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STRESS DISTRIBUTION AROUND HOLES

by G. N. Savin

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"Naukova Dumka" Press
Kiev, 1968

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STRESS DISTRIBUTION AROUND HOLES

By G. N. Savin

Translation of "Raspredeleniye Napryazheniy Okolo Otverstiy."
"Naukova Dumka" Press, Kiev, 1968

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FOREWORD

G. N. Savin's monograph *Stress Distribution Around Holes* is a significant development of his monograph *Concentration of Stress Around Holes* which appeared in 1951 and was highly valued in its time. This monograph has been widely distributed and has been translated into several foreign languages.

Regardless of the powerful computer technology now available which allows numerical solution of many problems, including problems of elasticity theory, the development of methods of approximate solution for individual classes of problems which would provide solutions in the form of certain relatively simple analytic expressions is very important.

The present monograph prefers the construction of solutions (approximate) of different classes of problems of elasticity theory which reduce to simple formulas for calculating stress and transfer components.

As is known, the results on solving problems of the plane theory of elasticity using the methods of complex-variable function theory, regardless of their effectiveness, had initially a more theoretical nature, therefore it became essential to give these results a form required by engineering practice.

G. N. Savin's great contribution is that he has filled this gap to a considerable degree through his work. The research results cited in the present monograph are of interest in the above sense as well as theoretically. The author's comparison of theoretical solutions with experimental data available in the world literature is especially valuable.

There is no doubt that G. N. Savin's monograph will be very useful and will be warmly received by readers, especially those working in the area of applications.

Academician N. I. Muskhelishvili

15 July 1966
Tbilisi
Mathematical Institute of the Academy of Sciences Georgian SSR

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FROM THE AUTHOR

The present monograph is a continuation and further expansion of the book *Concentration of Stress Around Holes* published by the State Publishing House of Technical-Theoretical Literature in 1951. During the following 15 years a great deal of attention was devoted to the problem of stress concentration around holes: new problems have been posed and effective solutions have been obtained both for new classes of problems (nonlinear problems, stresses around holes in envelopes and others), as well as for problems which were formulated earlier, in particular, for multiply-connected regions of an isotropic and anisotropic media.

The monograph includes 12 chapters and an appendix; it contains primarily the results of research of the author and his students done during the past 15 years, but it, just as the preceding work, includes the most important and interesting results obtained by other authors in this specialization.

The monograph *Concentration of Stress Around Holes* consisted of eight chapters. Many problems were solved by the same method, namely, by reduction to finding two analytical functions of a complex variable -- the complex potentials of G. V. Kolosov and N. I. Muskhelishvili with further application (in most cases) of conformal mappings of the outside or inside of an identical circle onto the regions under consideration and of Cauchy-type integrals (or Schwartz formulas), since single-connected finite or infinite regions were primarily investigated.

In connection with the considerable expansion of the classes of problems examined in the present monograph it was not possible to hold to the former unification of methods of solving the problems cited. It was necessary to use three methods to solve these problems:

1) the method of complex Kolosov-Muskhelishvili potentials and Cauchy-type integrals or Shwartz formulas; this method was used to solve the problems examined in Chapters II, III, IV (§1 and 2), V, VII and VIII;

2) the method of perturbation of the form of boundaries for an isotropic medium (when the problem reduces to solving a non-biharmonic equation) and for an anisotropic medium (when the hole is not elliptical); this method was used to solve problems which comprised the content of Chapters III, IV (§3), VI, X and XI;

3) the method based on V. Voltaire's principle, for solving of the influence of the tensile-elastic properties of material on stress concentration around holes (Chapter XII).

Of the 12 chapters of the present monograph, Chapters IV, VI, VII, VIII, IX, X, XI and XII are completely new (in comparison with the former work), and Chapters II, III and V have been essentially redone.

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The book contains the most important results of research on the concentration of stresses around holes and fissures in a form convenient for application and allows making conclusions, sometimes without going to great theoretical depth. The monograph provides an acquaintance with the methods of solving problems of the concentration of stresses around holes, and also with the basic literature on this problem. It is designed so that all chapters can be read independently. Many research results have not been cited with the completeness with which they should be explained. The primary obstacle to this was the unreasonable size of the book. We have tried to refer as completely as possible to original works, a list of which is given in alphabetic order at the end of each chapter. The most interesting results of research which, in our opinion, can be applied directly in engineering practice or help to master the complex picture of the stressed state around a hole under consideration and to establish the influence of some factors (rounding the corners of holes, distance between holes, rigidness of a fastening ring and others), are given in the form of graphs, tables and so forth.

A great deal of help in choosing material for the corresponding chapters of the present monograph, and critical analysis and verification of the solutions of their own and those of other authors and compilation of graphs and tables were provided by my students -- Doctor of Technical Sciences A. N. Guz', G. A. Van Fo Fy, A. G. Ugodchikov, D. V. Grilitskiy, V. V. Panasyuk, I. A. Prusov, N. P. Fleyshman, Yu. A. Shevlyakov, Ya. S. Podstrigach, Candidates of Physical-Mathematical Sciences Ya. F. Kayuk, A. A. Kaminskiy, Yu. I. Koyfman, V. D. Kubenko, Yu. N. Nemish, Yu. N. Podil'chuk, I. Yu. Khoma, L. P. Khoroshun, I. A. Tsurpal, for which I express my deep thanks to them.

I consider it my pleasant duty to express my deep thanks to Doctor of Technical Sciences Professor Ye. F. Burmistrov, Doctor of Physical-Mathematical Sciences Professor I. I. Vorovich, Doctor of Physical-Mathematical Sciences Professor D. D. Ivlev, Corresponding Member of AS UkrSSR Doctor of Technical Sciences Professor A. S. Kosmodamianskiy, Doctor of Physical-Mathematical Sciences Professor L. A. Tolokonnikov and Candidate of Physical-Mathematical Sciences Docent V. G. Gromov, who expressed a number of critical remarks in reading the corresponding chapters, which promoted improving the book to a considerable degree.

A great deal of help was rendered in the huge work associated with preparing the manuscript for press by engineers of the Department of Rheology of the Institute of Mechanics of the AS UkrSSR I. O. Guberman, Ts. B. Pinskaya, I. Yu. Babich, Technicians S. S. Kirichenko, N. A. Chaykun, S. G. Tsegel'n and other workers and aspirants of the department. I take the opportunity to express to them my deep recognition of the work they have done.

G. N. Savin

25 January 1966
Kiev
Institute of Mechanics of the AN UkrSSR,
Department of Rheology

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STRESS DISTRIBUTION AROUND HOLES

G. N. Savin

CHAPTER I. BASIC EQUATIONS OF THE PLANE PROBLEM OF ELASTICITY THEORY

ABSTRACT. This chapter deals with the principal equations of the plane problem of the theory of elasticity in a linear formulation, valid both for the isotropic and anisotropic elastic media. The method is presented for solving the principal plane boundary value problems for the simple connected domains. Some ways are shown for obtaining the mapping functions of the exterior (interior) of a unit circle on the exterior of a curvilinear hole whose contour is an arbitrary smooth curved line.

§1. The Plane Problem of the Linear Elasticity Theory of an Isotropic Medium[1]

The plane problem of the theory of elasticity of an isotropic body, as is known, combines two physically different problems: the plane stressed state and the plane deformation. /9*

The stressed state in any point of an elastic body for the plane problem is fully defined by three stresses -- σ_x, σ_y, τ_{xy}, which in the absence of volumetric forces satisfies the two equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{I.1}$$

and the compatibility condition

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{I.2}$$

The problem reduces to the integration of these equations with defined boundary conditions.

Two basic problems are isolated depending upon what is assigned on contour L of region S occupied by the body.

---

[1]For more detail see N. I. Muskhelishvili's monograph [1] (here and below asterisks by a word denote references, and in all other cases -- indices).

*Numbers in the margin indicate pagination in the foreign text.

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For the first basic problem, i.e., when external forces X_n and Y_n are given on contour L of region S, of the boundary, conditions may be written in the form

$$\sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) = X_n,$$
$$\tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) = Y_n. \tag{I.3}$$

Here n is the outward normal to contour L.

For the second basic problem, i.e., when displacements are assigned on contour L of region S, the boundary conditions take on the form

$$u = g_1(s), \quad v = g_2(s), \tag{I.4}$$

where $g_1(s)$ and $g_2(s)$ are assigned shifts of points of contour L which are given functions of the arc s of the contour, taken from an arbitrary point.

In addition to these two basic problems there are different variations of combined-type problems. We shall encounter below the simplest of these problems, usually called the third problem, namely when stress is placed upon one part of contour L of region S and displacement is given on the rest of this contour.

Equation systems (I.1) and (I.2) can be reduced to a single biharmonic equation /10

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0, \tag{I.5}$$

by introducing the stress function U(x, y), which is related to the stresses by the relations

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{I.6}$$

It can be shown here that the contour conditions for the function U(x, y), in the case of the first basic problem, acquire the form

$$\frac{\partial U}{\partial y} = \int_0^s X_n ds + C_1, \quad \frac{\partial U}{\partial x} = -\int_0^s Y_n ds + C_2, \tag{I.7}$$

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where C_1 and C_2 are two arbitrary real constants, which for the one-connected region, can be assumed to be equal to zero.

Thus the planar problem of the theory of elasticity reduced to the determination of the biharmonic function U(x, y) which satisfies contour conditions (I.4) or (I.7).

N. I. Muskhelishvili demonstrated [1] that the solution of the equation (I.5) can be written as follows:

$$U(x, y) = \mathrm{Re}\,[z\varphi_1(z) + \chi_1(z)]. \tag{I.8}$$

Here Re is the symbol for the real part of the expression contained within the brackets; $\phi_1(z)$ and $\chi_1(z)$ are some analytical functions of the complex variable $z = x + iy$.

Consequently, the solution of the planar problem reduces to the determination of two analytic functions $\phi_1(z)$ and $\psi_1(z) = d\chi_1/dz$ which, on contour L, satisfy definite conditions.

The representation of the solution of equation (I.5) in the form (I.8) affords the possibility to express the boundary conditions through two functions $\phi_1(z)$ and $\psi_1(z)$.

For the first basic problem condition (I.7) acquires the form

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = \\ = i\int_0^s (X_n + iY_n)\,ds + C = f_1 + if_2 + \text{const} \quad \text{on} \quad L. \tag{I.9}$$

If, however, the displacements are known, we find the following boundary condition on contour L:

$$2\mu(u + iv) = \varkappa\varphi_1(z) - z\overline{\varphi_1'(z)} - \overline{\psi_1(z)} = 2\mu(g_1 + ig_2) \quad \text{on} \quad L, \tag{I.10}$$

where $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stressed state and $\kappa = 3 - 4\nu$ for plane deformations; ν is Poisson coefficient; E is Young's modulus; $\mu = G = E/2(1 + \nu)$ is shear modulus. /11

We shall further combine the boundary conditions (I.9) and (I.10) and write them in the form

$$\varkappa_1\varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = F(z) \quad \text{on} \quad L, \tag{I.11}$$

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where

$$\varkappa_1 = 1, \qquad F = i\int_0^s (X_n + iY_n)\,ds + \text{const}$$

for the first basic problem;

$$\varkappa_1 = -\varkappa, \qquad F = -2\mu'(g_1 + ig_2) \tag{I.12}$$

for the second basic problem.

If the functions $\phi_1(z)$ and $\psi_1(z)$ are known, the stress components σ_x, σ_y and τ_{xy} may be found directly through $\phi_1(z)$ and $\psi_1(z)$ according to the Kolosov-Muskhelishvili[1] formulas

$$\begin{aligned}\sigma_x + \sigma_y &= 2[\varphi_1'(z) + \overline{\varphi_1'(z)}] = 4\operatorname{Re}\varphi_1'(z),\\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2[\bar{z}\varphi_1''(z) + \psi_1'(z)].\end{aligned} \tag{I.13}$$

From equation (I.13) we easily find formulas for maximal tangential stress τ_{max} and principal stresses σ_1 and σ_2 expressed through complex potentials $\phi_1(z)$ and $\psi_1(z)$:

$$\begin{aligned}\tau_{max} &= |\bar{z}\varphi_1''(z) + \psi_1'(z)|,\\ \sigma_1 &= \varphi_1'(z) + \overline{\varphi_1'(z)} + |\bar{z}\varphi_1''(z) + \psi_1'(z)|,\\ \sigma_2 &= \varphi_1'(z) + \overline{\varphi_1'(z)} - |\bar{z}\varphi_1''(z) + \psi_1'(z)|.\end{aligned} \tag{I.14}$$

In solving the first basic problem it is sometimes more convenient to use a boundary condition obtained from equation (I.9) by differentiating by $z \equiv t$ (where t is the value of variable z on contour L of region S):

$$\Phi_1(t) + \overline{\Phi_1(t)} - e^{-2i\theta}[t\overline{\Phi_1'(t)} + \overline{\Psi_1(t)}] = F'(t) = (X_n + iY_n)e^{-i\theta} = N + iT. \tag{I.15}$$

Here and later in formulas (I.17)-(I.21) $\Phi_1(z) = \phi_1'(z)$, $\Psi_1(z) = \psi_1'(z)$; N is projection of stress applied to an arc of the contour onto the direction of the /12

[1]A complex representation of the solution of the plane problem in the form first published by G. V. Kolosov (1909), as is indicated by I. N. Veku and N. I. Muskhelishvili [1], was given even earlier (1900) by S. A. Chapl'gin [1]. However, these works of S. A. Chapl'gin were published only after his death (1950).

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external normal in relation to the body; T is projection of the same stress on a tangent directed to the left, if looking along the positive direction of the normal; θ is angle between the normal and axis x.

If the region is bounded by a circle (circular disk, plane with a round opening, round ring), then

$$F'(t) = N + iT = \sigma_r + i\tau_{r\theta}, \tag{I.16}$$

where σ_r and $\tau_{r\theta}$ are the normal and tangential stress in a polar coordinate system ($z = re^{i\theta}$), respectively.

The first, second and combined problems of the plane theory of elasticity for many regions are solved very simply and effectively by reducing them to the Hilbert boundary problem or to the problem of linear stress. A detailed exposition of this problem may be found in N. I. Muskhelishvili's monograph [1]. Here we shall cite only some of the formulas which will be needed later.

In a polar coordinate system (r, θ) formulas (I.13) take on the form

$$\begin{aligned} \sigma_r + \sigma_\theta &= 2[\Phi_1(z) + \overline{\Phi_1(z)}], \\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} &= 2e^{2i\theta}[\bar{z}\Phi_z'(z) + \Psi_1(z)], \\ 2\mu(v_r + iv_\theta) &= [\varkappa\varphi(z) - z\overline{\varphi'(z)} - \psi(z)]e^{-i\theta}. \end{aligned} \tag{I.17}$$

From the first two formulas of (I.17) it follows that

$$\sigma_r - i\tau_{r\theta} = \Phi_1(z) + \overline{\Phi_1(z)} - e^{2i\theta}[\bar{z}\Phi_1'(z) + \Psi_1(z)]. \tag{I.18}$$

For a region bounded by a circle

$$\Psi_1(z) = \frac{1}{z^2}\Phi_1(z) + \frac{1}{z^2}\overline{\Phi_1}\left(\frac{1}{z}\right) - \frac{1}{z}\Phi_1'(z); \tag{I.19}$$

$$\sigma_r + i\tau_{r\theta} = \Phi_1(z) - \Phi_1\left(\frac{1}{z}\right) + \bar{z}\left(\bar{z} - \frac{1}{z}\right)\overline{\Psi_1(z)}; \tag{I.20}$$

$$2\mu\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) = iz\left[\varkappa\Phi_1(z) + \Phi_1\left(\frac{1}{z}\right) - \bar{z}\left(\bar{z} - \frac{1}{z}\right)\overline{\Psi_1(z)}\right]. \tag{I.21}$$

Formulas (I.20) and (I.21) are very convenient for solving many important problems of the plane theory of elasticity.

Solution of the Basic Boundary Problems for Simply-Connected (Infinite) Regions. In this case the region of change of complex variable z is the outside (including the infinitely distant point) of a certain sufficiently smooth contour L, i.e., an infinite plane with a single hole. Placing the origin of the

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coordinates in the hole (outside the region) it is possible to represent the functions $\phi_1(z)$ and $\psi_1(z)$ in the form /13

$$\varphi_1(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\ln z + \sum_{-\infty}^{+\infty} a_n' z^n,$$
$$\psi_1(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln z + \sum_{-\infty}^{+\infty} b_n' z^n. \qquad (I.22)$$

Here X and Y are components of the resultant vector of all external forces applied to contour L of the hole.

For the stresses to remain bounded throughout region S, it is necessary to set all coefficients a_n' and b_n' for $n \geqslant 2$ equal to zero in functions $\phi_1(z)$ and $\psi_1(z)$ (I.22). With these restrictions

$$\varphi_1(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\ln z + \Gamma z + \varphi_1^0(z),$$
$$\psi_1(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln z + \Gamma' z + \psi_1^0(z). \qquad (I.23)$$

Here the constants $\Gamma = B + iC$ and $\Gamma' = B' + iC'$ are defined by the stressed state at infinity, and $\phi_1^0(z)$ and $\psi_1^0(z)$ are functions which have expansions of the form

$$\varphi_1^0(z) = a_0' + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots,$$
$$\psi_1^0(z) = b_0' + \frac{b_1'}{z} + \frac{b_2'}{z^2} + \ldots. \qquad (I.24)$$

with large enough $|z|$. If the contour of the hole is free of external forces, it follows that X = Y = 0 in formula (I.23).

Denoting by N_1 and N_2 the values of the principal stresses at infinity, we obtain for constants $\Gamma = B + iC$ and $\Gamma' = B' + iC'$ the following expressions:

$$B = \frac{N_1+N_2}{4}, \qquad B' = -\frac{N_1-N_2}{2}\cos 2\alpha, \qquad C' = \frac{N_1-N_2}{2}\sin 2\alpha. \qquad (I.25)$$

Here α is angle comprised by force N_1 at infinity with axis Ox.

The stresses at infinity

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$$\sigma_x^{\infty} = 2B - B', \qquad \sigma_y^{\infty} = 2B + B', \qquad \tau_{xy}^{\infty} = C'. \tag{I.25a}$$

The constant C' is associated with rigid body rotation of the infinitely distant part of the plane and does not influence stress distribution, and therefore may be set equal to zero.

Let us now map region S onto the outside (or inside) of a unit circle γ with the help of mapping function $z = \omega(\zeta)$. Let us denote the value of variable ζ on the unit circle by σ. /14

By replacing variable z on contour L of region S through $\omega(\zeta)$ condition (I.11) reduces to the form

$$\varkappa_1 \varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = F(\sigma) \quad \text{on} \quad \gamma, \tag{I.26}$$

where the functions $\phi(\sigma)$ and $\psi(\sigma)$ are equal to $\phi_1[\omega(\sigma)]$ and $\psi_1[\omega(\sigma)]$, respectively.

After replacing variable z by $\omega(\zeta)$ formulas (I.13) take on the form

$$\begin{aligned} \sigma_\varrho + \sigma_\vartheta &= 2[\Phi(\zeta) + \overline{\Phi(\zeta)}] = 4\mathrm{Re}\,\Phi(\zeta), \\ \sigma_\vartheta - \sigma_\varrho + 2i\tau_{\varrho\vartheta} &= \frac{2\zeta^2}{\varrho^2\overline{\omega'(\zeta)}}[\overline{\omega(\zeta)}\,\Phi'(\zeta) + \omega'(\zeta)\,\Psi(\zeta)], \end{aligned} \tag{I.27}$$

where

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}, \tag{I.28}$$

and σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ are equal, respectively, to σ_x, σ_y and τ_{xy} in a moving rectangular coordinate system, in which the coordinate origin is located in the point under consideration and axis Oy is a tangent to the curve ρ = const, and axis Ox is directed along the outward normal to this curve in the given point.

In solving specific problems it is sometimes more convenient in the practical sense to define the functions $\Phi(\zeta)$ and $\Psi(\zeta)$, satisfying on contour γ the condition

$$\varkappa_1\Phi(\sigma) + \overline{\Phi(\sigma)} - \frac{\sigma^{-2}}{\omega'(\sigma)}[\omega(\sigma)\overline{\Phi'(\sigma)} + \overline{\omega'(\sigma)}\,\overline{\Psi(\sigma)}] = \frac{F'(\sigma)}{\omega'(\sigma)} \text{ on } \gamma. \tag{I.29}$$

Condition (I.29) is obtained from (I.26) by differentiating by σ.

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As was shown in the works of I. N. Kartsivadze [1], Ye. F. Burmistrov [2], if the function $\Phi(\zeta)$ is defined, this is enough for a complete solution to the problem, since the function $\Psi(\zeta)$ is connected with the function $\Phi(\zeta)$ by the formula[1]

$$\Psi(\zeta)=\frac{\overline{\omega'}\left(\frac{1}{\zeta}\right)}{\zeta^2\omega'(\zeta)}\left[\varkappa_1\overline{\Phi}\left(\frac{1}{\zeta}\right)+\Phi(\zeta)\right]-\frac{1}{\omega'(\zeta)}\left[\overline{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta)+\frac{1}{\zeta^2}\overline{F'}\left(\frac{1}{\zeta}\right)\right] \qquad (I.30)$$

Substituting formula (I.30) into (I.27), we obtain the following expressions for stresses on the contour: /15

$$\begin{aligned}\sigma_\rho&=\mathrm{Re}\left[(1-\varkappa_1)\Phi(\sigma)+\frac{\overline{F'(\sigma)}}{\overline{\omega'(\sigma)}}\right],\\ \sigma_\theta&=\mathrm{Re}\left[(3+\varkappa_1)\Phi(\sigma)-\frac{\overline{F'(\sigma)}}{\overline{\omega'(\sigma)}}\right],\\ \tau_{\rho\theta}&=\mathrm{Im}\left[(1-\varkappa_1)\Phi(\sigma)-\frac{\overline{F'(\sigma)}}{\overline{\omega'(\sigma)}}\right].\end{aligned} \qquad (I.31)$$

In particular, if the contour is free from external forces, the stresses are found by formulas

$$\sigma_\theta=4\mathrm{Re}\,\Phi(\sigma),\qquad \sigma_\rho=\tau_{\rho\theta}=0. \qquad (I.32)$$

In the case of the second basic problem (an absolutely rigid ring or disk is soldered into the hole)

$$\sigma_\rho=\frac{4}{1+\nu}\mathrm{Re}\,\Phi(\sigma),\qquad \sigma_\theta=\nu\sigma_\rho,\qquad \tau_{\rho\theta}=\frac{4}{1+\nu}\mathrm{Im}\,\Phi(\sigma). \qquad (I.33)$$

Later we will use mapping both onto the outside and the inside of a unit circle, accepting in the first case the mapping function in the form

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[1]This expression may be obtained from the formula indicated by A. G. Ugodchikov by simple differentiation by ζ:

$$\psi=-\varkappa_1\overline{\varphi}\left(\frac{1}{\zeta}\right)-\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta)+\overline{F}\left(\frac{1}{\zeta}\right).$$

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$$z = \omega(\zeta) = R\left(\zeta + \sum_{k=0}^{n} c_k \zeta^{-k}\right) \qquad (|\zeta| \geqslant 1), \tag{I.34}$$

and in the second -- in the form

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + \sum_{k=0}^{n} c_k \zeta^{k}\right) \qquad (|\zeta| < 1). \tag{I.35}$$

It is obvious that after replacing variable z by ζ with the help of correlation (I.34), functions (I.23) will have the form

$$\begin{aligned} \varphi(\zeta) &= -\frac{X+iY}{2\pi(1+\varkappa)} \ln\zeta + R(B+iC)\zeta + \varphi_0(\zeta), \\ \psi(\zeta) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)} \ln\zeta + R(B'+iC')\zeta + \psi_0(\zeta); \end{aligned} \tag{I.36}$$

where $\varphi_0(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^{-k}$ and $\psi_0(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^{-k}$ are two holomorphic functions when $\zeta = \infty$.

Substituting functions $\phi(\zeta)$ and $\Psi(\zeta)$ (I.36) into the contour conditions (I.26) and comparing coefficients of identical powers of ζ in the right and left parts, we obtain a system of algebraic equations. Solving it, we find coefficients a_k and b_k. This approach, however, can be accomplished only in the simplest cases. In general, it is extremely cumbersome and inconvenient[1]. /16

N. I. Muskhelishvili [1] showed that, using the Harnak theorem, it is possible to replace the contour conditions (I.11), where instead of z the value of the mapping function on the circuit $z = \omega(\sigma)$ (I.35) is introduced, by two equivalent functional equations:

$$\begin{aligned} \varphi_0(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \frac{\overline{\varphi_0'(\sigma)}}{\sigma-\zeta}\, d\sigma + \bar{b}_0 &= \frac{1}{2\pi i}\int_\gamma \frac{f_1^0 + if_2^0}{\sigma-\zeta}\, d\sigma, \\ \psi_0(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \cdot \frac{\varphi_0'(\sigma)}{\sigma-\zeta}\, d\sigma &= \frac{1}{2\pi i}\int_\gamma \frac{f_1^0 - if_2^0}{\sigma-\zeta}\, d\sigma, \end{aligned} \tag{I.37}$$

where

[1] Formulas (I.31)-(I.33) were obtained by Ye. F. Burmistrov [1].

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$$f_1^0 + if_2^0 = f_1 + if_2 - \frac{X+iY}{2\pi}\ln\sigma - \frac{BR}{\sigma} - \\ - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\left\{\frac{X-iY}{2\pi(1+\varkappa)}\sigma - B\overline{R}\sigma^2\right\} - (B' - iC')\overline{R}\sigma - \tag{I.38}$$

reduced contour conditions for functions $\phi_0(z)$ and $\psi_0(z)$ and f_1^0 - if_2^0 is an expression conjugate with expression (I.38).

Analogous correlations are obtained for the second basic problem. It may be shown[1] that if $z = \omega(\zeta)$ is a rational function, it is possible to determine the function $\phi_0(\zeta)$ from the first equation of system (I.37), and then from the second equation of the system (I.37) -- $\psi_0(\zeta)$. If $z = \omega(\zeta)$ is an irrational function, system (I.37) may be reduced to integral Fredholm equations[2].

Substituting the functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ found into expression (I.36), we obtain the final form of functions $\phi(\zeta)$ and $\psi(\zeta)$ with a given stressed state on infinity. Introducing into formulas (I.27) the functions $\phi(\zeta)$ and $\psi(\zeta)$ (I.36), as well as the mapping function $\omega(\zeta)$ from expression (I.35) which is appropriate for the given form of hole, and isolating on them the real and imaginary parts, we find formulas for stress components σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ in a curvilinear orthogonal system of coordinates (ρ, ϑ), corresponding to mapping function $\omega(\zeta)$ (I.35).

In reality, it is relatively seldom necessary to concern ourselves with unbounded regions which are weakened by some hole or series of holes. In most cases of engineering practice the problem reduces to investigating the concentration of stresses around holes in a plane of finite dimensions, exceeding the greatest hole dimensions by a small number (3-5) times, i.e., multiply-connected finite regions are encountered. An application of these methods to the solution of specific problems may be found, for example, in the works of N. F. Gur'yev [1, 2], M. P. Sheremet'yev [1], N. I. Muskhelishvili [1]. These /17
methods make it possible to solve any problem completely; however, here it is necessary to carry out a very large number of calculations, where the solution obtained will still be approximate. Naturally, the question arises, particularly for finite double-connected regions, with what correlations of the dimensions of regions and holes in them may simple solutions for singly-connected regions be used with an assigned degree of accuracy of the final results. To solve the latter problem the relatively simple approximate solution of A. G. Ugodchikov [8] for finite double-connected regions may be used, which makes it possible to satisfy an assigned boundary condition on one of the contours identically, and on the other -- with some error. We apparently may

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[1]See N. I. Muskhelishvili [1], §85, pp. 329-333.
[2]See N. I. Muskhelishvili [1], §96, pp. 363-386.

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judge the accuracy of the approximate solution found by the magnitude of this error[1].

§2. Mapping Functions

It follows from §1 that a necessary condition for the effective solution of the plane problem is the preliminary construction of functions which conformally map the inside or outside of a unit circle onto the regions in which we are interested.

We will examine finite simply-connected and doubly-connected and infinite simply-connected regions which are the outside of some curve.

Let us accept that the boundaries of our regions are simple piecewise-smooth Jordan curves in parametric equations $x(t)$, $y(t)$, for which the interval $t_1 \leqslant t \leqslant t_2$ of change of parameter t may be segmented into a finite number of intervals such that in each of them the functions $x(t)$ and $y(t)$ have continuous derivatives $x'(t)$, $y'(t)$, which do not turn simultaneously to zero -- the tangent and curve change continuously.

The question arises whether any region may be conformally mapped onto a circle (circular ring). An exhaustive answer is given to this question by the Reimann theorem, the formulation of which is cited in the books of E. Gurs [2], V. I. Smirnov [1], I. I. Privalov [1], and G. M. Goluzin [1].

If the boundary (boundaries) of a given simply-connected (doubly-connected) region is a smooth curve with a continuously changing tangent, then a conformal transformation of a circle (circular ring) onto the assigned simply-connected (doubly-connected) region existed and (with defined norming conditions) the /18
boundary of the circle (circular ring) and boundary of the simply-connected (doubly-connected) region flow into one another identically and continuously.

Thus, if the curve is piecewise-smooth, the conformity of the transformation will be disrupted in the corner points. The Reimann theorem establishes only the existence of a function $z = \omega(\zeta)$ which accomplishes conformal mapping, but does not indicate how to construct this function for an a priori assigned region. In other words, the effective design of a function which performs conformal mapping is often a rather complex problem. Even if this function is known "exactly," it (except for the simplest cases[2]) is represented by a complex analytic expression which leads in practice to such calculation difficulties and inconveniences that it is necessary to avoid exact expressions for the function $z = \omega(\zeta)$ and replace them with more convenient expressions from

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[1]Chapters II and III will examine problems of the concentration of stress around several holes, i.e., problems of the plane theory of elasticity for multiply-connected regions of both isotropic as well as anisotropic media; the appropriate methods of solving these problems will also be cited there (in sufficient amount), therefore we will not pause on these problems here.

[2]See example 2 below.

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well-studied functions. Polynomials should be included among the latter first of all.

This circumstance promoted the creation of convenient approximate methods of designing mapping functions: analytical[1], graphoanalytical -- methods of P. V. Melent'yev [1], B. F. Shilov [1] and M. Kikukawa [1-4], experimental-analytic[2] and numerical[3].

In solving specific problems of elasticity theory, use may be made of known mapping functions which have a convenient form.

Example 1. An infinite region with an elliptical hole. From the theory of functions of a complex variable it is known that a region S which is the outside of an elliptical hole is mapped onto the inside of a unit circle by function

$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} + m\zeta\right) = Re^{i\delta}\left(\frac{1}{\zeta} + m\zeta\right), \tag{I.39}$$

where $c = Re^{i\delta}$ is a complex constant influencing only the dimensions and position of a hole. The constant $|m| \leqslant 1$ characterizes the eccentricity of the ellipse. When $m = 0$ the ellipse becomes a circle, and with a real $c = R(\delta = 0)$ and when $m = +1$ or $m = -1$ -- it becomes a straight-line segment of length $4R$ along axis Ox or Oy, respectively. Giving m different values from -1 to +1, we may obtain an elliptical hole with any correlation of its axes. By changing R we may attain any dimensions of the elliptical opening. Changing δ, we rotate (in the positive direction) the major axis of the elliptical hole relative to axis x by the same angle.

With real c, i.e., when $\delta = 0$, the equation of the contour of the hole has the form[4]

$$x = R(1 + m)\cos\vartheta, \qquad y = -R(1 - m)\sin\vartheta, \tag{I.40}$$

Here, if the half-axes of the ellipse a and b are assigned, denoting by k the ratio of half-axis b of the ellipse lying on axis Oy to half-axis a lying on axis Ox, we find /19

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[1]See G. Goluzin, L. Kantorovich and others [1], L. V. Kantorovich and V. I. Krylov [2], V. I. Makhovikov [1, 2] and others.
[2]See G. N. Polozhiy [1], O. V. Tozoni [1, 2], Yu. G. Tolstov [1, 2], A. G. Ugodchikov [1-4], V. Ye. Shamanskiy [1], P. F. Fil'chakov [4] and others.
[3]See Yu. V. Blagoveshchenskiy [1, 2], P. F. Fil'chakov [1, 6, 7] and others.
[4]In mapping a plane with a hole onto the inside of a unit circle the circuit of the boundary of the circle in a counter clockwise direction corresponds to a clockwise circuit of the contour of the hole in question because of the conformity of the mapping.

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$$k = \frac{b}{a} = \frac{1-m}{1+m}.$$

Whence

$$m = \frac{1-k}{1+k} = \frac{a-b}{a+b}, \quad R = \frac{a+b}{2}. \tag{I.41}$$

Thus, still another form may be given to mapping function (I.39) with a real c:

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right) = \frac{a+b}{2}\frac{1}{\zeta} + \frac{a-b}{2}\zeta. \tag{I.42}$$

In addition, the following approach may be used: take any function $z = \omega(\zeta)$, which is holomorphic and smooth within a unit circle $|\zeta| \leq 1$ with boundary γ, and see to what circuit L on plane z will correspond circle γ. If it turns out here that the circuit of contour L will coincide (in both cases the region is to the left) with the direction of the circuit of the unit circle γ, the function $z = \omega(\zeta)$ will yield a conformal mapping of region S onto the region of a unit circle and back[1].

For example, the function

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + \frac{1}{9}\zeta^3\right) \tag{I.43}$$

yields a mapping of a unit circle onto the outside of a square with rounded corners, and the function

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + \frac{1}{4}\zeta^2\right) \tag{I.44}$$

is a mapping of a unit circle onto the outside of an equilateral triangle with rounded corners. However, these approaches can very seldom be used, and in most cases it is necessary to construct a mapping function, using some approximate method. The function obtained will provide a mapping of the unit circle not onto the assigned region S with boundary L, but onto a region S' with boundary L', which is close to the assigned. The degree of deviation of contours L and L' may serve as a criterion for the accuracy of the function constructed.

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[1]See N. I. Muskhelishvili [1].

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Using the Kristoffel-Schwartz Integral. To design mapping functions of singly-connected regions bounded by polygons use may be made, as is known, of the so-called Kristoffel-Schwartz integral[1]. Using this integral, an approximate value of the mapping function may be obtained in the form of a polynomial or in another form which makes it possible to approximate the assigned contour with the required accuracy.

Let it be required to find a mapping of the outside of unit circle $|\zeta| \geqslant 1$ onto the outside of a polygon, i.e., onto a region including an infinitely remote point and bounded by a contour of a form interesting us (rectangle, square, triangle). The Kristoffel-Schwartz integral which provides a solution /20 to the problem has in this case the form

$$z = \omega(\zeta) = c\int_1^{\zeta} (t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2-1}\ldots(t-a_n)^{\alpha_n-1}\frac{dt}{t^2} + d, \qquad (I.45)$$

where c and d are essentially, generally speaking, complex constants characterizing the position of the polygon and its dimensions; a_1, a_2, ..., a_n are so-called constants of the Kristoffel'-Schwartz integral -- points of unit circle γ of plane ζ, corresponding to the vertices of polygon A_1, A_2, ..., A_n on plane z; α_1, α_2, ..., α_n are real positive constants showing what part of π is comprised by the outside angles of the polygon; t is the point of the region outside of the circle $|t| = 1$.

The mapping accomplished by the Kristoffel-Schwartz function (I.45) is continuous up to the contour, excepting the angle points a_1, a_2, ..., a_n, where $\omega'(\zeta)$ becomes zero. Consequently, the conformity of the mapping in these points (corners) is disrupted. Noting that $\alpha_1 + \alpha_2 + \ldots + \alpha_n = n + 2$, we transform the integrand in formula (I.45):

$$\omega(\zeta) = c\int_1^{\zeta}\left(1-\frac{a_1}{t}\right)^{\alpha_1-1}\left(1-\frac{a_2}{t}\right)^{\alpha_2-1}\ldots\left(1-\frac{a_n}{t}\right)^{\alpha_n-1}dt. \qquad (I.46)$$

since $|a_n| = 1$, $|t| > 1$, expanding the integrand in formula (I.46) into a series in the region of the infinitely remote point and integrating it, we obtain

$$\omega(\zeta) = c\left\{\zeta - [(\alpha_1-1)a_1 + (\alpha_2-1)a_2 + \ldots + (\alpha_n-1)a_n]\ln\zeta + \frac{e_1}{\zeta} + \frac{e_2}{\zeta^2} + \ldots\right\}. \qquad (I.47)$$

[1]See I. I. Privalov [1].

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To obtain a mapping of a simply connected region onto a simply connected region it is necessary and sufficient that

$$(\alpha_1 - 1)a_1 + (\alpha_2 - 1)a_2 + \ldots + (\alpha_n - 1)a_n = 0. \tag{I.48}$$

It follows from expression (I.47) that when condition (I.48) is observed it is possible to construct a function $\omega(\zeta)$ which provides a conformal mapping (onto the outside of a unit circle) of schlicht region S', which is as close as desired to assigned region S (infinite plane with a given opening).

Example 2. A region with a rectangular hole[1]. Let region S be a plane with a rectangular opening, the ratio of the sides $(A_1A_4)/(A_1A_2) = \lambda$ (Figure I.1). Let us take the coordinate origin xOy in the center of the rectangle, i.e., outside region S.

Points a_1, a_2, a_3, a_4 on the unit circle corresponding to the vertices of rectangle A_1, A_2, A_3 and A_4 may be selected such that the assigned ratio of the sides of the rectangle is maintained. Three of the four points, for example a_1, a_2, a_3 on the unit circle may be selected such that they are located symmetrically relative to coordinate axes $\xi O\eta$, i.e., such that points a_1 and a_2 are symmetrical relative to axis $O\xi$, and points a_2 and a_3 -- relative to axis $O\eta$. We conclude further, because of the symmetry principle, that a_4 must be symmetrical to point a_3 relative to axis $O\xi$ and point a_1 relative to axis $O\eta$. /21

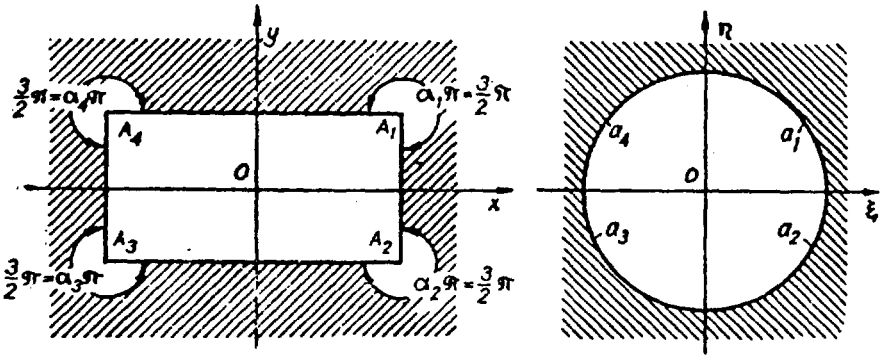


Figure I.1.

If we denote by k the magnitude characterizing the ratio of the sides of a rectangular hole,

[1]The mapping is done with the help of elliptical functions (for example, see V. I. Smirnov [1], p. 325).

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$$a_1 = e^{k\pi i},\ a_2 = e^{(2-k)\pi i},\ a_3 = e^{(1+k)\pi i},\ a_4 = e^{(1-k)\pi i};$$

when $k < 1/4$ the rectangular opening will be located as is shown in Figure I.1, i.e., the longer sides are parallel to axis Ox; when $k > 1/4$ the long sides will be parallel to axis Oy; the value $k = 1/4$ corresponds to a square opening.

In the case of a rectangular opening $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 3/2$. Substituting the values found for $a_1, \ldots, a_4$ and $\alpha_1, \ldots, \alpha_4$ into (I.47) and inverting (using $1/\zeta$ instead of ζ), we obtain a function $z = \omega(\zeta)$ which maps region S, i.e., a region with a rectangular opening[1] onto the outside of the unit circle:

$$z = \omega(\zeta) = c\left\{\frac{1}{\zeta} + \frac{a+\bar{a}}{2}\zeta + \frac{(a-\bar{a})^2}{24}\zeta^3 + \frac{(a^2-\bar{a}^2)(a-\bar{a})}{80}\zeta^5 + \right.$$
$$+ \frac{5(a^4+\bar{a}^4) - 4(a^2+\bar{a}^2) - 2}{896}\zeta^7 + \frac{7(a^5+\bar{a}^5) - 5(a^3+\bar{a}^3) - 2(a+\bar{a})}{2304}\zeta^9 + \qquad \text{(I.49)}$$
$$\left. + \frac{21(a^6+\bar{a}^6) - 14(a^4+\bar{a}^4) - 5(a^2+\bar{a}^2) - 4}{11\,264}\zeta^{11} + \ldots\right\},$$

where $a = e^{2k\pi i}$, $\bar{a} = e^{-2k\pi i}$.

It is clear that if we take only a finite number of members of the series in expression (I.49), we obtain an opening which is not in the form of a perfect rectangle, but a rectangle with curvilinear sides and rounded corners. Since $|a| = 1$, it is not difficult to see that the series (I.49) obtained at the beginning converges rather rapidly, and we always can select the numbers of members of this series so that the required accuracy of deviation of the opening contour from a rectangle is satisfied. Here it is possible a priori to assign a certain magnitude r_0 and determine the number of members of series (I.49) which is necessary so that radius r of rounding of the corners of the rectangle is less than r_0. After this, it follows to check again the degree of /22 deviation of the rectangle sides from straight-line, and if it is now necessary to increase the number of members of series (I.49), condition $r < r_0$ will be fulfilled.

Assuming in (I.49) $a = e^{i\pi/2}$ and $\bar{a} = e^{-i\pi/2}$, we obtain a function mapping a region with a square hole onto a circle:

$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 - \frac{1}{176}\zeta^{11} + \frac{1}{384}\zeta^{15} - \frac{7}{4864}\zeta^{19} + \ldots\right). \qquad \text{(I.50)}$$

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[1] See G. N. Savin [1, 2].

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Assuming in (I.49) $a = e^{i\pi/3}$, i.e., taking $k = 1/6$, we obtain a function $\omega(\zeta)$ which maps a region with a rectangular opening, the ratio of the sides of which is 3.2:1, in the form

$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} + \frac{1}{2}\zeta - \frac{1}{8}\zeta^3 - \frac{3}{80}\zeta^5 - \frac{3}{896}\zeta^7 + \frac{5}{768}\zeta^9 + \frac{57}{11\,264}\zeta^{11} + \ldots\right). \qquad (I.51)$$

When $a = e^{i10/36\pi}$ we obtain a mapping function $\omega(\zeta)$ for a region with a rectangular opening, the ratio of the sides of which is 5:1, in the form[1]

$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} + 0{,}643\zeta - 0{,}098\zeta^3 - 0{,}038\zeta^5 - 0{,}011\zeta^7\right). \qquad (I.52)$$

Example 3. A region with an opening in the form of a right polygon. Considerations analogous to those cited for the case of a rectangular opening show that the points of the unit circle a_1, a_2, ..., a_n on plane $\zeta = \xi + i\eta$, corresponding to vertices of polygon A_1, ..., A_n of plane $z = x + iy$, will divide the circle into even parts. Mapping the circle onto a circle of the same radius, we can consider that point $a_1 = 1$ corresponds to the vertex A_1; then

$$a_1 = 1,\ a_2 = e^{\frac{2\pi i}{n}},\ \ldots,\ a_n = e^{\frac{2(n-1)}{n}\pi i},\ \alpha_1 + \ldots + \alpha_n = 1 + \frac{2}{n}, \qquad (I.53)$$

where n is the number of sides of the right polygon.

Substituting instead of a_1, ..., a_n and α_1, ..., α_n their values from (I.53) into (I.45), we obtain (considering that the center of gravity of the polygon corresponds to the center of the unit circle)

$$\omega(\zeta) = c\int_1^{\zeta} (t - e^{\frac{2\pi}{n}i})^{\frac{2}{n}}(t - e^{\frac{4\pi}{n}i})^{\frac{2}{n}} \ldots (t - e^{\frac{2(n-1)}{n}\pi i})^{\frac{2}{n}} \frac{dt}{t^2}.$$

Expanding the integrand into a series in the region of the infinitely remote point, integrating it and inverting, we obtain a function which conformally maps the unbounded plane with an opening in the form of a right polygon onto the inside of a unit circle[2]

---

[1]Other ratios of the sides of a rectangle and trapezoid with a priori assigned curves in the corner points of these figures are examined in the works of V. M. Gur'yanov and O. S. Kosmodamianskiy [1, 2].

[2]See G. N. Savin [1, 2].

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$$z = \omega(\zeta) = c\left\{\frac{1}{\zeta} + \frac{2}{n(n-1)}\zeta^{n-1} + \frac{n-2}{n^2(2n-1)}\zeta^{2n-1} - \right.$$
$$- \frac{(n-2)(2n-2)}{2n^3(3n-1)}\zeta^{3n-1} + \frac{(n-2)(2n-2)(3n-2)}{12n^4(4n-1)}\zeta^{4n-1} -$$
$$\left. - \frac{(n-2)(2n-2)(3n-2)(4n-2)}{60n^5(5n-1)}\zeta^{5n-1} + \ldots\right\}, \quad n = 3, 4, 5, \ldots \tag{I.54}$$

To obtain a parametric equation of the opening contour which corresponds to mapping function (I.49) or (I.54), it follows to place $\zeta = \rho e^{i\vartheta}$ when $\rho = 1$ in these functions and to separate the real and imaginary parts. /23

Assuming n = 3 in (I.54) we obtain a mapping function $\omega(\zeta)$ for a plane with an opening in the form of an equilateral triangle:

$$\omega(\zeta) = c\left(\frac{1}{\zeta} + \frac{1}{3}\zeta^2 + \frac{1}{45}\zeta^5 + \frac{1}{162}\zeta^8 + \frac{7}{2673}\zeta^{11} + \ldots\right). \tag{I.55}$$

If we put n = 4 in (I.54), we obtain a mapping function for a plane with a square opening (turned by $\pi/4$ relative to the square examined in example 2):

$$\omega(\zeta) = c\left(\frac{1}{\zeta} + \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 + \frac{1}{176}\zeta^{11} + \ldots\right). \tag{I.56}$$

To obtain a mapping function for any position of an opening under examination, it follows to put $c = Re^{i\delta}$ in functions (I.49) or (I.54), where R is a real constant, and δ is the angle by which the opening must be turned from its initial position.

Inverting in functions $\omega(\zeta)$ (I.49) and (I.54), i.e., putting $\zeta = 1/\zeta'$ ($|\zeta'| \geqslant 1$), we obtain a mapping onto the outside of the unit circle.

Use of the Kristoffel'-Schwartz integral for the approximate construction of mapping functions is, first of all, made difficult by the fact that determining the constants of the Kristoffel'-Schwartz integral, i.e., points $a_1, a_2, \ldots, a_n$ of the unit circle which correspond to the vertices of the polygon is associated, except for the simplest cases, with laborious calculations[1]; secondly, when the mapping function is constructed in the form of a polynomial $z = \omega_n(\zeta)$ for finite single-connected regions, the number of polynomial members must be selected large enough to obtain good coincidence of

[1]Works devoted to this question may be found in the collection of G. Goluzin, L. Kantorovich et al [1], see also G. N. Polozhiy [1], P. F. Fil'chakov [3], I. S. Khara [1].

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assigned boundary L and boundary L' obtained with a conformal mapping of a circle using an approximate function.

To improve the convergence of the process V. I. Makhovikov [1, 2] proposes constructing an approximate function in the form of a sum consisting of a finite irrational part (the main part of the Kristoffel'-Schwartz integral) and a rapidly converging series. From the viewpoint of conformal mapping this provides a good result, although the form of the function is less convenient for solving the problems of elasticity theory. For example, a function which performs conformal mapping of a circle onto a region bounded by a square may be obtained here in the form

$$z = \omega(\zeta) = c\left[-\frac{\zeta}{2}(1+\zeta^4)^{\frac{1}{2}} + 1{,}5\left(\zeta + 0{,}1\zeta^5 - \frac{1}{72}\zeta^9 + \frac{1}{208}\zeta^{13} - 0{,}0023\zeta^{17}\right)\right]. \qquad (I.57)$$

The corresponding boundary L' is shown in Figure I.2a.

