p_e Z_q, \]
and so on.

Some insight into the interdependence of the variables and the associated derivatives can be gained by comparing the expansions for a force or moment in the two sets, \( \omega \) and \( \Omega \). Consider a case where the derivatives \( Z_\phi, Z_\theta, Z_\psi, Z_\xi, Z_\eta, Z_\zeta \) are zero, and likewise all derivatives with respect to \( \dot{\xi}, \dot{\eta}, \dot{\zeta} \), etc. and higher order derivatives. In addition suppose that the forces and moments are independent of \( x_0, y_0, z_0 \) and \( \Phi, \Theta, \Psi \). The expansions for \( Z \) are then written (see (17.4) and (17.6)):
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\[ Z = Z_d + Z_hh' + Z_uu' + Z_ww' + Z_\phi \phi' + \]
\[ + Z_{p\theta}' + Z_{q\phi}' + Z_{r\phi}' + Z_{\phi\phi}' , \]
\[ Z = Z_d + Z_hh' + Z_xx^+ + Z_yy^+ + Z_zz^+ + \]
\[ + Z_\phi \phi^+ + Z_\theta \theta^+ + Z_\psi \psi^+ + \]
\[ + Z_\phi \dot{\phi} + Z_\theta \dot{\theta} + Z_\psi \dot{\psi} + Z_{\phi\phi}' . \]

Here \( Z_\phi, Z_\theta, Z_\psi \) denote \((Z_\phi)_o\), etc. since \((Z_\phi)_o, \) etc. are zero.

It is natural to regard \( x^+, \phi \) roughly as substitutes for \( u', p' \), and it is somewhat disconcerting to find additional terms \( Z_xx^+, Z_\phi \phi \), etc. in the second form of expansion when we have specifically excluded any dependence of \( Z \) on the displacements. These terms are in a sense spurious, since they are present merely when the displacement set of variables is chosen rather than the velocity set, and not because of the physical assumptions. When the forces and moments really do not depend on the displacements, equations of motion in terms of the displacement set of variables are unnecessarily complicated, as shown by stability polynomials of higher degree having some coefficients which turn out to be zero.

In the more general case, and with the assumptions stated earlier in this Section, all the derivatives (including \( Z_x \), etc.) are functions of \( x_o, y_o, z_o \). In the event that the ground effect might be simple enough to represent in terms of derivatives with respect to \( x_o, y_o, z_o \), the increment in \( Z \) due to perturbations in displacement would consist of two parts: one would be the sum of the genuine ground effect terms \((Z_{x_o} x_o, \) etc.), and the other would consist of the 'spurious' terms already discussed. This treatment is of course illustrated by the angular displacement terms: the increments due to genuine ground effect are \((Z_\phi)_o \phi \), etc. and the other terms are \((w_\varphi Z_\varphi - v_\varphi Z_\psi + r_\varphi Z_\theta - q_\varphi Z_\phi \phi \), etc.

It would be possible formally to transform \( Z_x x^+ \) terms into \( x_o^+, y_o^+, \) \( z_o^+ \) terms by means of the matrix relation \( \{x^+\} = S\{x_o^+\} \), but there seems to be no advantage since \( x_o^+ \) and \( x_o \) are quite different quantities, namely \( (x^+)_o \) and \( (x_o)_o \). The same considerations do not apply to the attitude terms, since perturbations in attitude are interpreted more conveniently in terms of \( \phi, \theta, \psi \) than \( \Phi', \Theta', \Psi' \), and both the genuine and spurious incremental terms would be expressed as \( \phi, \theta, \psi \) terms.

The remarks about genuine and spurious terms are similar to those made in Section 20.2, where it is shown that additional terms appear in the expressions for attitude derivatives when earth axes are in use.

Acknowledgements.

This Report includes many ideas arising out of earlier deliberations of an R.A.E. Panel (C. H. E. Warren, A. K. Weaver, S. Neumark, H. R. Hopkin) and out of more recent discussions with H. H. B. M. Thomas and others.
Oxyz and $O_1x_1y_1z_1$ are body axes
$Ox_0y_0z_0$ are earth axes

Fig. 5. Relative positions of earth axes and two systems of body axes.