If simple series expansion is used, with eleven members of the series, i.e., when members are considered up to ζ^{41} inclusive

$$z = \omega(\zeta) = c\left(\zeta + \sum_{k=1}^{10} \delta_k \zeta^{4k+1}\right), \qquad (I.58)$$

boundary L' takes on a form shown in Figure I.2b.

Using Other Methods. It becomes necessary in connection with solving applied problems to construct approximate expressions of mapping functions for a priori assigned regions.

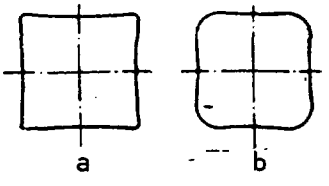


Figure I.2

A. G. Ugodchikov [1, 2] proposed a method of constructing mapping functions in the form of polynomials for a priori assigned single-connected and double-connected regions in which the correspondence of the boundary points of the mapped regions was determined (in null approximation) with the help of electromodeling, and the methodology of P. V. Melent'yev [1, 2] and B. F. Shilov [1] was used for calculating the coefficients and constructing sequential approximations. This method was later extended to infinite single-connected regions[1]. /24

It should be noted that P. V. Melent'yev's method [1] of calculating coefficients C_k of polynomial $z = \omega_n(\zeta)$, based on formulas of trigonometric

[1]See A. G. Ugodchikov, I. I. Serebrennikov [1], A. G. Ugodchikov, A. Ya. Krylov [1].

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interpolation of the real (or imaginary) part of function z/ζ, is such that polynomial $z = \omega_n(\zeta)$ does not coincide with the assigned value of the mapping function even at the points of interpolation.

The work of P. F. Fil'chakov [4] proposed a new method of constructing mapping functions for singly-connected regions in which it was proposed to find the initial data graphically or with the help of electromodeling, and the trigonometric interpolation method was used for calculating the coefficients of polynomial $z = \omega_n(\zeta)$, which insured coincidence of the polynomial and the assigned values of the mapping function in the points of interpolation. The process of constructing sequential approximations using gradual doubling of the number of points and alternation of even and uneven points is described in P. F. Fil'chakov's article [5].

A. G. Ugodchikov's works [5, 6] proposed a method according to which the constructed polynomial $z = \omega_n(\zeta)$ also coincides with the assigned values of the mapping function in the points of interpolation; clarification of the initial data is done by sequential approximation by alternating the initial and intermediate points. This method, which was extended by A. G. Ugodchikov to finite double-connected regions as well as to infinite and semi-infinite single-connected regions is based upon the formulas of Lagrange interpolation polynomials, which allowed obtaining simple enough expressions for coefficients C_k of interpolation polynomial $z = \omega_n(\zeta)$.

In subsequent works of P. F. Fil'chakov [6, 7] the trigonometric interpolation method was used to obtain for single-connected regions formulas of calculating coefficients C_k which coincide with the formulas which were obtained earlier by A. G. Ugodchikov [5, 6].

Since these methods are described in detail with many examples in the above works and monographs of A. G. Ugodchikov [7] and P. F. Fil'chakov [6], we will not cite them in detail here; however, we note that these methods make it possible, using computers, to construct mapping functions with an a priori assigned accuracy of coincidence of assigned boundary L and boundary L', corresponding to the constructed polynomial $z = \omega_n(\zeta)$. /25

§3. Basic Equations of the Linear Theory of Elasticity of an Anisotropic Medium

An elastic medium is called anisotropic if its elastic properties at a particular point are different in different directions. For an elastic characterization of an isotropic medium, as is known, only two elastic constants are needed, for example, constants λ and μ, or two modulos E and G, and others, at the same time, there must be 21 of these constants in the general case for an anisotropic medium. From here it is clear that the basic system of equations of the theory of elasticity of an anisotropic medium will differ from the same system of equations for an isotropic medium only in those equations which establish the connection between deformations and stresses, i.e., the difference of the systems of equations will consist of the generalized Hooke's law.

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We shall write out the basic equations of the statics of an elastic body, since they will be needed later[1].

Equilibrium equations

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0,\\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0,\\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0,
\end{aligned} \tag{I.59}$$

where X, Y and Z are projections on the coordinate axis of volumetric source reduced to a unit of volume.

Contour conditions

$$\begin{aligned}
X_n &= \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, z),\\
Y_n &= \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, z),\\
Z_n &= \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_z \cos(n, z).
\end{aligned} \tag{I.60}$$

Compatibility conditions /26

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \quad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),\\
&\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}, \quad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right),\\
&\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \quad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right)\\
&(\gamma_{yz} = 2\varepsilon_{yz}; \quad \gamma_{zx} = 2\varepsilon_{zx}; \quad \gamma_{xy} = 2\varepsilon_{xy}).
\end{aligned} \tag{I.61}$$

Generalized Hooke's law

[1]A detailed derivation of these equations may be found in any course of elasticity theory (for example, see N. I. Muskhelishvili [1], P. F. Papkovich [1], A. Lyav [1], S. G. Lekhnitskiy [1, 2].

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$$
\begin{aligned}
\sigma_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y + A_{13}\varepsilon_z + A_{14}\gamma_{yz} + A_{15}\gamma_{zx} + A_{16}\gamma_{xy},\\
\sigma_y &= A_{21}\varepsilon_x + A_{22}\varepsilon_y + A_{23}\varepsilon_z + A_{24}\gamma_{yz} + A_{25}\gamma_{zx} + A_{26}\gamma_{xy},\\
\sigma_z &= A_{31}\varepsilon_x + \ldots\ldots\ldots\ldots + A_{36}\gamma_{xy},\\
\tau_{yz} &= A_{41}\varepsilon_x + \ldots\ldots\ldots\ldots + A_{46}\gamma_{xy},\\
\tau_{zx} &= A_{51}\varepsilon_x + \ldots\ldots\ldots + A_{56}\gamma_{xy},\\
\tau_{xy} &= A_{61}\varepsilon_x + \ldots\ldots\ldots + A_{66}\gamma_{xy}\\
A_{ik} &= A_{ki} \quad (i, k = 1, 2, \ldots, 6)
\end{aligned}
\tag{I.62}
$$

or

$$
\begin{aligned}
\varepsilon_x &= c_{11}\sigma_x + c_{12}\sigma_y + c_{13}\sigma_z + c_{14}\tau_{yz} + c_{15}\tau_{zx} + c_{16}\tau_{xy},\\
\varepsilon_y &= c_{21}\sigma_x + c_{22}\sigma_y + c_{23}\sigma_z + c_{24}\tau_{yz} + c_{25}\tau_{zx} + c_{26}\tau_{xy},\\
\varepsilon_z &= c_{31}\sigma_x + \ldots\ldots\ldots\ldots + c_{36}\tau_{xy},\\
\gamma_{yz} &= c_{41}\sigma_x + \ldots\ldots\ldots\ldots + c_{46}\tau_{xy},\\
\gamma_{zx} &= c_{51}\sigma_x + \ldots\ldots\ldots\ldots + c_{56}\tau_{xy},\\
\gamma_{xy} &= c_{61}\sigma_x + \ldots\ldots\ldots\ldots + c_{66}\tau_{xy}\\
c_{ik} &= c_{ki} \quad (i, k = 1, 2, \ldots, 6),
\end{aligned}
\tag{I.63}
$$

i.e., the stress σ_x, σ_y, ..., τ_{xy} and deformation ε_x, ε_y, ..., γ_{xy} components in a given point of the body are linear one-to-one functions.

The magnitudes $A_{ik} = A_{ki}$ (21 in number) in equations (I.62), as well as $c_{ik} = c_{ki}$ in (I.63) are called the elastic constants of material in the sense that the elastic properties of a given body are fully characterized by these magnitudes and the values of these constants do not depend upon the stressed or deformed state of a given elastic body.

If the elastic constants A_{ik} are functions of the point coordinates of an elastic body, this body is called heterogeneous in the sense of elastic properties. If A_{ik} constants do not depend upon the point coordinates of the elastic body and are constant values for a fixed system of coordinate axes, the body is called homogeneous. /27

The Generalized Plane Stressed State. Let us assume that we have a thin anisotropic plate 2h in thickness. We take the mean plane of the sheet for plane xOy; let the external forces X_n and Y_n be applied to the side surface of the sheet so that the resultant of these forces in terms of height 2h lies in plane xOy. The remaining faces of the sheet -- the "upper" z = +h and the

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"lower" z = -h will be considered free of external forces, i.e. $\sigma_z = \tau_{yz} = \tau_{xz} = 0$ when $z = \pm h$.

Let us suppose that the thickness of the sheet 2h is extremely small; then, with a large degree of approximation, we assume $\sigma_z = \tau_{yz} = \tau_{xz} = 0$ everywhere within the sheet. Instead of the real stress components σ_x, σ_y and τ_{xy}, we will consider their mean values by height:

$$\sigma_x^* = \frac{1}{2h}\int_{-h}^{+h}\sigma_x dz; \quad \sigma_y^* = \frac{1}{2h}\int_{-h}^{+h}\sigma_y dz; \quad \tau_{xy}^* = \frac{1}{2h}\int_{-h}^{h}\tau_{xy} dz.$$

Let us multiply the first two equilibrium equations (I.59) by 1/2h dz and integrate by sheet height from -h to +h:

$$\begin{aligned}\frac{\partial\sigma_x^*}{\partial x} + \frac{\partial\tau_{xy}^*}{\partial y} + X^* = 0,\\ \frac{\partial\tau_{xy}^*}{\partial x} + \frac{\partial\sigma_y^*}{\partial y} + Y^* = 0.\end{aligned} \tag{I.64}$$

Now let us turn to Hooke's law (I.63) and look at the mean values of the deformation components by height. We multiply the first, second and sixth equation of (I.63) by 1/2h dz and add them according to sheet height:

$$\begin{aligned}\varepsilon_x^* &= c_{11}\sigma_x^* + c_{12}\sigma_y^* + c_{16}\tau_{xy}^*,\\ \varepsilon_y^* &= c_{12}\sigma_x^* + c_{22}\sigma_y^* + c_{26}\tau_{xy}^*,\\ \gamma_{xy}^* &= c_{16}\sigma_x^* + c_{26}\sigma_y^* + c_{66}\tau_{xy}^*.\end{aligned} \tag{I.65}$$

It is clear that the mean values of the deformation components ε_x^*, ε_y^* and γ_{xy}^* will be functions of only x and y.

Let us apply the same averaging operation to the fifth compatibility equation (I.61), i.e., multiply it by 1/2h dz and integrate from -h to +h:

$$\frac{\partial^2\varepsilon_x^*}{\partial y^2} + \frac{\partial^2\varepsilon_y^*}{\partial x^2} = \frac{\partial^2\gamma_{xy}^*}{\partial x\partial y}. \tag{I.66}$$

Thus, equations (I.64)-(I.66) are the basic equations for the generalized plane stressed state.

Plane Deformation. Plane Deformation Conditions. Let us assume that /28
external forces are applied to the side surface of an anisotropic body of cylindrical or prismatic form (finite or infinite cylinder) which satisfy the following conditions:

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1) the external forces (stresses) are perpendicular to the elements (or, in other words, to the axis of this body);

2) the external forces are constant along each element (but, generally speaking, change when moving from one element to another).

Let us use the axis of this cylinder as axis Oz, and any cross section of it as plane xOy.

If the body were isotropic, these conditions would be sufficient for the deformation to be plane. In the case of an anisotropic elastic body, as follows from the generalized Hooke's law (I.63), stresses σ_x, σ_y, τ_{xy} and σ_z can cause the appearance of deformations γ_{xz} and γ_{yz}, i.e., distortion of the cross sections.[1] For the deformation to be plane in this case the elasticity coefficients c_{ik} must satisfy certain conditions. The latter, for the general case of a heterogeneous anisotropic body, were first obtained by S. G. Mikhlin [1]:

$$\begin{gathered}
A_{14}c_{13} + A_{24}c_{23} + A_{34}c_{33} = 0,\\
A_{14}c_{12} + A_{24}c_{22} + A_{34}c_{23} - A_{15}c_{13} - A_{25}c_{23} - A_{35}c_{33} = 0,\\
A_{14}c_{11} + A_{24}c_{12} + A_{34}c_{13} - A_{15}c_{12} - A_{25}c_{22} - A_{34}c_{23} = 0,\\
A_{15}c_{11} + A_{25}c_{12} + A_{35}c_{13} = 0,\\
\frac{\partial A_{14}}{\partial x} + \frac{\partial A_{15}}{\partial y} = 0, \quad \frac{\partial A_{24}}{\partial x} + \frac{\partial A_{25}}{\partial y} = 0, \quad \frac{\partial A_{34}}{\partial x} + \frac{\partial A_{35}}{\partial y} = 0.
\end{gathered} \tag{I.67}$$

An especially simple case occurs when

$$A_{46} = A_{14} = A_{24} = A_{34} = A_{15} = A_{25} = A_{35} = A_{56} = 0. \tag{I.68}$$

---

[1] If the anisotropic body is homogeneous, i.e., c_{ik} = const for all points of this body, then, as S. G. Lekhnitskiy [3] showed, the problem in the general case (with 21 elastic constants) may be reduced to plane in the sense that the stress components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} τ_{xz} and deformations ε_x, ε_y, ε_z, ..., γ_{xz} will be functions only of coordinates x, y. Actually, in the general case of anisotropy, i.e., for a homogeneous body with 21 elastic constants under the influence of external forces subject to the first and second conditions the deformation will not be plane and the cross sections will be distorted, but these distortions because of the homogeneity of the body and constancy of the external load along the elements of the cylinder will be identical for all sections.

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Equations (I.68) mean that the cross sections of a cylinder, i.e., plane x0y and its parallels are elastic symmetry planes.

Thus, deformation of an anisotropic body will be plane if its cross sections (of cylindrical form) are elastic symmetry planes.

Let us assume that the plane deformation conditions are satisfied. Let us take any cross section which is sufficiently distant from the ends of the cylinder for plane x0y. If the deformation is plane, then w = 0, and components u and v will be functions of only x and y. It follows that $\varepsilon_z = 0$, $\gamma_{xz} = \gamma_{zy} = 0$. /29

Since $c_{44}c_{55} - c_{45}^2 \neq 0$, Hooke's law (I.63) may be represented in the form

$$\begin{aligned}
\varepsilon_x &= c_{11}\sigma_x + c_{12}\sigma_y + c_{13}\sigma_z + c_{16}\tau_{xy},\\
\varepsilon_y &= c_{12}\sigma_x + c_{22}\sigma_y + c_{23}\sigma_z + c_{26}\tau_{xy},\\
0 &= c_{13}\sigma_x + c_{23}\sigma_y + c_{33}\sigma_z + c_{36}\tau_{xy},\\
\gamma_{xy} &= c_{16}\sigma_x + c_{26}\sigma_y + c_{36}\sigma_z + c_{66}\tau_{xy},
\end{aligned} \tag{I.69}$$

for $\tau_{yz} = \tau_{xz} = 0$ at all points of the body.

Determining σ_z from the third equation of (I.69) and substituting it in the remaining equations of (I.69), we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy},\\
\frac{\partial v}{\partial y} &= \varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy},\\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \gamma_{xy} = a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy},
\end{aligned} \tag{I.70}$$

where

$$\begin{aligned}
a_{11} &= \frac{c_{11}c_{33} - c_{13}^2}{c_{33}}, & a_{12} &= \frac{c_{12}c_{33} - c_{13}c_{23}}{c_{33}},\\
a_{16} &= \frac{c_{16}c_{33} - c_{13}c_{36}}{c_{33}}, & a_{22} &= \frac{c_{22}c_{33} - c_{23}^2}{c_{33}},\\
a_{26} &= \frac{c_{26}c_{33} - c_{23}c_{36}}{c_{33}}, & a_{66} &= \frac{c_{66}c_{33} - c_{36}^2}{c_{33}}.
\end{aligned} \tag{I.71}$$

The first two equilibrium equations of (I.59) in this case will have the form

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$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + X = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0, \tag{I.72}$$

and the third equation of (I.59) becomes Z = 0, and, consequently, with plane deformation parallel to plane Oxy, the volumetric force component on axis Oz should become zero.

The five compatibility equations (I.61) are likewise satisfied and a single equation remains.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{I.73}$$

Thus, equations (I.70), (I.72) and (I.73) will be basic for plane deformation (remembering that any cross section of our anisotropic body must be an elastic symmetry plane). Comparing them with the appropriate equations for the generalized plane deformed state we see that they are the same in form, but instead of the mean values of the stress and deformation components in the first case (the generalized plane deformed state) we find the exact values of these magnitudes in the second case (plane deformation). A greater difference between them is the fact that whereas in the first case the elastic constants c_{ik} enter directly into (I.65), in the second case the magnitudes a_{ik} determined according to the c_{ik} given from formula (I.71) enter into (I.70). /30

The Stress Function. Complex Representation of the General Solution of the Plane Problem. Let us write out the basic equations of the plane problem:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0; \tag{I.74}$$

$$\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}; \end{aligned} \tag{I.75}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{I.76}$$

We remember that in the case of the generalized plane deformed state the mean values of the stress and deformation components will enter into all of equations (I.74)-(I.76), and a_{ik} must be set equal to c_{ik} in equations (I.75).

The solution to the plane problem is considerably simplified in the case of the absence of volumetric forces. In static problems gravitational forces are usually the volumetric forces. In most problems they can be simply

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disregarded[1], but they may be very easily eliminated in the general case, just as for an isotropic medium[2]. For this it is necessary to find any partial solution of the basic equation system (I.74)-(I,76).

Let us assume that axis Oy is directed vertically upwards; then X = 0, Y = -ρg, where g is acceleration of gravity and ρ is density, which we consider constant, and equations (I.74) take the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \varrho g. \tag{I.77}$$

Obviously, these equations are satisfied if /31

$$\sigma_x = \tau_{xy} = 0 \text{ and } \quad \sigma_y = \varrho g y. \tag{I.78}$$

From equations (I.75) and (I.78) we have

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_{12}\varrho g y, \qquad \varepsilon_y = \frac{\partial v}{\partial y} = a_{22}\varrho g y, \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_{26}\varrho g y. \tag{I.79}$$

It is easy to see that the values of ε_x, ε_y and γ_{xy} (I.79) satisfy equation (I.76).

From the first two equations of (I.79) we have

$$u = \int a_{12}\varrho g y dx + f_1(y), \qquad v = \int a_{22}\varrho g y\, dy + f_2(x).$$

Substituting the values found for u and v into the last equation of (I.79), we obtain

$$f_2'(x) + f_1'(y) + a_{12}\varrho g x = a_{26}\varrho g y.$$

---

[1] If the dimensions of the body are small and the applied external forces considerably exceed the weight of the body, then the mass forces X and Y (with a great degree of accuracy) may obviously be disregarded, since they have practically no influence on the stressed state.

[2] See N. I. Muskhelishvili [1].

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The latter equation may be satisfied, assuming, for example,

$$f_2(x) = -\frac{1}{2} a_{12} \varrho g x^2; \quad f_1(y) = \frac{1}{2} a_{26} \varrho g y^2.$$

Thus, for displacements

$$u = a_{12} \varrho g x y + \frac{1}{2} a_{26} \varrho g y^2,$$
$$v = \frac{1}{2} a_{22} \varrho g y^2 - \frac{1}{2} a_{12} \varrho g x^2. \tag{I.80}$$

Since the basic equations system (I.74)-(I.76) is linear, then, setting $\sigma_x = \sigma_x^0 + \sigma_x'$, $\sigma_y = \sigma_y^0 + \sigma_y'$, $\tau_{xy} = \tau_{xy}^0 + \tau_{xy}'$, and also $u = u^0 + u'$, $v = v^0 + v'$, we find that σ_x', σ_y', τ_{xy}', u' and v' will be the solution of the basic equations system (I.74)-(I.76) in the absence of mass forces. σ_x^0, σ_y^0, τ_{xy}^0, u^0, v^0 denote, respectively, the stress (I.78) and displacement (I.80) components, i.e., a partial solution of basic equations system (I.74)-(I.76).

Thus, from now on we will assume that mass forces do not enter into the basic equations (I.74)-(I.76) of the plane problem.

Just as in the case of an isotropic medium, the equilibrium equations (I.77) may be satisfied if the Airy function is introduced: /32

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{I.81}$$

Substituting in (I.75) instead of σ_x, σ_y and τ_{xy} their values expressed through the stress function U(xy), from equation (I.76), we obtain the basic equation of the plane problem of elasticity theory first found by S. G. Lekhnitskiy [5], which should be satisfied by the stress function U(x, y):

$$a_{22} \frac{\partial^4 U}{\partial x^4} - 2a_{26} \frac{\partial^4 U}{\partial x^3 \partial y} + (2a_{12} + a_{66}) \frac{\partial^4 U}{\partial x^2 \partial y^2} - 2a_{16} \frac{\partial^4 U}{\partial x \partial y^3} + a_{11} \frac{\partial^4 U}{\partial y^4} = 0. \tag{I.82}$$

Equation (I.82) is a generalization of the known biharmonic equation (I.5).

It should be noted again that in the case of the generalized plane deformed state the coefficients of equation (I.82) $a_{ik} = c_{ik}$; in the case of plane deformation the elastic constants a_{ik} are determined according to formulas (I.71).

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If the anisotropic material has three planes of elastic symmetry, equation (I.82) is simplified, for in this case $c_{36} = c_{16} = c_{26} = 0$, and consequently, $a_{16} = a_{26} = 0$ (see (I.71)), and equation (I.82) takes the form

$$a_{22}\frac{\partial^4 U}{\partial x^4} + (2a_{12} + a_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} + a_{11}\frac{\partial^4 U}{\partial y^4} = 0. \tag{I.83}$$

The common integral of equation (I.82) depends upon the roots of the characteristic equation

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26} + a_{22} = 0 \tag{I.84}$$

and in the case of unequal[1] roots it has the form

$$U(x, y) = F_1(x + s_1 y) + F_2(x + s_2 y) + F_3(x + s_3 y) + F_4(x + s_4 y). \tag{I.85}$$

On the basis of energy considerations, S. G. Lekhnitskiy [5] proved that equation (I.84) cannot have real roots.

Let us denote the roots of equation (I.84) by s_1, s_2, s_3 and s_4 (considering them unequal). Let

$$\begin{aligned} s_1 &= \alpha_1 + i\beta_1, \quad & s_2 &= \alpha_2 + i\beta_2, \\ s_3 &= \alpha_1 - i\beta_1, \quad & s_4 &= \alpha_2 - i\beta_2, \end{aligned} \tag{I.86}$$

where α_1, α_2, β_1, β_2 are real constants, and it is always possible that $\beta_1 > 0$, $\beta_2 > 0$.

We shall consider below that /33

$$\beta_1 > 0, \quad \beta_2 > 0, \quad \beta_1 \neq \beta_2.$$

Since U(x, y) is a real function from x, y, denoting

[1] In the case of divisible roots of equation (I.84) equation (I.82) is transformed into a generalized biharmonic equation, the solution method of which is provided by S. G. Mikhlin [1], by replacing the independent variable. However, as will be apparent below, the case of equal roots is very easily obtained from the general case for all partial problems, which we will examine. Therefore, we will not examine separately the case of equal roots of equation (I.84).

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$$z_1 = x + s_1 y = x + \alpha_1 y + i\beta_1 y,$$
$$z_2 = x + s_2 y = x + \alpha_2 y + i\beta_2 y, \tag{I.87}$$

we represent the common integral (I.85) in the form

$$U(x, y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}, \tag{I.88}$$

where $F_1(z_1)$ and $F_2(z_2)$ are two analytical[1] functions of their arguments; $\overline{F_1(z_1)}$ and $\overline{F_2(z_2)}$ are functions which are conjugate respectively to functions $F_1(z_1)$ and $F_2(z_2)$.

We denote:

$$\frac{dF_1}{dz_1} = \varphi(z_1), \qquad \frac{dF_2}{dz_2} = \psi(z_2). \tag{I.89}$$

Then, obviously,

$$\frac{\overline{dF_1}}{d\bar{z}_1} = \overline{\varphi(z_1)}, \qquad \frac{\overline{dF_2}}{d\bar{z}_2} = \overline{\psi(z_2)}.$$

---

[1] This follows directly from the fact that $2\mathrm{Re}[F_1(z_1)] = U_1(\xi_1, \eta_1)$ satisfies the equation

$$\frac{\partial^2 U_1}{\partial \xi_1^2} + \frac{\partial^2 U_1}{\partial \eta_1^2} = 0,$$

where $\xi_1 = x + \alpha_1 y$, $\eta_1 = \beta_1 y$.

The same is true of the function $2\mathrm{Re}[F_2(z_2)] = U_2(\xi_1, \eta_1)$ which satisfies the equation

$$\frac{\partial^2 U_2}{\partial \xi_2^2} + \frac{\partial^2 U_2}{\partial \eta_2^2} = 0,$$

where

$$\xi_2 = x + \alpha_2 y, \qquad \eta_2 = \beta_2 y.$$

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If we substitute stress function U(x, y) from (I.88) into (I.81) and take into consideration definition (I.89), we obtain general expressions for the stress components through the two analytical functions $\phi(z_1)$ and $\psi(z_2)$:

$$\begin{aligned}\sigma_x &= 2\mathrm{Re}\,[s_1^2\varphi'(z_1) + s_2^2\psi'(z_2)],\\ \sigma_y &= 2\mathrm{Re}\,[\varphi'(z_1) + \psi'(z_2)],\\ \tau_{xy} &= -2\mathrm{Re}\,[s_1\varphi'(z_1) + s_2\psi'(z_2)],\end{aligned} \tag{I.90}$$

where Re is the symbol for the real part of the adjacent expression:

$$\varphi'(z_1) = \frac{d\varphi}{dz_1}, \qquad \psi'(z_2) = \frac{d\psi}{dz_2};$$

s_1 and s_2 are roots of equation (I.84)

Substituting the values of σ_x, σ_y and τ_{xy} from (I.90) into equations (I.70) and integrating them, we obtain general expressions for the displacement components: /34

$$\begin{aligned}u(x, y) &= 2\mathrm{Re}\,[p_1\varphi(z_1) + p_2\psi(z_2)] - \gamma_0 y + \alpha_0,\\ v(x, y) &= 2\mathrm{Re}\,[q_1\varphi(z_1) + q_2\psi(z_2)] + \gamma_0 x + \beta_0,\end{aligned} \tag{I.91}$$

where, for brevity, we define the following:

$$\begin{aligned}p_1 &= a_{11}s_1^2 + a_{12} - a_{16}s_1, \qquad p_2 = a_{11}s_2^2 + a_{12} - a_{16}s_2,\\ q_1 &= \frac{a_{12}s_1^2 + a_{22} - a_{26}s_1}{s_1}, \qquad q_2 = \frac{a_{12}s_2^2 + a_{22} - a_{26}s_2}{s_2},\end{aligned} \tag{I.92}$$

and γ_0, α_0, β_0 are arbitrary real constants.

The additional terms $\gamma_0 y + \alpha_0$ and $\gamma_0 x + \beta_0$ in (I.91) describe body displacement and may be discarded when considering elastic equilibrium. Equations (I.90) and (I.91) were first obtained by S. G. Lekhnitskiy [5].

Stresses and displacements in polar coordinates :

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$$\left.\begin{aligned}
\sigma_r &= 2\mathrm{Re}\,[(\sin\theta - s_1\cos\theta)^2\varphi'(z_1) + (\sin\theta - s_2\cos\theta)^2\psi'(z_2)],\\
\sigma_\theta &= 2\mathrm{Re}\,[(\cos\theta + s_1\sin\theta)^2\varphi'(z_1) + (\cos\theta + s_2\sin\theta)^2\psi'(z_2)],\\
\tau_{r\theta} &= 2\mathrm{Re}\,[(\sin\theta - s_1\cos\theta)(\cos\theta + s_1\sin\theta)\varphi'(z_1) +\\
&\quad + (\sin\theta - s_2\cos\theta)(\cos\theta + s_2\sin\theta)\psi'(z_2)],\\
\sigma_r + \sigma_\theta &= 2\mathrm{Re}\,[(1 + s_1^2)\varphi'(z_1) + (1 + s_2^2)\psi'(z_2)];
\end{aligned}\right\} \qquad (I.93)$$

$$\left.\begin{aligned}
v_r &= 2\mathrm{Re}\,[(p_1\cos\theta + q\sin\theta)\varphi(z_1) + (p_2\cos\theta + q_2\sin\theta)\psi(z_2)],\\
v_\theta &= 2\mathrm{Re}\,[(q_1\cos\theta - p_1\sin\theta)\varphi(z_1) + (q_2\cos\theta - p_2\sin\theta)\psi(z_2)].
\end{aligned}\right\} \qquad (I.94)$$

Contour Conditions for Functions $\phi(z_1)$ and $\psi(z_2)$. In addition to the basic equation (I.82) the stress function U(x, y) should, depending upon the conditions of the problems, satisfy definite conditions on the contour of the region under consideration. Let us look at these conditions for each basic problem separately.

The first basic problem. Projections of external forces X_n, Y_n applied on the coordinate axis xOy are given on contour L of region S (n is positive normal[1]).

The contour conditions in this case, as is known, have the form /35

$$X_n = \sigma_x\cos(n, x) + \tau_{xy}\cos(n, y),$$
$$Y_n = \tau_{xy}\cos(n, x) + \sigma_y\cos(n, y).$$

Taking into consideration the fact that

$$\cos(n, x) = \frac{dy}{ds}, \qquad \cos(n, y) = -\frac{dx}{ds},$$

and expressing the stress components σ_x, σ_y and τ_{xy} by stress function U(x, y) (I.81), we obtain

$$X_n = \frac{\partial^2 U}{\partial y^2}\cdot\frac{dy}{ds} + \frac{\partial^2 U}{\partial x\partial y}\cdot\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial U}{\partial y}\right),$$
$$Y_n = -\frac{\partial^2 U}{\partial x\partial y}\cdot\frac{dy}{ds} - \frac{\partial^2 U}{\partial x^2}\cdot\frac{dx}{ds} = -\frac{d}{ds}\left(\frac{\partial U}{\partial x}\right)$$

[1]The positive normal to the contour, as usual, will be considered the normal to contour L directed to the right when circuiting the region in a positive direction. The positive direction of circuiting a region is the direction with which the region remains to the left.

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or

$$\frac{\partial U}{\partial x} = -\int_0^s Y_n ds + C_1, \qquad \frac{\partial U}{dy} = \int_0^s X_n ds + C_2, \tag{I.95}$$

where C_1 and C_2 are arbitrary real constants; s is an arc measured from an arbitrary point of contour L of region S.

Substituting function U(x, y) from (I.88) into the left parts of equations (I.95) and recalling definitions (I.89), we obtain final expressions of the contour conditions for functions $\phi(z_1)$ and $\psi(z_2)$:

$$\begin{gathered}\varphi(z_1) + \overline{\varphi(z_1)} + \psi(z_2) + \overline{\psi(z_2)} = -\int_0^s Y_n ds + C_1 = f_1, \\ s_1\varphi(z_1) + \bar{s}_1\overline{\varphi(z_1)} + s_2\psi(z_2) + \bar{s}_2\overline{\psi(z_2)} = \int_0^s X_n ds + C_2 = f_2.\end{gathered} \tag{I.96}$$

The second basic problem. Displacement components u(x, y) and v(x, y) are given on contour L of region S. The contour conditions for functions $\phi(z_1)$ and $\psi(z_2)$ are obtained directly from equations (I.90):

$$\begin{gathered}p_1\varphi(z_1) + \bar{p}_1\overline{\varphi(z_1)} + \bar{p}_2\overline{\psi(z_2)} + p_2\psi(z_2) = g_1(s), \\ q_1\varphi(z_1) + \bar{q}_1\overline{\varphi(z_1)} + q_2\psi(z_2) + \bar{q}_2\overline{\psi(z_2)} = g_2(s),\end{gathered} \tag{I.97}$$

where $g_1(s)$ and $g_2(s)$ are displacement components given as functions of arc s of contour L on contour L measured from an arbitrary point.

The third basic (combined[1]) problem. On one part $L^{(1)}$ of contour L of region S are given displacement components u and v, and on the remaining part /36 $L^{(2)}$ of contour $L = L^{(1)} + L^{(2)}$ -- projections of external forces X_n and Y_n on the coordinate axis. Obviously, in this problem the contour conditions for functions $\phi(z_1)$ and $\psi(z_2)$ on $L^{(1)}$ will be (I.97), and on $L^{(2)}$ -- (I.96).

Thus, the solution of the basic problems reduces to determining functions $\phi(z_1)$ and $\psi(z_2)$ according to contour conditions (I.96) and (I.97).

---

[1]Of no less interest are other combined problems, particularly, when on contour L or on a certain portion of it $L^{(k)}$ is given one of the stress components and one of the displacement vector components $\bar{u} = u \pm iv$. This is the so-called mixed problem of the "contact" type; some of them will be examined in Chapters III and IV.

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Expressions for the Resultant Vector and Resultant Moment. Later we will need expressions for the resultant moment and resultant vector of the forces acting on profile AB from the side of the positive normal.

To obtain a formula for the resultant vector, let us look at formula (I.95).

The main resultant vector of forces applied to element ds.

$$(X_n + iY_n)\,ds = d\left[\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right] = -id\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right].$$

Denoting the resultant force vector applied to contour AB by X + iY, we obtain

$$X + iY = \int_{AB} (X_n + iY_n)\,ds = -i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B, \tag{I.98}$$

where $[\ldots]_A^B$ denotes the increment of the expressions enclosed by the bracket when moving along arc AB from A to B. However, the expression $\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$ according to (I.95) and (I.96) may be given in the form

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = (1 + is_1)\varphi(z_1) + (1 + i\bar{s}_1)\overline{\varphi(z_1)} + (1 + is_2)\psi(z_2) + (1 + i\bar{s}_2)\overline{\psi(z_2)}.$$

Introducing this expression into equation (I.98), we obtain the final expression for the resultant vector:

$$X + iY = \int_A^B (X_n + iY_n)\,ds = \tag{I.99}$$

$$= -i\left[(1 + is_1)\varphi(z_1) + (1 + i\bar{s}_1)\overline{\varphi(z_1)} + (1 + is_2)\psi(z_2) + (1 + i\bar{s}_2)\overline{\psi(z_2)}\right]_A^B.$$

Let us now derive the formula for the resultant moment M of these forces applied to element ds of contour L relative to the coordinate origin: /37

$$M = \int_{AB} (xY_n - yX_n)\,ds. \tag{I.100}$$

Instead of X_n and Y_n let us substitute their values from (I.95):

$$M = -\int_{AB} \left\{x\,d\left(\frac{\partial U}{\partial x}\right) + y\,d\left(\frac{\partial U}{\partial y}\right)\right\}$$

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Integrating the latter equation by parts, we obtain

$$M = -\left[x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right]_A^B + \int_{AB}^{} \left(\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy\right),$$

or

$$M = -\left[x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right]_A^B + [U]_A^B.$$

Since

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \mathrm{Re}\left[z\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right)\right],$$

which can be proved by direct checking and taking into consideration the fact that

$$\frac{\partial U}{\partial x} = \varphi(z_1) + \overline{\varphi(z_1)} + \psi(z_2) + \overline{\psi(z_2)},$$
$$\frac{\partial U}{\partial y} = s_1\varphi(z_1) + \bar{s}_1\overline{\varphi(z_1)} + s_2\psi(z_2) + \bar{s}_2\overline{\psi(z_2)},$$

we finally obtain:

$$M = 2\mathrm{Re}[F_1(z_1) + F_2(z_2)]_A^B - \mathrm{Re}[z\{(1 - is_1)\varphi(z_1) + \\ + (1 - i\bar{s}_1)\overline{\varphi(z_1)} + (1 - is_2)\psi(z_2) + (1 - i\bar{s}_2)\overline{\psi(z_2)}\}]_A^B. \qquad \text{(I.101)}$$

In a singly-connected region (functions $\phi(z_1)$ and $\psi(z_2)$ in the case of a multiply-connected region will be examined below) the functions $\phi(z_1)$, $\psi(z_2)$, $F_1(z_1)$ and $F_2(z_2)$ are single valved. If point A coincides with point B, i.e., the profile AB is a closed contour, the value of all functions entering into formulas (I.99) and (I.101) at points A and B will coincide, and we obtain

$$X = Y = M = 0 \qquad \text{(I.102)}$$

This is the condition that the set of all external forces applied to the closed contour are statically equivalent to zero.

Degree of Definiteness of Functions $\phi(z_1)$ and $\psi(z_2)$. The functions $\phi(z_1)$ and $\psi(z_2)$ (I.89), through which the components of both stresses σ_x, σ_y and τ_{xy}

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(I.90) as well as displacements u(x, y) and v(x, y) (I.91) were introduced as the solution of basic equation (I.82).

On the other hand, the solution of the basic problems of elasticity theory was reduced to contour problems, i.e., to problems of determining the two analytical functions $\phi(z_1)$ and $\psi(z_2)$ according to given values of certain combinations (I.96) and (I.97) of these functions on a contour of the region under consideration. /38

The question arises whether these functions are uniquely determined. Are there any functions $\phi^*(z_1)$ and $\psi^*(z_2)$ which are identically not equal to zero, the addition of which to functions $\phi(z_1)$ and $\psi(z_2)$ does not change the stressed state of the body? What is the most general form of functions $\phi^*(z_1)$ and $\psi^*(z_2)$? What other conditions must be imposed on functions $\phi(z_1)$ and $\psi(z_2)$ so that they will be uniquely defined?

On the basis of the way in which the functions $\phi(z_1)$ and $\psi(z_2)$ were introduced we know [1] that there is a certain arbitrariness in their selection; furthermore, the elastic equilibrium of a body defined by these functions is the same. However, it is more natural in establishing the definiteness of the functions $\phi(z_1)$ and $\psi(z_2)$ to follow a different path -- to establish the form of functions $\phi(z_1)$ and $\psi(z_2)$ which satisfy contour conditions (I.96) in the case where contour L of region S is free of external forces. According to the uniqueness theorem in this case the body will be in the unstressed state, i.e., at all points of the body

$$\sigma_x = \sigma_y = \tau_{xy} = 0.$$

Thus, in contour conditions (I.96) we set $X_n = Y_n = 0$ and determine the form of functions $\phi(z_1)$ and $\psi(z_2)$ which satisfy the equations

$$\begin{aligned} \varphi(z_1) + \overline{\varphi(z_1)} + \psi(z_2) + \overline{\psi(z_2)} &= C_1, \\ s_1\varphi(z_1) + \bar{s}_1\overline{\varphi(z_1)} + s_2\psi(z_2) + \bar{s}_2\overline{\psi(z_2)} &= C_2, \end{aligned} \tag{I.103}$$

where C_1 and C_2 are two real constants.

We take from both parts of (I.103) the operator

---

[1] This is apparent from the fact that $\Delta_1 \mathrm{Re}\{F_1(z_1)\} \neq 0$ and $\Delta_2 \mathrm{Re}\{F_2(z_2)\} = 0$ (see footnote to formula (I.88)).

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$$\Delta_2 = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}.$$

Taking into consideration the fact that

$$z_2 = x + \alpha_2 y + i\beta_2 y = x_2 + iy_2,$$
$$z_1 = x + \alpha_1 y + i\beta_1 y = x_1 + iy_1,$$

we obtain

$$\varphi''(z_1)\left[1 + \left(\frac{\alpha_1 - \alpha_2}{\beta_2} + i\frac{\beta_1}{\beta_2}\right)^2\right] + \overline{\varphi''(z_1)}\left[1 + \left(\frac{\alpha_1 - \alpha_2}{\beta_2} - i\frac{\beta_1}{\beta_2}\right)^2\right] = 0,$$
$$s_1\left[1 + \left(\frac{\alpha_1 - \alpha_2}{\beta_2} + i\frac{\beta_1}{\beta_2}\right)^2\right]\varphi''(z_1) + \bar{s}_1\left[1 + \left(\frac{\alpha_1 - \alpha_2}{\beta_2} - i\frac{\beta_1}{\beta_2}\right)^2\right]\overline{\varphi''(z_1)} = 0. \quad (I.104)$$

The determinant of system (I.104) is different from zero (since $\beta_1 \neq 0$ and $\beta_2 = 0$ and, furthermore, we agreed to select them so that $\beta_1 > 0$ and $\beta_2 > 0$) and /39

$$-i2\beta_1\left\{1 + 2\left(\frac{\alpha_1 - \alpha_2}{\beta_2}\right)^2 + 2\frac{\beta_1^2}{\beta_2^2} + \left[\left(\frac{\alpha_1 - \alpha_2}{\beta_2}\right)^2 + \frac{\beta_1^2}{\beta_2^2}\right]^2\right\} \neq 0.$$

Consequently,

$$\varphi''(z_1) \equiv 0. \quad (I.105)$$

Now we take the operator

$$\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

from both equations of (I.103) and, proceeding analogously, we obtain

$$\psi''(z_2) \equiv 0. \quad (I.106)$$

It follows from (I.105) and (I.106) that functions (z_1) and (z_2) may be only of the form

$$\varphi(z_1) = A_1 z_1 + B_1, \quad \psi(z_2) = A_2 z_2 + B_2, \quad (I.107)$$

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where $A_1 = A_1' + iA_1''$; $A_2 = A_2' + iA_2''$; $B_1 = B_1' + iB_1''$; $B_2 = B_2' + iB_2''$ are arbitrary complex constants.

However, functions $\phi(z_1)$ and $\psi(z_2)$ (I.107) must satisfy equation system (I.103) or (according to the uniqueness theorem), exactly the same, equations (I.90):

$$\begin{aligned}
&\varphi'(z_1) + \overline{\varphi'(z_1)} + \psi'(z_2) + \overline{\psi'(z_2)} = 0,\\
&s_1^2\varphi'(z_1) + \bar{s}_1^2\overline{\varphi'(z_1)} + s_2^2\psi'(z_2) + \bar{s}_2^2\overline{\psi'(z_2)} = 0,\\
&s_1\varphi'(z_1) + \bar{s}_1\overline{\varphi'(z_1)} + s_2\psi'(z_2) + \bar{s}_2\overline{\psi'(z_2)} = 0.
\end{aligned} \tag{I.108}$$

Substituting the values of functions $\phi(z_1)$ and $\psi(z_2)$ from (I.107) into equation system (I.108), we obtain

$$\begin{aligned}
&A_1' + A_2' = 0,\\
&\alpha_1 A_1' + \alpha_2 A_2' - \beta_1 A_1'' - \beta_2 A_2'' = 0,\\
&(\alpha_1^2 - \beta_1^2) A_1' - 2\alpha_1\beta_1 A_1'' + (\alpha_2^2 - \beta_2^2) A_2' - 2\alpha_2\beta_2 A_2'' = 0.
\end{aligned} \tag{I.109}$$

It follows from system (I.109) that one of the four constants A_1', A_1'', A_2', A_2'' remains undefined and may be given arbitrarily. Let us take A_1'' as this undefined value and express all the others in terms of it. Solving equation system (I.109), we obtain:

$$\left.\begin{aligned}
A_1' &= \frac{2\beta_1\beta_2(\alpha_1 - \alpha_2)}{\beta_2(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2) - 2\alpha_2\beta_2(\alpha_1 - \alpha_2)} A_1'',\\
A_2' &= -\frac{2\beta_1\beta_2(\alpha_1 - \alpha_2)}{\beta_2(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2) - 2\alpha_2\beta_2(\alpha_1 - \alpha_2)} A_1'',\\
A_2'' &= \frac{2\alpha_1\beta_1(\alpha_1 - \alpha_2) - \beta_1(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2)}{\beta_2(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2) - 2\alpha_2\beta_2(\alpha_1 - \alpha_2)} A_1''.
\end{aligned}\right| \tag{I.110}$$

Consequently, /40

$$\varphi'(z_1) = A_1''i\left[1 - i\frac{2\beta_1\beta_2(\alpha_1 - \alpha_2)}{\beta_2(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2) - 2\alpha_2\beta_2(\alpha_1 - \alpha_2)}\right] = iA_1''\alpha,$$

$$\psi'(z_2) = A_1''i\left[\frac{\beta_1(\alpha_2^2 - \beta_2^2 - \alpha_1^2 + \beta_1^2) + 2\beta_1(\alpha_1 - \alpha_2)(\alpha_1 + i\beta_2)}{\beta_2(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2) - 2\alpha_2\beta_2(\alpha_1 - \alpha_2)}\right] = iA_1''\beta,$$

where α and β denote the expressions enclosed in the brackets.

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As will be apparent later, for many materials $s_1 = i\beta_1$; $s_2 = i\beta_2$, i.e., $a_1 = \alpha_2 = 0$, then $\psi'(z_1) = iA_1''$, i.e., $\alpha = 1$, and $\psi'(z_2) = -i \cdot \beta_1/\beta_2 \cdot A_1''$, i.e., $\beta = -\beta_1/\beta_2$.

Thus, the functions $\phi(z_1)$ and $\psi(z_2)$ (I.107) will have the form

$$\varphi(z_1) = iA_1'\alpha z_1 + B_1, \quad \psi(z_2) = iA_1'\beta z_2 + B_2. \tag{I.111}$$

The constants $B_1 = B_1' + iB_1''$ and $B_2 = B_2' + iB_2''$ remain arbitrary.

Thus, in the case of the first basic problem the functions $\phi(z_1)$ and $\psi(z_2)$ contain five arbitrary real constants -- A_1'', B_1', B_1'', B_2' and B_2'', to which may be assigned any desired value.[1]

If the coordinate origin is within the region under consideration, these constants may be given as follows:

$$\operatorname{Im}\{\varphi'(0)\} = 0, \quad \varphi(0) = 0, \quad \psi(0) = 0, \tag{I.112}$$

where Im is the symbol of the imaginary part of the adjacent expression.

Let us now see how many undefined constants remain in functions $\phi(z_1)$ and $\psi(z_2)$ (I.111) for the second basic problem. In this case contour conditions (I.97) are assigned exactly. Consequently, of the five values A_1'', B_1', B_1'', B_2' and B_2'' we may arbitrarily fix only three, and the remaining are determined from the contour conditions (I.97) themselves. Considering that the coordinate origin is within the region, these constants may be written thus:

$$\begin{gathered}\operatorname{Im}\{\varphi'(0)\} = 0, \\ \varphi(0) = 0.\end{gathered} \tag{I.113}$$

It is obvious that conditions (I.112) or (I.113) exclude arbitrariness in selecting the functions $\phi(z_1)$ and $\psi(z_2)$.

---

[1] It should be considered that if the constants C_1 and C_2 in contour conditions (I.96) are assigned in any manner, one of the constants B_1 or B_2 cannot be assigned, for B_1 and B_2 are associated by correlations $B_1 + \overline{B}_1 + B_2 + \overline{B}_2 = C_1$ and $s_1 B_1 + \overline{s}_1 \overline{B} + s_2 B_2 + \overline{s}_2 \overline{B}_2 = C_2$, as follows from the same contour condition (I.103).

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The Form of Functions $\phi(z_1)$ and $\psi(z_2)$ in the Case of a Multiply-Connected Region. A finite multiply-connected region. Let us assume that elastic anisotropic material occupies not all of a plane, but, as shown in Figure I.3, only the portion of it which is shaded (a sheet with various holes). In this case region S, occupied by the body, is multiply-connected. /41

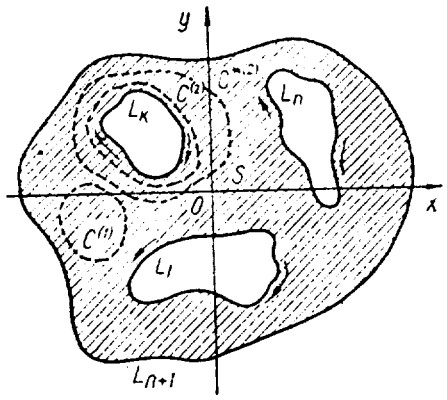


Figure I.3.

Let us place the coordinate region within the region and denote by L_1, L_2, ..., L_n, L_{n+1} the contours which bound the region under consideration.

We shall consider that: 1) L_1, L_2, ..., L_n, L_{n+1} are closed smooth[1] contours which do not intersect; 2) contour L_{n+1} encompasses all of the preceding.

Each contour L_n has applied to it external forces under the action of which the body is in the elastic equilibrium state. Let us denote the resultant vector of the external forces applied to contour L_k by $X_k + iY_k$.

Let us mentally isolate some part of the region (in Figure I.3 the dotted lines denote these contours $C^{(1)}$ and $C^{(2)}$) and determine according to formula (I.99) the resultant vector of the external forces applied to the contour of the cut out portion of region S. If the region thus cut out is singly-connected (for example, in Figure I.3 the region bounded by contour $C^{(1)}$), the resultant vector will be equal to zero, since the functions $\phi(z_1)$ and $\psi(z_2)$ within this region are identical. If the region is taken which is bounded by contour $C^{(2)}$ the resultant vector of the external forces applied to contour $C^{(2)}$ will, in general, be different from zero and equal to $-(X_k + iY_k)$. The latter affirmation comes directly from the fact that the entire body under the action of the given forces is in equilibrium, and consequently, any part of it is in equilibrium. However, since contour L_k is under the influence of forces, the resultant vector of which is $X_k + iY_k$, consequently, contour $C^{(2)}$ is subject to external forces, the resultant vector of which is equal to $-(X_k + iY_k)$. Instead of contour $C^{(2)}$, obviously, any other contour $C^{*(2)}$ may be taken which embraces L_k, with the condition that it does not encompass or intersect the other contours.

---

[1]More accurately, let us assume that the contours have a continuously changing tangent.

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It is obvious that the resultant vector of the external forces applied to contours $C^{(2)}$, $C^{*(2)}$ and so forth will be the same. From here it follows that /42 $\phi(z_1)$ and $\psi(z_2)$ in the case of a multiply-connected region should be sought in the form

$$\varphi(z_1) = \sum_{k=1}^{n} A_k \ln(z_1 - z_{1,k}) + \varphi^*(z_1),$$
$$\psi(z_2) = \sum_{k=1}^{n} B_k \ln(z_2 - z_{2,k}) + \psi^*(z_2), \tag{I.114}$$

where $z_{1,k}$, $z_{2,k}$ are points arbitrarily fixed within contours L_k; $\phi^*(z_1)$ and $\psi^*(z_2)$ are single valved functions in region S; A_k and B_k are certain complex constants subject to definition.

Constants A_k and B_k, obviously, must be determined from the conditions: 1) the resultant vector of the forces applied to contour $C^{(2)}$ (or $C^{*(2)}$), is equal to $-(X_k + iY_k)$; 2) displacements u and v, determined by formulas (I.91) are identical, i.e., from equations[1]

$$(1 + is_1)A_k - (1 + i\bar{s}_1)\bar{A}_k + (1 + is_2)B_k - (1 + i\bar{s}_2)\bar{B}_k = -\frac{X_k + iY_k}{2\pi}$$
$$(p_1 + iq_1)A_k - (\bar{p}_1 + i\bar{q}_1)\bar{A}_k + (p_2 + iq_2)B_k - (p_2 + iq_2)\bar{B}_k = 0. \tag{I.115}$$

Let us add to these equations their conjugate equations

$$(1 + is_1)A_k - (1 + i\bar{s}_1)\bar{A}_k + (1 + is_2)B_k - (1 + i\bar{s}_2)\bar{B}_k = -\frac{X_k + iY_k}{2\pi},$$
$$(p_1 + iq_1)A_k - (\bar{p}_1 + i\bar{q}_1)\bar{A}_k + (p_2 + iq_2)B_k - (\bar{p}_2 + i\bar{q}_2)\bar{B}_k = 0,$$
$$-(1 - is_1)A_k + (1 - i\bar{s}_1)\bar{A}_k - (1 - is_2)B_k + (1 - i\bar{s}_2)\bar{B}_k = -\frac{X_k - iY_k}{2\pi}, \tag{I.116}$$
$$-(p_1 - iq_1)A_k + (\bar{p}_1 - i\bar{q}_1)\bar{A}_k - (p_2 - iq_2)B_k + (\bar{p}_2 - i\bar{q}_2)\bar{B}_k = 0.$$

[1]The second equation of (I.115) is derived from (I.91) and represents an increment of the expression [u + iv] when passing around the contour L_k.

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The determinant of system (I.116)

$$D = \frac{16a_{11}a_{22}\beta_1\beta_2}{(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)}\{2[(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2](\beta_2+\beta_1)^2+(\alpha_2-\alpha_1)^4\}, \qquad (I.117)$$

is obviously different from zero.

For the particular values of $s_1 = i\beta_1$ and $s_2 = i\beta_2$, i.e., when $\alpha_1 = \alpha_2 = 0$,

$$D = \frac{16a_{11}a_{22}}{\beta_1\beta_2}(\beta_1^2-\beta_2^2)^2.$$

Solving system (I.116) we find fully defined values of A_k and B_k. If the resultant vector $X_k + iY_k$ of the external forces applied to each contour (see Figure I.3) is equal to zero, then setting $X_k = Y_k = 0$ in (I.116), we find /43

$$A_k = B_k = 0.$$

Functions $\phi(z_1)$ and $\psi(z_2)$ (I.114) in this case will be regular and single-valued functions in region S.

An infinite multiply-connected region. Of great practical interest[1] is the case in which contour L_{n+1} approaches infinity, i.e., the case of an infinite multiply-connected region (Figure I.4).

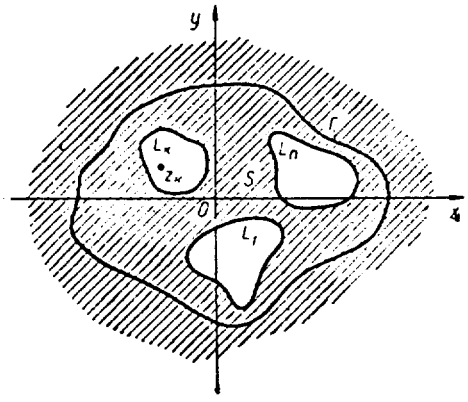


Figure I.4

Let us mentally draw contour Γ, which would encompass all of contours L_1, L_2, ..., L_n. For greater clarity it is convenient to introduce two more planes z_1 and z_2 (Figures I.5 and I.6) which are obtained from z by affine transformation:

$$x_1 = x + \alpha_1 y, \quad y_1 = \beta_1 y, \qquad (I.118)$$
$$x_2 = x + \alpha_2 y, \quad y_2 = \beta_2 y$$

[1] A very large number of practical problems concerning stress concentration may be cited in connection with this problem. If the sizes of the holes are small, a plate of finite dimensions may be taken as infinite under the condition that the holes are not located close to its edges.

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With this transformation region S (Figure I.4) enters into regions $S^{(1)}$ and $S^{(2)}$ of the same connectedness as the assigned region (Figure I.5-I.6). However, the circuit direction of regions $S^{(1)}$ and $S^{(2)}$ may be both forward and reverse.

The circuit direction of regions $S^{(1)}$ and $S^{(2)}$ is maintained if the corresponding functional determiners[1] are

$$D\begin{bmatrix} x_1, & y_1 \\ x, & y \end{bmatrix} = \beta_1 > 0, \quad D\begin{bmatrix} x_2, & y_2 \\ x, & y \end{bmatrix} = \beta_2 > 0. \tag{I.119}$$

We agreed earlier to select the values β_1 and β_2 in order to maintain the circuit direction of regions S, $S^{(1)}$ and $S^{(2)}$.

The function $\phi(z_1)$ will a function of a simple complex variable z_1 in region $S^{(1)}$, and the function $\psi(z_2)$ -- in region $S^{(2)}$.

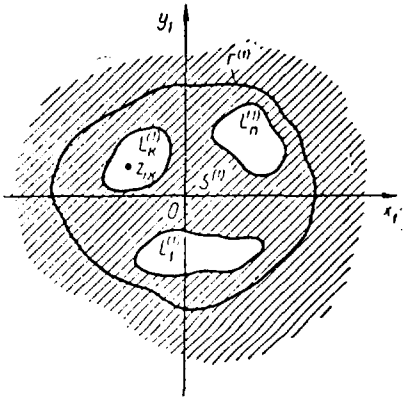


Figure I.5.

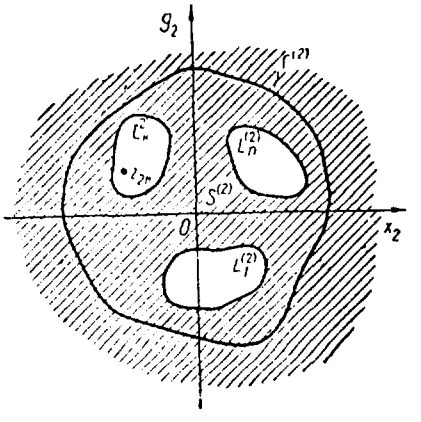


Figure I.6.

Obviously, for any point z located outside Γ (Figure I.4), points z_1 corresponding to it are found outside $\Gamma^{(1)}$ (Figure I.5) and z_2 outside $\Gamma^{(2)}$ (Figure I.6), for which, obviously, when $|z| > |z_k|$, $|z_1| > |z_{1,k}|$ and $|z_2| > |z_{2,k}|$ (k = 1, 2, ..., n), where $z_{1,k}$, $z_{2,k}$ are points inside contours $L_k^{(1)}$, $L_k^{(2)}$ (Figures I.5 and I.6) and, consequently, /44

[1] See E. Gursa [2], p. 288.

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$$\ln(z_1 - z_{1,k}) = \ln z_1 + \ln\left(1 - \frac{z_{1,k}}{z_1}\right) = \ln z_1 - \frac{z_{1,k}}{z_1} -$$

$$- \frac{1}{2}\left(\frac{z_{1,k}}{z_1}\right)^2 - \ldots = \ln z_1 + \text{function holomorphic outside } \Gamma^{(1)};$$

$$\ln(z_2 - z_{2,k}) = \ln z_2 + \ln\left(1 - \frac{z_{2,k}}{z_2}\right) = \ln z_2 - \frac{z_{2,k}}{z_2} - \frac{1}{2}\left(\frac{z_{2,k}}{z_2}\right)^2 - \ldots = \ln z_2 +$$

$$+ \text{function holomorphic outside } \Gamma^{(2)}.$$

Substituting the expressions $\ln(z_1 - z_{1,k})$ and $\ln(z_2 - z_{2,k})$ in (I.114), we obtain

$$\varphi(z_1) = A \ln z_1 + \varphi^{**}(z_1),$$
$$\psi(z_2) = B \ln z_2 + \psi^{**}(z_2), \tag{I.120}$$

where $\phi^{**}(z_1)$ and $\psi^{**}(z_2)$ denote functions which are respectively holomorphic outside $\Gamma^{(1)}$ and $\Gamma^{(2)}$, except, perhaps, for the infinitely remote point; $A = \Sigma A_k$ and $B = \Sigma B_k$ are constants including, obviously, the resultant vector of all external forces applied to contour $L = L_1 + L_2 + \ldots + L_n$ of region S (Figure I.4).

Functions $\phi^{**}(z_1)$ and $\psi^{**}(z_2)$ outside $\Gamma^{(1)}$ and $\Gamma^{(2)}$ may be expanded (each in its own plane) into a Loran series: /45

$$\varphi^{**}(z_1) = \sum_{-\infty}^{+\infty} a_n z_1^n,$$
$$\psi^{**}(z_2) = \sum_{-\infty}^{\infty} a_n' z_2^n. \tag{I.121}$$

We assume that $\sigma_x^{(\infty)} = \text{const}$, $\sigma_y^{(\infty)} = \text{const}$, $\tau_{xy}^{(\infty)} = \text{const}$, i.e., we introduce the condition that the stress components σ_x, σ_y and τ_{xy} remain bounded[1] throughout region S.

Substituting the values of functions $\phi(z_1)$ and $\psi(z_2)$ from (I.120) into the formulas for the stress components (I.90) and taking into consideration the expansion (I.121), we obtain

[1]The more general case easily reduces to that under consideration, for example, in §1 of Chapter III.

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$$\sigma_x = \frac{s_1^2 A}{z_1} + \sum_{-\infty}^{+\infty}{}' s_1^2 n a_n z_1^{n-1} + \frac{\bar{s}_1^2 \bar{A}}{\bar{z}_1} + \sum_{-\infty}^{+\infty}{}' n \bar{s}_1^2 \bar{a}_n \bar{z}_1^{n-1} +$$
$$+ \frac{s_2^2 B}{z_2} + \sum_{-\infty}^{+\infty}{}' s_2^2 n a_n' z_2^{n-1} + \frac{\bar{s}_2^2 \bar{B}}{\bar{z}_2} + \sum_{-\infty}^{+\infty}{}' \bar{s}_2^2 n \bar{a}_n' z_2^{n-1};$$
$$\sigma_y = \frac{A}{z_1} + \sum_{-\infty}^{+\infty} n a_n z_1^{n-1} + \frac{\bar{A}}{\bar{z}_1} + \sum_{-\infty}^{+\infty} n \bar{a}_n \bar{z}_1^{n-1} + \qquad \text{(I.122)}$$
$$+ \frac{B}{z_2} + \sum_{-\infty}^{+\infty} n a_n' z_2^{n-1} + \frac{\bar{B}}{\bar{z}_2} + \sum_{-\infty}^{+\infty} n \bar{a}_n' \bar{z}_2^{n-1};$$
$$-\tau_{xy} = \frac{A s_1}{z_1} + \sum_{-\infty}^{+\infty} s_1 n a_n z_1^{n-1} + \frac{\bar{A} \bar{s}_1}{\bar{z}_1} + \sum_{-\infty}^{+\infty} \bar{s}_1 n \bar{a}_n \bar{z}_1^{n-1} +$$
$$+ \frac{B s_2}{z_2} + \sum_{-\infty}^{+\infty} s_2 n a_n' z_2^{n-1} + \frac{\bar{B} \bar{s}_2}{\bar{z}_2} + \sum_{-\infty}^{+\infty} \bar{s}_2 n \bar{a}_n' \bar{z}_2^{n-1}.$$

Let us represent $z_1 = r_1 e^{i\theta_1}$; $z_2 = r_2 e^{i\theta_2}$ and rewrite the preceding formulas in the form

$$\sigma_x = \left(\frac{s_1^2 A}{z_1} + \frac{\bar{s}_1^2 \bar{A}}{\bar{z}_1}\right) + \sum_{-\infty}^{+\infty} r_1^{n-1} n \left(s_1^2 a_n e^{i\theta_1(n-1)} + \bar{s}_1^2 \bar{a}_n e^{-i\theta_1(n-1)}\right) +$$
$$+ \left(\frac{s_2^2 B}{z_2} + \frac{\bar{s}_2^2 \bar{B}}{\bar{z}_2}\right) + \sum_{-\infty}^{+\infty} r_2^{n-1} n \left(s_2^2 a_n' e^{i\theta_2(n-1)} + \bar{s}_2^2 \bar{a}_n' e^{-i\theta_2(n-1)}\right);$$
$$\sigma_y = \left(\frac{A}{z_1} + \frac{\bar{A}}{\bar{z}_1}\right) + \sum_{-\infty}^{+\infty} n r_1^{n-1} \left(a_n e^{i\theta_1(n-1)} + \bar{a}_n e^{-i\theta_1(n-1)}\right) +$$
$$+ \left(\frac{B}{z_2} + \frac{\bar{B}}{\bar{z}_2}\right) + \sum_{-\infty}^{+\infty} n r_2^{n-1} \left(a_n' e^{i\theta_2(n-1)} + \bar{a}_n' e^{-i\theta_2(n-1)}\right); \qquad \text{/46}$$
$$-\tau_{xy} = \left(\frac{s_1 A}{z_1} + \frac{\bar{s}_1 \bar{A}}{\bar{z}_1}\right) + \sum_{-\infty}^{+\infty} n r_1^{n-1} \left(s_1 a_n e^{i\theta_1(n-1)} + \bar{s}_1 \bar{a}_n e^{-i\theta_1(n-1)}\right) +$$
$$+ \left(\frac{s_2 B}{z_2} + \frac{\bar{s}_2 \bar{B}}{\bar{z}_2}\right) + \sum_{-\infty}^{+\infty} n r_2^{n-1} \left(s_2 a_n' e^{i\theta_2(n-1)} + \bar{s}_2 \bar{a}_n' e^{-i\theta_2(n-1)}\right).$$

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For the stress components σ_x, σ_y and τ_{xy} to be bounded throughout the plane with any θ_1, θ_2 and r_1, r_2 it is obviously necessary that

$$\left.\begin{aligned} s_1^2 a_n &= \bar{s}_1^2 \bar{a}_n \equiv 0, & s_2^2 a_n' &= \bar{s}_2^2 \bar{a}_n' \equiv 0, \\ s_1 a_n &= \bar{s}_1 \bar{a}_n \equiv 0, & s_2 a_n' &= \bar{s}_2 \bar{a}_n' \equiv 0, \\ a_n &= \bar{a}_n \equiv 0, & a_n' &= \bar{a}_n' \equiv 0. \end{aligned}\right\} \text{for } n \geqslant 2.$$

All of these equations will be satisfied if

$$a_n = \bar{a}_n \equiv 0, \quad a_n' = \bar{a}_n' \equiv 0 \text{ for } n \geqslant 2. \tag{I.123}$$

Thus, functions $\phi(z_1)$ and $\psi(z_2)$ (I.120) have the form

$$\begin{aligned} \varphi(z_1) &= A \ln z_1 + (B^* + iC^*) z_1 + \varphi_0(z_1), \\ \psi(z_2) &= B \ln z_2 + (B'^* + iC'^*) z_2 + \psi_0(z_2), \end{aligned} \tag{I.124}$$

where $a_1 = B^* + iC^*$, $a_1' = B'^* + iC'^*$; $\phi_0(z_1)$ and $\psi_0(z_2)$ are functions holomorphic at infinity, i.e.,

$$\begin{aligned} \varphi_0(z_1) &= a_0 + \frac{a_{-1}}{z_1} + \frac{a_{-2}}{z_1^2} + \frac{a_{-3}}{z_1^3} + \ldots, \\ \psi_0(z_2) &= a_0' + \frac{a_{-1}'}{z_2} + \frac{a_{-2}'}{z_2^2} + \frac{a_{-3}'}{z_2^3} + \ldots. \end{aligned} \tag{I.125}$$

The constants B*, C*, B'* and C'* in (I.124) may be expressed through the stress components at infinity, considering the stressed state at the infinitely remote part of the plane to be equivalent.

Let at infinity be given the stresses

$$\sigma_x^{(\infty)} = \text{const}, \quad \sigma_y^{(\infty)} = \text{const}, \quad \tau_{xy}^{(\infty)} = \text{const}. \tag{I.126}$$

Substituting functions $\phi(z_1)$ and $\psi(z_2)$ (I.124) into equations (I.90) and directing $z_1 \to \infty$ and $z_2 \to \infty$, we obtain /47

$$\begin{aligned} s_1^2(B^* + iC^*) + \bar{s}_1^2(B^* - iC^*) + s_2^2(B'^* + iC'^*) + \bar{s}_2^2(B'^* - iC'^*) &= \sigma_x^{(\infty)}, \\ 2B^* + 2B'^* &= \sigma_y^{(\infty)}, \\ s_1(B^* + iC^*) + \bar{s}_1(B^* - iC^*) + s_2(B'^* + iC'^*) + \bar{s}_2(B'^* - iC'^*) &= -\tau_{xy}^{(\infty)}. \end{aligned} \tag{I.127}$$

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We see from (I.127) that one of the values B*, C*, B'* and C'* may be given[1] arbtrarily.

Let us assume C* = 0. The determinant[2] of system is

$$D = -8\beta_2 I\,(\alpha_1 - \alpha_2)^2 + (\beta_2^2 - \beta_1^2)].$$

Solving system (I.127), we obtain

$$B^* = \frac{\sigma_x^{(\infty)} + (\alpha_2^2 + \beta_2^2)\,\sigma_y^{(\infty)} + 2\alpha_2\tau_{xy}^{(\infty)}}{2\,[(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2)]},$$

$$B'^* = \frac{(\alpha_1^2 - \beta_1^2)\,\sigma_y^{(\infty)} - 2\alpha_1\alpha_2\sigma_y^{(\infty)} - \sigma_x^{(\infty)} - 2\alpha_2\tau_{xy}^{(\infty)}}{2\,[(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2)]}, \qquad \text{(I.128)}$$

$$C'^* = \frac{(\alpha_1 - \alpha_2)\,\sigma_x^{(\infty)} + [\alpha_2\,(\alpha_1^2 - \beta_1^2) - \alpha_1\,(\alpha_2^2 - \beta_2^2)]\,\sigma_y + [(\alpha_1^2 - \beta_1^2) - (\alpha_2^2 - \beta_2^2)]\,\tau_{xy}^{(\infty)}}{2\beta_2\,[(\alpha_2 - \alpha_1)^2 - (\beta_2^2 - \beta_1^2)]}.$$

Formulas (I.128) take on an especially simple form when[3]

---

[1]The fourth constant may be associated with revolution of the infinitely remote part of plane xOy. Substituting the values u and v from (I.91) into the revolution equation $e = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ and directing x and y to infinity, we obtain an inadequate equation. Adding the equation obtained to system (I.127) we obtain a system of four equations, the determinant of which when $s_1 \neq s_2$ is different from zero:

$$D = -\frac{16a_{22}\beta_1\beta_2}{(\alpha_1^2 + \beta_1^2)(\alpha^2 + \beta_2^2)}\,[(\alpha_2 - \alpha_1)^4 + (\beta_1^2 - \beta_2^2)^2 + 2\,(\alpha_2 - \alpha_1)^2\,(\beta_1^2 + \beta_2^2)].$$

Setting the value of $\varepsilon^{(\infty)}$, we obtain from this system of four equations fully defined values for B*, C*, B'* and C'*. However, from a purely practical viewpoint it is more convenient to set one of the constants C* or C'* in advance.

[2]The case of equal roots is excluded. Consequently, it is always possible to select values such that $\beta_2 - \beta_1 > 0$. If $\beta_2 - \beta_1 < 0$, then setting C'* = 0 in (I.127), we obtain

$$D = -8\beta_1\,[(\alpha_1 - \alpha_2)^2 + (\beta_1^2 - \beta_2^2)].$$

[3]For most materials with three elastic symmetry planes s_1 and s_2 have this form only if the direct intersections of these elastic symmetry planes serve as the chorded axes.

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$$s_1 = i\beta_1, \quad s_2 = i\beta_2.$$

Assuming $\alpha_1 = \alpha_2 = 0$ in (I.128), we obtain /48

$$B^* = -\frac{\sigma_x^{(\infty)} + \beta_2^2\sigma_y^{(\infty)}}{2(\beta_1^2 - \beta_2^2)},$$

$$B'^* = \frac{\sigma_x^{(\infty)} + \beta_1^2\sigma_y^{(\infty)}}{2(\beta_1^2 - \beta_2^2)}, \qquad \text{(I.129)}$$

$$C'^* = \frac{\tau_{xy}^{(\infty)}}{2\beta_2}.$$

Substituting in functions $\phi(z_1)$ and $\psi(z_2)$ (I.124) the values found for B*, B'*, C'* from (I.128) or (I.129) and C* = 0, we obtain the final form of these functions.

According to the above, we may always consider that $a_0 = 0$ and $a_0' = 0$, i.e., $\phi_0(\infty) = \psi_0(\infty) = 0$, and, in addition, as was assumed before, C* = 0.

An infinite singly-connected region. Let us assume that the infinite region under consideration is not multiply-connected, but singly-connected, or in other words, an infinite sheet with a single hole. It is obvious that if we place the coordinate origin outside of our region, i.e., within the hole, all of the above concerning functions $\phi(z_1)$ and $\psi(z_2)$ will remain valid, assuming that the role of contour Γ (Figure I.4) is played by contour L of the hole.

Let us substitute functions $\phi(z_1)$ and $\psi(z_2)$ (I.124) into the formulas for the displacement coefficients (I.91). For the displacements at infinity to remain bounded, these conditions must be satisfied: 1) the principal vector of the external forces applied to contour L of the hole must equal zero, i.e., X = Y = 0; 2) the stresses and rotation at infinity must also equal zero, i.e., B* = B'* = C* = C'* = 0.

Later, values are needed for coefficients A and B defined as the solution of equation system (I.116) for an orthotropic sheet with a round hole.

Directing coordinate axes x and y along the principal directions of elasticity of the sheet material, we obtain

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$$A = \frac{X}{4\pi}\cdot\frac{\beta_1(a_{22}-a_{12}\beta_2^2)}{a_{22}(\beta_1^2-\beta_2^2)} + i\frac{Y}{4\pi}\cdot\frac{a_{11}\beta_2^2-a_{12}}{a_{11}(\beta_1^2-\beta_2^2)},$$

$$B = \frac{X}{4\pi}\cdot\frac{\beta_2(a_{12}\beta_1^2-a_{22})}{a_{22}(\beta_1^2-\beta_2^2)} - i\frac{Y}{4\pi}\cdot\frac{a_{11}\beta_1^2-a_{12}}{a_{11}(\beta_1^2-\beta_2^2)}. \qquad (I.130)$$

On the basis of the preceding formulas we cite the values of constants A and B for the case of one concentrated force P applied normally to the contour of a round opening at a point determined by angle α, measured in the positive direction of axis Ox counter clockwise: /49

$$A = \frac{P\cos\alpha}{4\pi}\cdot\frac{\beta_1(a_{22}-a_{12}\beta_2^2)}{a_{22}(\beta_1^2-\beta_2^2)} + i\frac{P\sin\alpha}{4\pi}\cdot\frac{a_{11}\beta_2^2-a_{12}}{a_{11}(\beta_1^2-\beta_2^2)},$$

$$B = \frac{P\cos\alpha}{4\pi}\cdot\frac{\beta_2(a_{12}\beta_1^2-a_{22})}{a_{22}(\beta_1^2-\beta_2^2)} + i\frac{P\sin\alpha}{4\pi}\cdot\frac{a_{12}-a_{11}\beta_1^2}{a_{11}(\beta_1^2-\beta_2^2)} \qquad (I.131)$$

Above, the solution of the basic contour problems of elasticity theory reduced to determining the functions $\phi(z_1)$ and $\psi(z_2)$ by contour conditions (I.96) and (I.97). Functions $\phi(z_1)$ and $\psi(z_2)$, generally speaking, in the case of a multiply connected region (both finite and infinite) are ambiguous. This is because the resultant vector of the external forces applied to contour L_k was different from zero. Using a very simple approach it is always possible to reduce the solution of the problem posed to the case where the principal external force vector is equal to zero[1] on each of boundaries L_k individually.

Let us examine more thoroughly the first basic problem. Functions $\phi(z_1)$ and $\psi(z_2)$ for this case have the form of (I.114). Let us substitute functions $\phi(z_1)$ and $\psi(z_2)$ (I.114) into equations (I.96)

$$\varphi^*(z_1) + \overline{\varphi^*(z_1)} + \psi^*(z_2) + \overline{\psi^*(z_2)} = f_1 - \sum_{k=1}^{n} A_k \ln(z - z_{1,k}) -$$

$$-\sum_{k=1}^{n}\overline{A}_k\ln(\bar{z}_1-\bar{z}_{1,k}) - \sum_{k=1}^{n}[B_k\ln(z_2-z_{2,k}) + \overline{B}_k\ln(\bar{z}_2-\bar{z}_{2,k})] = f_1^*; \qquad (I.132)$$

[1] In this case, as follows from (I.116), functions $\phi(z_1)$ and $\psi(z_2)$ will be regular in region S.

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$$s_1\varphi^*(z_1) + \overline{s_1}\overline{\varphi^*(z_1)} + s_2\psi^*(z_2) + \overline{s_2}\overline{\psi^*(z_2)} = f_2 - \sum_{k=1}[A_k s_1 \ln(z_1 - z_{1,k}) +$$

$$+ \overline{A_k}\overline{s_1}\ln(\overline{z_1} - \overline{z_{1,k}})] - \sum_{k=1}^{n}[B_k s_2 \ln(z_2 - z_{2\,k}) + \overline{B_k}\overline{s_2}\ln(\overline{z_2} - \overline{z_{2,k}})] = f_2^*,$$

where $\phi^*(z_1)$ and $\psi^*(z_2)$ are unique functions in region S.

Now it is easy to see from (I.132), that f_1^* and f_2^* are unique functions. Actually, when circuiting contour L_k in the positive direction the increment is $[f_2^*] = X_k - X_k = 0$; likewise, increment $[f_1^*] = -Y_k + Y_k = 0$. When circuiting the external contour L_{n+1} we obtain X = Y = 0, i.e., the resultant vector of all external forces applied to contour $L = L_1 + L_2 + \ldots + L_n$ is equal to zero.

In the case of the second basic problem this approach makes it possible to reduce the problem to the solution of functions $\phi^*(z_1)$ and $\psi(z_2)$ which are in region S by contour conditions (I.97). In the latter case it is necessary to know the resultant vector of the external forces applied to each contour L_K individually. /50

§4. Table of Some Cauchy-type Integrals

We present a table of Cauchy-type integrals (without delving into the method of their calculation) which will be encountered in the following chapters.

The following definitions are used here.

γ is circle of unit radius in the plane of complex variable ζ;
ζ is arbitrary point of the plane of a complex variable;
ζ_0 is point of application of concentrated force or moment;
$\overline{\zeta_0}$ is point conjugate with point ζ_0;
t, σ, σ_1 are points located on unit circle γ.

I. When $|\zeta| > 1$, $|\zeta_0| > 1$

1) $\frac{1}{2\pi i}\int_\gamma \frac{\ln\frac{1-\sigma\bar{\zeta}_0}{\sigma}}{\sigma-\zeta}\,d\sigma = -\ln\frac{1-\zeta\bar{\zeta}_0}{\zeta} + \ln(-\bar{\zeta}_0)$;

2) $\frac{1}{2\pi i}\int_\gamma \frac{d\sigma}{(1-\sigma\bar{\zeta}_0)(\sigma-\zeta)} = -\frac{1}{1-\zeta\bar{\zeta}_0}$;

3) $\frac{1}{2\pi i}\int_\gamma \frac{\sigma d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)} = 0$;

4) $\frac{1}{2\pi i}\int_\gamma \frac{\ln(\sigma-\zeta_0)}{\sigma-\zeta}\,d\sigma = 0$;

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5) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma(\sigma-\zeta_0)\,d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=-\frac{1-\bar{\zeta}_0\zeta_0}{\bar{\zeta}_0^2(1-\bar{\zeta}_0\zeta)}$;

6) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(1-\sigma\bar{\zeta}_0)(\sigma-\zeta)}=-\frac{1}{\bar{\zeta}_0(1-\bar{\zeta}_0\zeta)}$;

7) $\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(1-\sigma\bar{\zeta}_0)^2(\sigma-\zeta)}=-\frac{1}{(1-\bar{\zeta}_0\zeta)^2}$;

8, 9) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=0.$

II. When $|\zeta|>1$, $|\zeta_0|<1$ /51

10) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(\sigma-\zeta_0)}{\sigma-\zeta}d\sigma=\ln(\zeta-\sigma_1)-\ln(\zeta-\zeta_0)+\text{const};$

11, 12, 13) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(1-\bar{\zeta}_0\sigma)}{\sigma-\zeta}d\sigma=\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma(\sigma-\zeta_0)\,d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=$

$$=\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=0;$$

14) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma^2 d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=-\frac{\zeta\zeta_0}{\zeta-\zeta_0}$;

15) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=-\frac{\zeta_0}{\zeta-\zeta_0}$.

III. When $|\zeta|<1$, $|\zeta_0|>1$

16) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln\frac{1-\bar{\zeta}_0\sigma}{\sigma}}{\sigma-\zeta}d\sigma=\ln(-\bar{\zeta}_0);$

17) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(\sigma-\zeta_0)}{\sigma-\zeta}d\sigma=\ln(\zeta-\zeta_0);$

18) $\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=\frac{1}{\zeta-\zeta_0}$;

19, 20) $\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(1-\bar{\zeta}_0^2\sigma)(\sigma-\zeta)}=0;$

21) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=-\frac{1}{\bar{\zeta}_0}$;

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22) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=\frac{\zeta}{\zeta-\zeta_0}$;

23) $\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)}=\frac{1}{\zeta-\zeta_0}$.

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IV. When $|\zeta|<1$, $|\zeta_0|<1$

24) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(\sigma-\zeta_0)}{\sigma-\zeta}d\sigma=\ln(\sigma_1-\zeta)$;

25) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=\frac{\zeta}{1-\bar{\zeta}_0\zeta}$;

26) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma(\sigma-\zeta_0)d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)}=\frac{\zeta(\zeta-\zeta_0)}{1-\bar{\zeta}_0\zeta}$;

27) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(1-\bar{\zeta}_0\sigma)}{-\sigma-\zeta}d\sigma=\ln(1-\bar{\zeta}_0\zeta)$.

V. When $|\zeta|>1$

28) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(\sigma-\sigma_1)}{\sigma-\zeta}d\sigma=0$;

29) $\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{\sigma(\sigma-\zeta)}=-\frac{1}{\zeta}$.

VI. When $|\zeta|<1$

30) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln\sigma}{\sigma-\zeta}d\sigma=\ln(\sigma_1-\zeta)$;

31) $\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma d\sigma}{\sigma-\zeta}=\zeta$;

32) $\frac{1}{2\pi i}\int_{\gamma}\frac{\ln(\sigma-\sigma_1)}{\sigma-\zeta}d\sigma=\ln(\sigma_1-\zeta)+\text{const.}$

VII. When $|\zeta_0|>1$

33) $\frac{1}{\pi i}\int_{\gamma}\frac{dt}{(1-\bar{\zeta}_0 t)(t-\sigma)}=-\frac{1}{1-\bar{\zeta}_0\sigma}$;

34) $\frac{1}{\pi i}\int_{\gamma}\frac{tdt}{(1-\bar{\zeta}_0 t)(t-\sigma)}=-\frac{2-\bar{\zeta}_0\sigma}{\bar{\zeta}_0(1-\bar{\zeta}_0\sigma)}$;

35) $\frac{1}{\pi i}\int_{\gamma}\ln\frac{1-\bar{\zeta}_0 t}{t}\frac{dt}{t-\sigma}=2\ln(-\bar{\zeta}_0)-\ln\frac{1-\bar{\zeta}_0\sigma}{\sigma}$;

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36) $\frac{1}{\pi i}\int_{\gamma} \ln(t-\zeta_0)\frac{dt}{t-\sigma} = \ln(\sigma-\zeta_0)$;

37, 38) $\frac{1}{\pi i}\int_{\gamma} \frac{\ln\frac{1-\bar{\zeta}_0\sigma}{\sigma}}{\sigma^2}\,d\sigma = \frac{1}{\pi i}\int_{\gamma} \frac{\ln(\bar{\sigma}-\bar{\zeta}_0)}{\sigma^2}\,d\sigma = 0$;

39, 40) $\frac{1}{\pi i}\int_{\gamma} \frac{d\sigma}{\sigma(1-\bar{\zeta}_0\sigma)} = \frac{1}{\pi i}\int_{\gamma} \frac{d\sigma}{\sigma^2(1-\bar{\zeta}_0\sigma)} = 0$;

41) $\frac{1}{2\pi i}\int_{\gamma} \frac{\ln(\sigma-\zeta_0)}{\sigma}\,d\sigma = \ln(-\zeta_0)$;

42, 43) $\frac{1}{2\pi i}\int_{\gamma} \frac{\ln(\sigma-\zeta_0)}{\sigma^2}\,d\sigma = \frac{1}{2\pi i}\int_{\gamma} \frac{a\sigma}{\sigma(\sigma-\zeta_0)} = -\frac{1}{\zeta_0}$;

44) $\frac{1}{\pi i}\int_{\gamma} \frac{dt}{(t-\zeta_0)(t-\sigma)} = \frac{1}{\sigma-\zeta_0}$.

VIII. When $|\zeta_0| < 1$

45) $\frac{1}{\pi i}\int_{\gamma} \frac{t(t-\zeta_0)dt}{(1-\bar{\zeta}_0 t)(t-\sigma)} = \frac{\sigma(\sigma-\zeta_0)}{1-\bar{\zeta}_0\sigma}$;

46) $\frac{1}{\pi i}\int_{\gamma} \frac{\ln(1-\bar{\zeta}_0 t)\,dt}{t-\sigma} = \ln(1-\bar{\zeta}_0\sigma)$;

47) $\frac{1}{\pi i}\int_{\gamma} \frac{t^2 dt}{(t-\zeta_0)(t-\sigma)} = \frac{\sigma(\sigma-2\zeta_0)}{\sigma-\zeta_0}$;

48) $\frac{1}{\pi i}\int_{\gamma} \frac{\ln(t-\zeta_0)}{t-\sigma}\,dt = 2\ln(\sigma-\sigma_1) - \ln(\sigma-\zeta_0)$;

49) $\frac{1}{\pi i}\int_{\gamma} \frac{(1-\bar{\zeta}_0 t)\,dt}{t(t-\zeta_0)(t-\sigma)} = -\frac{1-\bar{\zeta}_0\sigma}{\sigma(\sigma-\zeta_0)}$;

50) $\frac{1}{\pi i}\int_{\gamma} \frac{tdt}{(1-\bar{\zeta}_0 t)(t-\sigma)} = \frac{\sigma}{1-\bar{\zeta}_0\sigma}$;

51) $\frac{1}{\pi i}\int_{\gamma} \frac{dt}{(t-\zeta_0)(t-\sigma)} = -\frac{1}{\sigma-\zeta_0}$;

52) $\frac{1}{\pi i}\int_{\gamma} \frac{d\sigma}{(1-\bar{\zeta}_0\sigma)\sigma} =$ [illegible]

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53) $\frac{1}{\pi i}\int_{\gamma}\frac{(\sigma-\zeta_0)\,d\sigma}{\sigma(1-\bar{\zeta}_0\sigma)}=-2\zeta_0;$

54) $\frac{1}{\pi i}\int_{\gamma}\frac{\ln(1-\bar{\zeta}_0\sigma)}{\sigma^2}\,d\sigma=-2\bar{\zeta}_0;$

55) $\frac{1}{\pi i}\int_{\gamma}\frac{(1-\bar{\zeta}_0\sigma)\,d\sigma}{\sigma^2(\sigma-\zeta_0)}=0;$

56) $\frac{1}{\pi i}\int_{\gamma}\frac{\ln(\sigma-\zeta_0)}{\sigma^2}\,d\sigma=-\frac{2}{\sigma_1}.$

IX. Type of Integrals

57) $\frac{1}{\pi i}\int_{\gamma}\frac{d\sigma}{\sigma}=2;$

58) $\frac{1}{\pi i}\int_{\gamma}\frac{t\,dt}{t-\sigma}=\sigma;$

59) $\frac{1}{\pi i}\int_{\gamma}\frac{t^2dt}{t-\sigma}=\sigma^2;$

60) $\frac{1}{\pi i}\int_{\gamma}\frac{dt}{t(t-\sigma)}=-\frac{1}{\sigma};$

61) $\frac{1}{\pi i}\int_{\gamma}\frac{\ln t\,dt}{t-\sigma}=2\ln(\sigma-\sigma_1)-\ln\sigma;$

62) $\frac{1}{\pi i}\int_{\gamma}\frac{\ln(t-\sigma_1)}{t-\sigma}\,dt=\ln(\sigma-\sigma_1);$

63, 64) $\frac{1}{\pi i}\int_{\gamma}\frac{\ln\sigma\,d\sigma}{\sigma^2}=\frac{1}{\pi i}\int_{\gamma}\frac{\ln(\sigma-\sigma_1)}{\sigma^2}\,d\sigma=2.$

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*Translator's Note: The references here and at the end of the other chapters are presented in the original Cyrillic alphabetical order.

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CHAPTER II. STRESS DISTRIBUTION IN A PLANE ISOTROPIC FIELD WEAKENED BY ANY HOLE OR SERIES OF HOLES

ABSTRACT. Chapter II deals with methods of solving the problem of a stressed state of the isotropic medium weakened by one or several curvilinear holes. The author considers a great number of examples for the medium with one curvilinear hole whose forms correspond to the majority of those that find direct application in current practice. Various forces applied to the mentioned medium are accounted for. Different problems are considered on the stress concentration in a beam of an infinite length and for a half-plane, weakened by a circular hole, and for an elastic plane with a finite series of identical circular and curvilinear holes as well as with two different holes.

§1. Solution Method

Statement of the Problem. Let us consider an unbounded elastic isotropic plane which is in some stressed state (tension, compression, pure deflection, deflection with constant shear force and so forth). Let us denote by $U_0(x,y)$ the stress function corresponding to this stressed state. /58

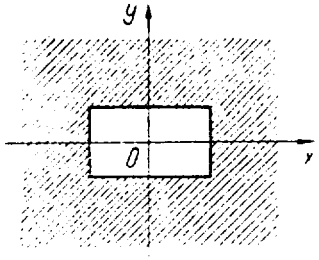


Figure II.1.

If a hole of any form (Figure II.1) is made in the plane under consideration, the distribution of stresses in it will change and instead of the stress state σ_x^0, σ_y^0 and τ_{xy}^0, which was present in a plane not weakened by a hole, we will have a new distribution of stresses characterized by new values of the stress components -- σ_x, σ_y, τ_{xy}.

Let us place the origin of the coordinate system xOy at any point within the hole (in the case of a rectangular hole -- at its geometric center) and call the stressed state in a plane without a hole the basic stressed state.

A new stressed state in the same plane, but weakened by any hole, may be represented in the form

$$\begin{aligned} \sigma_x &= \sigma_x^0 + \sigma_x^*, \\ \sigma_y &= \sigma_y^0 + \sigma_y^*, \\ \tau_{xy} &= \tau_{xy}^0 + \tau_{xy}^*, \end{aligned} \tag{II.1}$$

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where σ_x^*, σ_y^*, τ_{xy}^* are additional stress components which arise because of the presence of a hole.

For the basic stressed state σ_x^0, σ_y^0 and τ_{xy}^0, as well as for the stressed state σ_x^*, σ_y^* and τ_{xy}^*, according to (I.6) and (I.8), may be found according to N. I. Muskhelishvili's functions [1] of the complex variable z -- $\phi^0(z)$, $\psi^0(z)$ and $\phi^*(z)$, $\psi^*(z)$. Thus, functions corresponding to the stressed state (II.1) will have the form

$$\varphi_1(z) = \varphi^0(z) + \varphi^*(z),$$
$$\psi_1(z) = \psi^0(z) + \psi^*(z). \qquad \text{(II.2)}$$

The functions $\phi^*(z)$ and $\psi^*(z)$ are unknown and must be defined. /59

As experimental research shows (see Appendix), the influence of a hole on the stressed state pattern in the plane under consideration is of a local nature, i.e., the stress components σ_x^*, σ_y^*, τ_{xy}^* fade rapidly as the distance from the hole increases. It follows that functions $\phi^*(z)$ and $\psi^*(z)$ which characterize this stressed state are essentially holomorphic functions in the region outside the hole, i.e., functions of form (I.24).

Let us use conformal mapping of the inside of unit circle γ onto the region S under consideration (the plane outside the opening). If we switch to the transformed region using function $\omega(\zeta)$ (I.39), (I.49) or (I.54), then depending upon the shape of the hole in question, the stress functions (II.2) will acquire the form

$$\varphi_1[\omega(\zeta)] = \varphi^0[\omega(\zeta)] + \varphi^*[\omega(\zeta)],$$
$$\psi_1[\omega(\zeta)] = \psi^0[\omega(\zeta)] + \psi^*(\zeta)]. \qquad \text{(II.3)}$$

We introduce the definitions

$$\varphi_1[\omega(\zeta)] = \varphi(\zeta), \qquad \psi_1[\omega(\zeta)] = \psi(\zeta),$$
$$\varphi^0[\omega(\zeta)] = \varphi^1(\zeta), \qquad \psi^0[\omega(\zeta)] = \psi^1(\zeta),$$
$$\varphi^*[\omega(\zeta)] = \varphi_0(\zeta), \qquad \psi^*[\omega(\zeta)] = \psi_0(\zeta).$$

Then functions (II.3) acquire the form

$$\varphi(\zeta) = \varphi^1(\zeta) + \varphi_0(\zeta),$$
$$\psi(\zeta) = \psi^1(\zeta) + \psi_0(\zeta), \qquad \text{(II.4)}$$

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where

$$\varphi_0(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n; \qquad \psi_0(\zeta) = \sum_{n=0}^{\infty} \beta_n \zeta^n. \tag{II.5}$$

To find equations for determining the sought functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$, let us substitute functions (II.4) into contour conditions (I.26). Multiplying both parts of condition (I.26) (and its conjugate expressions) by $1/2\pi i \cdot d\sigma/\sigma - \zeta$, where ζ denotes a point within the unit circle γ, and integrating by γ, we obtain two functional equations[1] for determining the functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ (II.5):

$$\begin{aligned} \varphi_0(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_0'(\sigma)}\,\frac{d\sigma}{\sigma-\zeta} + \bar{\beta}_0 &= \frac{1}{2\pi i}\int_{\gamma}\frac{f_1^0 + if_2^0}{\sigma-\zeta}\,d\sigma, \\ \psi_0(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\varphi_0'(\sigma)\,\frac{d\sigma}{\sigma-\zeta} &= \frac{1}{2\pi i}\int_{\gamma}\frac{f_1^0 - if_2^0}{\sigma-\zeta}\,d\sigma, \end{aligned} \tag{II.6}$$

/60

where $f_1^0 - if_2^0$ denote the contour conditions cited for functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$:

$$f_1^0 + if_2^0 = f_1 + if_2 - \left[\varphi^1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi^{1'}(\sigma)} + \overline{\psi^1(\sigma)}\right], \tag{II.7}$$

and $f_1^0 - if_2^0$ is an expression conjugate with (II.7).

By determining the functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ from the functional equations (II.6) and substituting them in (II.4), we find the functions $\varphi(\zeta)$ and $\psi(\zeta)$.

To determine stress components σ_ρ, σ_ϑ and $\sigma_{\rho\vartheta}$ in the curvilinear orthogonal (given by conformal mapping) coordinate system, it is necessary to substitute the functions $\phi(\zeta)$ and $\psi(\zeta)$ (II.4) into equations (I.23) and separate the real and imaginary parts. If the contour of the hole is free of external forces, the stresses along the contour of the hole can be found from the first equation of (I.23), assuming $\sigma_\rho = 0$ where $\rho = 1$:

$$\sigma_\vartheta = 4\mathrm{Re}\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)}\right], \tag{II.8}$$

where $\sigma = e^{i\vartheta}$ is the value of the variable $\zeta = \rho e^{i\vartheta}$ on the contour of the hole.

Functions $\phi_0(z)$, $\psi_0(z)$ for Basic Stress State. From equation (I.8),

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[1]See N. I. Muskhelishvili [1], pp. 246-251.

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$$\Delta U_0(x, y) = \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 4\,\mathrm{Re}[\varphi^{0'}(z)]. \tag{II.9}$$

Hence the real part of the function $\phi^{0'}(z)$ is completely defined. By denoting $[\phi^0(z)]' = P(x, y) + iQ(x, y)$, we obtain from (II.9)

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 4P(x, y). \tag{II.10}$$

The imaginary part of $[\phi^0(z)]'$, i.e., the function Q(x, y) is found from the known differential relations:

$$\frac{\partial P(x, y)}{\partial x} = \frac{\partial Q(x, y)}{\partial y}, \qquad \frac{\partial P(x, y)}{\partial y} = -\frac{\partial Q(x, y)}{\partial x}. \tag{II.11}$$

Since the function Q(x, y) is found from a differential equation of the first order, the function $[\phi^0(z)]'$ is defined with an accuracy up to an imaginary constant iC_1. The function $\phi^0(z)$, however, is defined with an accuracy up to an expression of the form $iC_1z + C_2$. However, without changing the stress state, as follows from (I.13), the arbitrary constants C_1 and C_2 can be found /61 arbitrarily. In the following discussion we will assume them to be equal to zero. Thus, the function $\phi^0(z)$ is completely defined. The function $\psi^0(z)$, however, or the function $\chi^0(z)$, which is the same thing, since $\psi^0(z) = [\chi^0(z)]'$, is found from equation (I.8). By denoting

$$\chi^0(z) = R(x, y) + iS(x, y),$$

we find

$$R(x, y) = U_0(x, y) - \frac{1}{2}[\bar{z}\varphi^0(z) + z\overline{\varphi^0(z)}]. \tag{II.12}$$

The function S(x, y) is found from the relations (II.11). Consequently,

$$\chi^0(z) = R(x, y) + iS(x, y) + iC_3, \tag{II.13}$$

i.e., the function $\chi^0(z)$ is defined with an accuracy up to an imaginary constant.

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And so, for the given basic stress state σ_x^0, σ_y^0, τ_{xy}^0, both functions $\phi^0(z)$ and $\psi^0(z)$ are defined. Depending on the shape of the hole, it will be necessary in proceeding to the transformed region to introduce, instead of z, the corresponding representative function $\omega(\zeta)$ into the functions $\phi^0(z)$ and $\psi^0(z)$ that we have found; by this means we will define the functions $\phi^1(\zeta)$ and $\psi^1(\zeta)$ that appear in equations (II.4) and (II.7).

Since the function $\omega(\zeta)$ has the form of an infinite series (see formulas (I.49) and (I.59)) for the examined holes, which are in the form of a rectangle, square, or regular polygon, then, by taking a certain finite number of terms of this series, we find the solutions for the regions that differ somewhat from those with identical holes. However, as pointed out in §2 of Chapter I, it is always possible to select from the series of the function $\omega(\zeta)$, a number of terms such that predetermined conditions will be satisfied.

By taking from the series of the function $\omega(\zeta)$ a different number of terms, we can obtain a clear picture of the effect of the rounding off of the corners on the stress distribution around the hole. However, by changing the value of δ in the expression $C = Re^{i\delta}$ (see formulas (I.39), (I.49), (I.54)), we determine the effect of the location of the hole on the stress distribution around it.

The data obtained in this chapter can be used with accuracy sufficient for practice (see Appendix) for the design of plates and beams of finite dimensions, with the condition that the dimensions of the holes that weaken them are sufficiently small[1] in comparison with the dimensions of the plate (beam), and that it is located at a sufficient distance from its edges.

In examining individual problems, it will be interesting to consider the perturbation that a given hole imparts to a given basic stress state, as well as the corresponding stress state factors. We will define the concentration factor as the ratio of some tensor component of stress at some point located in the zone of perturbation near the hole, to the same tensor component of stress at the same point of the plate, but without a hole, under the effect of the same system of external forces as the plate with the hole. Therefore, for the plane problem, there are three concentration factors at each point: k_1 for σ_ρ, k_2 for σ_ϑ and k_3 for $\tau_{\rho\vartheta}$. But since the greatest stresses in the zone of concentration around the hole are located on the contour of the hole, then only one of the three concentration factors remains, namely $k_2 = k$ for σ_ϑ since no external forces are applied to the contour of the hole. /62

[1]The hole may also be rather large in comparison with the cross section dimensions of a beam, approaching, according to Z. Tuzi [1], 0.6 of its height in the case of a round hole, and 1/3 of its height, according to G. N. Savin [13], with a square hole in any position in a beam under deflection (see Appendix).

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We will examine certain cases of the stress state of a plane, weakened by different holes, considering the contour of a hole to be free of external forces, i.e., assuming $f_1 = f_2 = 0$ in (II.7).

§2. Uniaxial Tension or Compression[1]

During the stretching of a solid plate by forces p in the direction constituting angle α with the Ox axis, the basic stress state of the plate is characterized, as we know, by the stress components

$$\begin{aligned} \sigma_x &= p\cos^2\alpha, \\ \sigma_y &= p\sin^2\alpha, \\ \tau_{xy} &= p\sin\alpha\cos\alpha, \end{aligned} \tag{II.14}$$

and the function of the stresses is

$$U_0(x, y) = \frac{p}{2}(x\sin\alpha - y\cos\alpha)^2. \tag{II.15}$$

We will find the functions $\phi^0(z)$ and $\psi^0(z)$. From (II.15) and (II.10), $P(x, y) = \frac{1}{4}p$, while from (II.11), we find $Q(x, y) = 0$. Hence

$$\frac{d\varphi^0(z)}{dz} = \frac{1}{4}p. \tag{II.16}$$

Consequently,

$$\varphi^0(z) = \frac{1}{4}pz. \tag{II.17}$$

We will find the function $\psi^0(z)$ from (II.12) (we will omit the intermediate calculations):

$$\psi^0(z) = -\frac{pe^{-2i\alpha}}{2}z. \tag{II.18}$$

If, in the given plate, a hole is made, the center of which coincides with the origin xOy of the coordinate system, and the contour of which is free of external forces, then /63

[1]We will agree to consider as positive the stresses that are identical to those at an infinitely remote point. The lines of equal but opposite stresses are indicated in the figures by a shaded area.

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$$\varphi_1(z) = \frac{p}{4} z + \varphi^*(z),$$
$$\psi_1(z) = -\frac{p}{2} z e^{-2i\alpha} + \psi^*(z). \tag{II.19}$$

Converting to the region transformed by formulas (I.39), (I.49) or (I.54), we find

$$\varphi(\zeta) = \frac{p}{4}[\omega(\zeta)] + \varphi_0(\zeta),$$
$$\psi(\zeta) = -\frac{p}{2}[e^{-2i\alpha}[\omega(\zeta)] + \psi_0(\zeta). \tag{II.20}$$

We write the adduced contour conditions (II.7) for the given case of the basic stress state of the plate:

$$f_2^0 + if_2^0 = -\frac{p}{2}[\omega(\sigma) - e^{-2i\alpha}\overline{\omega(\sigma)}],$$
$$f_1^0 - if_2^0 = -\frac{p}{2}[\overline{\omega(\sigma)} - e^{-2i\alpha}\omega(\sigma)]. \tag{II.21}$$

From (II.6), for (II.21), we find the stress functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$. By substituting them into (II.20), we find the functions $\phi(\zeta)$ and $\psi(\zeta)$ that correspond to tension of the plate with the given hole. If the stress functions $\phi(\zeta)$ and $\psi(\zeta)$ (II.20) are known, the components of stresses σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$, are defined by formulas (I.27). The stress components near the hole are conveniently calculated from the coordinate lines ρ = const, which give a conformal representation.

In the examples given below, the components of stresses σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ were calculated for ρ equal to 1.0, 0.9, 0.8, 0.7, 0.5, and 0.3 every 5 or 10°. The values[1] σ_{max}, σ_{min} and τ_{max} were then calculated by the known formulas

$$\sigma_{\substack{max\\min}} = \frac{\sigma_\rho + \sigma_\vartheta}{2} \pm \sqrt{\left(\frac{\sigma_\rho - \sigma_\vartheta}{2}\right)^2 + \tau_{\rho\vartheta}^2},$$
$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_\rho - \sigma_\vartheta}{2}\right)^2 + \tau_{\rho\vartheta}^2} \tag{II.22}$$

[1]The law of distribution of these stresses will be given later (see Figures II.3-II.35).

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The curves of equal principal and tangential stresses were constructed through the points; the corresponding stresses are shown on the graphs, where the stress at the infinitely remote point is assumed to be unity. Moreover, the trajectories of the principal stresses, i.e., the curves whose tangentials coincide at each point with the direction of the principal elements at this point, were constructed by the formula /64

$$\tan 2\alpha = \frac{2\tau_{\rho\vartheta}}{\sigma_\rho - \sigma_\vartheta} \tag{II.23}$$

Square Hole[1]. We will use the function $\omega(\zeta)$ (I.56) in the form (Figure II.2)

$$\omega(\zeta) = R\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3\right). \tag{II.24}$$

The equations of the contour of the hole are found from (II.24) for $\rho = 1$, by dividing the real and imaginary parts[2]:

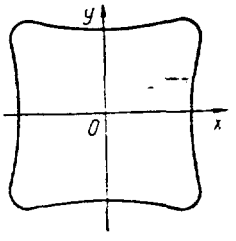


Figure II.2.

$$x = R\left(\cos\vartheta - \frac{1}{6}\cos 3\vartheta\right),$$
$$y = -R\left(\sin\vartheta + \frac{1}{6}\sin 3\vartheta\right). \tag{II.25}$$

By expressing R through the length of the side of the square hole, we find $R = 3/5 \cdot a$, where a is the length of the side of a curvilinear square hole, measured along the Ox or Oy axes. The radius of rounding of the corners of the curvilinear square hole (II.25) is

$$r_{\vartheta=45^\circ} = \frac{1}{10}R = \frac{3}{50}a.$$

The stress functions for this case, according to (II.20), are

[1]The solutions for square or rectangular holes with the lateral ratio a/b = 5 and a/b = 3.2, and for a triangular hole, were given by G. N. Savin [1]. P. A. Sokolov [1] found a somewhat different solution for a square and triangle with rounded corners. The solution of this problem for various values of the factor for ζ^3 in (II.24) was found by H. Czudek [1].

[2]The contour in Figure II.2 was constructed by equations (II.25).

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$$\varphi(\zeta) = \frac{pR}{4}\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3\right) + \varphi_0(\zeta),$$
$$\psi(\zeta) = -\frac{pR}{2}\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3\right)e^{-2i\alpha} + \psi_0(\zeta). \qquad \text{(II.26)}$$

The adduced contour conditions (II.21) are

$$f_1^0 + if_2^0 = -\frac{pR}{2}\left(\frac{1}{\sigma} + \frac{e^{2i\alpha}}{6}\cdot\frac{1}{\sigma^3} - \frac{1}{6}\sigma^3 - e^{2i\alpha}\sigma\right),$$
$$f_1^0 - if_2^0 = \frac{pR}{2}\left(\frac{1}{6}\cdot\frac{1}{\sigma^3} + \frac{e^{-2i\alpha}}{\sigma} - \sigma - \frac{e^{-2i\alpha}}{6}\sigma^3\right).$$

From the given function $\omega(\zeta)$ (II.24) we determined the expressions /65

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \text{ and } \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)},$$

found in functional equations (II.6):

$$\frac{\omega(\sigma)}{\overline{\omega'(\omega)}} = \frac{1}{6}\sigma - \frac{13\sigma}{6(2\sigma^4+1)},$$
$$\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \frac{1}{6}\cdot\frac{1}{\sigma} - \frac{13\sigma^3}{6(2+\sigma^4)}. \qquad \text{(II.27)}$$

We will find the values of the integrals in the right hand sides of system (II.6):

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1^0 + if_2^0}{\sigma - \zeta}\,d\sigma = \frac{pR}{2}(\zeta^2 + 6e^{2i\alpha})\zeta,$$
$$\frac{1}{2\pi i}\int_\gamma \frac{f_1^0 - if_2^0}{\sigma - \zeta}\,d\sigma = -\frac{pR}{12}(e^{-2i\alpha}\zeta^2 + 6)\zeta. \qquad \text{(II.28)}$$

We now compute the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_0'(\sigma)}\,\frac{d\sigma}{\sigma - \zeta},$$

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in which the function $\phi_0(\zeta)$ has the form (II.5):

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0'(\sigma)}\frac{d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\int_{\gamma}\left[\frac{\sigma}{6}-\frac{13\sigma}{2(6\sigma^4+3)}\right]\times$$
$$\times\left[\bar{\alpha}_1+\frac{2\bar{\alpha}_2}{\sigma}+\frac{3\bar{\alpha}_3}{\sigma^2}+\ldots\right]\frac{d\sigma}{\sigma-\zeta}=\frac{\bar{\alpha}_1}{6}\zeta+\frac{\bar{\alpha}_2}{3}. \tag{II.29}$$

By substituting the values of the integrals from (II.28) and (II.29) into the first equation (II.6) and assuming the factors to be equal for identical degrees of ζ, we find the equation system for determining the factors of the function $\phi_0(\zeta)$:

$$\alpha_2=0,$$
$$\alpha_3=\frac{pR}{12},$$
$$\alpha_1+\frac{\bar{\alpha}_1}{6}=\frac{pR}{2}e^{2i\alpha}, \tag{II.30}$$

whence

$$\alpha_1=pR\left(\frac{3}{7}\cos 2\alpha+i\frac{3}{5}\sin 2\alpha\right),$$
$$\alpha_2=0;\qquad \alpha_3=\frac{pR}{12}. \tag{II.31}$$

Consequently, /66

$$\varphi_0(\zeta)=pR\left[\left(\frac{3}{7}\cos 2\alpha+i\frac{3}{5}\sin 2\alpha\right)\zeta+\frac{1}{12}\zeta^3\right] \tag{II.32}$$

The function $\psi_0(\zeta)$ (II.5) is found from the second equation of (II.6) by substituting the corresponding expressions from (II.27), (II.32) and (II.28). Omitting the intermediate calculations, we write its final form:

$$\psi_0(\zeta)=-\frac{pR}{12}\left[e^{2i\alpha}\zeta^3+\frac{13\zeta-26\left(\frac{3}{7}\cos 2\alpha+i\frac{3}{5}\sin 2\alpha\right)\zeta^3}{2+\zeta^4}\right]. \tag{II.33}$$

The expressions for the functions $\phi(\zeta)$ and $\psi(\zeta)$ are found from (II.26) by substituting the values obtained for (II.32) and (II.33):

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$$\varphi(\zeta) = pR\left[\frac{1}{4\zeta} + \left(\frac{3}{7}\cos 2\alpha + i\frac{3}{5}\sin 2\alpha\right)\zeta + \frac{1}{24}\zeta^3\right],$$

$$\psi(\zeta) = -pR\left[\frac{e^{-2i\alpha}}{2\zeta} + \frac{13\zeta - 26\left(\frac{3}{7}\cos 2\alpha + i\frac{3}{5}\sin 2\alpha\right)\zeta^3}{12(2+\zeta^4)}\right]. \qquad \text{(II.34)}$$

Stress functions (II.34) for $\alpha = 0$, i.e., for the condition $\sigma_x^{(\infty)} = p$ and $\sigma_y^{(\infty)} = \tau_{xy}^{(\infty)} = 0$, will have the form

$$\varphi(\zeta) = pR\left[\frac{1}{4\zeta} + \frac{3}{7}\zeta + \frac{1}{24}\zeta^3\right],$$

$$\psi(\zeta) = -pR\left[\frac{1}{2\zeta} + \frac{91\zeta - 78\zeta^3}{84(\zeta^4+2)}\right]. \qquad \text{(II.35)}$$

By substituting the corresponding values of the derivatives of functions $\omega(\zeta)$ (II.24) and $\phi(\zeta)$ (II.35) into formula (II.8), assuming $\zeta = \rho e^{i\vartheta}$ for $\rho = 1$, we find the formula for determining the stresses on the contour of the square hole:

$$\sigma_\vartheta = \frac{8p}{5 + 4\cos 4\vartheta}\left(\frac{3}{8} - \frac{9}{7}\cos 2\vartheta\right). \qquad \text{(II.36)}$$

By assigning ϑ various values from 0 to 360° (due to the complete symmetry in the given case we may confine ourselves to the range 0-90°), we calculate by formula (II.36) the stresses at the points of the contour of the given hole.

When $\alpha = \pi/4$, stress functions (II.34) become

$$\varphi(\zeta) = pR\left(\frac{1}{4\zeta} + i\frac{3}{5}\zeta + \frac{1}{24}\zeta^3\right),$$

$$\psi(\zeta) = pR\left(\frac{i}{2\zeta} - \frac{65\zeta - i78\zeta^3}{60(2+\zeta^4)}\right). \qquad \text{(II.37)}$$

The stresses at the points of the contour of the hole are /67

$$\sigma_\vartheta = \frac{8p}{5 + 4\cos 4\vartheta}\left(\frac{3}{8} - \frac{3}{5}\sin 2\vartheta\right). \qquad \text{(II.38)}$$

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For any value of α, the stress is

$$\sigma_{\vartheta} = \frac{8p}{5+4\cos 4\vartheta}\left(\frac{3}{8} - \frac{9}{7}\cos 2\alpha \cos 2\vartheta - \frac{3}{5}\sin 2\alpha \sin 2\vartheta\right).$$

From the functions $\phi(\zeta)$ and $\psi(\zeta)$ (II.35), we determine the stress components in the vicinity of the plate around the square hole. The lines of equal σ_{max} and σ_{min} are shown on Figure II.3 and the lines of equal τ_{max}, on Figure II.4. Figure II.5 shows the trajectories of the principal stresses.

We now write the function $\omega(\zeta)$ (I.56) in the form (Figure II.6)

$$\omega(\zeta) = R\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7\right). \tag{II.39}$$

An increase in the number of terms of the function $\omega(\zeta)$ results in a decrease in the radius of rounding of the angles of the hole and less deviation of the sides from rectangular (see Figures II.2 and II.6, where these contours are constructed by their equations). The radius of rounding of the angles of the curvilinear square for a side equal to a is $r_{\vartheta=45^\circ} = 0.0245\,a$ in this case, while the functions are

$$\begin{aligned}
\varphi(\zeta) &= pR\left[\frac{0.25}{\zeta} + (0.426\cos 2\alpha + i0.608\sin 2\alpha)\zeta + \right.\\
&\left. + 0.046\zeta^3 + (0.008\cos 2\alpha - i0.011\sin 2\alpha)\zeta^5 - 0.004\zeta^7\right];\\
\psi(\zeta) &= -pR\left[\frac{0.5e^{-2i\alpha}}{\zeta} + \frac{0.548\zeta - (0.457\cos 2\alpha + i0.672\sin 2\alpha)\zeta^3}{1+0.5\zeta^4 - 0.125\zeta^8} - \right.\\
&\left. - \frac{0.026\zeta^5 + (0.029\cos 2\alpha - i0.068\sin 2\alpha)\zeta^7}{1+0.5\zeta^4 - 0.125\zeta^8}\right].
\end{aligned} \tag{II.40}$$

The functions $\phi(\zeta)$ and $\psi(\zeta)$ (II.40) for $\alpha = 0$ are

$$\begin{aligned}
\varphi(\zeta) &= pR\left[\frac{0.25}{\zeta} + 0.426\zeta + 0.046\zeta^3 + 0.008\zeta^5 - 0.004\zeta^7\right];\\
\psi(\zeta) &= -pR\left[\frac{0.5}{\zeta} + \frac{0.548\zeta - 0.457\zeta^3 - 0.026\zeta^5 - 0.029\zeta^7}{1+0.5\zeta^4 - 0.125\zeta^8}\right].
\end{aligned} \tag{II.41}$$

For $\alpha = \pi/4$, functions (II.40) are

$$\begin{aligned}
\varphi(\zeta) &= pR\left[\frac{0.25}{\zeta} + i0.608\zeta + 0.046\zeta^3 - i0.011\zeta^5 - 0.004\zeta^7\right];\\
\psi(\zeta) &= pR\left[\frac{0.5i}{\zeta} - \frac{0.548\zeta - i0.672\zeta^3 - 0.026\zeta^5 + i0.068\zeta^7}{1+0.5\zeta^4 - 0.125\zeta^8}\right].
\end{aligned} \tag{II.42}$$

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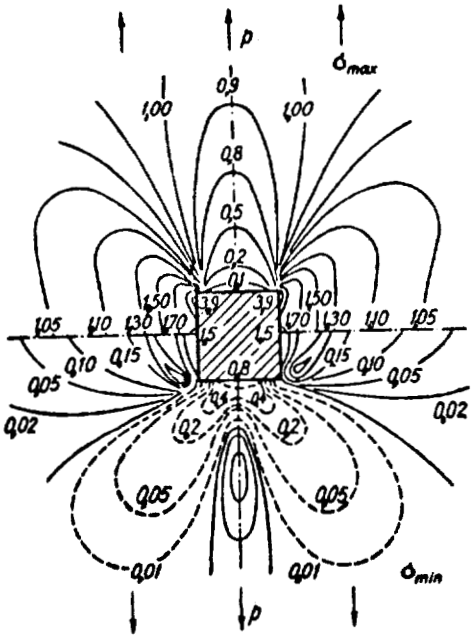


Figure II.3.

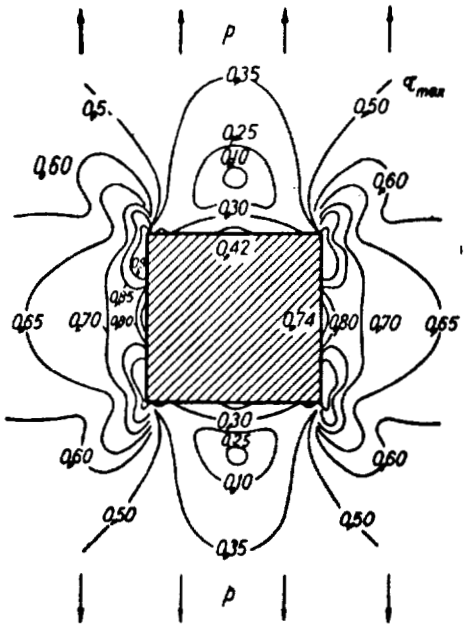


Figure II.4.

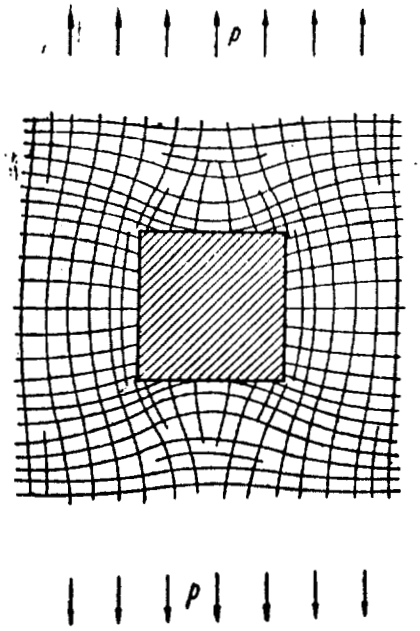


Figure II.5.

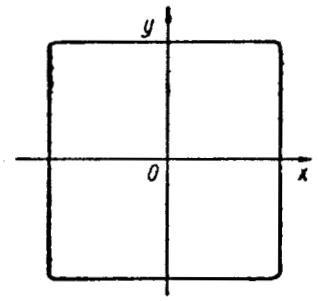


Figure II.6.

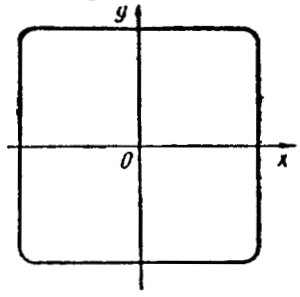


Figure II.7.

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To evaluate the effect of rounding of the angles on the stress concentration along the contour of the curvilinear square hole, we present in Table II.1 the values of stresses σ_ϑ in fractions of p for two cases: 1) for $\alpha = 0$ in accordance with functions (II.35) and (II.41) and 2) for $\alpha = \pi/4$ in accordance with functions (II.37) and (II.42). /69

TABLE II.1

$\vartheta°$	(II. 35)	(II. 41)	(II. 37)	(II. 42)
0	—0,808	—0,936	0,333	0,412
35	—0,268	—0,544	3,880	6,564
40	0,980	0,605	6,223	9,672
45	3,000	4,368	7,800	11,516
50	3,860	4,460	6,223	9,672
55	3,366	2,888	3,880	6,564
90	1,472	1,760	0,333	0,412

Tr. Note: Commas indicate decimal points.

The numerical data of Table II.1 indicate the following.

1. The maximum stress concentration occurs near the angles of the square.

2. As the radius of rounding of the angles gets smaller the concentration of stresses near the hole increases greatly.

3. If we retain four terms in the function $\omega(\zeta)$ (I.56), i.e., take it in the form (Figure II.7)

$$\omega(\zeta) = R\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 + \frac{1}{176}\zeta^{11}\right), \tag{II.43}$$

the radius of rounding of the angles of the curvilinear square is $r_{\vartheta=45°} = 0.014\ a$.

The stress functions $\phi(\zeta)$ and $\psi(\zeta)$, for the condition $\alpha = \pi/2$, i.e., when

$$\sigma_x^{(\infty)} = 0,\ \sigma_y^{(\infty)} = p,\ \tau_{xy}^{(\infty)} = 0,$$

will have the form

$$\begin{aligned}\varphi(\zeta) &= pR\left[\frac{0{,}25}{\zeta} - 0{,}4254\zeta + 0{,}0476\zeta^3 - 0{,}0086\zeta^5 - \right.\\ &\left. - 0{,}0060\zeta^7 + 0{,}0024\zeta^9 + 0{,}0014\zeta^{11}\right];\\ \psi(\zeta) &= pR\left[\frac{0{,}5}{\zeta} - \frac{0{,}479\zeta - 0{,}457\zeta^3 + 0{,}269\zeta^5 - 0{,}037\zeta^7}{1 + 0{,}5\zeta^4 - 0{,}125\zeta^8 + 0{,}063\zeta^{12}} - \right.\\ &\left. - \frac{0{,}073\zeta^9 - 0{,}017\zeta^{11} - 0{,}031\zeta^{13}}{1 + 0{,}5\zeta^4 - 0{,}124\zeta^8 + 0{,}063\zeta^{12}}\right].\end{aligned} \tag{II.44}$$

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The values of the stresses σ_ϑ in fractions of p along the contour of the hole for $\alpha = \pi/2$, corresponding to functions (II.41), are presented in Table II.2. /70

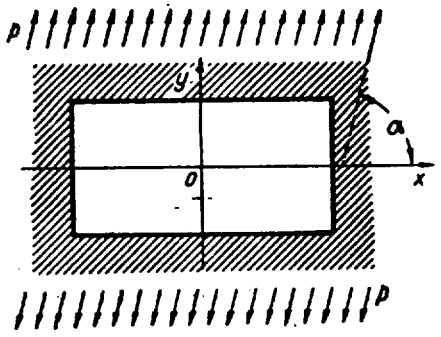


Figure II.8.

TABLE II.2

$\vartheta°$	σ_ϑ/p	$\vartheta°$	σ_ϑ/p
0	1,616	50	0,265
15	1,802	60	—0,702
30	1,932	75	—0,901
40	4,230	90	—0,871
45	5,763	—	—

Tr. Note: Commas indicate decimal points.

The problems of the stress state near a square hole for various loads on its contour and conditions at infinity for different numbers of terms of series $\omega(\zeta)$ (I.56) were examined by D. V. Vaynberg [1], V. M. Gur'yanov and A. S. Kosmodamiansk [2], Barton Cliff Smith [1], H. Czudek [1], I. J. Geerlings [1], H. G. Hahn [1], Isida Makoto [3], Kunert Karlheinz [1], Takeuti Yoitiro, Yurikowa Takasi [1], Villagio Piero [1], Isida [1], and others.

Rectangular Hole with Ratio of Sides a/b = 5. We will confine ourselves to five terms of expansion in the representative function $\omega(\zeta)$ (I.49) and state that

$$\omega(\zeta) = R\left[\frac{1}{\zeta} + 0{,}643\zeta - 0{,}098\zeta^3 - 0{,}038\zeta^5 - 0{,}011\zeta^7\right]. \qquad \text{(II.45)}$$

Omitting all intermediate calculations analogous to those described above, we write the functions $\phi(\zeta)$ and $\psi(\zeta)$ for certain values of α (Figure II.8). For $\alpha = \pi/3$

$$\varphi(\zeta) = pR\left[\frac{0{,}25}{\zeta} - (0{,}358 - i0{,}500)\,\zeta + (0{,}045 + i0{,}023)\,\zeta^3 + \right.$$
$$\left. + (0{,}015 + i0{,}006)\,\zeta^5 + 0{,}003\zeta^7\right];$$
$$\psi(\zeta) = pR\left[\frac{0{,}25 + i0{,}433}{\zeta} + \frac{0{,}040\zeta^7 + 0{,}127\zeta^5 - 0{,}096\zeta^3 - 0{,}832\zeta}{1 - 0{,}643\zeta^2 + 0{,}293\zeta^4 + 0189\zeta^6 + 0{,}078\zeta^8} + \right. \qquad \text{(II.46)}$$
$$\left. + i\,\frac{0{,}280\zeta + 0{,}574\zeta^3 + 0{,}102\zeta^5 + 0{,}034\zeta^7}{1 - 0{,}643\zeta^2 + 0{,}293\zeta^4 + 0{,}189\zeta^6 + 0{,}078\zeta^8}\right].$$

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For $\alpha = \pi/6$ /71

$$\varphi(\zeta) = pR\left[\frac{0{,}25}{\zeta} + (0{,}090 - i0{,}500)\,\zeta + (0{,}026 - i0{,}023)\,\zeta^3 + \right.$$
$$\left. + (0{,}011 - i0{,}006)\,\zeta^5 + 0{,}003\zeta^7\right];$$
$$\psi(\zeta) = pR\left[-\frac{0{,}25 + i\,0{,}433}{\zeta} - \frac{0{,}564 - 0{,}236\zeta^3 + 0{,}066\zeta^5 - 0{,}034\zeta^7}{1 - 0{,}643\zeta^2 + 0{,}293\zeta^4 + 0{,}189\zeta^6 + 0{,}078\zeta^8} + \right. \tag{II.47}$$
$$\left. + \frac{i\,(0{,}280\zeta + 0{,}567\zeta^3 + 0{,}102\zeta^5 + 0{,}032\zeta^7}{1 - 0{,}643\zeta^2 + 0{,}293\zeta^4 + 0{,}189\zeta^6 + 0{,}078\zeta^8}\right].$$

For $\alpha = 0$, i.e., for $\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = \tau_{xy}^{(\infty)} = 0$

$$\varphi(\zeta) = pR\left[\frac{0{,}25}{\zeta} + 0{,}323\zeta + 0{,}016\zeta^3 + 0{,}008\zeta^5 + 0{,}003\zeta^7\right];$$
$$\psi(\zeta) = -pR\frac{0{,}5 + 0{,}101\zeta^2 + 0{,}414\zeta^4 + 0{,}064\zeta^6 + 0{,}021\zeta^8}{\zeta - 0{,}643\zeta^3 + 0{,}293\zeta^5 + 0{,}189\zeta^7 + 0{,}078\zeta^9}. \tag{II.48}$$

For $\alpha = \pi/2$

$$\varphi(\zeta) = pR\left[\frac{0{,}25}{\zeta} - 0{,}586\zeta + 0{,}055\zeta^3 + 0{,}018\zeta^5 + 0{,}003\zeta^7\right];$$
$$\psi(\zeta) = pR\frac{0{,}5 - 1{,}256\zeta^2 - 0{,}172\zeta^4 + 0{,}252\zeta^6 + 0{,}100\zeta^8}{\zeta - 0{,}643\zeta^3 + 0{,}293\zeta^5 + 0{,}189\zeta^7 + 0{,}078\zeta^9}. \tag{II.49}$$

By substituting the functions $\phi(\zeta)$ and $\psi(\zeta)$ (II.46)-(II.49) into equations (I.27) and separating the real and imaginary parts, we find the stress components σ_ρ, σ_ϑ, $\tau_{\rho\vartheta}$.

The values of the stress σ_ϑ computed by formula (II.8), in fractions of p for a/b = 5 around the contour of the square hole, are presented in Table II.3.

TABLE II.3

$\vartheta°$	$\alpha=\pi/3$	$\alpha=\pi/6$	$=0$	$\alpha=\pi/2$
0	1,641	0,033	—0,768	—2,420
20	9,070	—0,452	—0,152	8,050
25	12,556	—2,519	2,692	7,030
30	5,541	—2,264	2,812	1,344
40	1,214	—0,278	1,558	—0,644
90	—0,412	0,653	1,192	—0,940
140	—1,889	1,877	1,558	—0,644
150	—2,078	7,466	2,812	1,344
160	2,115	5,096	—0,152	8,050
180	1,641	0,033	—0,768	2,420

Tr. Note: Commas indicate decimal points.

Figures II.9-II.11 show the lines of equal σ_{max}, σ_{min}, and τ_{max}, while Figure II.12 shows the trajectory of the principal stresses for the case $\alpha = \pi/3$.

Figures II.13-II.15 show the lines of equal σ_{max}, σ_{min},

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and τ_{max}, and Figure II.16 shows the trajectories of the principal stresses for the case $\alpha = \pi/6$.

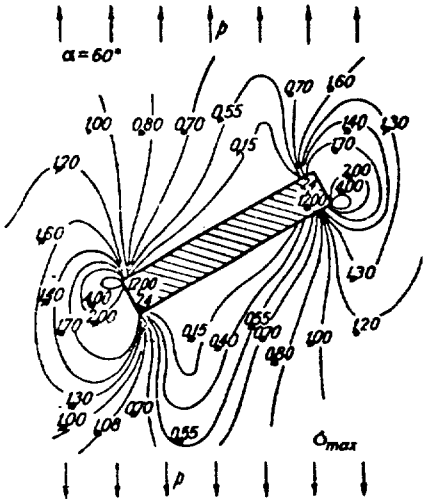


Figure II.9.

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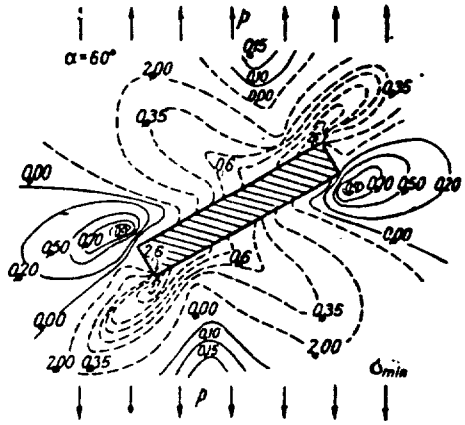


Figure II.10.

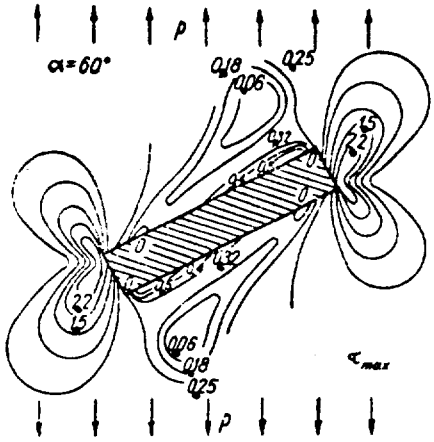


Figure II.11.

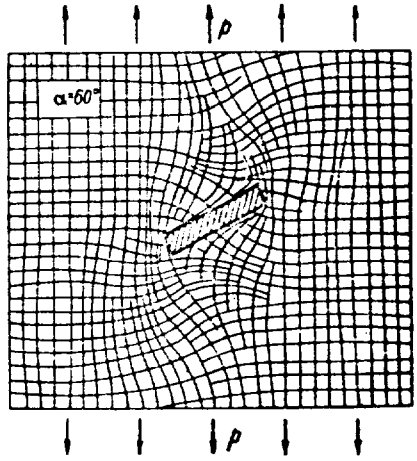


Figure II.12.

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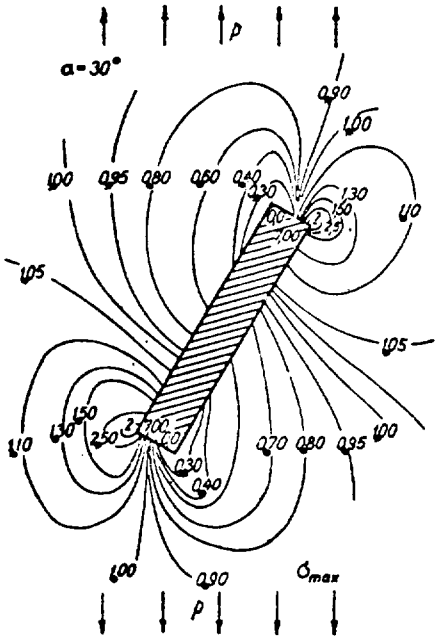


Figure II.13.

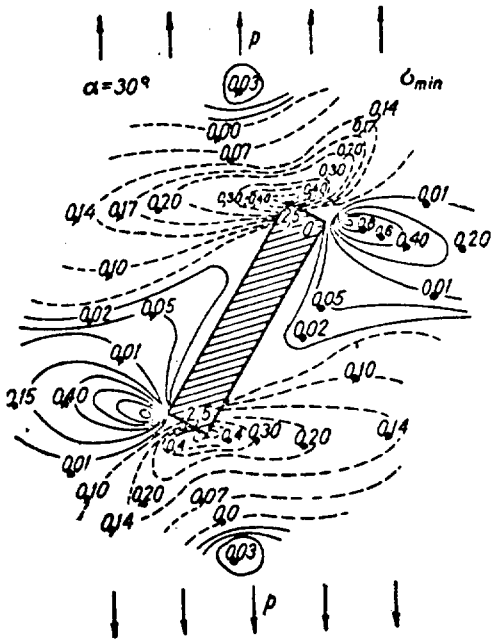


Figure II.14.

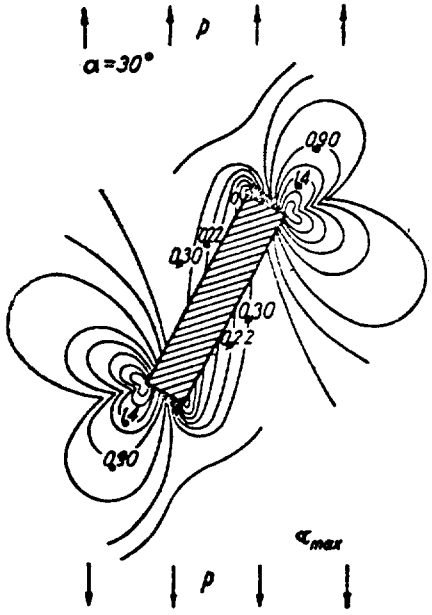


Figure II.15.

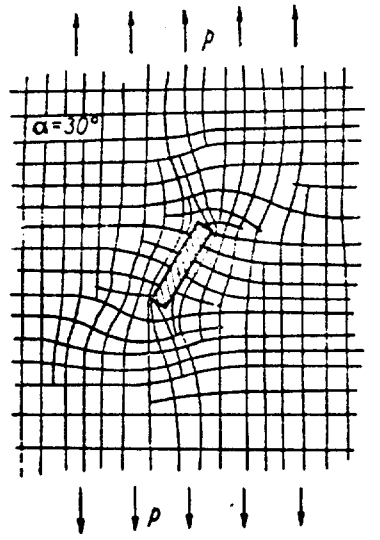


Figure II.16.

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Figures II.17 and II.18 show the lines of equal σ_{max}, σ_{min} and τ_{max} for the case $\alpha = 0$. /74

Figures II.19 and II.20 show the lines of equal σ_{max}, σ_{min} and τ_{max} for the case $\alpha = \pi/2$.

Rectangular Hole with Ratio of Sides a/b ≃ 3.2. We are limited here to four terms of expansion in the representative function $\omega(\zeta)$ (I.49), and we assume the function corresponding to the given ratio of the sides to be in the form

$$\omega(\zeta) = R\left[\frac{1}{\zeta} + 0{,}5\zeta - 0{,}125\zeta^3 - 0{,}038\zeta^5\right]. \tag{II.50}$$

By substituting, as earlier, we find the functions $\phi(\zeta)$, $\psi(\zeta)$.

For $\alpha = 0$, i.e., for $\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = \tau_{xy}^{(\infty)} = 0$,

$$\begin{aligned}\varphi(\zeta) &= pR\left[\frac{0{,}25}{\zeta} + 0{,}338\zeta + 0{,}023\zeta^3 + 0{,}0095\zeta^5\right];\\ \psi(\zeta) &= -pR\left[\frac{0{,}5}{\zeta} + \frac{0{,}432\zeta - 0{,}549\zeta^3 - 0{,}029\zeta^5}{1 - 0{,}5\zeta^2 + 0{,}375\zeta^4 + 0{,}019\zeta^6}\right].\end{aligned} \tag{II.51}$$

The lines of equal σ_{max} and σ_{min} are illustrated on Figure II.21, and the lines of equal τ_{max} are shown on Figure II.22. The trajectories of the principal stresses calculated by formulas (II.22) and (II.23) with consideration of (II.51) are shown on Figure II.23.

When $\alpha = \pi/2$, i.e., when $\sigma_y^{(\infty)} = p$, $\sigma_x^{(\infty)} = \tau_{xy}^{(\infty)} = 0$,

$$\begin{aligned}\varphi(\zeta) &= pR\left[\frac{0{,}25}{\zeta} - 0{,}538\zeta + 0{,}038\zeta^3 + 0{,}0095\zeta^5\right];\\ \psi(\zeta) &= pR\left[\frac{0{,}5}{\zeta} + \frac{0{,}826\zeta + 0{,}360\zeta^3 + 0{,}510\zeta^5}{1 - 0{,}5\zeta^2 + 0{,}375\zeta^4 + 0{,}190\zeta^6}\right].\end{aligned} \tag{II.52}$$

The lines of equal σ_{max} and σ_{min} are shown on Figure II.24, and the lines of equal σ_{max} are shown on Figure II.25. Figure II.26 shows the trajectory of the principal stresses, calculated with consideration of functions (II.52).

The stresses σ_ϑ/p on the contour of a rectangular hole, calculated from fuctions (II.51) and (II.52) where ϑ is determined from the middle of the short side of the rectangle, are given in Table II.4.

Ye. P. Anikin [1] studied the problems of stress concentration in an infinite plate with a rectangular hole when a/b = 1-5.15 in the case of tension

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(compression) as a function of the ratio of the radius of rounding of the angle of the hole to its short side. This problem was also examined in the works of V. M. Gur'yanov and A. S. Kosmodamianskiy [2], V. N. Kozhevnikova [1], S. R. Heller [1], and also S. R. Heller, Jr., S. S. Brock and R. Bart [1], Boyd H. Phillips, F. Asce and Ira E. Allen [1], and A. I. Sobey [1].

Triangular Hole. We confine ourselves in the function $\omega(\zeta)$ (I.55) to only two terms:

$$\omega(\zeta) = R\left[\frac{1}{\zeta} + \frac{1}{3}\zeta^2\right]. \tag{II.53}$$

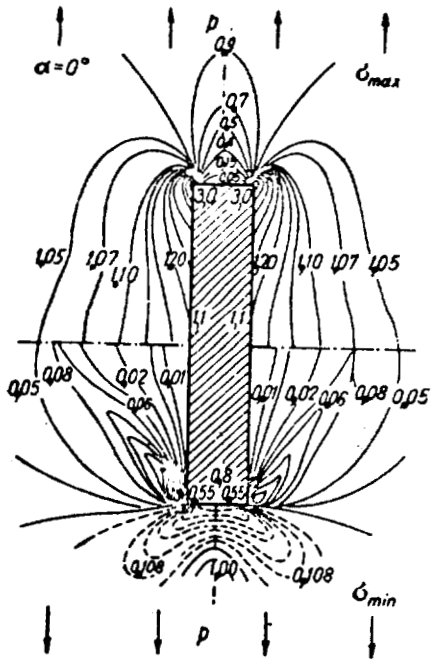


Figure II.17.

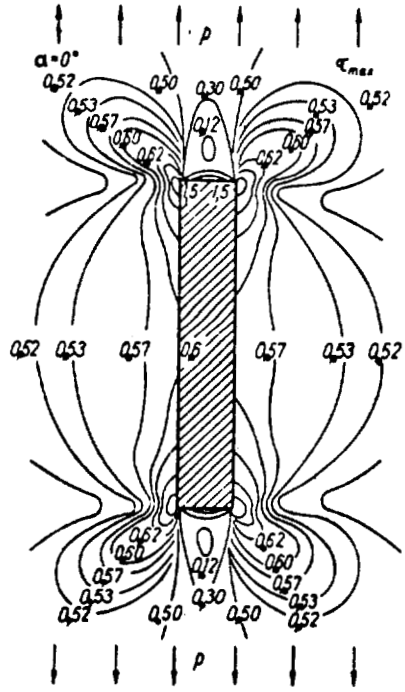


Figure II.18.

The equation of the contour of the given hole is found from (II.53) by assuming $\zeta = \rho e^{i\vartheta}$ for $\rho = 1$ and separating the real and imaginary parts: /77

$$x = R\left(\cos\vartheta + \frac{1}{3}\cos 2\vartheta\right),$$
$$y = -R\left(\sin\vartheta - \frac{1}{3}\sin 2\vartheta\right). \tag{II.54}$$

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The radius of curvature of the angles of the given hole is r = 1/21 R. By determining this radius through height H of the triangular hole, we find r = 1/14 H. The contour of the hole constructed by equations (II.54) is shown in Figure II.27.

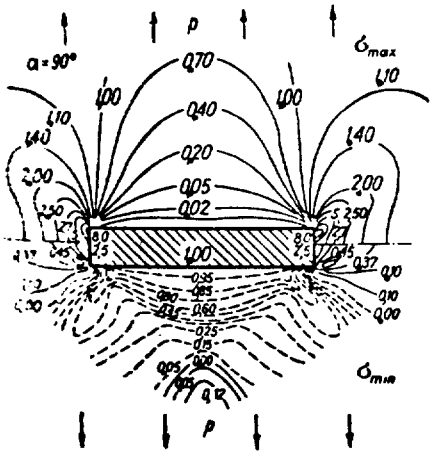


Figure II.19.

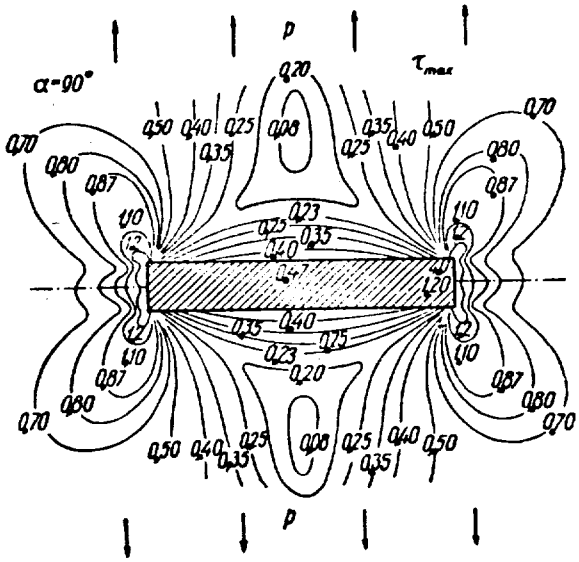


Figure II.20.

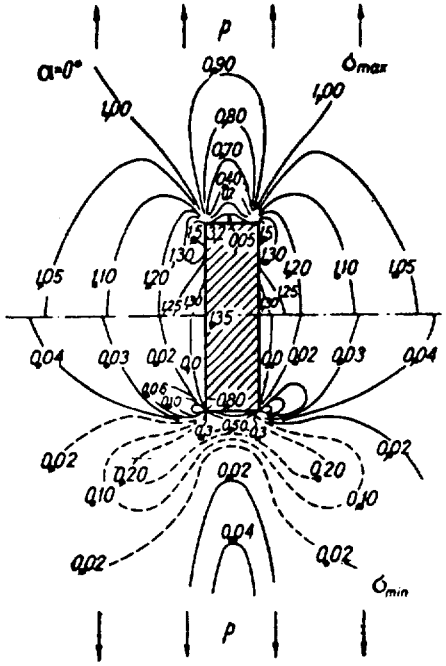


Figure II.21.

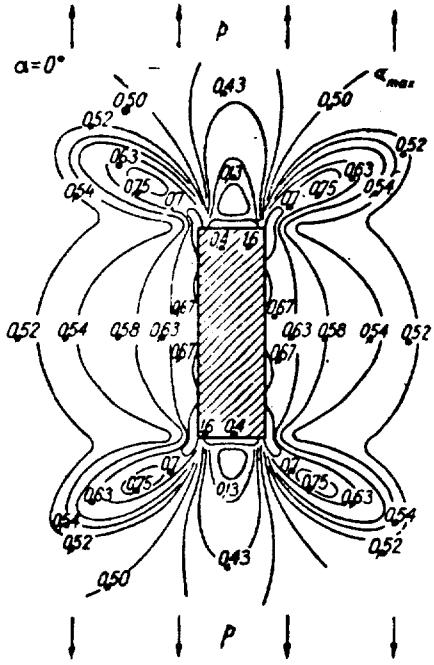


Figure II.22.

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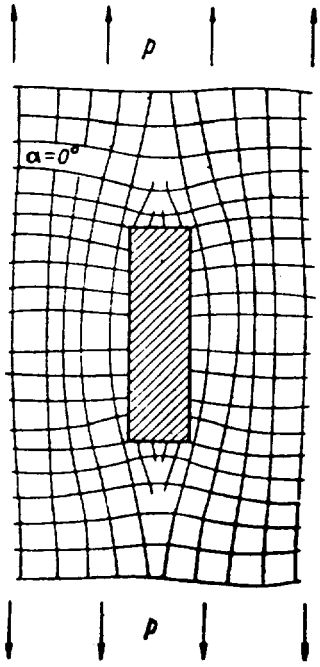


Figure II.23.

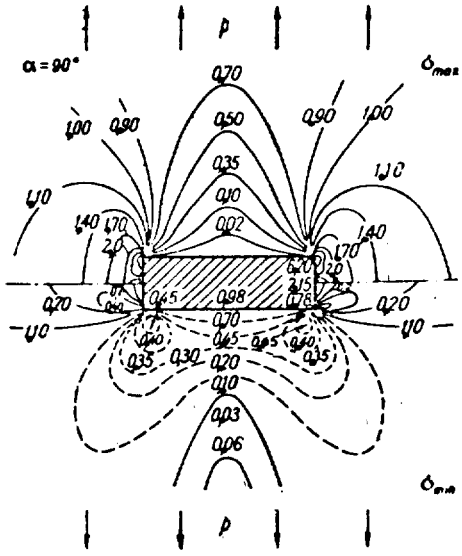


Figure II.24.

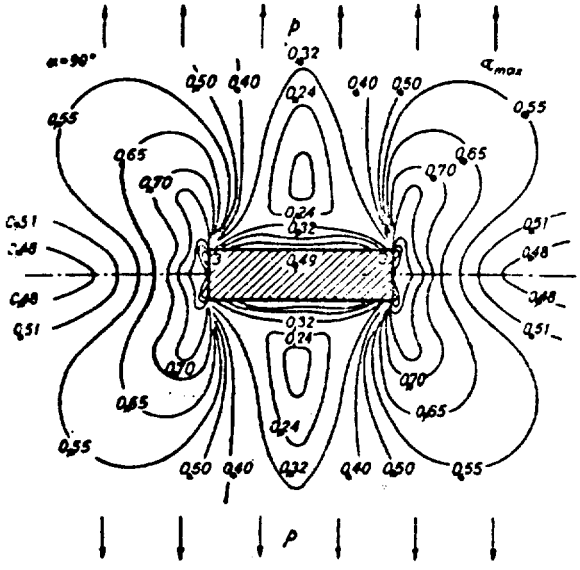


Figure II.25.

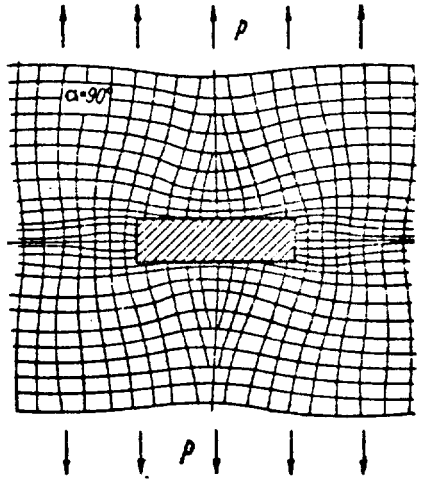


Figure II.26.

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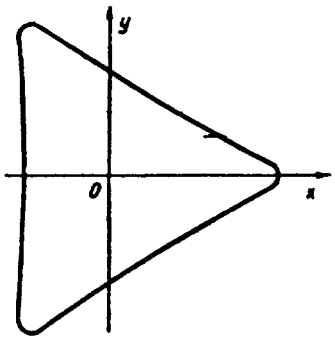


Figure II.27.

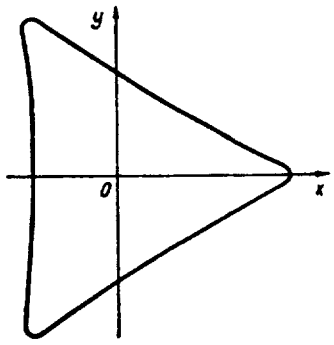


Figure II.28.

TABLE II.4.

$\vartheta°$	$\alpha=0$	$\alpha=\pi/2$
0	—0,770	2,152
10	—0,807	2,520
20	—0,686	4,257
25	—	6,204
30	2,610	5,512
35	3,181	—
40	2,392	—0,193
90	1,342	—0,980

Tr. Note: Commas indicate decimal points.

For this case

$$\varphi(\zeta) = pR\left[\frac{1}{4\zeta} + \frac{e^{2i\alpha}}{2}\zeta - \frac{1}{12}\zeta^2\right),$$
$$\psi(\zeta) = -pR\left(\frac{e^{-2i\alpha}}{2\zeta} + \frac{3e^{2i\alpha} + 9e^{2i\alpha}\zeta^3 - 11\zeta}{12\zeta^3 - 18}\right); \tag{II.55}$$

in particular, for $\alpha = 0$

$$\varphi(\zeta) = pR\left[\frac{1}{2}\zeta - \frac{1}{12}\zeta^2 + \frac{1}{4\zeta}\right],$$
$$\psi(\zeta) = -pR\left[\frac{1}{2\zeta} + \frac{9\zeta^3 - 11\zeta + 3}{12\zeta^3 - 18}\right]. \tag{II.56}$$

If, on the other hand, $\alpha = \pi/2$, then /78

$$\varphi(\zeta) = pR\left(\frac{1}{4\zeta} - \frac{1}{2}\zeta - \frac{1}{12}\zeta^2\right),$$
$$\psi(\zeta) = pR\left[\frac{1}{2\zeta} + \frac{9\zeta^3 + 11\zeta + 3}{12\zeta^3 - 18}\right]. \tag{II.57}$$

If in the function $\omega(\zeta)$ (I.55) we use not two terms of expansion, but three, i.e., we use it in the form (Figure II.28)

$$\omega(\zeta) = R\left(\frac{1}{\zeta} + \frac{1}{3}\zeta^2 + \frac{1}{45}\zeta^5\right), \tag{II.58}$$

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then the functions of the stresses for any uniaxial tension or compression, i.e., for any α, are

$$\varphi(\zeta) = pR\left(\frac{1}{4\zeta} + \frac{675}{1348}e^{2i\alpha}\zeta - \frac{1}{12}\zeta^2 + \frac{15e^{-2i\alpha}}{1348}\zeta^3 - \frac{1}{180}\zeta^5\right);$$

$$\psi(\zeta) = -pR\left[\frac{e^{-2i\alpha}}{2\zeta} + \frac{112\frac{1}{3}\zeta^7 + 31725e^{2i\alpha}\zeta^6}{181980(\zeta^6 + 6\zeta^3 - 9)} + \right. \tag{II.59}$$

$$\left. + \frac{546675e^{-2i\alpha}\zeta^5 - 31004\zeta^4 + 101475e^{-2i\alpha}\zeta^3 + 19035e^{-2i\alpha}\zeta^2 - 909989\zeta}{181980(\zeta^6 + 6\zeta^3 - 9)}\right].$$

The values of σ_ϑ/p along the contour of the triangular hole, that correspond to the functions (II.56), (II.57), (II.59) for $\alpha = 0$ and $\alpha = \pi/2$ are presented in Tables II.5 and II.6

TABLE II.5

$\vartheta°$	(II. 56)	(II. 59)	$\vartheta°$	(II. 56)	(II. 59)
0	—1,000	—1,600	115	8,160	8,863
10	—0,037	0,056	120	8,000	12,800
15	0,222	0,222	125	3,032	1,628
30	0,492	0,439	135	—0,772	—1,060
90	1,770	1,632	180	—1,000	—1,086

Tr. Note: Commas indicate decimal points.

We see by comparing the data in Tables II.5 and II.6 that as the radius of rounding off of the angles decreases the concentration of stresses in the angles of the triangle increase considerably. The solution of this problem for the factors for ζ^2 equal to 1/4, 3/8, and 1/2 is found in the work of H. Czudek [1]. This problem is also discussed in the work of Villagio Piero [1].

Elliptic Hole[1]. The function that conformally represents the interior of a unit circle γ on the exterior of an elliptic hole with semi-axes a and b has the form /79

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right), \tag{II.60}$$

where

$$R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b} = \frac{1-k}{1+k}, \quad k = \frac{b}{a}.$$

By omitting the intermediate calculations, we write in final form:

[1]The solution of this problem is given by N. I. Muskhelishvili [4]; see also the solutions of G. V. Kolosov [1], C. E. Inglis [1] and T. Poschl [1].

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$$\varphi(\zeta) = \frac{pR}{4}\left[\frac{1}{\zeta} + (2e^{2i\alpha} - m)\,\zeta\right],$$

$$\psi(\zeta) = -\frac{pR}{2}\left[\frac{e^{-2i\alpha}}{\zeta} + \frac{\zeta^3 e^{2i\alpha} + (me^{2i\alpha} - m^2 - 1)\,\zeta}{m\zeta^2 - 1}\right]. \tag{II.61}$$

The stresses for the points of the contour of the elliptic hole are found from (II.8) in the form

$$\sigma_\vartheta = pRe\left[\frac{(2e^{2i\alpha} - m)\,\sigma^2 - 1}{m\sigma^2 - 1}\right] = p\,\frac{1 - m^2 + 2m\cos 2\alpha - 2\cos 2(\vartheta + \alpha)}{1 - 2m\cos 2\vartheta + m^2}. \tag{II.62}$$

By substituting m in (II.62) as shown above, we obtain /80

$$\sigma_\vartheta = p\,\frac{(1+k)^2\sin^2(\vartheta + \alpha) - \sin^2\alpha - k^2\cos^2\alpha}{\sin^2\vartheta + k^2\cos^2\vartheta}. \tag{II.63}$$

Then, by substituting in (II.62) the various values of m, we find the formulas for stresses along the contour of any elliptic hole. In (II.62) we assume $\alpha = 0$; then

$$\sigma_\vartheta = p\,\frac{1 - m^2 + 2m - 2\cos 2\vartheta}{1 - 2m\cos 2\vartheta + m^2} = p\,\frac{\sin^2\vartheta + 2k\sin^2\vartheta - k^2\cos^2\vartheta}{\sin^2\vartheta + k^2\cos^2\vartheta}. \tag{II.64}$$

The greatest stresses on the contour of any elliptic hole, i.e., for any m, will be, for $\vartheta = \pm\pi/2$:

$$\sigma_\vartheta = p\left(1 + 2\,\frac{b}{a}\right), \tag{II.65}$$

where a is the semi-axis of the ellipse lying on the Ox axis, and b, on the Oy axis. The small stresses on the contour of any elliptic hole, i.e., for any m, for $\vartheta = 0$, will be:

$$\sigma_\vartheta = -p. \tag{II.66}$$

If b = a, i.e., m = 0, the ellipse is converted to a circle, and we find from (II.65) the known result: $\sigma_\vartheta = 3p$.

For an ellipse for which the ratio of the semi-axes is $k = b/a = 2/3$, according to formulas (II.22) and (II.23), the lines of equal σ_{max} and σ_{min} are shown in Figure II.29, and the lines of equal τ_{max} are shown in Figure II.30. The trajectories of the principal stresses are illustrated in Figure II.31.

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TABLE II.6

$\vartheta°$	(II. 57)	(II. 59)	$\vartheta°$	(II. 57)	(II. 59)
0	11,000	17,600	115	—1,064	—2,173
10	3,871	2,543	120	2,000	3,200
30	0,277	0,021	125	4,064	5,062
90	—1,000	—0,912	135	2,987	2,478
100	—1,404	—1,989	180	1,400	1,657

Tr. Note: Commas indicate decimal points.

For an ellipse with the semi-axial ratio k = b/a = 3/2, the lines of equal σ_{max} and σ_{min} are shown in Figure II.32 and the lines of equal τ_{max} are shown in Figure II.33. The trajectories of the principal stresses are shown in Figure II.34. /81

The stresses σ_ϑ on the contour of an elliptic hole when ϑ is calculated from the end of the semi-axis lying on the Ox axis, counter clockwise, are presented in Table II.7.

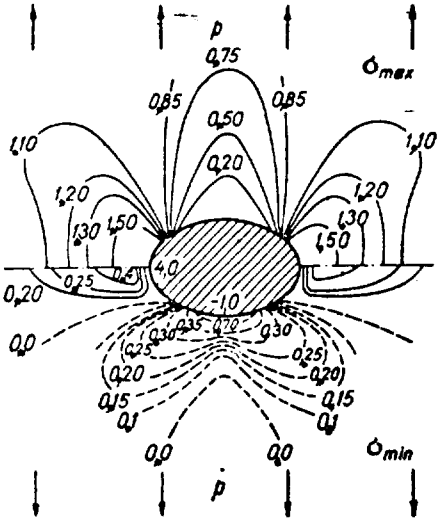


Figure II.29.

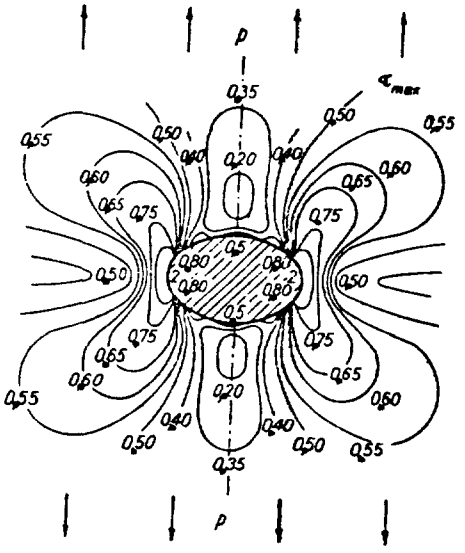


Figure II.30.

The values of σ_ϑ for other m (II.60) are given in the works of I. I. Vorovich and A. S. Kosmodamianskiy [1], H. Czudek [1], K. Hirschfeld [1], Isida [2], and Villagio Piero [1].

Circular Hole[1]. If in (II.61) we assume m = 0 and α = 0, we obtain the functions of the stresses for a plate with a circular hole of radius R, under tension along its Ox axis by forces p:

[1]The solution of this problem is given in the work of G. Kirch [1].

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$$\varphi(\zeta) = \frac{pR}{4}\left(\frac{1}{\zeta} + 2\zeta\right),$$
$$\psi(\zeta) = -\frac{pR}{2}\left(\frac{1}{\zeta} + \zeta - \zeta^3\right). \qquad \text{(II.67)}$$

The stresses are found from formula (I.27) by substituting the corresponding values of the functions (II.67):

$$\sigma_\varrho = \frac{p}{2}[(1-\varrho^2) + (1 - 4\varrho^2 + 3\varrho^4)\cos 2\vartheta],$$
$$\sigma_\vartheta = \frac{p}{2}[(1+\varrho^2) - (1 + 3\varrho^4)\cos 2\vartheta], \qquad \text{(II.68)}$$
$$\tau_{\varrho\vartheta} = \frac{p}{2}(1 + 2\varrho^2 - 3\varrho^4)\sin 2\vartheta.$$

The stresses along the contour of the hole are found from (II.62) when m = 0 or from (II.68) when $\rho = 1$:

$$\sigma_\vartheta = p(1 - 2\cos 2\vartheta). \qquad \text{(II.69)}$$

From (II.69) we see that the maximum values of σ_ϑ will occur when cos 2 ϑ= = -1, i.e., when $\vartheta = \pm\pi/2$

$$(\sigma_\vartheta)_{max} = 3p.$$

The lines of equal σ_{max}, and σ_{min} constructed by formulas (II.22), (II.23), and (II.68) are shown in Figure II.35, and the lines of equal τ_{max} are shown in Figure II.36. The trajectories of the principal stresses are shown in Figure II.37.

Narrow Slit[1]. In analyzing stress concentration near a narrow slit, it is necessary to distinguish between cases of tension of a surface with a slit and compression, since the solution for tension of a surface with a slit that is free of external forces will be inapplicable for compression. When the edge of /83 the slit is under compression, contact can occur, and consequently, the boundary conditions (along the contour of the slit) in these cases will be different. Therefore, in the following we will examine individually the cases of tension and compression of a plane with a slit.

[1] Investigation of stress concentration near variously oriented rectilinear and curvilinear slits (cracks) is discussed in Chapter VIII.

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Tension of an elastic plane with a slit. The function of stresses for an elastic surface with a narrow slit that is free of external forces is found directly from the functions (II.61), assuming that m = ±1 in them.

For a surface with a slit of length 2a, distributed along the Ox axis, we have m = 1, and for a slit of length 2b distributed along the Oy axis, m = -1.

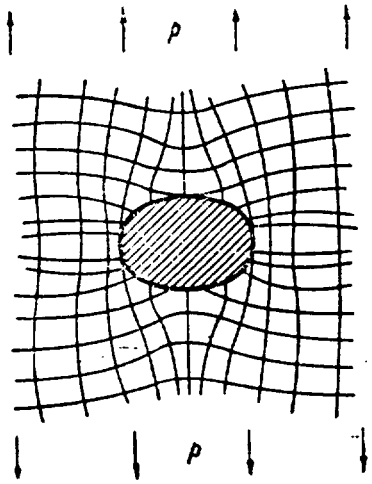


Figure II.31.

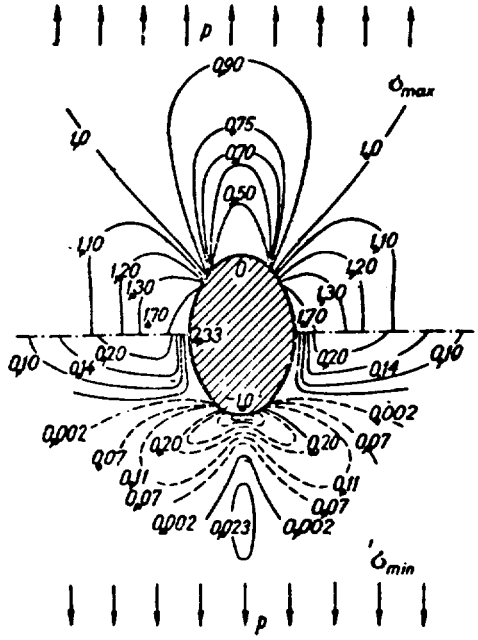


Figure II.32.

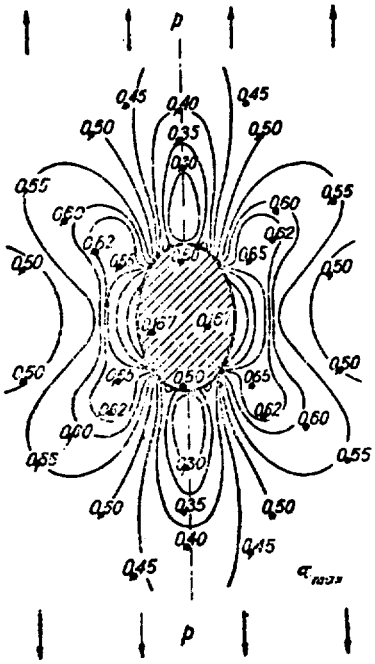


Figure II.33.

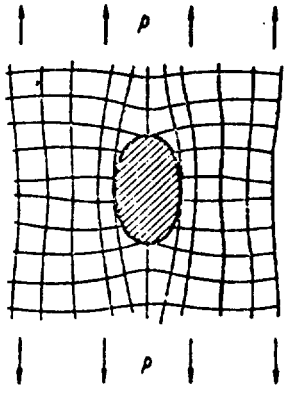


Figure II.34.

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TABLE II.7

$\vartheta°$	$b/a=3/2$	$b/a=2/3$	$\vartheta°$	$b/a=3/2$	$b/a=2/3$
0	—1,00	—1,00	50	0,93	1,54
10	—0,93	—0,78	60	1,86	1,90
20	—0,72	—0,23	70	2,85	2,15
30	—0,35	0,43	80	3,67	2,29
40	0,19	1,04	90	4,00	2,33

Tr. Note: Commas indicate decimal points.

The stresses σ_ϑ at the point of intersection of the contour of an elliptic hole (on the semi-axis $b \neq 0$) with the Ox axis is found from formula (II.63), assuming that $\vartheta = 0$ or $\vartheta = 180°$ in it: /84

$$(\sigma_\vartheta)_{\substack{\vartheta=0° \\ \vartheta=180°}} = p\left(2\frac{a}{b}\sin^2\alpha - \cos 2\alpha\right).$$

If the slit is distributed along the Ox axis, then $b = 0$, and from the latter formula we know that the stresses σ_ϑ for all $\alpha \neq 0$, on the ends of the slit, increase infinitely. In other words, on the ends of the slit there will be plastic deformations or (in the case of brittle destruction of the material of the plate), cracks.

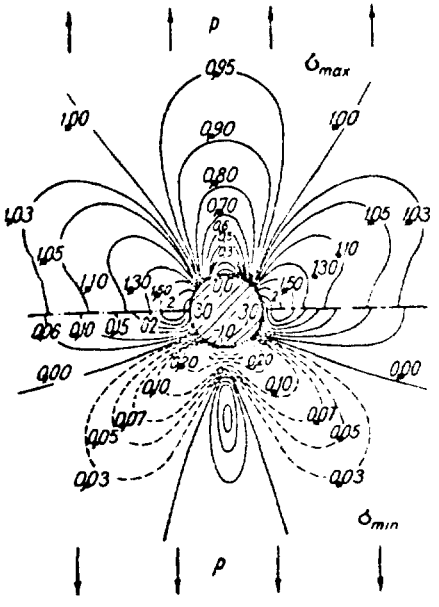


Figure II.35.

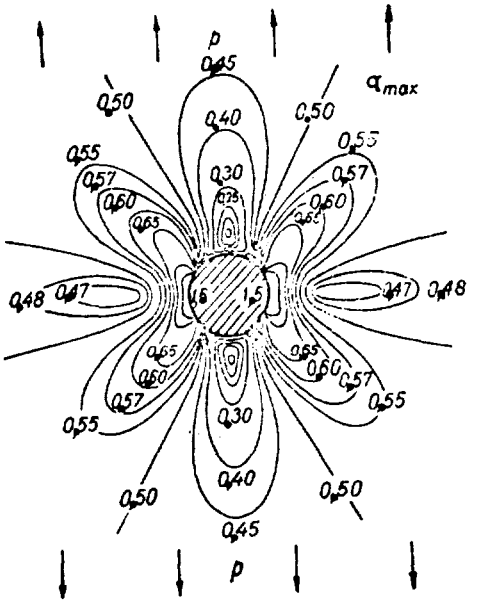


Figure II.36.

Thus, at the ends of any slit in a plate under tension by forces p that constitute angle $\alpha \neq 0$ with the slit, plastic deformations[1] or, when $p = p_{cr}$, cracks[2] begin to occur. Only a slit coinciding with the direction of the forces p of tension will have no effect on the stress state of the plate.

[1]See §8, Chapter V.
[2]See §6, Chapter VIII.

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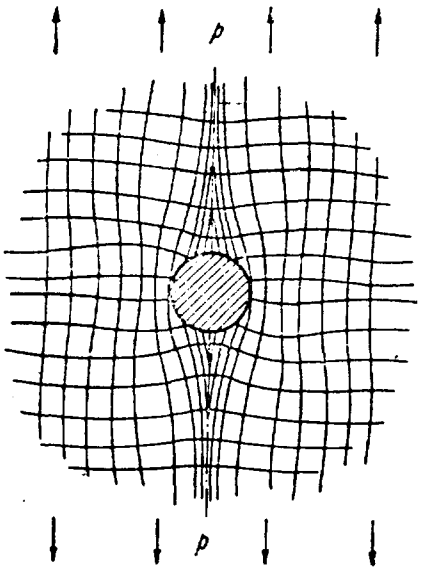


Figure II.37.

Compression of an elastic plane with a slit[1]. We will assume that an elastic isotropic plane that is weakened by a rectilinear slit of length 2a is compressed at infinity by uniformly distributed forces p that form angle α with the direction of the slit. We will select the axes of the Oxy coordinates such that the Ox axis coincides with the axis of the slit, and the origin of the Oxy coordinates will be placed at its center.

We will distinguish two types of slit: the "mathematical" slit, whose width is so small that it can be disregarded and assumed to be equal to zero, and the "physical" slit, whose width (up to deformation) δ is commensurate with the magnitude of elastic deformations of the material of the plate.

We will first examine the case of the "mathematical" slit. When external compressive forces p act upon the plate at angle $\alpha \neq 0$, the edges of the slit come into contact for all values $p \neq 0$.

By assuming the forces of friction (which can occur in the points of contact of the edges of the slit) to be governed by Coulomb's law, we write the boundary conditions on the contour of the slit (i.e., for t changing in the range $-a < t < +a$: /85

$$\sigma_y^{(+)}(t) = \sigma_y^{(-)}(t),\ \tau_{xy}^{(+)}(t) = k_s\,\sigma_y^{(+)}(t),\ \tau_{xy}^{(-)}(t) = k_s\sigma_y^{(-)}(t),\ v^{(+)} = v^{(-)},$$

where k_s is the coefficient of friction; t is an arbitrary point on the Ox axis with which the points of the "upper" and "lower" edges of the "mathematical" slit coincide.

The stresses along the slit are defined by the formulas:

$$\sigma_y^{(+)} = -p\sin^2\alpha,\ \tau_{xy}^{(+)} = -k_s p\sin^2\alpha = k_s\sigma_y^{(+)},$$

$$\sigma_x^{(+)} = p\left[(\sin 2\alpha + k_s\cos 2\alpha - k_s)\frac{t}{\sqrt{a^2-t^2}} - \cos^2\alpha\right].$$

[1]The solution of this problem was found by V. I. Mossakovskiy and P. A. Zagubizhenko in works [1, 2], the basic results of which are discussed below. Tension and deflection of plates with various slits are examined in a series of works by V. V. Panasyuk and B. L. Lozoviy; the basic results of these works are presented in §4 and 7 of Chapter VIII. A rather complete bibliography on cracks is available in the Candidate Dissertation of B. L. Lozoviy [1].

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From these formulas we see that σ_y and τ_{xy} along the slit remain constants, while the stresses σ_x on the ends of the slit increase infinitely. Here we must expect either plastic deformation or the occurrence of cracks due to the brittle destruction of the material from which the plate is made.

Let $y = f_1(t)$ and $y = f_2(t)$ be the equations of the "upper" and "lower" edges, respectively, of a "physical" crack. By definition, the width of this slit $\delta = f_1(t) - f_2(t)$ in the nondeformed state is commensurate with the elastic deformations. Under compression, the edges of a surface with such a slit may come into contact at some section $(-\tau_0, +\tau_0)$, such that after the deformation, the contour of the slit will consist of three sections: two free $(-a, -\tau_0)$ and $(+\tau_0, +a)$, and one section $(-\tau_0, +\tau_0)$ of contact in the middle. The lengths of these sections are not known beforehand and must be determined during the solution of the problem.

The boundary conditions in this problem, as we will see (on the contour of the slit) will be:

on the average section of contact $(-\tau_0, +\tau_0)$

$$\sigma_y^{(+)} = \sigma_y^{(-)},\ \tau_{xy}^{(+)} = k_s\sigma_y^{(+)},\ \tau_{xy}^{(-)} = k_s\sigma_y^{(-)},$$
$$v^{(+)} - v^{(-)} = f_1(t) - f_2(t) = f(t);$$

on the segments free of forces $(-a, -\tau_0)$ and $(+\tau_0, +a)$

$$\sigma_y^{(+)} = \sigma_y^{(-)} = 0,\ \tau_{xy}^{(+)} = \tau_{xy}^{(-)} = 0.$$

The stresses along the contour of the slit in this case are defined by the formulas

$$\sigma_y^{(+)} = -\frac{\sqrt{\tau_0^2 - t^2}}{\sqrt{a^2 - t^2}}\, p\sin^2\alpha,\ \tau_{xy}^{(+)} = -k_s p\frac{\sqrt{\tau_0^2 - t^2}}{\sqrt{a^2 - t^2}}\sin^2\alpha,$$
$$\sigma_x^{(+)} = p\left[(\sin 2\alpha + k_s\cos 2\alpha - k_s)\frac{t}{\sqrt{a^2 - t^2}} - \frac{\sqrt{\tau_0^2 - t^2}}{\sqrt{a^2 - t^2}}\sin^2\alpha - \cos 2\alpha\right],$$

while the length of the section of contact is defined by the formula /86

$$E\left(\frac{\pi}{2}, k\right) - \lambda^2 K\left(\frac{\pi}{2}, k\right) = \frac{2\mu\delta}{(\varkappa + 1)\, ap\sin^2\alpha},$$

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where $\lambda = \tau_0/a$; $k = \sqrt{1 - \lambda^2}$, while K and E are complete elliptical integrals of the first and second kind, respectively.

If $\lambda = 0(\tau_0 = 0)$, then $E(\pi/2, 1) = 1$, while $\lambda^2 K(\pi/2, 1) = 0$. Hence the "upper" and "lower" edges of the "physical" slit come into contact under the condition

$$p \geqslant \frac{2\mu\delta}{(\varkappa + 1)\, a \sin^2 \alpha}.$$

For the "mathematical" slit $\delta = 0$, and from the above inequality we see that its edges come into contact for any value $p \neq 0$, which was assumed for the solution of the problem for this slit.

Semicircular Hole.[1] The function that conformally represents the interior of unit circle γ on the exterior of an infinite elastic plane with a hole close to a semicircle is found from the function

$$\omega(\zeta) = R\left(\zeta + \sum_{k=1}^{n} a_k \zeta^{-k}\right) \tag{II.70}$$

for $n = 5$, $a_1 = -0.31250$; $a_2 = -0.15625$; $a_3 = -0.05078$; $a_4 = 0.00195$; $a_5 = 0.015113$; R is a constant that affects the dimensions of the hole and its position relative to the axes of the Oxy coordinates.

By assuming in (II.70) that $\zeta = \rho e^{i\vartheta}$ for $\rho = 1$, we find the equation for the contour of the hole

$$\begin{aligned} x &= R\,[0{,}68750\cos\vartheta - 0{,}15625\cos 2\vartheta - 0{,}05078\cos 3\vartheta + \\ &\quad + 0{,}00195\cos 4\vartheta + 0{,}01513\cos 5\vartheta]; \\ y &= R\,[1{,}31250\sin\vartheta + 0{,}15625\sin 2\vartheta + 0{,}05078\sin 3\vartheta - \\ &\quad - 0{,}00195\sin 4\vartheta - 0{,}01513\sin 5\vartheta]. \end{aligned} \tag{II.71}$$

The stresses along the contour of the hole in a plate under tension along the Ox axis ($\alpha = 0$) are defined by the formula

$$\begin{aligned} \sigma_\vartheta = \frac{p}{L(\vartheta)}\,(&0{,}16542 - 0{,}92141\cos\vartheta - 1{,}75649\cos 2\vartheta + \\ &+ 0{,}03923\cos 3\vartheta + 0{,}07549\cos 4\vartheta), \end{aligned} \tag{II.72}$$

[1]The solution for holes in the shape of a semicircle and arch is given by Ye. F. Burmistrov [1-3].

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and under tension along the Oy axis ($\alpha = \pi/2$), by the formula /87

$$\sigma_\vartheta = \frac{p}{L(\vartheta)}(1{,}38129 + 0{,}32298\cos\vartheta + 1{,}86243\cos 2\vartheta + \\ + 0{,}02376\cos 3\vartheta - 0{,}03928\cos 4\vartheta), \qquad (II.73)$$

where

$$L(\vartheta) = 1{,}2243 + 0{,}28930\cos\vartheta + 0{,}69230\cos 2\vartheta + 0{,}57280\cos 3\vartheta + \\ + 0{,}25740\cos 4\vartheta - 0{,}1560\cos 5\vartheta - 0{,}1524\cos 6\vartheta. \qquad (II.74)$$

The concentration factors of stresses $k = \sigma_\vartheta/p$, found by formulas (II.72) and (II.73), are presented in Table II.8.

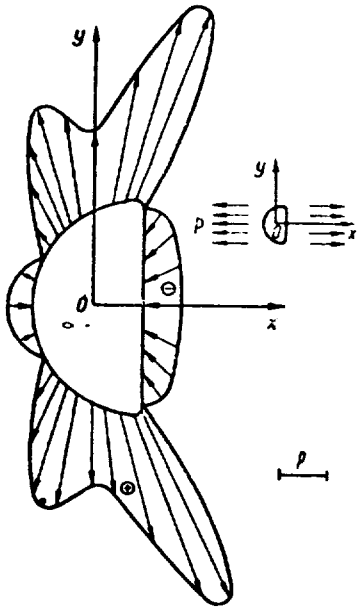


Figure II.38.

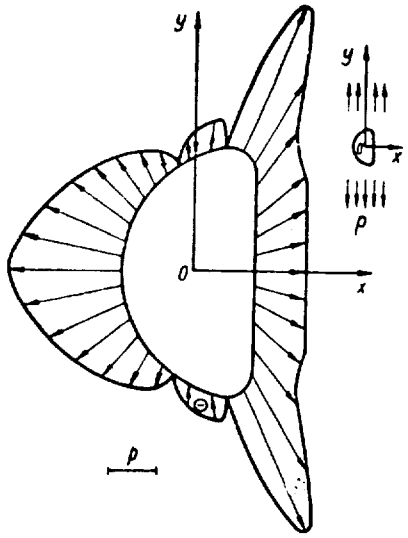


Figure II.39.

The distribution of stresses σ_ϑ along the contour of the given hole is illustrated in Figures II.38 and II.39. From these graphs we see that the greatest concentration of stresses σ_ϑ will occur under tension along the Ox axis.

The case where the edge of the semicircle is subjected to uniform compression is examined in the works of G. S. Grushko [1], and the case of tension of a semi-infinite plate with a semicircular notch is discussed in the works of S. F. Yeung [1] and L. H. Mitchell [1].

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Arch-Shaped Hole. The contour of the hole is given by the equations[1]

$$x = R[1{,}14153\cos\vartheta + 0{,}06795\cos 2\vartheta - 0{,}10130\cos 3\vartheta + \\ + 0{,}03873\cos 4\vartheta + 0{,}00194\cos 5\vartheta];$$
$$y = R[0{,}85847\sin\vartheta - 0{,}06795\sin 2\vartheta + 0{,}10130\sin 3\vartheta - \\ - 0{,}03873\sin 4\vartheta - 0{,}00194\sin 5\vartheta]. \qquad \text{(II.75)}$$

/88

Stresses along the contour of the hole free of external forces, under tension along the Ox axis ($\alpha = 0$) are

$$\sigma_\vartheta = \frac{p}{L(\vartheta)}[1{,}11878 + 0{,}40289\cos\vartheta - 2{,}41694\cos 2\vartheta + \\ + 0{,}11960\cos 3\vartheta + 0{,}01519\cos 4\vartheta], \qquad \text{(II.76)}$$

and under tension along the Oy axis ($\alpha = \pi/2$),

$$\sigma_\vartheta = \frac{p}{L(\vartheta)}[0{,}57096 - 0{,}08668\cos\vartheta + 1{,}66426\cos 2\vartheta - \\ - 0{,}16301\cos 3\vartheta - 0{,}02026\cos 4\vartheta], \qquad \text{(II.77)}$$

where

$$L(\vartheta) = 1{,}1549 - 0{,}1353\cos\vartheta - 0{,}3329\cos 2\vartheta - 0{,}2253\cos 3\vartheta + \\ + 0{,}6105\cos 4\vartheta - 0{,}3098\cos 5\vartheta - 0{,}0194\cos 6\vartheta. \qquad \text{(II.78)}$$

The values of σ_ϑ/p along the contour of the given hole are presented in Table II.9.

The distribution of stresses σ_ϑ calculated by formulas (II.76) and (II.77) is illustrated in Figures II.40 and II.41. We see from these graphs that the highest values of σ_ϑ will occur when the plate is under tension along the Oy axis (Figure II.41).

Investigation of stress concentration near an arch-shaped hole is also discussed in the works of I. S. Khara [1, 2], who retained ten terms in function (II.70). /89

[1]This corresponds to six terms in function (II.70) for $a_1 = 0.14153$, $a_2 = 0.06795$, $a_3 = -0.10130$, $a_4 = 0.03873$, $a_5 = 0.00194$.

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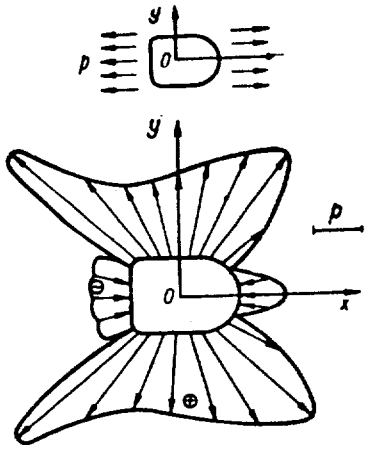


Figure II.40.

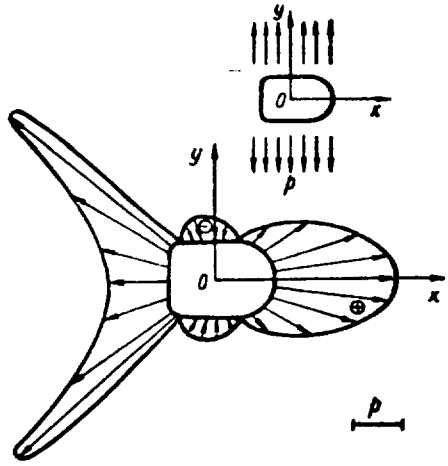


Figure II.41.

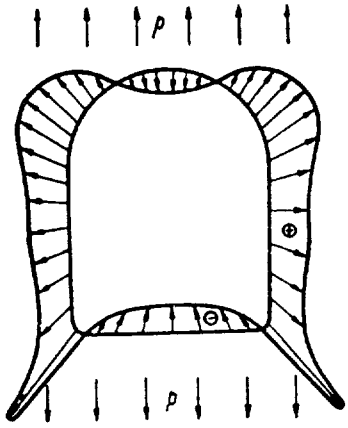


Figure II.42.

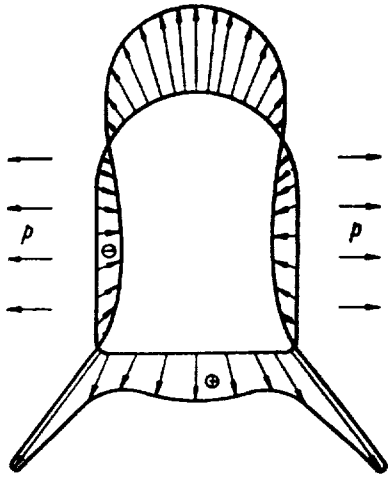


Figure II.43.

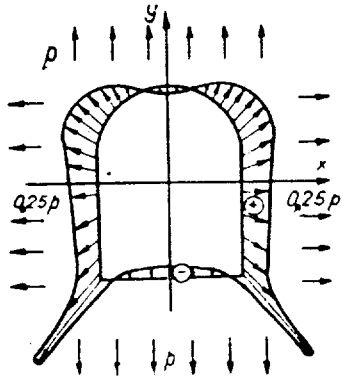


Figure II.44.

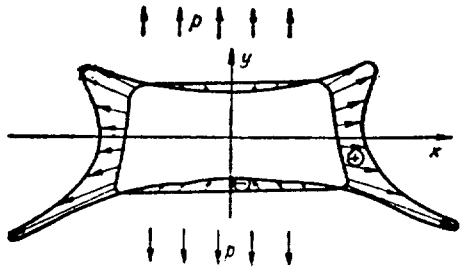


Figure II.45.

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TABLE II.8

$\vartheta°$	σ_ϑ/p		$\vartheta°$	σ_ϑ/p	
	$\alpha=0$	$\alpha=90°$		$\alpha=0$	$\alpha=90°$
0	—0,836	1,238	90	2,123	—0,553
10	—0,831	1,249	100	1,876	—0,422
20	—0,826	1,279	110	1,742	—0,132
30	—0,899	1,518	120	2,818	0,621
40	—0,829	1,752	130	0,939	0,832
50	—0,154	0,819	140	0,419	1,226
55	0,525	3,463	150	0,035	1,537
60	0,617	0,740	160	0,244	1,892
65	5,349	1,907	165	—0,334	1,937
70	4,691	0,150	170	—0,452	2,269
80	3,831	—0,624	180	—0,539	2,421

Tr. Note: Commas indicate decimal points.

The values of σ_ϑ/p along the contour of the given hole under tension along the Ox ($\alpha = 0$) and Oy ($\alpha = \pi/2$) axes, found by formulas (II.76) and (II.77), respectively, are given in Figures II.42-II.44. The stress-strain diagram of stress σ_ϑ along the contour of hole (II.75) under biaxial tension at infinity is shown in Figure II.44:

$$\sigma_x^{(\infty)} = p,\ \sigma_y^{(\infty)} = \frac{1}{4}p,\quad \tau_{xy}^{(\infty)} = 0.$$

Trapezoidal Hole[1]. In the function (II.70) that represents the interior of a trapezoidal hole on the exterior of the unit circle, I. S. Khara [1] used six terms, i.e., he assumed $n = 5$ in (II.70) for the corresponding values of the factors a_i ($i = 1, 2, \ldots, 5$).

The stress-strain diagrams of the concentration factors $k = \sigma_\vartheta/p$ along the contour of this hole are illustrated in Figures II.45-II.47, both under uniaxial ($\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = 0$, $\tau_{xy}^{(\infty)} = 0$; $\sigma_x^{(\infty)} = 0$, $\sigma_y^{(\infty)} = p$, $\tau_{xy}^{(\infty)} = 0$), and under biaxial ($\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = 0.25p$, $\tau_{xy}^{(\infty)} = 0$) stress.

In the work of A. A. Boyma [1], the function $\omega(\zeta)$ (II.70) was used in the form[2]

$$\omega(\zeta) = R\left[\zeta - 0{,}0086 + \frac{0{,}0167}{\zeta} + \frac{0{,}058}{\zeta^2} - \frac{0{,}1405}{\zeta^3} - \frac{0{,}0109}{\zeta^4}\right]. \qquad \text{(II.79)}$$

Figures II.48-II.50 represent the stress-strain diagrams of σ_ϑ/p and σ_ρ/p, /91 both for a free (curves II for σ_ϑ/p) and fortified (by an absolutely rigid ring) (curves I for σ_ρ/p) holes defined by functions (II.79) for $\rho = 1$, in the case of plane deformation when Poisson's ratio is $\nu = 0.2$. Figure II.48 shows the case of tension of a plate (at infinity) along the Ox axis by forces p, and Figure II.49, for tension along the Oy axis by forces 0.25p. Figure II.50

[1]The solution of this problem is found in the works of A. A. Boym [1], I. S. Khara [1], V. M. Gur'yanov and A. S. Kosmodamianskiy [1].

[2]The origin of the coordinate system is placed at the center of gravity of the given hole.

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shows the case of biaxial stress by forces $\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = 0.25p$ and $\tau_{xy}^{(\infty)} = 0$.

TABLE II.9

$\vartheta°$	σ_θ/p		$\vartheta°$	σ_θ/p	
	$\alpha=0$	$\alpha=90°$		$\alpha=0$	$\alpha=90°$
0	—1,024	2,846	100	1,627	—0,519
10	—0,829	2,450	110	1,765	—0,493
20	—0,389	2,081	120	2,252	—0,384
30	0,245	1,557	130	3,377	0,540
40	1,120	1,065	140	2,044	4,444
50	2,711	0,628	145	—0,056	3,547
60	2,403	—0,141	150	—0,809	2,682
70	2,231	—0,441	160	—0,790	1,515
80	1,888	—0,518	170	—0,936	1,355
90	1,677	—0,526	180	—0,858	1,177

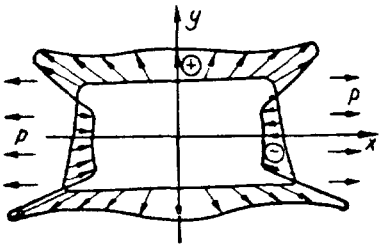


Figure II.46.

Tr. Note: Commas indicate decimal points.

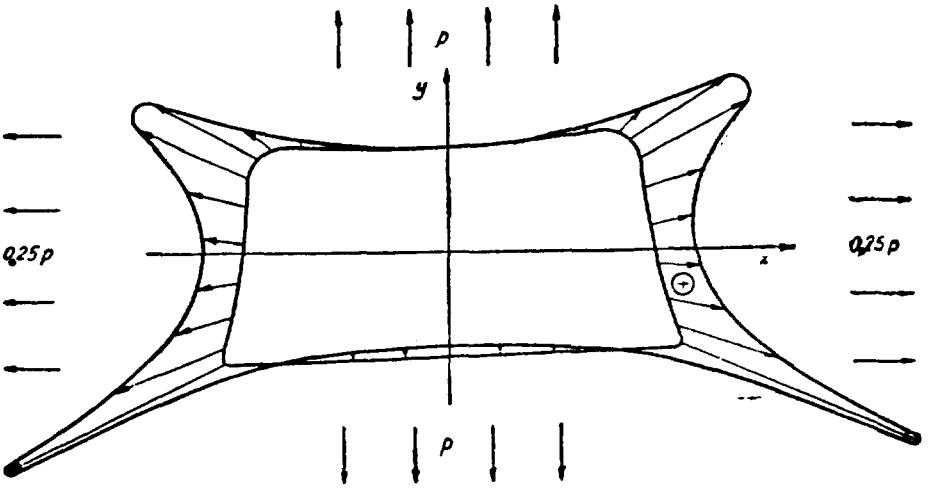


Figure II.47.

A hole close in shape to an equilateral trapezoid is analyzed in the work of V. M. Gur'yanov and A. S. Kosmodamianskiy [1]. In this work the effect of rounding off of the corners of the hole is examined in the general case for holes given by the functions (II.70).

The curvature of contour of a curvilinear hole given by the function $\omega(\zeta)$ for $\rho = 1$ at any of its points is

$$K = Re\frac{[\sigma\omega''(\sigma)+\overline{\omega'(\sigma)}]\overline{\omega'(\sigma)}}{|\omega'(\sigma)|^3} \qquad (II.80)$$

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This curvature can be represented in the form

$$\tilde{K} = KR \qquad \text{(II.81)}$$

and in the following will be called the adduced curvature.

The values of σ_ϑ/p in the angular points of the trapezoidal contour of the hole with the least radius of rounding as a function of the adduced curvature $\tilde{K}$ of form I, II or III of the basic stress state, are presented in Table II.10. /92

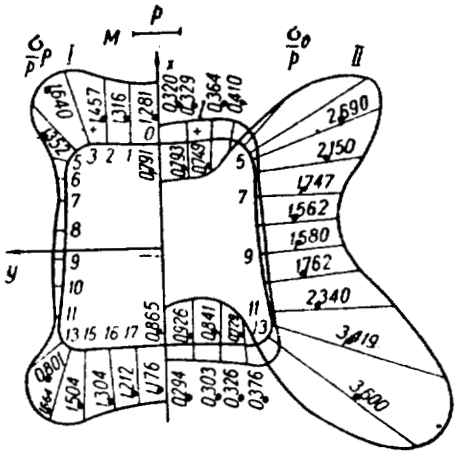


Figure II.48.

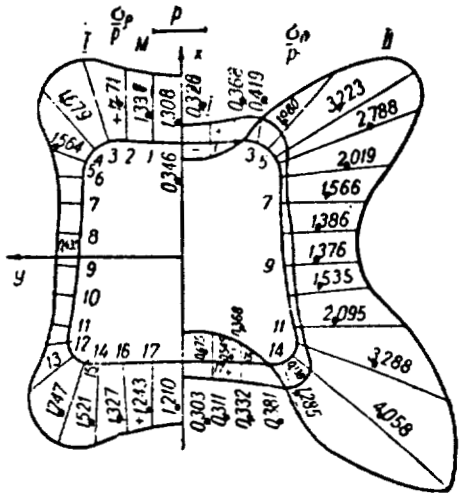


Figure II.49.

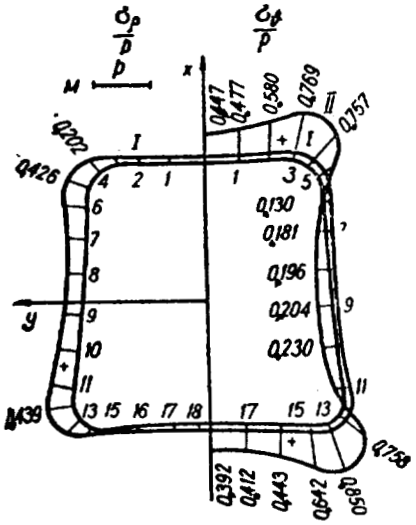


Figure II.50.

$\tilde{K}$	σ_ϑ/p		
	I	II	III
15	7,67	4,70	2,97
30	10,58	6,95	3,63
45	12,91	8,03	4,87
60	14,59	9,19	5,40

Tr. Note: Commas indicate decimal points

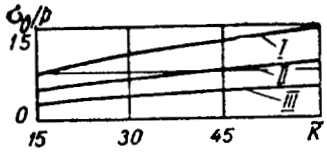


Figure II.51.

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Figure II.11.

$\vartheta°$	σ_ϑ/p		
	I	II	III
0	1,00	—0,83	1,83
15	1,03	—0,72	1,76
30	2,49	—0,46	2,95
45	3,24	2,62	0,63
60	0,95	1,48	—0,53
75	0,81	1,60	—0,79
90	0,78	1,67	—0,89
105	0,98	1,97	—0,99
120	2,39	3,41	—1,02
135	3,59	—0,096	3,69
150	0,88	—0,87	1,75
165	0,69	—0,99	1,69
180	0,59	—0,95	1,54

Tr. Note: Commas indicate decimal points.

The values of σ_ϑ/p at the acute angles of the trapezoidal contour with an adduced curvature equal to 60, as a function of the parameter ϑ and form of load, I, II, or III, are presented in Table II.11.

The following definitions are used in Tables II.10 and II.11: I, universal tension by forces p; II, uniaxial tension by forces p along the axis of symmetry of the hole, i.e., along the Ox axis (Figure II.48); III, uniaxial tension by forces p transverse to the axis of symmetry of the hole, i.e., along the Oy axis.

The graph of the dependence of σ_ϑ/p on ϑ for the given hole is shown in Figure II.51 (the definitions are the same as those used in Tables II.10 and II.11).

Other Forms of Holes. We cite some brief reports concerning certain works on stress concentration near holes of other shapes.

The concentration of stresses in disks with holes of special shape was examined by D. V. Vaynberg and A. A. Sinyavskiy [1]; with a star-shaped hole, by B. A. Obodovskiy [1]; with an oval hole, by E. Ye. Khachiyan [1], Ye. F. Burmistrov [2]; with a lemniscate hole, by A. M. Sen-Gupta [1], W. Showdon [1], with a cross-shaped hole, by H. C. Hahn [1], and with a hypotrochoidal hole, by Villagio Piero [1]; near a hole with a serrated contour, by I. V. Baklashov [1], and in the shape of a rhombus, by A. S. Avetisyan [1, 2][1].

Concluding Comments. On the basis of the tables and figures presented herein, several features of the law of distribution of stresses near holes in the uniaxial stress state can be established. /93

It is necessary first of all to note the heavy concentration of stresses near the edge of a hole in the vicinity of the points where the tangent to the contour is parallel to the direction of the forces of tension.

For circular hole (II.69), the greatest concentration factor is equal to three and relates to points lying on the ends of the diameter that is perpendicular to the forces of tension. For the ellipse under tension (or compression), parallel to one of the axes of symmetry of the ellipse, the concentration factor of stresses is determined by formula (II.65).

[1]See also the review of G. Neyber and G. Khan [1] of theoretical and experimental research on stress concentration near holes in linear and nonlinear statements, published in foreign press.

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For holes with the shape of a rectangle, square, semicircle, trapezoid, or equilateral triangle, the greatest stress concentration occurs at the angles, and the concentration factor k depends both on the radius of rounding of the corners, and, particularly, on the position of the hole in relation to the direction of the forces of tension.

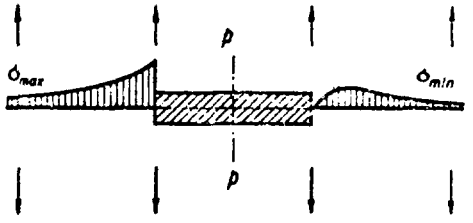


Figure II.52.

In the appendix to Table II.10, below, /94 are given the values of the concentration factors of stresses k' for rectangular holes with different side ratios, corresponding to the middle of the sides parallel to the forces of tension, as well as of the concentration factors of stresses k'' in the corners[1] of these holes under tension parallel to one of the axes of symmetry of the rectangle ($\alpha = 0$ or $\alpha = \pi/2$):

a/b	1:5	1:3	1:1	3:1	5:1
k'	2,5	2,2	1,5	1,35	1,2
k''	8,0	6,2	3,0	3,2	3,0

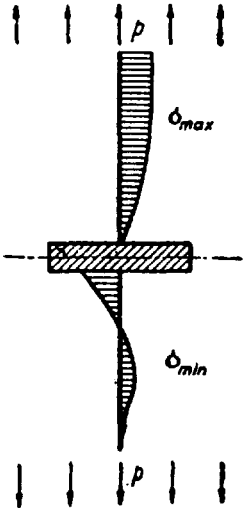


Figure II.53.

The coefficients of stress concentration k in the corners of the square are given below for various ratios of the radius of rounding r of these corners to the length of the side of the square under tension parallel to one of the sides:

r/a	1:9	1:20	1:40
k	3,0	4,6	7,0

The law of change of stresses σ_{max} and σ_{min} near a rectangular hole along the axis perpendicular to the forces of tension is illustrated in Figure II.52. Along the edge of the contour, where the tangent to the contour is perpendicular to the forces, or forms

[1]Since the holes examined above have rounded corners, the stress σ_ϑ in the angular point of the contour can be less than at the points of the contour located in the immediate vicinity of the corner (see, for instance, the first column of Table II.1, where $(\sigma_\vartheta)_{\vartheta=50^\circ} = 3.86p$, and $(\sigma_\vartheta)_{\vartheta=45^\circ} = 3.00p$.

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with it an angle close to 90°, stresses σ_{min} change signs. The region that is encompassed by the stresses of the opposite is saddle-shaped, as indicated in Figures II.3, II.10, II.14, II.17, II.19, II.21, II.24, II.29, II.32, and II.35, by the shaded areas. The law of change of stresses σ_{max} and σ_{min} near a rectangular hole along the axis parallel to the forces of tension is illustrated in Figure II.53.

§3. Biaxial Homogeneous Stress State

/95

Biaxial Tension or Compression. We will write the function of stresses in the form

$$U_0(x, y) = \frac{\sigma_T}{2}(\lambda_2 x^2 + \lambda_1 y^2), \tag{II.82}$$

where λ_1, λ_2 are dimensionless parameters, and σ_T is the yield point of the material under simple tension. To this function corresponds the basic stress state

$$\begin{aligned} \sigma_x^0 &= \frac{\partial^2 U_0}{\partial y^2} = \lambda_1 \sigma_T = p, \\ \sigma_y^0 &= \frac{\partial^2 U_0}{\partial x^2} = \lambda_2 \sigma_T = q, \\ \tau_{xy}^0 &= 0. \end{aligned} \tag{II.83}$$

When $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, we have the homogeneous biaxial stress state. By knowing[1] λ_1 and λ_2, we can find all possible cases of the homogeneous stress state.

The functions $\phi^0(z)$ and $\psi^0(z)$ (II.2), corresponding to the function (II.82) will have the form

$$\begin{aligned} \varphi^0(z) &= \frac{\sigma_T}{2}(\lambda_2 + \lambda_1) z = \frac{p+q}{4} z, \\ \psi^0(z) &= \frac{\sigma_T}{2}(\lambda_2 - \lambda_1) z = \frac{q-p}{2} z. \end{aligned} \tag{II.84}$$

The given contour conditions (II.7) for $f_1 = f_2 = 0$, i.e., for the case where the contour of the hole is free of external forces, will have the form

---

[1] The values λ_1 and λ_2 must be such that the material in the zone of stress concentration remains in the elastic state.

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$$f_1^0 + if_2^0 = -\frac{\sigma_\tau}{2}[(\lambda_1 + \lambda_2)\,\omega(\sigma) + (\lambda_2 - \lambda_1)\overline{\omega(\sigma)}],$$

$$f_1^0 - if_2^0 = -\frac{\sigma_\tau}{2}[(\lambda_1 + \lambda_2)\,\overline{\omega(\sigma)} + (\lambda_2 - \lambda_1)\,\omega(\sigma)].$$

Proceeding in the same manner as was described in detail in the preceding sections for the various hole shapes, we easily find the functions $\phi(\zeta)$ and $\psi(\zeta)$. For instance, for a plate with a circular hole,

$$\varphi(\zeta) = \frac{\sigma_\tau R}{4}\left[(\lambda_1 + \lambda_2)\frac{1}{\zeta} + 2(\lambda_1 - \lambda_2)\zeta\right],$$
$$\psi(\zeta) = \frac{\sigma_\tau R}{2}\left[(\lambda_2 - \lambda_1)\frac{1}{\zeta} - (\lambda_1 + \lambda_2)\zeta - (\lambda_2 - \lambda_1)\zeta^3\right]. \qquad \text{(II.85)}$$

By functions (II.85) we determine from (I.27) the stress components: /96

$$\sigma_\varrho = \frac{\lambda_1 + \lambda_2}{2}\sigma_\tau(1 - \varrho^2) - \lambda\sigma_\tau(1 - 4\varrho^2 + 3\varrho^4)\cos 2\vartheta,$$
$$\sigma_\vartheta = \frac{\lambda_1 + \lambda_2}{2}\sigma_\tau(1 + \varrho^2) + \lambda\sigma_\tau(1 + 3\varrho^4)\cos 2\vartheta, \qquad \text{(II.86)}$$
$$\tau_{\varrho\vartheta} = -\lambda\sigma_\tau(1 + 2\varrho^2 - 3\varrho^4)\sin 2\vartheta,$$

where

$$\lambda = \frac{\lambda_2 - \lambda_1}{2}.$$

However, from the point of view of simplicity of calculations, it is much more convenient to find directly the individual solutions for uniaxial stress states, and then, by using the corresponding solutions, to find the solutions for the desired biaxial stress states[1]. Figures II.54-II.57 show the results of such superposing of two solutions for uniaxial stress states obtained above, namely

$$\sigma_x^\infty = \frac{1}{4}p, \quad \sigma_y^{(\infty)} = 0, \quad \tau_{xy}^{(\infty)} = 0$$

and

$$\sigma_x^{(\infty)} = 0, \qquad \sigma_y^{(\infty)} = p, \quad \tau_{xy}^{(\infty)} = 0,$$

for an elliptic hole for which the ratio of the semiaxes is a/b = 3/2 and for

[1]Several such problems for p = q/4, for elliptic and rectangular holes (in particular, narrow, long slit), and for square holes, were analyzed in detail in the joint work of A. N. Dinnik, A. B. Morgayevskiy and G. N. Savin [1].

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a rectangular hole, for which the ratio of the sides is a/b = 3/2. The compressive stresses are represented in these figures by the shaded areas.

Pure Displacement. In the case where the basic stress state is pure displacement, the solution can be found by two methods: 1) by assuming in (II.82) $\lambda_1 = -\lambda_2$ and 2) by taking directly the function of stresses in the form

$$U_0(x, y) = -\tau xy. \tag{II.87}$$

It is easier to use function (II.87) and to obtain for it the functions $\phi(\zeta)$ and $\psi(\zeta)$, since this function is much simpler than (II.82).

Functions $\phi^0(z)$ and $\psi^0(z)$, corresponding to function (II.87), as is readily seen, have the form

$$\varphi^0(z) = 0, \qquad \psi^0(z) = i\tau z, \tag{II.88}$$

and the given contour conditions (II.7) are

$$\begin{aligned} f_1^0 + if_2^0 &= i\overline{\tau\omega(\sigma)}, \\ f_1^0 - if_2^0 &= -i\tau\omega(\sigma). \end{aligned} \tag{II.89}$$

The functions $\phi(\zeta)$, $\psi(\zeta)$ for the region with one hole under simple displacement /97 consequently, will be

$$\begin{aligned} \varphi(\zeta) &= \varphi_0(\zeta), \\ \psi(\zeta) &= \psi_0(\zeta) + i\tau\omega(\zeta). \end{aligned} \tag{II.90}$$

For a region with a square hole[1], we will use the representative function in the form (II.24). However, the functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ are found from equations (II.6) and from the adduced contour conditions (II.89). Finally:

$$\begin{aligned} \varphi(\zeta) &= i\frac{6}{5}R\tau, \\ \psi(\zeta) &= iR\tau\left[\frac{1}{\zeta} + \frac{13\zeta^3}{5(2+\zeta^4)}\right]. \end{aligned} \tag{II.91}$$

[1]The problem of pure displacement of a finite square plate with a central circular hole was solved by C. K. Wang [1].

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The stresses along the contour of the hole are found from (II.8):

$$\sigma_\vartheta = \frac{48}{5}\tau\frac{\sin 2\vartheta}{5+4\cos 4\vartheta}. \qquad \text{(II.92)}$$

The values of stresses σ_ϑ in fractions of τ, calculated by formula (II.92), are presented in Table II.12.

TABLE II.12

$\vartheta°$	σ_ϑ/τ	$\vartheta°$	σ_ϑ/τ	$\vartheta°$	σ_ϑ/τ
0	0,000	60	2,771	135	—9,600
20	1,084	80	0,407	140	—7,618
30	2,771	90	0,000	150	—2,771
40	7,618	100	—0,407	160	—1,084
45	9,600	110	—1,084	170	—0,407
50	7,618	130	—7,618	180	0,000

Tr. Note: Commas indicate decimal points.

Stress concentration with simple displacement in an infinite plate with a rectangular hole with sides a and b, depending on the value r/b (r is the radius of rounding off of the corner of the hole) for various values of a/b were examined by Ye. P. Anikin [1]. In this work are found the functions as functions of stresses and the representative parameters a/b and r/b.

The stresses along the contour of the hole are found from formulas (II.8). The values thus found for $k_{max} = \tau_{max}/\tau$ at the most stressed points of the contour of the hole, depending on parameters a/b and r/b, are presented in Table II.13[1]. /100

TABLE II.13

a/b	r/b									
	0,03	0,06	0,09	0,12	0,16	0,20	0,25	0,30	0,40	0,50
1,0	7,7	5,5	4,5	3,8	3,3	3,0	2,7	2,4	2,2	2,0
1,4	8,2	5,8	5,0	4,2	3,7	3,3	2,9	2,7	2,4	2,2
1,8	8,6	6,3	5,2	4,5	3,9	3,6	3,2	2,9	2,6	2,4
2,0	8,8	6,5	5,3	4,7	4,1	3,7	3,3	3,0	2,7	2,5
2,4	—	6,8	5,7	5,0	4,4	4,0	3,5	3,2	2,9	2,7
2,8	—	7,2	6,1	5,3	4,7	4,2	3,7	3,4	3,1	2,8
3,0	—	7,3	6,2	5,5	4,8	4,3	3,9	3,5	3,2	2,9
3,5	—	7,6	6,5	5,7	5,1	4,6	4,2	3,8	3,4	3,1
4,0	—	7,9	6,8	6,0	5,4	4,8	4,4	4,0	3,6	3,3
4,5	—	8,1	7,0	6,2	5,6	5,0	4,6	4,2	3,7	3,4
5,0	—	8,2	7,2	6,4	5,8	5,2	4,8	4,3	3,8	3,5

[1] From the data of this table it is easy to construct a system of curves $k_{max}(r/b)$ and $k_{max}(a/b)$ for the given values a/b = 1.0; 1.4; ... 5.0 and r/b = 0.03; ... 0.50, and, on the basis of these smooth curves, to find (approximately) k_{max} for all values of a/b and r/b within the ranges $(0.03 \leqslant r/b \leqslant 0.50)$ and $(1.0 \leqslant a/b \leqslant 5.0)$.

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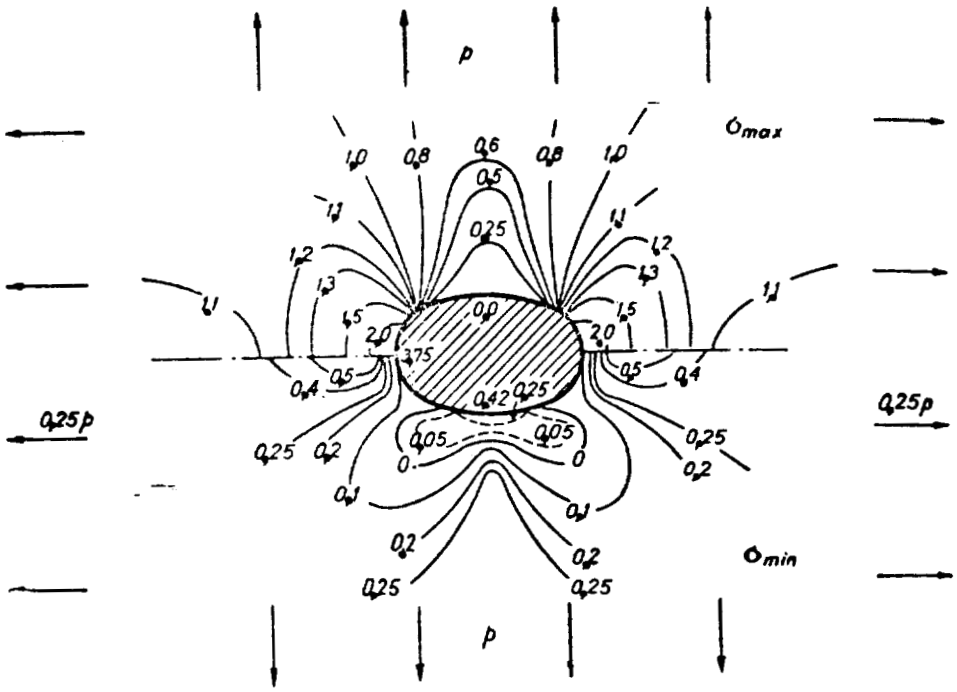


Figure II.54.

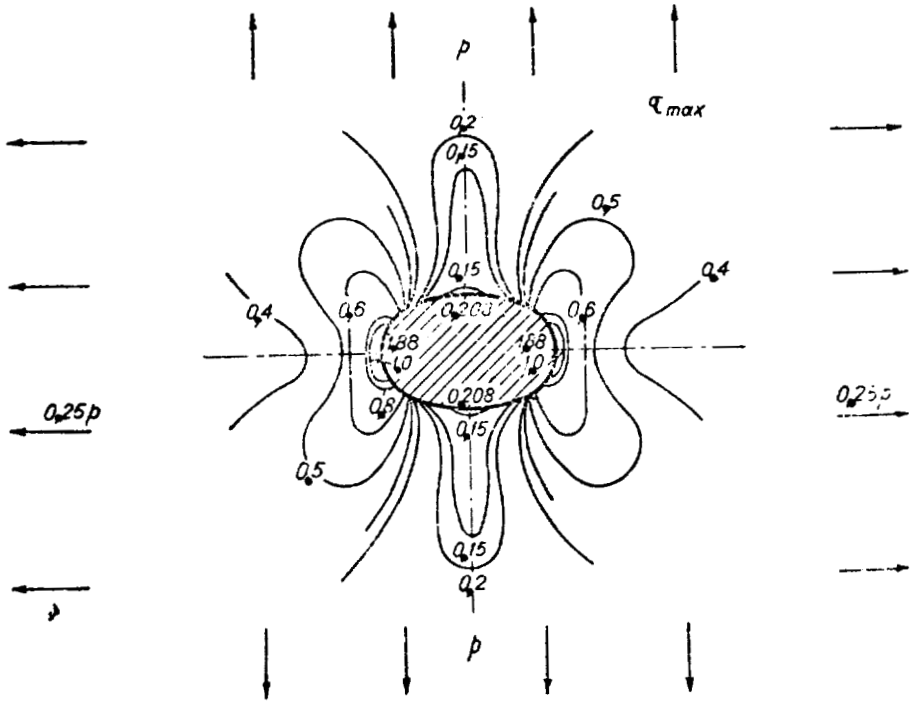


Figure II.55.

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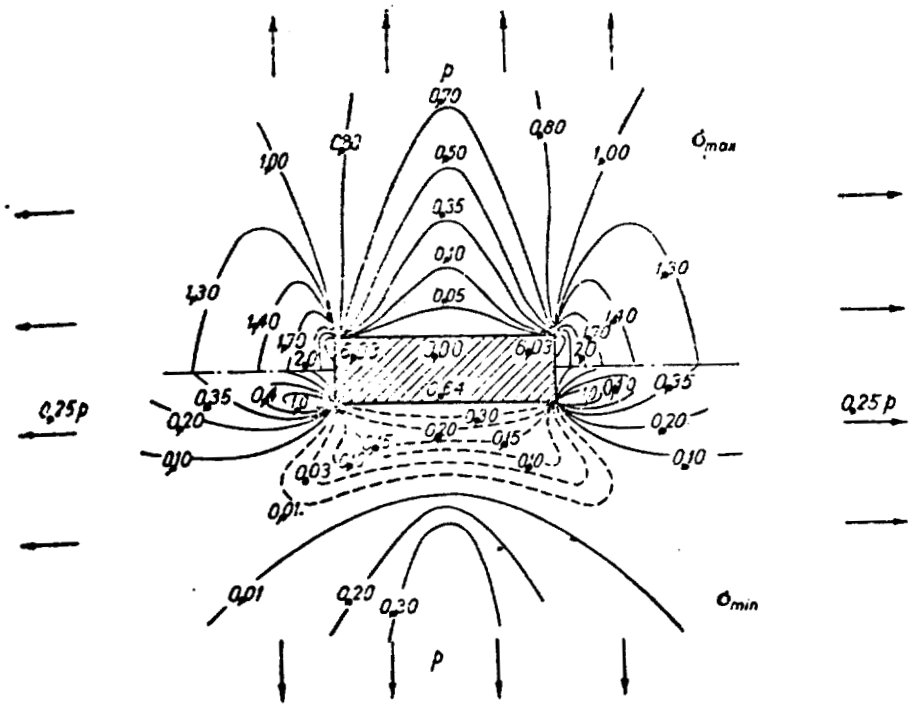


Figure II.56.

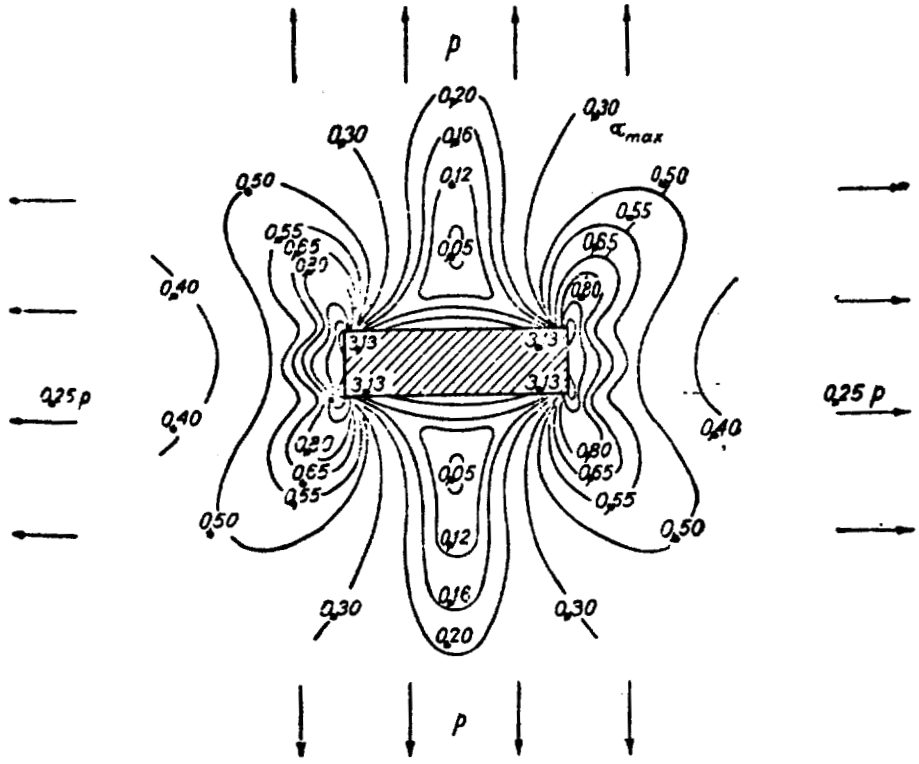


Figure II.57.

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The stress concentration in an infinite plane, weakened by a rhombic hole, was examined by A. S. Avetisyan [3], and by A. Sen-Gupta [1], in the case of a lemniscate hole. The effect of crescent shaped holes on the stresses under pure displacement was investigated by N. A. Savruk [6]. The stress concentration near a rectangular hole under pure displacement and under tension was examined by I. M. Budnyatskiy [1] and V. N. Kozhevnikova [2].

§4. Pure Deflection[1]

Stresses under pure deflection of a rod (beam) are defined by the formulas

$$\sigma_x^0 = -\frac{M}{J}y, \quad \sigma_y^0 = 0, \quad \tau_{xy}^0 = 0, \tag{II.93}$$

where M is the deflecting moment; J is the moment of inertia of the cross section of the beam. The direction of the axes of the coordinates and the effect of the deflecting moment M are illustrated in Figure II.58.

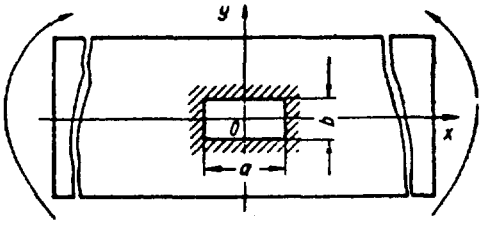


Figure II.58.

The Ox axis coincides with the neutral axis of the beam[2]. The stress function is /101

$$U_0(x, y) = -\frac{M}{6J}y^3. \tag{II.94}$$

We will determine the functions $\phi^0(z)$ and $\psi^0(z)$. From (II.9) and (II.10), we know that $P(x, y) = -\frac{M}{4J}y$. From (II.11), we see that $Q(x, y) = \frac{M}{4J}x$. Hence

$$\varphi^{0\prime}(z) = i\frac{M}{4J}(x + iy) = i\frac{M}{4J}z;$$

consequently, when $C_1 = C_2 = 0$,

$$\varphi^0(z) = i\frac{Mz^2}{8J}. \tag{II.95}$$

---

[1]The solutions presented in §4 and 5 of this chapter were presented by G. N. Savin in [2].

[2]If the center of the hole does not lie on the neutral line of the beam, but is displaced from it by a distance d, the problem of the distribution of stresses near holes reduces to the solution of the following types of problems: a) rod under tension in direction of neutral line by forces p = -Md/J and b) rod under pure deflection, with a hole whose center lies on the neutral line (for instance, see §1, Chapter III).

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The function $\psi^0(z)$ is found from (II.13). By omitting the intermediate calculations, we write, finally:

$$\psi^0(z) = -i\frac{Mz^2}{8J}. \tag{II.96}$$

Now let a hole of some shape be made in a rod (beam). The center of this hole coincides with the origin of the coordinate system (see Figure II.58). The dimensions of this hole are small in comparison with the height 2h of the beam, and the contour of the hole is free of external stresses. The functions of complex variable (II.1) have the form

$$\varphi_1(z) = \varphi^*(z) + i\frac{M}{8J}z^2,$$
$$\psi_1(z) = \psi^*(z) - i\frac{M}{8J}z^2. \tag{II.97}$$

Converting to the transformed zone, or, in other words, to the variable ζ, depending on the shape of the hole, from formulas (I.39), (I.49) or (I.54), we have

$$\Phi(\zeta) = \varphi_0(\zeta) + i\frac{M}{8J}[\omega(\zeta)]^2,$$
$$\Psi(\zeta) = \psi_0(\zeta) - i\frac{M}{8J}[\omega(\zeta)]^2. \tag{II.98}$$

The given contour conditions (II.7) (considering that the contour of the hole is free of external forces) for the given case of deflection of the beam will have the form /102

$$f_1^0 + if_2^0 = -i\frac{M}{8J}[\omega(\sigma) - \overline{\omega(\sigma)}]^2,$$
$$f_1^0 - if_2^0 = i\frac{M}{8J}[\overline{\omega(\sigma)} - \omega(\sigma)]^2. \tag{II.99}$$

Elliptic Hole[1]. By taking the function that conformally represents the interior of the unit circle γ on the exterior of the ellipse in the form (II.60) we obtain

[1]The solution of this problem is given by N. I. Muskhelishvili [4] and A. S. Lokshin [1].

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$$-\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} + (2m-1)\zeta^2\right],$$

$$\psi(\zeta) = i\frac{MR^2}{8J}\left[(1-2m)\zeta^2 + \frac{2(1-m)^2(m+\zeta^2)}{m\zeta^2-1}\zeta^2 - \frac{1}{\zeta^2}\right]. \qquad \text{(II.100)}$$

The stresses along the contour of the elliptic hole are found from the equation (II.8) by substitution of the corresponding derivatives of the functions $\phi(\zeta)$ (II.100) and $\omega(\zeta)$ (II.60) into it:

$$\sigma_\vartheta = \frac{MR}{J}\cdot\frac{(1+m-2m^2)\sin\vartheta + (m-1)\sin 3\vartheta}{1+m^2-2m\cos 2\vartheta}. \qquad \text{(II.101)}$$

By substituting into (II.101) the various values of m and R, we find the formula for stresses σ_ϑ along the contour of any elliptic hole. For instance, for an elliptic hole of any size with the ratio of the axes k = 1/3, i.e., for m = 1/2, we have

$$\sigma_\vartheta = \frac{MR}{J}\cdot\frac{4\sin\vartheta - 2\sin 3\vartheta}{5-4\cos 2\vartheta}. \qquad \text{(II.102)}$$

We find the greatest value of σ_ϑ on the contour of any elliptic hole from (II.101) for $\vartheta = \pi/2$:

$$\sigma_\vartheta = 2\frac{MR}{J}\cdot\frac{b}{a} = \frac{Mb}{b}\left(1+\frac{b}{a}\right). \qquad \text{(II.103)}$$

If b = 0 and a ≠ 0, i.e., in the case of a slit on the Ox axis of length 2a, $\sigma_\vartheta = 0$. This signifies that the slit of length 2a, located parallel to the central line of the beam, does not cause concentration of stresses in the rod (beam) under examination.

If b ≠ 0 and a = 0, i.e., in the case of a slit located on the Oy axis of length 2b, σ_ϑ will be infinitely great, i.e., zones of plastic deformation of the material or cracks will occur.

Round Hole[1]. If we assume m = 0 in (II.100) we find the stress functions for a circular hole /103

[1]Pure deflection of a beam, weakened by a hole in the form of a symmetrical circular lune, is discussed by G. N. Savin and M. A. Savruk [1, 2]. From these solutions, the solution of the given problem follows as a partial case.

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$$\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} - \zeta^2\right],$$

$$\psi(\zeta) = i\frac{MR^2}{8J}\left[\zeta^2 - 2\zeta^4 - \frac{1}{\zeta^2}\right]. \qquad \text{(II.104)}$$

We find the formula for stresses along the contour of a circular hole from (II.101) for m = 0:

$$\sigma_\vartheta = \frac{MR}{J}[\sin\vartheta - \sin 3\vartheta]. \qquad \text{(II.105)}$$

Square Hole I. We take the function $\omega(\zeta)$ in the same form (II.24) as in the case of tension[1]. The location of the hole relative to the coordinate axes is shown in Figure II.2; the Ox axis coincides with the central line of the beam. The final form of the functions for the given hole are:

$$\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} - \zeta^2 - \frac{1}{3}\zeta^4\right],$$

$$\psi(\zeta) = -i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} + \frac{5}{3}\zeta^2 - \frac{52\zeta^2 + 39}{5(\zeta^4 + 2)}\right]. \qquad \text{(II.106)}$$

The formula for stresses along the contour of square hole I is found from (II.8):

$$\sigma_\vartheta = \frac{MR}{J}\cdot\frac{14\sin\vartheta - 12\sin 3\vartheta - 2\sin 5\vartheta}{15 + 12\cos 4\vartheta}. \qquad \text{(II.107)}$$

Square Hole II. Rotated relative to the position of the preceding hole by angle $\pi/4$. By taking the representative function in the form

$$\omega(\zeta) = R\left(\frac{1}{\zeta} + \frac{1}{6}\zeta^3\right), \qquad \text{(II.108)}$$

we find the functions for a plane with square hole II:

$$\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} - \zeta^2 + \frac{1}{3}\zeta^4\right],$$

$$\omega(\zeta) = i\frac{MR^2}{8J}\cdot\frac{18 + 117\zeta^2 - 31\zeta^4 - 15\zeta^8}{9\zeta^2(\zeta^4 - 2)}.$$

---

[1]Deflection of a beam with square and triangular holes of somewhat different forms is examined by M. I. Neyman [1].

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The formula of stresses for the points of the contour of the given hole is found from (II.8): /104

$$\sigma_\theta = 2\frac{MR}{J}\cdot\frac{\sin\vartheta - 6\sin 3\vartheta + \sin 5\vartheta}{15 - 12\cos 4\vartheta}. \tag{II.109}$$

Triangular Hole. By taking the representative function $\omega(\zeta)$ in the same form (II.53) as in the case of tension (see Figure II.27), we find

$$\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} - \zeta^2 + \frac{2}{3}\zeta^3\right],$$
$$\psi(\zeta) = i\frac{MR^2}{8J}\cdot\frac{27 + 33\zeta^2 - 45\zeta^4 + 22\zeta^6 + 54\zeta^6 - 36\zeta^7}{9\zeta^2(2\zeta^3 - 3)}. \tag{II.110}$$

The stress along the contour of a triangular hole is

$$\sigma_\theta = 3\frac{MR}{J}\cdot\frac{\sin\vartheta - 3\sin 3\vartheta + \sin 4\vartheta}{13 - 12\cos 3\vartheta}. \tag{II.111}$$

Rectangular Hole[1] with Ratio of Sides a/b ≈ 3.2. We will take the representative function $\omega(\zeta)$ in the form (II.50):

$$\omega(\zeta) = R\left[\frac{1}{\zeta} + \frac{1}{2}\zeta - \frac{1}{8}\zeta^3 - \frac{3}{80}\zeta^5\right]. \tag{II.112}$$

The final form of the functions is

$$\varphi(\zeta) = i\frac{MR^2}{8J}\left[\frac{1}{\zeta^2} - \frac{1}{8}\zeta^2 - \frac{23}{80}\zeta^4 - \frac{3}{40}\zeta^6\right],$$
$$\psi(\zeta) = -i\frac{MR^2}{8J}\cdot\frac{20480 - 10240\zeta^2 - 2965{,}6\zeta^4}{2048(80\zeta^2 - 40\zeta^4 + 30\zeta^6 + 15\zeta^8)} + \tag{II.113}$$
$$+\frac{12012{,}4\zeta^6 + 7462{,}7\zeta^8 + 3027{,}3\zeta^{10}}{2048(80\zeta^2 - 40\zeta^4 + 30\zeta^6 + 15\zeta^8)}.$$

The stresses along the contour of the hole are

$$\sigma_\theta = \frac{MR}{J}\cdot\frac{1228\sin\vartheta - 490\sin 3\vartheta - 112\sin 5\vartheta - 48\sin 7\vartheta}{1825 - 1580\cos 2\vartheta + 720\cos 4\vartheta + 480\cos 6\vartheta}. \tag{II.114}$$

[1]The problem of the distribution of stresses near a rectangular hole in an infinite plate under deflection is also examined in the work of V. N. Kozhevnikova [2, 3].

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Concluding Comments. We will compare the effect of various holes on the pattern of the stress state in the beam under pure deflection with the aid of formulas (II.102), (II.105), (II.107), (II.109), (II.111) and (II.114).

All formulas for stresses σ_ϑ include as a factor the value MR/J, where the constant R characterizes the size of the hole. In order to obtain the correct representation of the effect of a given hole on the stress distribution near the holes, it is necessary to make this comparison for holes of identical sizes, i.e., when their dimensions constitute a certain part of the height of the beam. /105

The values of stresses σ_ϑ in fractions of MR/J along the lower part of the contour of the examined holes (see Figure II.8) are presented in Table II.14. In order to find, for instance, the stresses for an elliptic hole with semi-axes a = 6 cm and b = 2 cm, we find first, from (II.60), that R = 4. By multiplying all values from the third column of Table II.14 by 4M/J, we find the values of stresses σ_ϑ along the contour of the given hole.

TABLE II.14

$\vartheta°$	Shape of Hole					
	Circle	Ellipse ($\frac{a}{b}$=3)	Square (I)	Square (II)	Triangle	Rectangle ($\frac{a}{b}$=3,2)
0	0,000	0,000	0,000	0,000	0,000	0,000
10	—0,326	—0,245	—0,169	—0,709	—0,785	—0,143
20	—0,524	—0,188	—0,443	—0,599	—0,544	—0,291
30	—0,500	0,000	—0,667	—0,476	—0,377	0,472
45	0,000	0,282	0,942	—0,315	—0,197	0,575
60	0,866	0,494	1,528	0,000	0,000	0,409
70	1,440	0,590	1,141	0,583	0,186	0,384
80	1,851	0,647	0,946	2,350	0,464	0,436
90	2,000	0,666	0,888	5,333	0,923	0,454
100	1,851	0,647	0,946	2,350	1,811	0,436
115	1,165	0,547	1,294	0,221	5,685	0,377
120	0,866	0,494	1,528	0,000	5,196	0,409
125	0,560	0,433	1,670	—0,143	1,459	0,435
135	0,000	0,282	0,942	—0,315	—0,939	0,575
170	—0,326	—0,245	—0,169	—0,709	—0,252	—0,143
180	0,000	0,000	0,000	0,000	0,000	0,000

Tr. Note: Commas indicate decimal points.

For the case of deflection of a beam by a constant shear force, the stress concentration near the same holes as in the case of pure deflection, is discussed in §5, Chapter II of G. N. Savin's monograph [13]. Here, however, in §6, Chapter II, only the problem of stresses near a circular hole for rods (beams) under more complex forms of deflection is discussed. It should be mentioned that since the publication of G. N. Savin's monograph [13], many works have appeared on the subject of deflection of a beam (with various types of holes) by a constant shear force. We will mention a few of them. Thus, S. R. Heller [1], S. R. Heller, Jr., I. S. Brock- R. Bart [2], investigated stress distribution near square, rectangular, and oval holes in a cantilever

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under deflection. G. N. Savin and M. A. Savruk [2, 3] analyzed the problem of stress distribution near a hole in the form of a symmetrical circular lune in a cantilever beam (rod). A. M. Averin and Ye. F. Burmistrov [1] investigated the problem of stress distribution in beams with a common form of hole.

§5. Determination of Effect of Forces of Gravity on Stress Distribution in an Elastic Plane with a Hole

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In discussing the pattern of the stress state around holes, we did not take into account the volumetric forces of gravity. However, in practice it is often forbidden to disregard the effect of gravity, particularly when gravity is not an insignificant force in comparison with the actual surface forces, and particularly in those cases where gravity is the sole factor governing the stress state, and consequently, the concentration of stresses around holes (as, for instance, in the concentration of stresses around underground mine shafts, etc).

We will assume that the stress components

$$(\sigma_r)_0, \quad (\sigma_\theta)_0, \quad (\tau_{r\theta})_0 \tag{II.115}$$

are related to a polar system (r, θ) and define the stress state of an infinitely heavy elastic isotropic plane that is not weakened by a hole. As we know, these stress components (II.115) should satisfy the basic equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\cdot\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = \gamma \sin\theta,$$

$$\frac{1}{r}\cdot\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \gamma \cos\theta \tag{II.116}$$

and the equation of compatibility

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\sigma_r + \sigma_\theta) = 0. \tag{II.117}$$

In (II.116) $\gamma = \rho g$, where g is acceleration of gravity and ρ = const is density. Obviously, due to the linearity of equations (II.116) and (II.117), their general solution, i.e., the stress components $(\sigma_r)_0$, $(\sigma_\theta)_0$ and $(\tau_{r\theta})_0$ (II.115), can be represented in the form

$$\begin{aligned} (\sigma_r)_0 &= \sigma_r^0 + \sigma_r^\infty, \\ (\sigma_\theta)_0 &= \sigma_\theta^0 + \sigma_\theta^\infty, \\ (\tau_{r\theta})_0 &= \tau_{r\theta}^0 + \tau_{r\theta}^\infty, \end{aligned} \tag{II.118}$$

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where the stress components

$$\sigma_r^0, \quad \sigma_\theta^0, \quad \tau_{r\theta}^0 \tag{II.119}$$

are the partial solution of equation system (II.117) and (II.116), while the components

$$\sigma_r^{00}, \quad \sigma_\theta^{00}, \quad \tau_{r\theta}^{00} \tag{II.120}$$

are found from the solution of the homogeneous system of differential equations of equilibrium (II.116) for $\gamma = 0$ and equations of compatibility (II.117) of the plane problem of elasticity theory. /107

Circular Hole. If the heavy elastic plane under examination is weakened by a circular hole of radius R, with the center at the origin of the coordinate system, some zone of concentration of stresses, caused by the basic stress state (II.118), occurs near the hole.

We will denote the stress components in the plane with the circular hole through

$$\begin{aligned} \sigma_r &= \sigma_r^* + \sigma_r^{**}, \\ \sigma_\theta &= \sigma_\theta^* + \sigma_\theta^{**}, \\ \tau_{r\theta} &= \tau_{r\theta}^* + \tau_{r\theta}^{**}, \end{aligned} \tag{II.121}$$

where the first components of the right hand side correspond to the basic stress state (II.119), and the second, to the stress state (II.120). We will take the partial solution (II.119) of the basic equations (II.116) and (II.117) in the form

$$\begin{aligned} \sigma_r^0 &= \frac{\gamma r}{4}(3\sin\theta - \sin 3\theta), \\ \sigma_\theta^0 &= \frac{\gamma r}{4}(\sin\theta + \sin 3\theta), \\ \tau_{r\theta}^0 &= \frac{\gamma r}{4}(\cos\theta - \cos 3\theta). \end{aligned} \tag{II.122}$$

Direct checking shows that the components of stresses σ_r^*, σ_θ^* and $\tau_{r\theta}^*$ in the plane with the circular hole, caused by the stress state (II.122), will have the form

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$$\sigma_r^* = \left(\frac{3\gamma r}{4} + \frac{\alpha_1}{r} + \frac{\alpha_3}{r^3}\right)\sin\theta + \left(-\frac{\gamma r}{4} + \frac{\beta_3}{r^3} + \frac{\beta_5}{r^5}\right)\sin 3\theta,$$
$$\sigma_\theta^* = \left(\frac{\gamma r}{4} + \frac{\omega_1}{r} - \frac{\alpha_3}{r^3}\right)\sin\theta + \left(\frac{\gamma r}{4} - \frac{\beta_3}{5r^3} - \frac{\beta_5}{r^5}\right)\sin 3\theta, \qquad \text{(II.123)}$$
$$\tau_{r\theta}^* = \left(\frac{\gamma r}{4} - \frac{\omega_1}{r} - \frac{\alpha_3}{r^3}\right)\cos\theta + \left(-\frac{\gamma r}{4} - \frac{3\beta_3}{5r^3} - \frac{\beta_5}{r^5}\right)\cos 3\theta.$$

The unknown factors α_1, α_3, β_3, β_5, ω_1 in (II.123) are found from the boundary conditions

$$\sigma_r^* = 0 \text{ and } \tau_{r\theta}^* = 0 \text{ for } r = R \qquad \text{(II.124)}$$

and from the condition of identity of displacements[1]

$$\alpha_1(1-2\nu) + \omega_1(3-2\nu) = 0, \qquad \text{(II.125)}$$

where ν is Poisson's ratio.

After substitution of expressions (II.123) into conditions (II.124) and (II.125), we obtain the equation system from which we find /108

$$\alpha_1 = -\frac{3-2\nu}{4(1-\nu)}\gamma R^2, \qquad \alpha_3 = \frac{\nu}{4(1-\nu)}\gamma R^4,$$
$$\omega_1 = -\frac{1-2\nu}{4(1-\nu)}\gamma R^2, \qquad \beta_3 = \frac{5}{4}\gamma R^4, \qquad \beta_5 = -\gamma R^6. \qquad \text{(II.126)}$$

We now represent the stress components (II.123) in their final form:

$$\sigma_r^* = \gamma R\left\{\left[\frac{3}{4}\cdot\frac{r}{R} - \frac{3-2\nu}{4(1-\nu)}\frac{R}{r} + \frac{\nu}{4(1-\nu)}\cdot\frac{R^3}{r^3}\right]\sin\theta + \right.$$
$$\left. + \left[-\frac{1}{4}\cdot\frac{r}{R} + \frac{5}{4}\frac{R^3}{r^3} - \frac{R^5}{r^5}\right]\sin 3\theta\right\},$$
$$\sigma_\theta^* = \gamma R\left\{\left[\frac{1}{4}\cdot\frac{r}{R} + \frac{1-2\nu}{4(1-\nu)}\frac{R}{r} - \frac{\nu}{4(1-\nu)}\cdot\frac{R^3}{r^3}\right]\sin\theta + \right. \qquad \text{(II.127)}$$
$$\left. + \left[\frac{1}{4}\cdot\frac{r}{R} - \frac{1}{4}\frac{R_3}{r^3} + \frac{R^5}{r^5}\right]\sin 3\theta\right\},$$
$$\tau_{r\theta}^* = \gamma R\left\{\left[\frac{1}{4}\cdot\frac{r}{R} - \frac{1-2\nu}{4(1-\nu)}\cdot\frac{R}{r} - \frac{\nu}{4(1-\nu)}\cdot\frac{R^3}{r^3}\right]\cos\theta + \right.$$
$$\left. + \left[-\frac{1}{4}\cdot\frac{r}{R} - \frac{3}{4}\cdot\frac{R^3}{r^3} + \frac{R^5}{r^5}\right]\cos 3\theta\right\}.$$

[1]For example see P. F. Papkovich [1], p. 486.

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Accordingly, we obtain the expressions for the components of displacement, which we write out with an accuracy up to a rigid displacement of the plane as a whole:

$$v_r^* = \frac{\gamma R^2}{8\mu}\left\{\left[\frac{3-4\nu}{2}\cdot\frac{r^2}{R^2} - \frac{3-4\nu}{1-\nu}\ln r - \right.\right.$$
$$\left. - \frac{\nu}{2(1-\nu)}\frac{R^2}{r^2}\right]\sin\theta +$$
$$+\left[-\frac{1}{2}\cdot\frac{r^2}{R^2} - \frac{5-4\nu}{2}\cdot\frac{R^2}{r^2} + \frac{R^4}{r^4}\right]\sin 3\vartheta,$$
$$v_\theta^* = \frac{\gamma R^2}{8\mu}\left\{\left[\frac{1+4\nu}{2}\cdot\frac{r^2}{R^2} - \frac{3-4\nu}{1-\nu}\ln r - \right.\right. \qquad \text{(II.128)}$$
$$- \frac{1}{1-\nu} + \frac{\nu}{2(1-\nu)}\cdot\frac{R^2}{r^2}\Big]\cos\theta +$$
$$\left. +\left[-\frac{1}{2}\cdot\frac{r^2}{R^2} - \frac{1-4\nu}{2}\cdot\frac{R^2}{r^2} - \frac{R^4}{r^4}\right]\cos 3\vartheta\right\}.$$

Formulas (II.127) and (II.128) are derived for the case of plane deformation. In the case of the generalized plane stress state, these formulas will give the mean components of stresses and displacements through the thickness of the plate if we substitute in them the constant ν by the value $\nu^* = \nu/1 + \nu$. /109

Combining now the stresses (II.127) and stresses σ_r^{**}, σ_θ^{**} and $\tau_{r\theta}^{**}$, we obtain the desired total stresses (II.121) in the heavy plane with a circular hole under a given external load[1].

Semicircular Hole[2]. In a heavy half plane, at distance H from the surface, a hole is cut in the shape of a semicircle, the contour of which is described by equation (II.71). The stresses along the contour of the given hole are

$$\sigma_\vartheta = -\frac{\gamma H}{L(\vartheta)}[1{,}44405 - 0{,}85986\cos\vartheta - 1{,}40164\cos 2\vartheta +$$
$$+ 0{,}01850\cos 3\vartheta + 0{,}06799\cos 4\vartheta]. \qquad \text{(II.129)}$$

Here γ is the density of the medium; H is the depth (from boundary of the half plane) of the center of the hole, and the function $L(\vartheta)$ is defined by formula (II.74).

---

[1]See A. N. Dinnik, A. B. Morgayevskiy and G. N. Savin [1], where the problem of stress around a circular hole in a heavy half plane is examined; see also the works of P. A. Zhuravlev and A. F. Zakharevich [1] and S. A. Orlov [1].

[2]The solution of problems for holes in the shape of a semicircle and arch is given by Ye. F. Burmistrov [3].

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The values of stresses σ_ϑ (in fractions of γH), calculated by formula (II.129), are presented in Table II.15.

Figure II.59 shows the graph of distribution of stresses σ_ϑ along the contour of the hole for the material (concrete) with Poisson's coefficient $\nu = 0.16$ and $\gamma = 2.4 \cdot 9.81 \cdot 10^3$ n/m^3.

TABLE II.15

$\vartheta°$	$\sigma_\vartheta/\gamma H$	$\vartheta°$	$\sigma_\vartheta/\gamma H$
0	—0,255	100	2,715
10	—0,236	110	2,640
20	—0,171	120	4,789
30	—0,021	130	2,009
40	0,416	135	0,926
50	0,716	140	1,469
60	2,026	150	1,089
65	12,224	160	0,899
80	4,435	170	0,824
90	3,097	180	0,809

TABLE II.16

$\vartheta°$	$\sigma_\vartheta/\gamma H$	$\vartheta°$	$\sigma_\vartheta/\gamma H$
0	0,407	100	—1,488
10	0,262	110	—1,626
20	—0,107	120	—1,138
30	—0,628	130	—3,536
40	—1,308	135	—3,936
50	—2,780	140	—4,925
60	—2,304	150	0,133
70	—2,068	160	0,542
80	—1,712	170	0,604
90	—1,521	180	0,565

Tr. Note: Commas indicate decimal points.

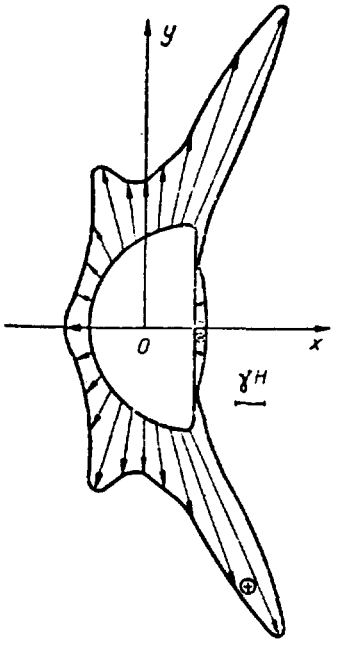


Figure II.59.

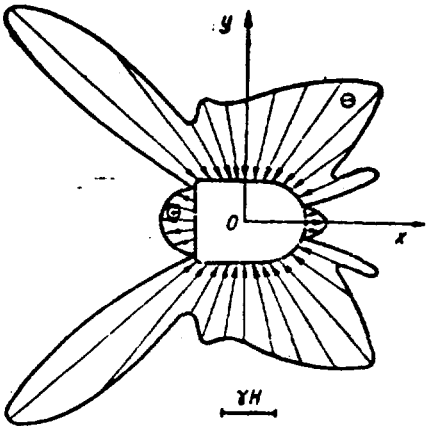


Figure II.60.

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Arch-Shaped Hole. The contour of the hole is given by formulas (II.75). /110
The stresses along the contour of the hole are

$$\sigma_\vartheta = -\frac{\gamma H}{L(\vartheta)}[1{,}22758 + 0{,}35174\cos\vartheta - 1{,}98149\cos 2\vartheta + \cdot$$
$$+ 0{,}08837\cos 3\vartheta + 0{,}01133\cos 4\vartheta], \qquad \text{(II.130)}$$

where the function $L(\vartheta)$ is defined by formula (II.78).

The values of $\sigma_\vartheta/\gamma H$, found by formula (II.130) are presented in Table II.16. The graph of distribution of stresses σ_ϑ along the contour of the hole for a half plane made of concrete, with Poisson's coefficient $\nu = 0.16$ and $\gamma = 2.4 \cdot 9.81 \cdot 10^3$ n/m^3, is illustrated in Figure II.60.

A. G. Ugodchikov and A. Ya. Krylov [1] studied the problem of the concentration of stresses around spectator galleries, both for the given form of hole, and for the basic stress state:

$$\sigma'_x = -\frac{2}{3}\gamma(\xi_0 - x), \qquad \sigma'_y = -\frac{\gamma}{6a}, \qquad \tau'_{xy} = \frac{\gamma}{3}y. \qquad \text{(II.131)}$$

The stress state caused by gravity near the hole (Figure II.61) in a dam, the outer contour of which is shaped like parabola $\xi = c\eta^2$, is analyzed in this work. In Figure II.61, the parameter $c = 0.055$ and its numerical value are selected such that its best approximation to actual profiles of dams can be selected. The hole is shaped like an arch with the following dimensions: $a = 2.5$ m, $h = 3.0$ m, $R_1 = 1.25$ m, $\xi_0 = 7.5$ m.

The function representing the interior of the unit circle on the exterior of a curve very close in contour to the given hole has the form

$$z = \omega(\zeta) = 1{,}50855\left[\frac{i}{\zeta} - 0{,}266786 + 0{,}0769579\zeta + \right.$$
$$\left. + 0{,}0764652\zeta^2 - 0{,}0885262\zeta^3 + 0{,}0245996\zeta^4 + 0{,}005903\zeta^5 + \ldots\right]. \qquad \text{(II.132)}$$

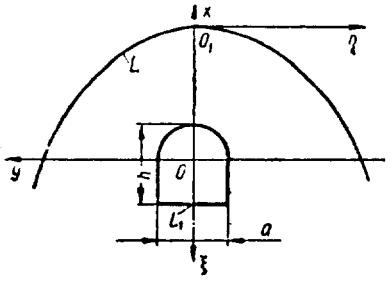


Figure II.61.

The problem is solved approximately by the method of series approximations. The solution for an infinite (heavy) plate under the basic stress state (II.131) is used as the zero approximation. This solution does not satisfy the conditions on the contour L_1 (see Figure II.61). According to those found from (II.131), on L_1 ($-X_n$) and ($-Y_n$) are located the corresponding complex Kolosov-Muskhelishvili /111

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potentials $\phi_1(z)$ and $\psi_1(z)$, which give the first corrections to the solution of (II.131). From these functions $\phi_1(z)$ and $\psi_1(z)$ are found the corresponding components of stresses σ''_x, σ''_y and τ''_{xy}.

The solution in the first approximation is found in the form

$$\sigma_x^{(I)} = \sigma'_x + \sigma''_x, \qquad \sigma_y^{(I)} = \sigma'_y + \sigma''_y,$$
$$\tau_{xy}^{(I)} = \tau'_{xy} + \tau''_{xy}. \tag{II.132a}$$

However, the solution of (II.132a) will not satisfy the conditions on the outer contour L (see Figure II.61). By eliminating the conditions on L, given by the solution of (II.132a), we find the solution of the problem in the second approximation, etc. If the hole is small and located sufficiently far from the outer contour L, then the first approximation yields quite satisfactory agreement.

The results of calculations (for $\gamma = 2.4 \cdot 9.81 \cdot 10^3$ n/m^3, $\nu = 1/6$) for a concrete dam in the first approximation for $\sigma_\vartheta^{(I)} = \sigma'_\vartheta + \sigma''_\vartheta$ and $\sigma_\rho^{(I)} = \sigma'_\rho + \sigma''_\rho$ on contour L_1 and through cross section $\vartheta = 0$ are presented in Tables II.17 and II.18, and in the form of stress-strain diagrams in Figure II.62, where the stress-strain diagrams for σ'_ρ and σ' of the basic stress state (II.131) are indicated in the cross section $\vartheta = 0$ by the shaded area.

TABLE II.17

ϑ	σ_ϑ /9,81·10³, n/m²	ϑ°	σ_ϑ /9,81·10³, n/m²
0	—12,49	105	—19,01
15	—13,26	120	—32,37
30	—16,96	135	—52,28
45	—23,89	150	—12,42
60	—24,90	165	— 2,99
75	—19,13	180	— 1,67
90	—16,42	—	—

TABLE II.18

ρ	σ_ρ /9,81·10³, n/m²	σ_ϑ/9,81·10³ n/m²
1,0	0,0	—12,49
0,8	—2,598	— 9,833
0,5	—5,386	— 8,151
0,42	—5,252	— 7,756

Tr. Note: Commas indicate decimal points.

The stress concentration in a heavy half plane near an arch-shaped hole, both free and fortified by an absolutely rigid ring, is analyzed in the works of A. A. Boyma [2, 3] and I. S. Khara [2].

Other Types of Holes. The problem of stress distribution in a heavy half plane near a free trapezoidal hole and near one that is fortified by an absolutely rigid ring is discussed by I. S. Khara [2]. The same problem, but with /112
a somewhat different representative function, was investigated by K. V. Ruppeneyt and Ts. Gomes [1]. The case of an elliptic hole is examined in the work of

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V. I. Vespalyy [1]. In the case where the elliptic hole is located sufficiently close to the boundary of the half plane, the problem is discussed by D. I. Sherman [10], and by L. N. Karpenko [1] by D. I. Sherman's method. The case of a rectangular hole in a heavy elastic plate was analyzed by V. N. Kozhevnikova [3].

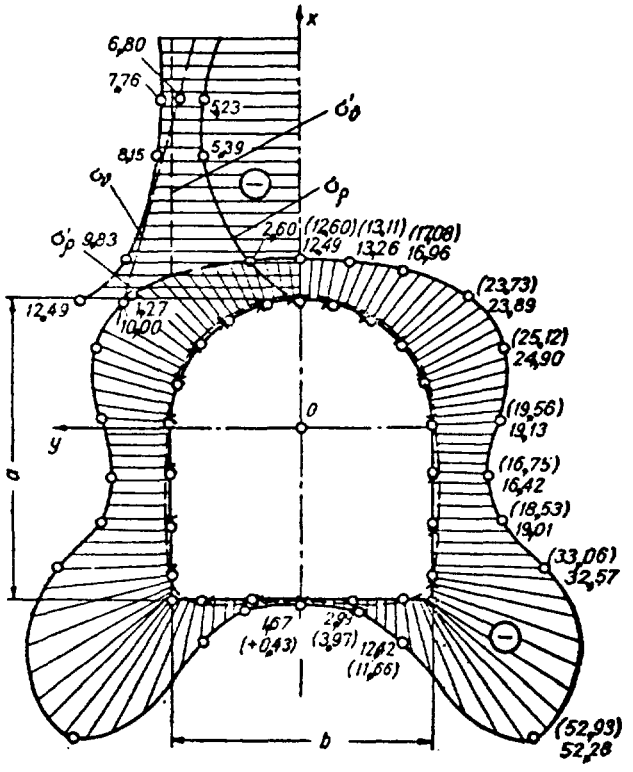


Figure II.62.

§6. Stresses in a Rod of Finite Width, Weakened by a Circular Hole

In discussing the problem of the effect of holes of various shape on the stress state in an elastic plane (plate, beam), we assumed this plane to be infinite. However, in engineering practice, problems concerning stress concentration near holes, in which the region weakened by a hole is infinite, are rarely encountered. In most cases the dimensions of these planes are small. Therefor the question arises as to which cases (in observance of a certain degree of precision) the solutions obtained in the preceding sections for infinite regions can be used for finite regions. The best answer to this question would be a comparison of the results of accurate solutions of problems for both infinite and finite regions. Unfortunately, this cannot be done for the problems at hand, since there are few reliable solutions of the corresponding problems for finite regions, even though the theoretical solutions of these problems for any coupling, both for isotropic, and for anisotropic media, have been available for quite a long time[1]. However, these solutions for finite regions are given in such a form that apart from the theorems of existence, nothing much has been obtained from them up to now because of the colossal computational difficulties[2].

/113

[1]See the works of N. I. Muskhelishvili [1, 2, 3], S. G. Mikhlin [5], D. I. Sherman [13], G. N. Savin [11, 12] and also the reviews of D. I. Sherman [12, 17].

[2]It should be pointed out that by using the method of electrosimulation (see References of Chapter I) for the construction of the representative function of a given region doubly-connected to a circular ring, it is possible to construct accurate or approximate (with a high degree of accuracy) solutions for doubly-connected regions, from which several important conclusions can be derived, particularly concerning the effect of the finiteness of the dimensions of the plate on stress concentration near a hole in it.

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Let us examine the problem of stress concentration near a circular hole for a rod of finite width and of infinite length.

Rod with Circular Hole under Tension[1]. We will assume that a thin infinite rod made of an elastic isotropic material of width 2b, thickness 2h, and with a circular hole of radius R = a < b, the contour of which is free of external forces, with the center lying on the axis of symmetry of the rod, is subjected to tension along this axis of symmetry by forces p = const (Figure II.63).

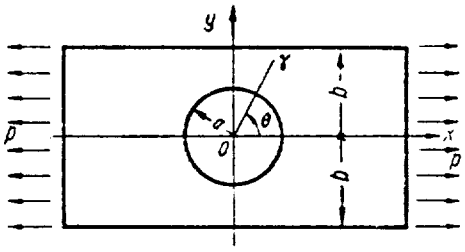


Figure II.63.

We will place the origin of the axes of coordinates xOy at the center of the circular hole, direct axis Ox along the axis of symmetry of the rod, and the Oy axis upward. We will determine the pattern of the stress state in this rod[2], weakened by a circular hole.

We will relate all dimensions to the half width of the rod, introducing the dimensionless coordinates: $\xi = x/b$, $\eta = y/b$, $\lambda = a/b$. Instead of the Cartesian coordinates ξ and η, we will use polar coordinates ρ, θ; $x = r\cos\theta$, $y = r\sin\theta$, $\rho = r/b$ (see Figure II.63). The polar axis coincides with the Ox axis, and the angles are read, as usual, from the Ox axis to the Oy axis.

The problem reduces to the determination of the biharmonic function $U(\xi, \eta)$ that satisfies equation (I.5) and conditions: /114

on the boundary of the rod, i.e., on the contour $\eta = \pm 1$,

$$\sigma_y = \frac{1}{b^2}\frac{\partial^2 U}{\partial \xi^2} = 0, \qquad \tau_{xy} = -\frac{1}{b^2}\cdot\frac{\partial^2 U}{\partial \xi \partial \eta} = 0; \tag{II.133}$$

on the contour of the hole r = a, i.e., for $\rho = \lambda$,

$$\sigma_\varrho = \frac{1}{b^2}\left[\frac{1}{\varrho^2}\cdot\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{\varrho}\cdot\frac{\partial U}{\partial \varrho}\right] = 0,$$

$$\tau_{\varrho\theta} = -\frac{1}{b^2}\cdot\frac{\partial}{\partial \varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial U}{\partial \vartheta}\right) = 0, \tag{II.134}$$

since the contour of the hole is free of external forces;

---

[1]The stress state in an infinite rod containing a circular inclusion is discussed in the work of R. G. Wilson [1].

[2]See R. Howland [1].

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at infinity for $x = \xi b \to \infty$

$$\sigma_x = \frac{1}{b^2} \cdot \frac{\partial^2 U}{\partial \eta^2} = p,$$

$$\sigma_y = \frac{1}{b^2} \cdot \frac{\partial^2 U}{\partial \xi^2} = 0, \qquad \text{(II.135)}$$

$$\tau_{xy} = -\frac{1}{b^2} \cdot \frac{\partial^2 U}{\partial \xi \partial \eta} = 0.$$

We will represent the function $U(\xi, \eta)$ in the form of the series

$$U(\xi, \eta) = U_0^{(1)}(\xi, \eta) + U_0(\xi, \eta) + U_1(\xi, \eta) + U_2(\xi, \eta) + \ldots, \qquad \text{(II.136)}$$

where the terms of this series are biharmonic functions with the following properties: the function $U^0(\xi, \eta) = U_0^{(1)}(\xi, \eta) + U_0(\xi, \eta)$ satisfies the condition at infinity (II.135) and on the contour of the hole (II.134), but does not satisfy the condition on the boundary of the rod (II.133), i.e., the function $U^0(\xi, \eta)$ is the solution of the problem for an infinite plane with a circular hole subjected to tension along the Ox axis by forces p.

From (II.67) and (II.8), we have the function

$$U^0(\varrho, \theta) = \frac{b^2 p}{4}\left[\varrho^2 - 2\lambda^2 \ln \varrho + \left(2\lambda^2 - \frac{\lambda^4}{\varrho^4} - \varrho^2\right)\cos 2\theta\right]. \qquad \text{(II.137)}$$

For simplicity we will break it down into two parts:

$$U_0^{(1)} = \frac{b^2 p}{4}\varrho^2(1 - \cos 2\theta)$$

and

$$U_0 = \frac{b^2 p}{4}\left[\left(2\lambda^2 - \frac{\lambda^4}{\varrho^4}\right)\cos 2\theta - 2\lambda^2 \ln \varrho\right].$$

The function $U_1(\xi, \eta)$ from (II.136) nullifies the stresses caused by the function $U_0(\xi, \eta)$ on the boundary of the rod $\eta = \pm 1$, but introduces stresses on the contour of the hole, etc.

Generally speaking, the function $U_{2n} + U_{2n+1}$ produces zero stresses on the boundary $\eta = \pm 1$, whereas the function $U_{2n-1} + U_{2n}$ produces zero stresses on the contour of the hole $\rho = \lambda$. /115

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The stress components in the rod near the circular hole for $\lambda = a/b \leqslant 0.5$ are given in the form of the series

$$\sigma_\rho = p\left\{\frac{1}{2}(1+\cos 2\vartheta) + 2m_0 - \frac{d_0}{\rho^2} + 2\sum_{n=1}^{\infty}\left[\frac{n(2n+1)\,d_{2n}}{\rho^{2n+2}} + \frac{(n+1)(2n-1)e_{2n}}{\rho^{2n}} + \right.\right.$$

$$\left.\left. + 2n(2n-1)\,l_{2n}\rho^{2n-2} + (n-1)(2n+1)\,m_{2n}\rho^{2n}\right]\cos 2n\vartheta\right\}; \qquad \text{(II.138)}$$

$$\sigma_\vartheta = p\left\{\frac{1}{2}(1-\cos 2\vartheta) + 2m_0 + \frac{d_0}{\rho^2} - 2\sum_{n=1}^{\infty}\left[\frac{n(2n+1)\,d_{2n}}{\rho^{2n+2}} + \frac{(n-1)(2n-1)e_{2n}}{\rho^{2n}} + \right.\right.$$

$$\left.\left. + n(2n-1)\,l_{2n}\rho^{2n-2} + (n+1)(2n+1)\,m_{2n}\rho^{2n}\right]\cos 2n\vartheta\right\},$$

$$\tau_{\rho\vartheta} = -p\left\{\frac{1}{2}\sin 2\vartheta + 2\sum_{n=1}^{\infty}\left[n(2n-1)\left(l_{2n}\rho^{2n-2} - \frac{e_{2n}}{\rho^{2n}}\right) + n(2n+1)\times\right.\right.$$

$$\left.\left.\times\left(m_{2n}\rho^{2n} - \frac{d_{2n}}{\rho^{2n+2}}\right)\right]\sin 2n\vartheta\right\}.$$

The values of the coefficients in formula (II.138) for certain values of λ are presented in Table II.19 .

The values of σ_θ/p along the contour of a circular hole in a rod, for various values of $\lambda = a/b$, are presented in Table II.20.

The values of σ_θ/p through the cross section of the rod ($\theta = \pi/2$), passing through the center of the circular hole (see Figure II.63), are presented in Table II.21.

The values $\sigma_\rho/p = \sigma_x/p$ and $\sigma_\theta/p = \sigma_y/p$ through cross section $\theta = 0$, i.e., through the Ox axis are presented in Table II.22 (in the columns for a finite rod) for the case where the diameter of the hole is equal to the height of the rod (when $\lambda = a/b = 0.5$). For comparison, the same values around such a hole, but for a plate of infinite dimensions, are presented in the same table.

The curves shown on Figure II.64 for various values of λ were constructed on the basis of data in Table II.20 : curve 1, for $\lambda = 0.5$; curve 2, for $\lambda = 0.4$; curve 3, for $\lambda = 0.3$ and curve 4, for $\lambda = 0$, i.e., for a plate of infinite dimensions.

[1]Tables II.19 and II.20 were borrowed from the work of R. Howland [1].
[2]The values of σ_θ/p are arranged radially as follows: positive, outside the circle, and negative, within the circle.

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TABLE II.19

k	λ = 0,1	0,2	0,3	0,4	0,5
d_0	$5{,}01\cdot10^{-3}$	$2{,}02\cdot10^{-2}$	$4{,}59\cdot10^{-2}$	$8{,}28\cdot10^{-2}$	$1{,}32\cdot10^{-1}$
d_2	$2{,}54\cdot10^{-5}$	$4{,}25\cdot10^{-4}$	$2{,}34\cdot10^{-3}$	$8{,}28\cdot10^{-3}$	$2{,}33\cdot10^{-2}$
d_4	$3{,}15\cdot10^{-11}$	$3{,}05\cdot10^{-8}$	$1{,}85\cdot10^{-6}$	$3{,}32\cdot10^{-5}$	$3{,}14\cdot10^{-4}$
d_6	$1{,}40\cdot10^{-15}$	$2{,}13\cdot10^{-11}$	$6{,}32\cdot10^{-9}$	$3{,}42\cdot10^{-7}$	$7{,}51\cdot10^{-6}$
d_8	$5{,}00\cdot10^{-20}$	$1{,}19\cdot10^{-14}$	$1{,}72\cdot10^{-11}$	$2{,}82\cdot10^{-9}$	$1{,}45\cdot10^{-7}$
e_2	$-5{,}08\cdot10^{-3}$	$-2{,}13\cdot10^{-2}$	$-5{,}21\cdot10^{-2}$	$-1{,}04\cdot10^{-1}$	$-1{,}89\cdot10^{-1}$
e_4	$-4{,}21\cdot10^{-9}$	$-1{,}02\cdot10^{-6}$	$-2{,}75\cdot10^{-5}$	$-2{,}79\cdot10^{-4}$	$-1{,}70\cdot10^{-3}$
e_6	$-1{,}68\cdot10^{-12}$	$-6{,}41\cdot10^{-10}$	$-8{,}45\cdot10^{-8}$	$-2{,}58\cdot10^{-6}$	$-3{,}64\cdot10^{-5}$
e_8	$-5{,}72\cdot10^{-18}$	$-3{,}41\cdot10^{-13}$	$-2{,}20\cdot10^{-10}$	$-2{,}02\cdot10^{-8}$	$-6{,}7\cdot10^{-7}$
l_2	$4{,}13\cdot10^{-3}$	$1{,}62\cdot10^{-2}$	$4{,}09\cdot10^{-2}$	$8{,}03\cdot10^{-2}$	$1{,}40\cdot10^{-1}$
l_4	$1{,}06\cdot10^{-3}$	$4{,}10\cdot10^{-3}$	$1{,}01\cdot10^{-2}$	$1{,}92\cdot10^{-2}$	$3{,}27\cdot10^{-2}$
l_6	$2{,}83\cdot10^{-4}$	$1{,}08\cdot10^{-3}$	$2{,}61\cdot10^{-3}$	$4{,}82\cdot10^{-3}$	$8{,}09\cdot10^{-3}$
l_8	$7{,}24\cdot10^{-5}$	$2{,}74\cdot10^{-4}$	$6{,}44\cdot10^{-4}$	$1{,}17\cdot10^{-3}$	$1{,}95\cdot10^{-3}$
l_{10}	$1{,}82\cdot10^{-5}$	$6{,}82\cdot10^{-5}$	$1{,}57\cdot10^{-4}$	$2{,}79\cdot10^{-4}$	$4{,}71\cdot10^{-4}$
l_{12}	$4{,}65\cdot10^{-6}$	$1{,}80\cdot10^{-5}$	$3{,}80\cdot10^{-5}$	$6{,}64\cdot10^{-5}$	$1{,}14\cdot10^{-4}$
l_{14}	$1{,}15\cdot10^{-6}$	$4{,}41\cdot10^{-6}$	$9{,}12\cdot10^{-6}$	$1{,}58\cdot10^{-6}$	$2{,}8\cdot10^{-5}$
m_0	$5{,}69\cdot10^{-4}$	$2{,}46\cdot10^{-3}$	$5{,}15\cdot10^{-3}$	$8{,}89\cdot10^{-3}$	$1{,}40\cdot10^{-2}$
m_2	$-1{,}13\cdot10^{-3}$	$-4{,}35\cdot10^{-3}$	$-1{,}09\cdot10^{-2}$	$-2{,}11\cdot10^{-2}$	$-3{,}62\cdot10^{-2}$
m_4	$-5{,}95\cdot10^{-4}$	$-2{,}29\cdot10^{-3}$	$-5{,}60\cdot10^{-3}$	$-1{,}06\cdot10^{-2}$	$-1{,}78\cdot10^{-2}$
m_6	$-2{,}28\cdot10^{-4}$	$-8{,}70\cdot10^{-4}$	$-2{,}10\cdot10^{-3}$	$-3{,}88\cdot10^{-3}$	$-6{,}47\cdot10^{-3}$
m_8	$-7{,}65\cdot10^{-5}$	$-2{,}90\cdot10^{-4}$	$-6{,}88\cdot10^{-4}$	$-1{,}25\cdot10^{-3}$	$-2{,}08\cdot10^{-4}$
m_{10}	$-2{,}39\cdot10^{-5}$	$-8{,}99\cdot10^{-5}$	$-2{,}10\cdot10^{-4}$	$-3{,}78\cdot10^{-4}$	$-6{,}30\cdot10^{-4}$
m_{12}	$-7{,}30\cdot10^{-6}$	$-2{,}86\cdot10^{-5}$	$-6{,}17\cdot10^{-5}$	$-1{,}10\cdot10^{-4}$	$-1{,}84\cdot10^{-4}$

Tr. Note: Commas indicate decimal points.

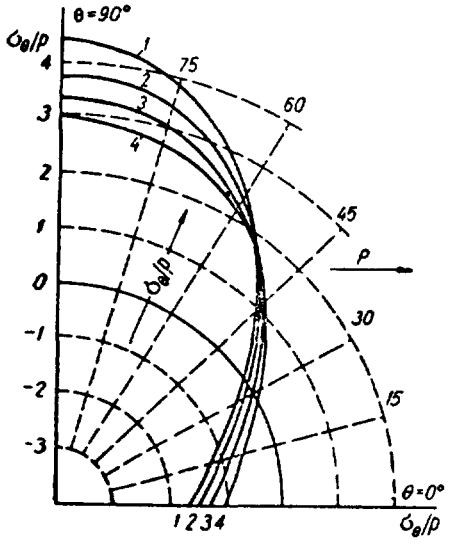


Figure II.64.

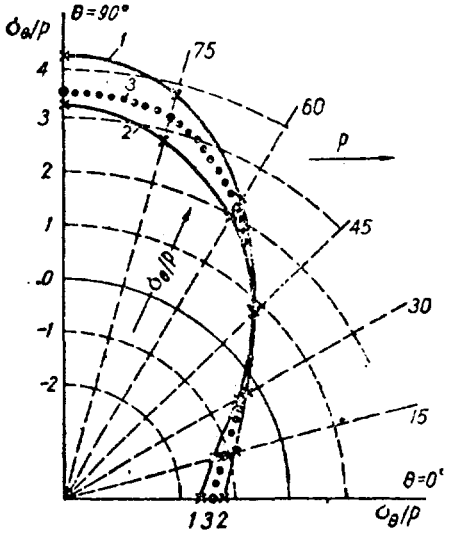


Figure II.65.

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TABLE II.20

θ°	λ					
	0	0,1	0,2	0.3	0,4	0.5
0	—1,00	—1,03	—1,11	—1,26	—1,44	—1,58
15	—0,73	—0,74	—0,82	—0,95	—1,12	—1,32
30	0,00	—0,01	—0,06	—0,15	—0,30	—0,51
45	1,00	1,00	1,00	0,98	0,91	0,77
60	2,00	2,01	2,07	2,15	2,25	2,32
75	2,73	2,74	2,85	3,03	3,32	3,72
90	3,00	3,03	3,14	3,36	3,74	4,32

Tr. Note: Commas indicate decimal points.

The stress-strain diagrams of the stresses are constructed in Figure II.65: curve 1 gives the values of σ_θ/p along the contour of the hole for $\lambda = 0.5$; curve 2 gives the values of σ_θ/p along the contour $\rho = 0.55$, determined by formulas (II.138) for $\lambda = 0.5$ and curve 3 (point) is based on data obtained by the method of photoelasticity on the contour of the hole. /117

TABLE II.21

ρ	λ				
	0.1	0.2	0.3	0.4	0.5
0,1	3,03	—	—	—	—
0,2	1,23	3,14	—	—	—
0,3	1,08	1,57	3,36	—	—
0,4	1,04	1,26	1,93	3,74	—
0,5	1,03	1,16	1,47	2,30	4,32
0,6	1,02	1,11	1,28	1,75	2,75
0,7	1,01	1,07	1,17	1,48	2,04
0,8	1,01	1,05	1,07	1,28	1,61
0,9	1,00	1,01	0,96	1,08	1,22
1,0	0,99	0,97	0,89	0,81	0,73

TABLE II.22

ξ	$\sigma_x/p=\sigma_\theta/p$		$\sigma_y/p=\sigma_\theta/p$	
	Finite rod	Infinite plate	Finite rod	Infinite plate
0,5	0,00	0,00	—1,6	—1,00
0,6	—0,09	—0,01	—0,7	—0,38
0,7	0,02	0,12	—0,4	—0,14
0,8	0,17	0,25	—0,1	0,03
0,9	0,31	0,37	0,0	0,01
1,0	0,44	0,47	0,1	0,03
π/2	0,81	0,76	—	—
2π/3	0,95	0,86	—	—
π	1,00	0,94	—	—

Tr. Note: Commas indicate decimal points.

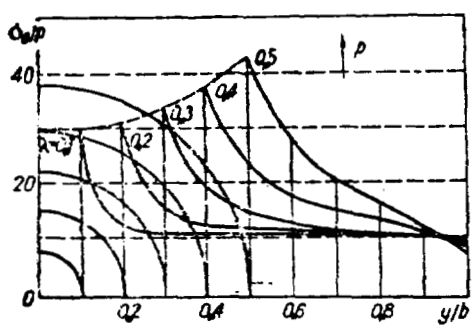


Figure II.66.

Figure II.66 shows the curves for σ_θ/p through the most dangerous cross section $\theta = \pi/2$ for various values of λ, constructed on the basis of the data in Table II.22. The broken curve shows the increase in the concentration factor of the stresses as λ increases.

Figure II.67 represents graphically data presented in the preceding tables for a rod with a circular hole, subjected to tension, for $\lambda = 0.5$: curve 1 gives the stresses σ_θ/p along the contour of the circular hole for $\lambda = 0.5$ (see Table II.20); curve 2 gives stresses σ_θ/p through the most dangerous cross section $\theta = \pi/2$ for $\lambda = 0.5$ (see Table II.21); curve 3 gives stresses σ_θ/p through cross section $\theta = 0$, i.e., along the axis of the rod for $\lambda = 0.5$ (see Table II.22); curve 4 gives stresses /118

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σ_y/p through cross section $\theta = 0$ for $\lambda = 0.5$ (see Table II.22). The broken curves were constructed on the basis of experimental data obtained by the optic method.

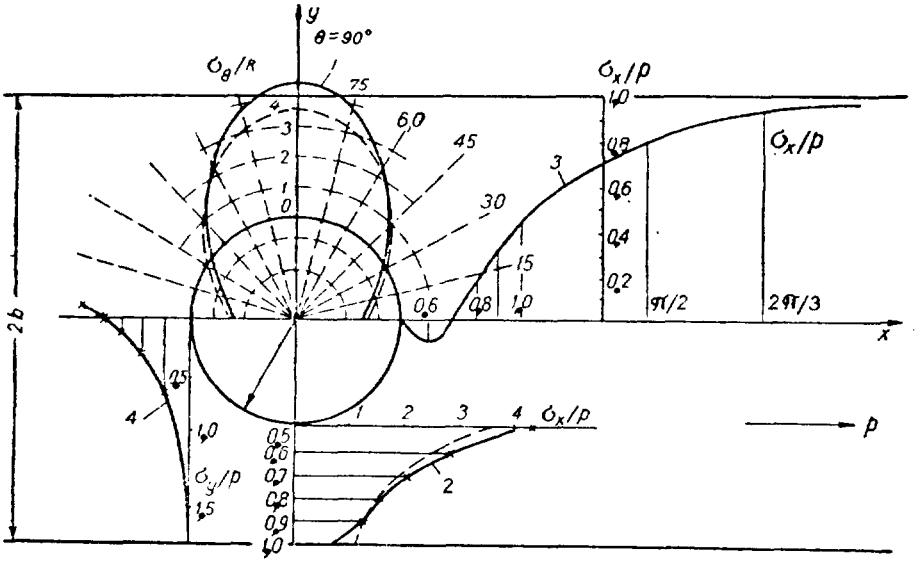


Figure II.67.

According to the data presented in the tables, it is possible to derive the following conclusions.

1. For a rod of given width, as the radius of the hole increases, the concentration factor increases considerably (see Table II.20). For instance, when $\lambda = 0.5$, this factor is equal to 4.32, whereas for a plate of infinite dimensions, it is only 3.0. For values of λ differeing from those in Figure II.66, the concentration factor can be taken directly from the broken curve of this figure.

2. Under the limitation of usual accuracy (up to 6%), then, as follows from the data in Tables II.20 and II.21, the solutions found for the infinite regions can be used for plates of finite dimensions if the ratio of the diameter of the hole (located centrally) to the smallest dimension of the plate, i.e., $\lambda = a/b$, will not be less than 0.2; in other words, the width of the plate should be at least five times greater than the diameter of the hole[1] /119

Pure Displacement of Rod with Circular Hole. By the same method used for the solution of the problem of a rod with a circular hole under tension, C. K. Wang [1] solved the problem of stress distribution in an isotropic rod

[1]This conclusion, derived for a plate with a circular hole under simple tension, will obviously be valid for the case of a plate weakened by a circular hole under pure deflection (see §2 and 4, Chapter II, and also §4, Chapter III).

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of width 2b (see Figure II.63) with a circular hole of radius R = a < b, where displacing forces $\tau_{xy} = -\tau = \text{const}$ are applied to boundaries y = ±b. The contour of the hole was free of external forces. The stress along the contour of the hole is found by C. K. Wang [1] in the form

$$\sigma_\theta = \tau(a_1 \sin 2\vartheta - a_2 \sin 4\vartheta + a_3 \sin 6\theta - a_4 \sin 8\vartheta + a_5 \sin 10\,\theta).$$

The numerical values of the coefficients a_1, a_2, a_3, a_4, a_5 for the various values of $\lambda = a/b$ are presented in Table II.23. The values $\tau_{\rho\theta}/\tau$ through the cross section $\theta = -\pi/2$ for various values of λ are presented in Table II.24.

TABLE II.23

λ=a/b	a_1	a_2	a_3	a_4	a_5
0,0	—4,0000	—	—	—	—
0,1	—4,1078	—0,0012	—	—	—
0,2	—4,4453	—0,0191	—0,0004		
0,3	—5,0577	—0,0996	—0,0053	—0,0003	—
0,4	—6,0367	—0,3292	—0,0301	—0,0021	—0,0001
0,5	—7,5563	—0,8700	—0,1193	—0,0105	—0,0011

Tr. Note: Commas indicate decimal points.

TABLE II.24

η=y/b	λ=a/b					
	0 (rod without hole)	0,1	0,2	0,3	0,4	0,5
0,0	1,000	—	—	—	—	—
0,1	1,000	0,000	—	—	—	—
0,2	1,000	1,348	0,000	—	—	—
0,3	1,000	1,217	1,448	0,000	—	—
0,4	1,000	1,143	1,464	1,527	0,000	—
0,5	1,000	1,103	1,383	1,712	1,716	0,000
0,6	1,000	1,079	1,311	1,666	2,051	2,114
0,7	1,000	1,062	1,251	1,570	2,023	2,577
0,8	1,000	1,046	1,190	1,447	1,847	2,459
0,9	1,000	1,028	1,115	1,275	1,533	1,973
1,0	1,000	1,000	1,001	1,003	1,006	1,028

Tr. Note: Commas indicate decimal points.

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Concluding Comments. We will cite several works concerning the investigation of the stress state in a plate of finite dimensions, weakened by a circular hole. The stress concentration near the circular hole in a finite rectangular plate under tension was examined by N. F. Gur'yev [1, 2], and in a square plate, by V. M. Rakivenenko and V. I. Makhovikov [1]. Tension and shear of an infinite isotropic rod, weakened by a circular hole, was examined by Nagai [1]. He analyzed the stress concentration near a hole as a function of the dimensions of the hole and its distribution through the width of the rod. Stress distribution in a square plate with a central circular hole under hydrostatic pressure along the contour of the hole was investigated in the work of G. A. O. Davies [1], D. K. Vaid, R. K. Varma, S. T. Awade [1]. In the latter work, the problem was solved by the method of relaxation with a fine screen. Many graphs were constructed for stress distribution for various geometrical ratios. The same problem was examined by Z. G. Aliyev [1] for a noncentral circular hole. M. Z. Narodetskiy [4], using the method of complex Kolosov-Muskhelishvili potentials, solved the problem of tension of an isotropic square plate with a central circular hole by forces uniformly distributed along the two opposite sides, where the contour of the hole is free of external forces (Figure II.68). /120

The contour of the square was replaced by a smooth contour, where the representative function was taken in the form

$$\omega(\zeta) = A\left[\zeta + \frac{m}{\zeta^3}\right], \qquad b = A(1-m), \qquad m = \frac{1}{9}, \tag{II.139}$$

$$\zeta = \rho e^{i\theta} \quad (\text{for } \rho = 1).$$

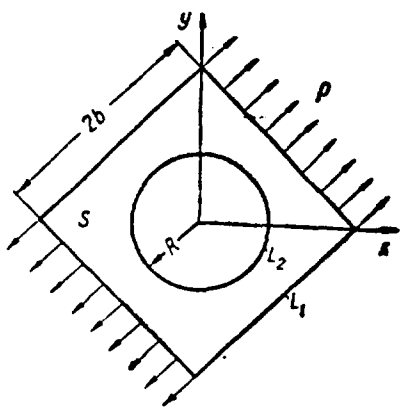


Figure II.68.

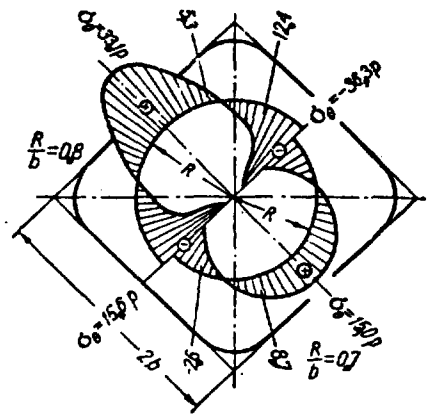


Figure II.69.

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Since the boundary conditions on the contour of the hole are satisfied precisely, $\sigma_r = 0$, $\tau_{r\theta} = 0$ on the contour of the hole, while /121

$$\sigma_\theta = 4p \sum_{k=0}^{s} \left(\frac{8}{9} c\right)^{4k} \{(4k+1)[a_{4k+1} \cos 4k\theta - b_{4k+1} \sin (4k+2)\theta] - \\ - (4k-3)[a_{4k+3} \sin (4k+2)\theta + b_{4k+3} \cos (4k+1)\theta]\}. \quad \text{(II.140)}$$

The stress-strain diagrams of stresses σ_θ at the points lying on the contour of the hole are presented in Figure II.69 for c = 0.7 and c = 0.8, where c = R/b (R is the radius of the hole). As we see, an increase in the parameter c by 0.1 (from 0.7 to 0.8) increases the largest stress σ_θ by a factor of 2.5.

§7. Stress Concentration in Regions Bounded by Two Circles or by a Circle and a Straight Line

In the general case where a region is bounded by two nonconcentric circles, the problem of determining the stress state within it on the basis of given external forces along the boundary is reduced by V. N. Zamyatina [1] to an integral equation. However, for the solution of partial problems, it is more convenient to change, as demonstrated by N. Ye. Zhukovskiy [1], to a curvilinear coordinate system $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, such that the boundary of the region can be represented in the form of the equation $\eta = \text{const}$ or $\xi = \text{const}$, i.e., so that it will coincide with one of the coordinates of the lines of our system. For this purpose we will use the following curvilinear coordinates:

$$x = \frac{a \sin \eta}{\operatorname{ch} \xi - \cos \eta}, \qquad y = \frac{a \operatorname{sh} \xi}{\operatorname{ch} \xi - \cos \eta}. \quad \text{(II.141)}$$

Then, on the plane xOy, the curves $\xi = \text{const}$ and $\eta = \text{const}$ represent circles (the centers of which lie, respectively, on the Ox and Oy axes), intersecting at a right angle.

For instance, we will consider the curve $\xi = \xi_0 = \text{const} > 0$. This will be the circle

$$(y - a \operatorname{cth} \xi_0)^2 + x^2 = \frac{a^2}{\operatorname{sh}^2 \xi_0} \quad \text{(II.142)}$$

with the center $C(0, a \operatorname{cth} \xi_0)$ lying on the Oy axis, and with radius

$$R = \frac{a}{\operatorname{sh} \xi_0} = a \operatorname{csch} \xi_0. \quad \text{(II.143)}$$

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When $\xi_0 \to \infty$, the radius (II.143) will approach zero while the center of this circle will approach the point $O_2(0, a)$ (Figure II.70).

Analogous considerations for $\xi = \xi_0 = \text{const} < 0$ lead to the same thing, namely that the limit circle for $\xi_0 < 0$ will be the point $O_1(0, -a)$. It is clear that for all $\xi > 0$, circles (II.142) will be located above the Ox axis, /122
while for $\xi < 0$, below it. The value of $\xi = 0$ will correspond to the straight line $y = 0$, i.e., the Ox axis.

Let us consider the curves $\eta = \eta_0 = \text{const} > 0$. By excluding the parameter ξ from equation (I.141), we find that the curve $\eta = \eta_0 = \text{const}$ is a circle

$$(x - a\cot\eta_0)^2 + y^2 = \frac{a^2}{\sin^2\eta_0} \qquad \text{(II.144)}$$

with the center $C_1(a, \cot\eta_0, 0)$ lying on the Ox axis, and with radius

$$R_1 = \frac{a}{\sin\eta_0}. \qquad \text{(II.145)}$$

It is obvious that all circles (II.144) pass through the points $O_1(0, -a)$ and $O_2(0, a)$. As η in (II.144) approaches zero, we see that the radii of the circles will increase infinitely, and since all these circles will pass through two points O_1 and O_2, the value $\eta = 0$ will correspond to the straight line $x = 0$, i.e., the Oy axis. For $\eta_0 > 0$, the centers of the circles (II.144) will be arranged on the right hand side of the Oy axis, and for $\eta < 0$, on the left hand side.

We will use the parameters ξ, η as curvilinear coordinates. The relationship between these curvilinear (orthogonal) coordinates, called bipolar coordinates, and the Cartesian coordinates, is given by formulas (I.141). On the plane xOy we will take some point $K(\xi_0, \eta_0)$ of intersection of the circles $\xi = \xi_0 = \text{const}$ and $\eta = \eta_0 = \text{const}$. Obviously, the coordinates of point A will be $\xi = \xi_0$ and $\eta = 0$, and of point B, $\xi = \xi_0$ and $\eta = \pi$. If we denote the central angle ACK through θ^* and calculate it as shown in Figure II.70, we easily find the relationship between angle θ and the parameter η:

$$\sin\theta^* = \frac{x}{R} = \frac{\operatorname{sh}\xi\sin\eta}{\operatorname{ch}\xi - \cos\eta}. \qquad \text{(II.146)}$$

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By using this system of curvilinear (orthogonal) coordinates, S. A. Chaplygin [1], S. G. Gutman [1] and G. B. Jeffery [1] solved several technically important problems concerning the stress concentration for regions bounded by two nonconcentric circles or a circle and straight line, and R. D. Mindlin [1] investigated stresses near a circular hole in a heavy half plane. Using another system of bipolar coordinates, C. B. Ling [1, 2] found the solution for problems of uniaxial tension of a plate weakened by two equal circular holes or by a hole, the contour of which consists of two equal arcs of a circle.

Cylinder under Internal Pressure p_1 = const. We will assume that the cross section of the cylinder is a doubly-connected region bounded by two nonconcentric circles of radius R_1 (internal circle) and R_2 (outer circle), as indicated in Figure II.71.

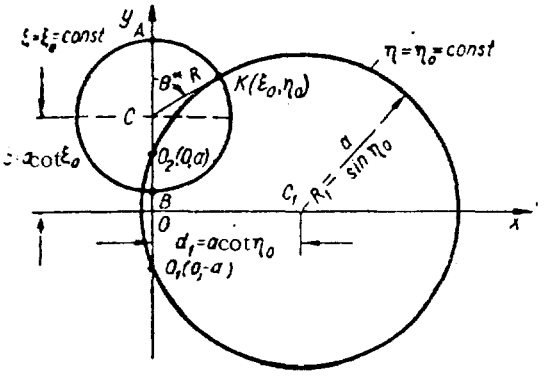


Figure II.70.

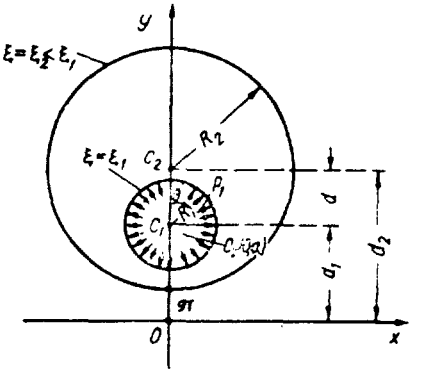


Figure II.71.

We will assume that a uniformly distributed pressure p_1 is applied to the inner circle of the cylinder, and that the outer contour (circle of radius R_2) is free of external forces. The distance between the centers of these circles is denoted through d (see Figure II.71). /123

Stresses along the contour of the outer circle of radius R_2 (in the elements perpendicular to the contour ξ = const) are

$$\sigma_\eta = 2p_1 \frac{R_1^2 [R_2^2 (R_2 - 2d\cos\eta)^2 - (R_1^2 - d^2)^2]}{(R_1^2 + R_2^2)[R_2^2 - (R_1 + d)^2][R_2^2 - (R_1 - d^2)]}. \qquad \text{(II.147)}$$

From (II.147) and inequality $d + R_1 < R_2$, we obtain the following.

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1. The largest numerical value of σ_η (II.147) will occur for $\eta = \pi$, i.e., at the point of intersection of the line of the centers (Oy axis) with the circle of radius R_2 (i.e., with the curve ξ_2 = const) in the thin part of the cylinder:

$$(\sigma_\eta)_{\eta=\pi} = 2p_1 \frac{R_1^2 (R_2^2 + R_1^2 + 2R_2 d - d^2)}{(R_1^2 + R_2^2)(R_2^2 - R_1^2 - 2R_2 d + d^2)}. \qquad \text{(II.148)}$$

2. If the distance between the centers of circles C_1 and C_2 is greater than half the radius of the outer circle, i.e., $d > 1/2\ R_2$, σ_η (II.147) acquires its least value at the point $\cos \eta = R_2/2d$. This stress σ_η is always negative (if p_1 is pressure):

$$\sigma_\eta = - \frac{2p_1 R_1^2 (R_1^2 - d^2)^2}{(R_1^2 + R_2^2)[R_2^2 - (R_1 + d)^2][R_2^2 - (R_1 - d)^2]}. \qquad \text{(II.149)}$$

The values of σ_η (II.149) are always numerically smaller than the maximum values of σ_η (II.148). Since in the range $0 < \eta < \pi$ the stresses σ_η change sign, then at the point $\eta = 0$, i.e., at the point of intersection of the line of the centers (Oy axis) with the circle of radius R_2 in the thick part of the cylinder, the stresses also acquire their extremal (maximum) value: /124

$$(\sigma_\eta)_{\eta=0} = 2p_1 \frac{R_1^2 (R_2^2 + R_1^2 - 2R_2 d - d^2)}{(R_1^2 + R_2^2)(R_2^2 - R_1^2 + 2R_2 d + d^2)}, \qquad \text{(II.150)}$$

although always $(\sigma_\eta)_{\eta=\pi} > (\sigma_\eta)_{\eta=0}$.

3. If the distance d between the centers of circles C_1 and C_2 (see Figure II.71) is less than half the radius of the outer circle, i.e., $d < 1/2\ R_2$, we have, in addition to maximum stresses σ_η (II.148), minimal values of σ_η for $\eta = 0$, given the formula (II.150). Greater maximums and minimums of σ_η will not occur in this case. It constantly decreases from maximum values of σ_η (II.148) in the thin part of the cylinder to the values of σ_η (II.150) in the thick part of the cylinder.

Stresses σ_η in the interior of the circle of radius R_1 (ξ_1 = const) will always be tension stresses:

$$\sigma_\eta = - p_1 + \frac{2p_1 R_2^2 [(R_2^2 - d^2)^2 - R_1^2 (R_1 + 2d \cos \eta)^2]}{(R_1^2 + R_2^2)[R_2^2 - (R_1 - d)^2][R_2^2 - (R_1 + d)^2]}. \qquad \text{(II.151)}$$

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From (II.151) we obtain the following.

1. If the distance d between the centers of circles ξ_1 = const and ξ_2 = = const is greater than half the radius of the inner circle, i.e., d > 1/2 R_1, the maximum values of σ_η in the interior of the circle will occur at the point cos $\eta = -R_1/2d$:

$$\sigma_\eta = -p_1\left\{1 - \frac{2R_2^2(R_2^2-d^2)^2}{(R_1^2+R_2^2)[R_2^2-(R_1-d)^2][R_2^2-(R_1+d)^2]}\right\}. \qquad \text{(II.152)}$$

2. If the distance d between the centers of the circles is less than half the radius of the inner circle, i.e., d < 1/2 R_1, the maximum value of σ_η will occur at the point $\eta = \pi$, i.e., at the point of intersection of the Oy axis with the circle of radius R_1 at the thin point of the cylinder:

$$\sigma_\eta = -p_1\left\{1 - \frac{2R_2^2(R_2^2+R_1^2-2R_1d-d^2)}{(R_1^2+R_2^2)(R_2^2-R_1^2-2R_1d-d^2)}\right\}. \qquad \text{(II.153)}$$

3. The minimum values of σ_η on the circle of radius R_1 will occur at the point $\eta = 0$, i.e., at the point of intersection of this circle with the line of the centers (Oy axis) in the thick part of the cylinder:

$$\sigma_\eta = -p_1\left\{1 - \frac{2R_2^2(R_2^2+R_1^2+2R_1d-d^2)}{(R_1^2+R_2^2)(R_2^2-R_1^2+2R_1d-d^2)}\right\}. \qquad \text{(II.154)}$$

Cylinder under External Pressure p_2 = const. We will assume that pressure p_2 = const is applied only to the outer circle of radius R_2, and that the inner circle $R = R_1$ is free of external forces (see Figure II.71).

The stresses along the internal contour, i.e., on the circle of radius R_1, /125 are

$$\sigma_\eta = -2p_2R_2^2\frac{(R_2^2-d^2)^2-R_1^2(R_1+2d\cos\eta)^2}{(R_1^2+R_2^2)[R_2^2-(R_1-d)^2][R_2^2-(R_1+d)^2]}, \qquad \text{(II.155)}$$

and along the outer contour, i.e., on the circle of radius R_2,

$$\sigma_\eta = -p_2\left\{1 + \frac{2R_1^2[R_2^2(R_2-2d\cos\eta)^2-(R_1^2-d^2)^2]}{(R_1^2+R_2^2)[R_2^2-(R_1-d)^2][R_2^2-(R_1+d)^2]}\right\}. \qquad \text{(II.156)}$$

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From (II.155) we obtain the following.

1. If the distance between the centers of the circles ξ_1 = const and ξ_2 = const is less than half the radius of the inner circle, i.e., $d < R_1/2$, the extremal values of σ_η will occur at the points $\eta = 0$ and $\eta = \pi$, and the greatest, in absolute value, σ_η (II.155), at the point $\eta = \pi$, i.e., at the point of intersection of the line of the centers and the inner circle of radius R_1 in the thin part of the cylinder.

2. If the distance between the centers of the circles ξ_1 = const and ξ_2 = const is greater than half the radius of the inner circle, i.e., $d > R_1/2$, the extremal values of σ_η will occur at the points $\eta = 0$ and $\eta = \pi$, and the greatest, in absolute value, compressive stresses σ_η (II.155), at the points $\cos \eta = -R_1/2d$. It is clear from (II.156) that for both $d < R_2/2$ and for $d > R_2/2$, the greatest, in absolute value, σ_η on the outer contour will occur at the point $\eta = \pi$. If in formulas (II.147), (II.151), (II.155) and (II.156), we assume $d = 0$, we find the known Lame formulas.

Half Plane with Circular Hole under Uniformly Distributed Pressure along the Edge. If we assume in the preceding formulas that $\xi_2 = 0$, $p_2 = 0$, $p_1 = p$ and $R_1 = R$, we obtain the solution for a half plane in which, at some distance d from a straight edge (Figure II.72), a circular hole of radius R is made, around which is applied a uniformly distributed pressure p. External forces, however, are not applied to the boundary of the half plane itself[1].

By denoting through x the distance measured along the Ax axis from point A, we obtain stresses σ_x along the contour of the straight boundary of the half plane, i.e., along the Ax axis,

$$\sigma_x = -4p\frac{R^2(x^2-d^2+R^2)}{(x^2+d^2-R^2)^2}. \qquad \text{(II.157)}$$

The greatest tension stresses will occur at point A:

$$(\sigma_x)_{max} = 4p\frac{R^2}{d^2-R^2}. \qquad \text{(II.158)}$$

[1]The case where a uniformly distributed pressure p = const is applied to the straight boundary of a half plane is examined in detail by S. G. Gutman [1] and D. P. Gupta [1, 2].

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At the points $x = \pm\sqrt{d^2 - R^2}$, stress σ_x is equal to zero, and as x increases further, it becomes compressive, reaching its greatest magnitude at the point $x = \pm\sqrt{3(d^2 - R^2)}$: /126

$$\sigma_x = -\frac{p}{2}\frac{R^2}{d^2 - R^2}.$$

Stresses σ_η along the contour of the circular hole can be found from simple geometric construction. By denoting through C the center of the circular hole, and through Q, the arbitrary point of the center of the hole, through CA, a perpendicular constructed from point C to the straight edge of the half plane, and through ϕ, angle QAC (see Figure II.72), we find stresses σ_η around the contour of the circular hole:

$$\sigma_\eta = p(1 + 2\tan^2\varphi). \tag{II.159}$$

It is clear from this construction that stresses σ_η will be identical at points Q and Q', which lie on a single straight line, passing from point A and intersecting the contour of the hole. The minimum tension stresses σ_η on the contour of the hole will obviously occur at points B and D, i.e., at the points that are closest to and farthest from the straight boundary of the half plane.

Assuming in (II.159) that $\phi = 0$, we find $\sigma_\eta = p$. From formula (II.159), it follows that the greatest values of σ_η will occur at the points of the contour of the hole for which $\phi = \phi_{max}$:

$$(\sigma_\eta)_{max} = p\frac{d^2 + R^2}{d^2 - R^2}; \tag{II.160}$$

these are the points of the contour of the round hole at which the straight line AQ will be tangent to the contour.

If $d = R\sqrt{3}$, then the greatest value of the tension stresses σ_η on the contour of the round hole will be equal to the maximum value of the tension stresses σ_x along the straight boundary of the half plane and, as follows from formulas (II.158) and (II.160), $(\sigma_\eta)_{max} = 2p$.

From formulas (II.158) and (II.160) we know that if the distance d of the center of the round hole from the straight edge is greater than $R\sqrt{3}$, then the greatest tension stresses σ_η will occur on the contour of the round hole.

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If, on the other hand, $d < R\sqrt{3}$, then the greatest stresses σ_η will be at point A of the straight edge of the half plane.

By introducing the definition $d = 2\lambda R$, formula (II.158) acquires the form

$$(\sigma_x)_{max} = p\frac{4}{4\lambda^2 - 1};$$

where $\lambda = 3/5$, i.e., if the distance AB = 0.2R, $(\sigma_x)_{max} = 9.1p$. If we divide AB in half, i.e., the distance AB = 0.1R, then $(\sigma_x)_{max} = 19.5p$.

Usually, in calculating riveted joints $\lambda = 1.5\text{-}2.5$, and from the latter formula we find that the stress at point A falls in the range /127

$$\frac{1}{6}p < (\sigma_x)_{x=0} < \frac{1}{2}p.$$

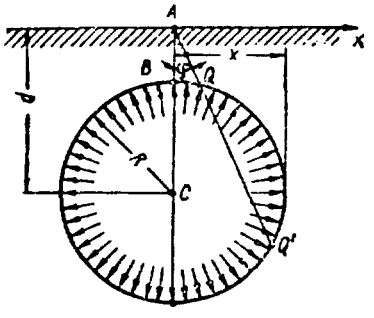


Figure II.72.

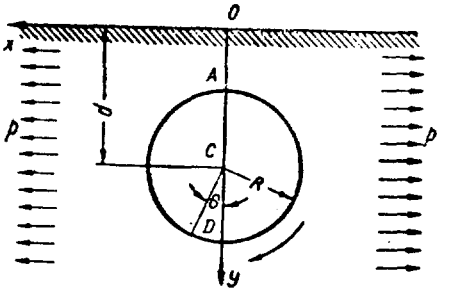


Figure II.73.

Half Plane, Weakened by a Circular Hole, under Tension[1]. We will assume that a circular hole of radius R is made in an isotropic half plane. The center of this hole is located at distance d from the straight boundary of the half plane (Figure II.73). This half plane is subjected to tension by forces p = const, parallel to the straight boundary of the half plane.

Stresses $\sigma_\eta = \sigma_x$ on the boundary of the half plane

$$\sigma_x = p\left[1 + (1 - \cos\eta)\sum_{n=1}^{\infty} P_n \cos n\eta\right]. \qquad \text{(II.161)}$$

[1]This case is also discussed by Yu. A. Ustinov [1] when the hole is located very close to the boundary of the half plane.

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The stresses on the contour of the circular hole[1] are

$$\sigma_\eta = 2p\left\{1 - \frac{2\mathrm{sh}^2\xi_1 \sin^2\eta}{(\mathrm{ch}\,\xi_1 - \cos\eta)^2} + (\mathrm{ch}\,\xi_1 - \cos\eta)\times\right.$$
$$\left.\times\left[\frac{1}{2\mathrm{sh}\,\xi_1} + 2e^{-2\xi_1}\cos\eta + \sum_{n=2}^{\infty} N_n \cos n\eta\right]\right\}, \qquad \text{(II.162)}$$

where the parameter ξ_1 is defined by the formula d = R ch ξ_1 or

$$\xi_1 = \ln\frac{d + \sqrt{d^2 - R^2}}{R}.$$

The relationship between angle θ (determined on the Oy axis in the direction indicated in Figure II.73 by the arrow) and the parameter η is defined by formula (II.146). It is readily seen from Figures II.70 and II.73 that the abscissas of the points of the straight boundary of the half plane are related to the parameter η by the relation x = a·1 - cos η/sin η, where a = R sh ξ_1 or a = d/cth ξ_1. The factors P_n and N_n in formulas (II.161) and (II.162) are presented in Tables II.25 and II.26, respectively, for several values of ξ_1. /128

TABLE II.25

P_n	ξ_1									
	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4
P_1	1,7241	1,0119	0,6261	0,3991	0,2590	0,1700	0,1124	0,0746	0,0497	0,0332
P_2	3,4481	2,0238	1,2521	0,7982	0,5180	0,3400	0,2247	0,1493	0,0994	0,0664
P_3	2,2403	1,0617	0,5110	0,2448	0,1160	0,0543	0,0251	0,0115	0,0053	0,0024
P_4	1,3549	0,4874	0,1699	0,0570	0,0185	0,0059	0,0018	0,0006	0,0002	0,0001
P_5	0,7601	0,1969	0,0474	0,0108	0,0024	0,0005	0,0001	—	—	—
P_6	0,3964	0,0713	0,0116	0,0018	0,0003	—	—	—	—	—
P_7	0,1934	0,0237	0,0026	0,0003	—	—	—	—	—	—
P_8	0,0891	0,0073	0,0005	—	—	—	—	—	—	—
P_9	0,0391	0,0022	0,0001	—	—	—	—	—	—	—
P_{10}	0,0165	0,0006	—	—	—	—		—	—	—

Tr. Note: Commas indicate decimal points.

[1] After correction of the error that crept, unfortunately, into G. B. Jeffrey's solution [1] (one term was omitted in the stress function), which was brought to my attention by D. G. Khlebnikov. This error in G. B. Jeffrey's solution [1] was probably first noticed by R. G. Mindlin [2]. Tables II.25 and II.27 were compiled with consideration of the stated corrections.

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TABLE II.26

N_n	ξ_1									
	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4
N_2	1,4649	0,77160	0,4139	0,2240	0,1219	0,0665	0,0364	0,0199	0,0109	0,0060
N_3	0,7457	0,26470	0,0914	0,0306	0,0100	0,0032	0,0010	0,0003	0,0001	—
N_4	0,3238	0,00719	0,0148	0,0029	0,0005	0,0001	—	—	—	—
N_5	0,1232	0,01620	0,0019	0,0002	—	—	—	—	—	—
N_6	0,0421	0,00320	0,0002	—	—	—	—	—	—	—
N_7	0,0131	0,00050	—	—	—	—	—	—	—	—
N_8	0,0038	0,00010	—	—	—	—	—	—	—	—

Tr. Note: Commas indicate decimal points.

For large values of ξ_1, with consideration of (II.146), formula (II.162) acquires the form

$$\sigma_\eta = 2p\left[1 - 2\sin^2\theta + \frac{1}{2}\right] = p(1 + 2\cos 2\theta),$$

i.e., the expected form (see §2, Chapter II).

The values of σ_x/p and σ_η/p at the points O, A and D (see Figure II.73) for various values of d/R, are presented in Table II.27.

TABLE II.27

ξ_1	d/R	σ_η/p		
		At Point 0	At point A	A point D
0,6	1,185	0,314	6,960	3,523
0,8	1,337	0,412	5,387	3,414
1,0	1,543	0,506	4,494	3,323
1,2	1,811	0,596	3,944	3,249
1,4	2,151	0,681	3,596	3,189
1,6	1,577	0,758	3,376	3,141
1,8	3,107	0,822	3,238	3,104
2,0	3,762	0,863	3,151	3,076
2,2	4,570	1,911	3,096	3,054
∞	∞	1,000	3,000	3,000

Tr. Note: Commas indicate decimal points.

Heavy Half Plane with Circular Hole[1]. The stresses in a heavy half plane, i.e., a half plane (see Figure II.73), under the effect of gravity $\gamma = \rho g = \text{const}$ (ρ is density and g is acceleration of gravity), weakened by a circular hole of radius R, the center of which is located at distance d from the straight edge of the half plane, are analyzed for three cases of the basic stress state of the impact mass: /129

[1]The problem of stresses in an anisotropic half plane weakened by two circular holes, where $\sigma_x^0 = \text{const}$, $\sigma_y^0 = \text{const}$, $\tau_{xy}^0 = 0$, is discussed by L. N. Nagibin [1].

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$$\text{I.}\quad \sigma_y^0 = -\gamma y, \quad \sigma_x^0 = -\gamma y, \quad \tau_{xy}^0 = 0;$$
$$\text{II.}\quad \sigma_y^0 = -\gamma y, \quad \sigma_x^0 = -\frac{\nu}{1-\nu}\gamma y, \quad \tau_{xy}^0 = 0;$$
$$\text{III.}\quad \sigma_y^0 = -\gamma y, \quad \sigma_x^0 = 0, \quad \tau_{xy}^0 = 0,$$

where ν is Poisson's ratio.

The stress components in bipolar coordinates are defined by formulas

$$\sigma_\xi = \frac{1}{a}\left\{(\operatorname{ch}\xi - \cos\eta)\frac{\partial^2 F}{\partial\eta^2} - \operatorname{sh}\xi\frac{\partial F}{\partial\xi} - \sin\eta\frac{\partial F}{\partial\eta} + F\operatorname{ch}\xi\right\} - \frac{a\gamma\operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\eta},$$
$$\sigma_\eta = \frac{1}{a}\left\{(\operatorname{ch}\xi - \cos\eta)\frac{\partial^2 F}{\partial\xi^2} - \operatorname{sh}\xi\frac{\partial F}{\partial\xi} - \sin\eta\frac{\partial F}{\partial\eta} + F\cos\eta\right\} - \frac{a\gamma\operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\eta}, \qquad \text{(II.163)}$$
$$\tau_{\xi\eta} = -\frac{1}{a}(\operatorname{ch}\xi - \cos\eta)\frac{\partial^2 F}{\partial\xi\partial\eta},$$

if the function F = UH is known, where U is the stress function in bipolar coordinates and H = 1/a(ch ξ - cos η) is the first differential parameter of the bipolar coordinate system.

For case I, R. G. Mindlin [1] gives the function $F(\xi, \eta)$ in the form /130

$$F(\xi,\eta) = -\frac{\gamma a^2 \operatorname{csch}^2\alpha}{2}\left\{\Phi\sin\eta - \frac{1-2\nu}{2(1-\nu)}\xi\operatorname{ch}\xi + \right.$$
$$+\frac{5-6\nu}{2(1-\nu)}\operatorname{cth}\alpha(\operatorname{ch}\xi - \cos\eta)\xi - \frac{3-4\nu}{4(1-\nu)}(\operatorname{ch}2\xi - 1)\cos\eta +$$
$$+\frac{1}{2}\left[\frac{5-6\nu}{2(1-\nu)}\operatorname{cth}\alpha - 1\right]\operatorname{sh}2\xi\cos\eta +$$
$$\left. + \sum_{n=2}^{\infty}\frac{2\operatorname{sh}\alpha e^{-n\alpha}}{\operatorname{sh}^2 n\alpha - n^2\operatorname{sh}^2\alpha}[\operatorname{sh}n\alpha\operatorname{sh}n\xi\operatorname{sh}(\xi-\alpha) - n\operatorname{sh}\alpha\operatorname{sh}\xi\operatorname{sh}n(\xi-\alpha)]\right\}\cos n\eta,$$

where $\alpha = \xi_0 > 0$ (see Figure II.70), $\tan\Phi = \frac{\operatorname{ch}\xi\cos\eta - 1}{\operatorname{sh}\xi\sin\eta}$.

Of greatest interest are stresses σ_η on the contour of a circular hole. These are expressed as follows:

$$[\sigma_\eta]_{\xi=\alpha} = \frac{2\gamma a(\operatorname{ch}\alpha - \cos\eta)}{\operatorname{sh}\alpha}\left\{\frac{1 - \operatorname{ch}\alpha\cos\eta}{(\operatorname{ch}\alpha - \cos\eta)^2} - \operatorname{cth}\alpha - \right.$$
$$\left. - \frac{(7-8\nu)\cos\eta}{4(1-\nu)\operatorname{sh}\alpha} + 2e^{-\alpha}\cos\eta - \sum_{n=2}^{\infty}R_n\cos n\eta\right\}. \qquad \text{(II.164)}$$

The values of R_n are presented in Table II.28.

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TABLE II.28

R_n	α								
	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
R_2	1,4663	0,5198	0,2043	0,0848	0,0364	0,0157	0,0069	0,0031	0,0014
R_3	0,7620	0,1852	0,0482	0,0127	0,0033	0,0034	0,0002	0,0001	—
R_4	0,3848	0,0615	0,0099	0,0016	0,0003	—	—	—	—
R_5	0,1884	0,0189	0,0018	0,0002	—	—	—	—	—
R_6	0,0893	0,0054	0,0003	—	—	—	—	—	—
R_7	0,0409	0,0015	—	—	—	—	—	—	—
R_8	0,0181	0,0004	—	—	—	—	—	—	—

Tr. Note: Commas indicate decimal points.

Figure II.74 shows the graph of stresses $(\sigma_\eta)_{\xi=\alpha}$, decreased by a factor of $2\gamma R$, acting on the contour of a hole of radius R and calculated by the formula (II.164) for $\nu = 1/2$, $\alpha = 0.4$ and $\alpha = 2.0$.

Figure II.75 represents similar graphs for $\xi = \alpha$, $\eta = \pi$, i.e., at the point A (see Figure II.73) for three values of Poisson's ratio, ν - 0, 1/4 and 1/2.

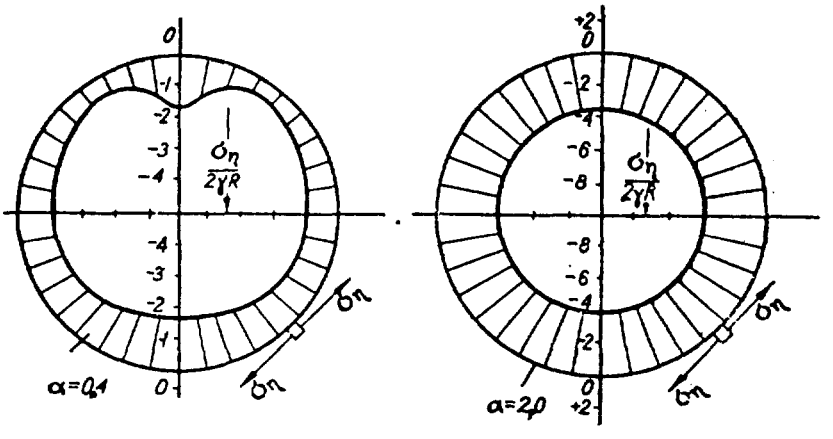


Figure II.74.

The data presented in Figures II.74 and II.75 show that: /131

a change in Poisson's ratio has a slight effect on the stresses, with the exception of those cases where the hole is very close ($d/R < 1.2$) to the straight line of the half plane;

stresses σ_η at point A (see Figure II.73) increase by a nearly linear law as the depth increases;

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stresses σ_η at point A increase only slightly as the hole is placed closer to the straight edge of the half plane;

the greatest value of stress σ_η on the contour of the hole for large values of α will occur at the point D (Figure II.73); for smaller values $\alpha(\alpha < 0.4)$, the greatest σ_η will occur at point A.

For cases II and III, the function F = UH acquires the form

$$F = M\gamma a^2 \left\{ \operatorname{ch}\xi + 2\sum_{n=1}^{\infty} (\operatorname{ch}\xi + n\operatorname{sh}\xi) e^{-n\xi} \cos n\eta + \right.$$

$$+ 3\operatorname{cth}\alpha \operatorname{ch}^2\alpha [\xi(\operatorname{ch}\xi - \cos\eta) + \operatorname{sh}\xi(\operatorname{ch}\xi\cos\eta - 1)] -$$

$$- \frac{3}{2}\operatorname{csch}^2\alpha(\operatorname{ch}2\xi - 1)\cos\eta + \sum_{n=2}^{\infty} (A_n[\operatorname{ch}(n+1)\xi - \operatorname{ch}(n-1)\xi] +$$

$$\left. + B_n[(n-1)\operatorname{sh}(n+1)\xi - (n+1)\operatorname{sh}(n-1)\xi])\cos n\eta \right\} - \qquad \text{(II.165)}$$

$$- \frac{1}{2}\gamma a^2 \operatorname{csch}^2\alpha \left\{ \Phi\sin\eta - \frac{1-2\nu}{2(1-\nu)}\xi\operatorname{sh}\xi + \frac{5-6\nu}{2(1-\nu)}\operatorname{cth}\alpha(\operatorname{ch}\xi - \cos\eta)\xi - \right.$$

$$- \frac{3-4\nu}{4(1-\nu)}(\operatorname{ch}2\xi - 1)\cos\eta + \frac{1}{2}\left(\frac{5-6\nu}{2(1-\nu)}\operatorname{cth}\alpha - 1\right)\operatorname{sh}2\xi\cos\eta + \qquad \text{/132}$$

$$\left. + \sum_{n=2}^{\infty} \frac{2e^{-n\alpha}\operatorname{sh}\alpha}{\operatorname{sh}^2 n\alpha - n^2\operatorname{sh}^2\alpha} [\operatorname{sh} n\alpha \operatorname{sh} n\xi \operatorname{sh}(\xi - \alpha) - n\operatorname{sh}\alpha \operatorname{sh}\xi \operatorname{sh} n(\xi - \alpha)] \right\} \cos n\eta,$$

where

$$A_n = -\frac{n(n^2-1)\operatorname{sh}^2\alpha}{\operatorname{sh}^2 n\alpha - n^2\operatorname{sh}^2\alpha}, \qquad B_n = \frac{n\operatorname{sh}\alpha(n\operatorname{sh}\alpha + \operatorname{ch}\alpha) + e^{-n\alpha}\operatorname{sh} n\alpha}{\operatorname{sh}^2 n\alpha - n^2\operatorname{sh}^2\alpha},$$

$$\tan\Phi = \frac{\operatorname{ch}\xi\cos\eta - 1}{\operatorname{sh}\xi\sin\eta};$$

for case II, M = (1 - 2ν)/6(1 - ν), and for case III, -M = 1/6. Of greatest interest are stresses σ_η on the contour of the round hole:

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$$(\sigma_\eta)_{\xi=\alpha} = \gamma Ma\,(\operatorname{ch}\alpha - \cos\eta)\left\{6\operatorname{cth}\alpha\operatorname{csch}\alpha + 6\operatorname{csch}^2\alpha\cos\eta + \right.$$

$$\left. + 4\operatorname{sh}\alpha\sum_{n=2}^{\infty} T_n\cos n\eta\right\} + 6\gamma Ma\operatorname{csch}\alpha\,[\cos\psi + 2\operatorname{ch}\alpha\cos 2\psi + \cos 3\psi] +$$

$$+ \frac{2\gamma a\,(\operatorname{ch}\alpha - \cos\beta)}{\operatorname{sh}\alpha}\left\{\frac{1 - \operatorname{ch}\alpha\cos\eta}{(\operatorname{ch}\alpha - \cos\eta)^2} - \operatorname{cth}\alpha - \frac{(7-8\nu)\cos\eta}{4\,(1-\nu)\operatorname{sh}\alpha} + \right.$$

$$\left. + 2e^{-\alpha}\cos\eta - \sum_{n=2}^{\infty} R_n\cos n\eta\right\}.$$

where

$$\cos\psi = \frac{\operatorname{ch}\alpha\cos\eta - 1}{\operatorname{ch}\alpha - \cos\eta}, \qquad \sin\psi = \frac{\operatorname{sh}\alpha\sin\eta}{\operatorname{ch}\alpha - \cos\eta}.$$

The factors R_n are presented in Table II.28, the values of α are presented in Table II.29 and the values of the factors T_n, in Table II.30.

TABLE II.29

α	d/R	α	d/R
0,2	1,02	1,4	2,15
0,4	1,08	1,6	2,58
0,6	1,19	1,8	3,11
0,8	1,34	2,0	3,76
1,0	1,54	—	—
1,2	1,81	—	—

Tr. Note: Commas indicate decimal points.

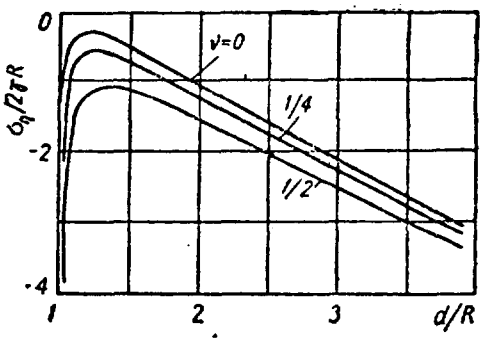


Figure II.75.

Figure II.76 shows the curves of stresses $(\sigma_\eta)_{\xi=\alpha}$, decreased by a factor of $2\gamma R$, for point A (see Figure II.73) for various values d/R, for both the case II (Figure II.76,a) and for case III (Figure II.76,b).

To the conclusions presented above, it is necessary to add the following.

For $d/R > 1.5$, as indicated by the data presented in Figures II.75 and II.76, the effect of the straight edge of the half plane is very slight as far as stresses σ_η are concerned. Therefore, for $d/R > 1.5$ (or $a > 1$), stresses σ_η on the contour of the hole can be calculated by the formulas that were

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derived under the assumption that the radius of the hole R is small in comparison with d. In this limiting case, the formulas for stresses σ_η on the contour of a round hole acquire the form:

Case I

/133

$$\sigma_\eta = -2d\gamma + \gamma R\frac{3-4\nu}{2(1-\nu)}\sin\Phi; \tag{II.166}$$

Case II

$$\sigma_\eta = \frac{\gamma R}{1-\nu}\left[\frac{1}{2}\sin\Phi + (1-2\nu)\sin 3\Phi\right] - \frac{\gamma d}{1-\nu}[1 + 2(1-2\nu)\cos 2\Phi], \tag{II.167}$$

where $\Phi = \frac{3}{2}\pi - \psi$ is an angle, which the tangent to the curve η = const makes with the Ox axis.

By placing in Figure II.73 the origin of the Cartesian coordinate system xOy at the center of the hole and directing Ox to the right and Oy upward, we write the stress components of the basic stress state in a heavy half plane that is not weakened by a hole:

$$\sigma_x^0 = \gamma\lambda(y-d), \quad \sigma_y^0 = \gamma(y-d), \quad \tau_{xy}^0 = 0,$$

where λ is any real number, or, in polar coordinates, in reading θ from the Ox axis counter clockwise:

$$\begin{aligned}(\sigma_r)_0 &= \frac{\gamma r}{4}[(3+\lambda)\sin\theta + (\lambda-1)\sin 3\theta] - \frac{\gamma d}{2}[(1+\lambda) - (1-\lambda)\cos 2\theta],\\ (\sigma_\theta)_0 &= \frac{\gamma r}{4}[(3\lambda+1)\sin\theta - (\lambda-1)\sin 3\theta] - \frac{\gamma d}{2}[(1+\lambda) + (1-\lambda)\cos 2\theta],\\ (\tau_{r\theta})_0 &= \frac{\gamma r}{4}(1-\lambda)(\cos\theta - \cos 3\theta) - \frac{\gamma d}{2}(1-\lambda)\sin 2\theta.\end{aligned} \tag{II.168}$$

By separating from stresses (II.168), the components σ_r^0, σ_θ^0 and $\tau_{r\theta}^0$ of the type (II.122), we obtain the components σ_r^{00}, σ_θ^{00}, $\tau_{r\theta}^{00}$ (II.120), to which corresponds the stress function

$$U_0(x, y) = \frac{\gamma\lambda}{6}y^3 - \frac{\lambda\gamma d}{2}y^2 - \frac{\gamma d}{2}x^2.$$

By this function, that characterizes stress state (II.120) in the intact plane,

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we determine[1] the stress components in the plane with a circular hole, the contour of which is free of external forces: /134

$$\sigma_r^{**} = \frac{\gamma\lambda R}{4}\left[\left(3\frac{r}{R}+\frac{R^3}{r^3}\right)\sin\theta+\left(-\frac{r}{R}+\frac{R^3}{r^3}-4\frac{R^5}{r^5}\right)\sin 3\theta\right]-$$
$$-\frac{\gamma d}{2}\left[(1+\lambda)\left(1+\frac{R^2}{r^2}\right)+(1-\lambda)\left(1+3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$
$$\sigma_\theta^{**} = \frac{\gamma\lambda R}{4}\left[\left(\frac{r}{R}-\frac{R^3}{r^3}\right)\sin\theta+\left(\frac{r}{R}-5\frac{R^3}{r^3}+4\frac{R^5}{r^5}\right)\sin 3\theta\right]- \quad \text{(II.169)}$$
$$-\frac{\gamma d}{2}\left[(1+\lambda)\left(1-\frac{R^2}{r^2}\right)+(1-\lambda)\left(-1+4\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)\cos 2\theta\right],$$
$$\tau_{r\theta}^{**} = \frac{\gamma\lambda R}{4}\left[\left(-\frac{r}{R}+\frac{R^3}{r^3}\right)\cos\theta+\right.$$
$$\left.+\left(\frac{r}{R}+3\frac{R^3}{r^3}-4\frac{R^5}{r^5}\right)\cos 3\theta\right]-\frac{\gamma d}{2}\times$$
$$\times(1-\lambda)\left(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)\sin 2\theta.$$

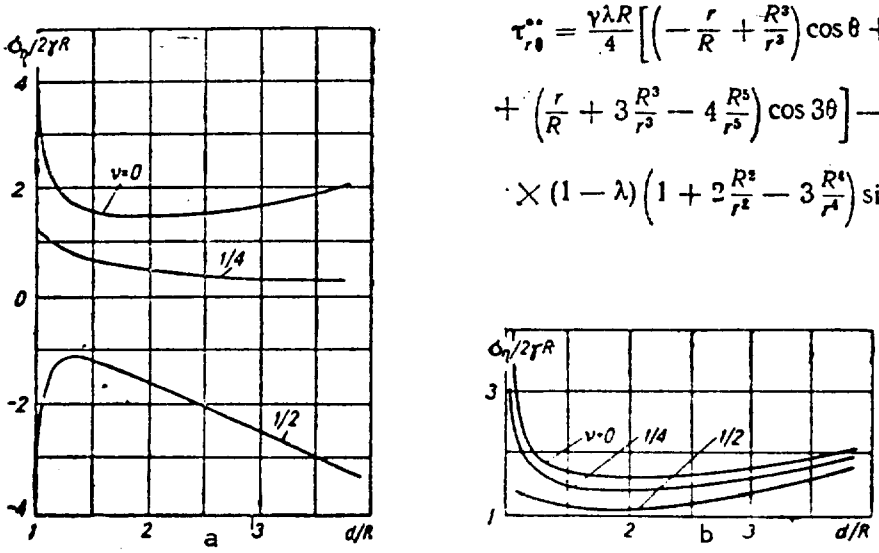


Figure II.76.

TABLE II.30

T_n	d								
	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
T_0	41,4132	10,1670	3,3052	1,2281	0,4924	0,2047	0,0876	0,0382	0,0168
T_2	33,2080	6,1642	1,4073	0,3146	0,0867	0,0219	0,0055	0,0026	0,0003
T_4	20,1894	3,1329	0,4583	0,0880	0,0099	0,0015	0,0001	—	—
T_5	16,2229	1,3750	0,1211	0,0103	0,0008	0,0001	—	—	—
T_6	10,1179	0,5326	0,0271	0,0012	0,0001	—	—	—	—
T_7	5,9144	0,1857	0,0053	0,0001	—	—	—	—	—
T_8	3,2620	0,0593	0,0005	—	—	—	—	—	—
T_9	1,7069	0,0176	0,0001	—	—	—	—	—	—
T_{10}	0,8523	0,0043	—	—	—	—	—	—	—

Tr. Note: Commas indicate decimal points.

[1]Here the method described in G. N. Savin's monograph [13], Chapter II, pp. 121-125, is used.

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Now, by combining stresses (II.169) with stresses (II.127), corresponding to the partial solution (II.122), we obtain the desired total stresses σ_r, σ_θ, $\tau_{r\theta}$ (II.121) in a heavy half plane with a circular hole. The stresses on the contour of the circular hole are /135

$$\sigma_\theta = \gamma R\left[\left(\frac{1-2\nu}{2(1-\nu)}+\lambda\right)\sin\theta+(1-\lambda)\sin 3\theta\right]- \\ -\gamma d[(1+\lambda)+2(1-\lambda)\cos 2\theta]. \qquad \text{(II.170)}$$

By assuming in (II.170) that $\lambda = 1$ or $\lambda = \nu/1 - \nu$, we obtain formulas that coincide with formulas (II.166) and (II.167). For the partial value $\lambda = \tan^2(45° - \phi/2)$, where ϕ is the angle of internal friction of the soil, some calculations were made by I. V. Rodin [1].

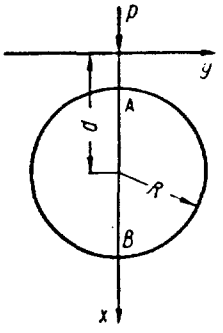


Figure II.77.

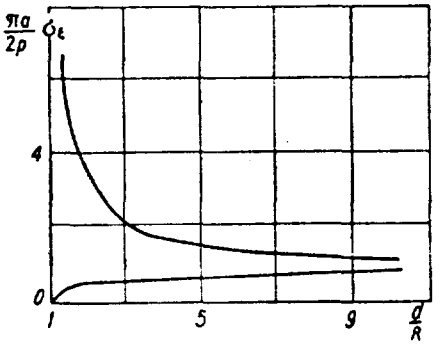


Figure II.78.

Effect of Concentrated Force or Couple Applied to Arbitrary Point of Straight Edge of Half Plane with Circular Hole. The problem of the stress state of a half plane weakened by a circular hole, where a concentrated force p, directed through the diameter of the hole (Figure II.77), is applied to its straight edge, was examined by Ya. S. Podstrigach [1]. The solution of this problem is found in coordinates (II.141).

The stresses along the contour of the hole are

$$\sigma_\xi = \frac{2p}{\pi a}[\operatorname{ch}\alpha-\cos\xi]\left[-\frac{\operatorname{ch}\alpha+\cos\xi}{\operatorname{sh}^2\alpha}+4\operatorname{sh}\alpha\sum_{n=1}^{\infty}\frac{n\operatorname{sh}2n\alpha\cos 2n\xi}{\operatorname{sh}^2 2n\alpha-4n^2\operatorname{sh}^2\alpha}\right];$$

and along the edge of the half plane

$$\sigma_\xi = \frac{2p}{\pi a}(1-\cos\xi)\left[\frac{\cos\xi}{\operatorname{sh}^2\alpha}-8\operatorname{sh}^2\alpha\sum_{n=1}^{\infty}\frac{n^2\cos 2n\xi}{\operatorname{sh}^2 2n\alpha-4n^2\operatorname{sh}^2\alpha}\right],$$

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where

$$a = \sqrt{d^2 - R^2}.$$

Figure II.78 shows the graphs of stresses σ_ξ at the points A and B (Figure II.77) along the contour of the hole for various values of d/R for constant a. /136

In the very same bipolar coordinates ξ and η (II.141), R. M. Evan-Iwanowski [1] found the expressions for stress functions in the half plate with a circular hole, under the effect of normal tangential forces applied to an arbitrary point of the boundary of the half plane.

The problem of stresses in a half plane with a circular hole under the effect of a concentrated couple applied to the straight boundary was examined by A. M. Sen-Gupta [3]. In this work the stresses σ_ξ are examined both on the straight, and on the curved boundaries.

Tension of Plate Weakened by a Hole, the Contour of Which Consists of Two Equal Arcs. We will assume that a hole is made in an infinite isotropic elastic plane. The contour of this hole consists of two equal arcs (Figure II.79). We will determine the stress state around this hole under the condition that its contour is free of external forces, while the stress state at infinity is bi- or uniaxial tension.

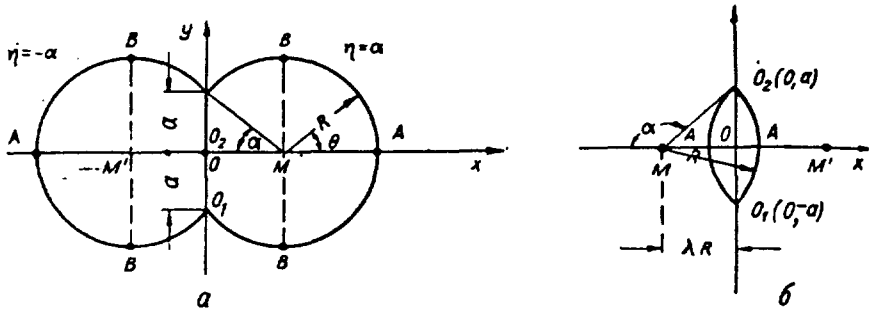


Figure II.79.

Figure II.79, a shows that when $\lambda > 1$, we have two nonintersecting circular holes, and when $\lambda = 1$, the contour of the hole consists of two equal intersecting circles. When $\lambda = 0$, the hole consists of two equal semicircles, i.e., of one circle of radius R. For holes shown in Figure II.79, b, $\lambda < 0$, and for $\lambda = -1$, the hole is reduced to a slit of length 2a on the Oy axis.

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If λR and R are known, then for the coordinate lines of the bipolar coordinates $\eta = d$ and $\eta = -\alpha$, coinciding with the contour of the hole, we may write $\lambda R = a \cot \alpha$, $a = R \sin \alpha$ or

$$\alpha = \arccos \lambda, \qquad a = \sqrt{1 - \lambda^2}. \tag{II.171}$$

The relationship between the Cartesian and bipolar coordinates is described by formulas (II.141). From Figure II.79, a, we find the relationship between ξ (on the line $\eta = \alpha = \text{const}$) and angle θ: /137

$$\operatorname{ch} \xi = \frac{1 + \lambda \cos \theta}{\lambda + \cos \theta}. \tag{II.172}$$

For the function $F(\xi, \eta) = H(\xi, \eta)U(\xi, \eta)$ we have

$$\begin{aligned} F(\xi, \eta) = 4apK \int_0^\infty \frac{[n \operatorname{ch} n(\alpha - \eta) \sin \eta + \operatorname{sh} n(\alpha - \eta) \cos \eta] \operatorname{sh} n\alpha \cos n\xi}{n(n^2 + 1)(\operatorname{sh} 2n\alpha + n \sin 2\alpha)} dn - \\ - 4apK \sin \alpha \int_0^\infty \frac{n \operatorname{ch} n\eta \sin(\alpha - \eta) + \operatorname{sh} n\eta \cos(\alpha - \eta)}{(n^2 + 1)(\operatorname{sh} 2n\alpha + n \sin 2\alpha)} \cos n\xi dn \pm \\ \pm 2ap \int_0^\infty \frac{n \operatorname{ch} n\eta \sin(\alpha - \eta) \sin \alpha - \operatorname{ch} n\alpha \operatorname{sh} n(\alpha - \eta) \sin \eta}{\operatorname{sh} 2n\alpha + n \sin 2\alpha} \cos n\xi dn, \end{aligned} \tag{II.173}$$

where K is found from the expression

$$4K \int_0^\infty \frac{(\operatorname{sh}^2 n\alpha - n^2 \sin^2 \alpha) dn}{n(n^2 + 1)(\operatorname{sh} 2n\alpha + n \sin 2\alpha)} = \delta \pm 2 \sin^2 \alpha \int_0^\infty \frac{ndn}{\operatorname{sh} 2n\alpha + n \sin 2\alpha}. \tag{II.174}$$

We know from the two signs (upper and lower) in (II.173) and (II.174) that we take only the upper sign for tension along the Ox axis, and the lower sign for tension along the Oy axis. The value δ in (II.174) will also have two values, depending on the conditions in the infinitely remote parts of the plate: for tension of the plate along the Oy axis, $\delta = 0$, but along the Ox axis, $\delta = 1$. The function $H(\xi, \eta) = (1/a)(\operatorname{ch} \xi - \cos \eta)$, and $U(\xi, \eta)$ is the stress function in bipolar coordinates. By knowing the function $F(\xi, \eta)$ (II.173), we can determine the stress components σ_ξ, σ_η and $\tau_{\xi\eta}$ from formulas (II.163), in which it is necessary to assume that $\gamma = 0$. The formula for stresses along the contour of the hole is very important:

$$\sigma_\xi = 4p(\operatorname{ch} \xi - \cos \alpha) \sin \alpha \int_0^\infty \frac{2K \mp n(n - \cot \alpha \operatorname{cth} n\alpha)}{\operatorname{sh} 2n\alpha + n \sin 2\alpha} \operatorname{sh} n\alpha \cos n\xi dn, \tag{II.175}$$

where K is defined by formula (II.174), and the double signs have the same meaning as before.

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The stresses σ_ξ on the contour of the hole under biaxial tension for $\lambda > 0$, i.e., for $\alpha < \pi/2$ (see Figure II.79,a), will be greatest at the points A. For $\lambda < 0$, i.e., for $\alpha > \pi/2$ (see Figure II.79,b), stresses σ_ξ at the points O_2 and O_1 are converted to infinity. When the tension is directed along the Ox axis for $\lambda > 0$, the greatest stresses σ_ξ will occur at the points located very close to point B, but for $\lambda < 0$, the stresses σ_ξ at the points O_2 and O_1 are converted to infinity. In the case of tension along the Oy axis, both for $\lambda > 0$ and for $\lambda < 0$, the greatest stresses σ_ξ will occur at the point A. /138

The stresses σ_ξ computed by formulas (II.174) and (II.175) by C. B. Ling [2] for three cases: universal tension of plate, tension along the Ox axis, and tension along the Oy axis, are presented in Table II.31.

TABLE II.31

λ	$\alpha°$	Biaxial tension σ_ξ/p at points A	Tension along Ox axis σ_ξ/p at points B	Tension along Oy axis σ_ξ/p at points A	Comments
1.0	0	2.894	2.569	3.869	Two circles in contact
0.5	60	2.524	2.630	3.493	One circular hole
0.0	90	2.000	3.00	3.000	
-0.5	120	--	--	2.101	Slit on Ox axis of length 2a
-1.0	180	--	--	1.000	

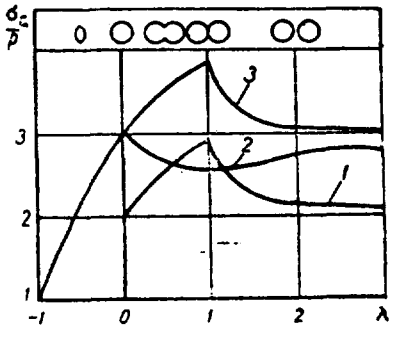


Figure II.80.

The stress-strain diagrams of the stresses, constructed from the data in Tables II.30 and II.31, are shown in Figure II.80: curve 1 represents σ_ξ/p at points A (see Figure II.79) when the plate is under biaxial tension, curve 2 represents σ_ξ/p at points B for tension along the Ox axis, and curve 3 represents σ_ξ/p at point A for tension along the Oy axis. The shapes of the holes corresponding to the indicated value of λ are shown at the top of the figure. Pure deflection of a plate weakened by a hole, the contour

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of which consists of two equal arcs, was examined by N. A. Savruk [2] and B. Karunes [1].

§8. Stresses in Elastic Plane Weakened by Identical Round Holes

We will assume that an elastic medium occupies plane xOy, weakened by identical round holes, the distances between the centers of which are also identical. External forces X_n and Y_n, which are statistically equivalent to zero on each contour, are applied to the contours of the holes. These conditions may vary on the different contours.

The problems related to the determination of the stress state of such a medium have been examined by many researchers.

In examining a medium with two round holes, C. B. Ling [1] used bipolar coordinates.

For holes located close to each other, where the holes have the greatest effect on the stress state of the medium near each of them, it is more convenient to use the theory of functions of complex variable. The solution of the given problem was found by D. I. Sherman [1, 5] as the solution of an infinite system of algebraic equations, which system is quasiregular for all distances between the holes. D. I. Sherman also used this method [2-8] for the solution of other problems related to the examination of multiply-connected regions. /139

Particularly noteworthy is A. S. Kosmodamianskiy's method [3, 5], who used the basic idea of the Bubnov-Galerkin method, which makes it possible to solve effectively many practically important problems rather easily.

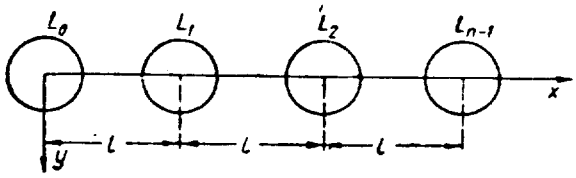


Figure II.81.

Let an elastic medium be weakened by n holes, the centers of which lie on a single straight line that coincides with the x axis (Figure II.81). The radius of the hole is r = 1, which does not contradict the generality of the examined problem. The distances between the centers of the holes are denoted through l. We will assume[1] that the elastic plane, at infinity, is loaded in such a way that Airy's stress function for a solid plate can be represented in the form of a whole polynomial with respect to variables x and y.

The problem of the stress state in a plane weakened by holes is reduced to the determination of functions $\phi_1(z)$ and $\psi_1(z)$ from boundary conditions (I.9) in the case of the first basic problem, and from conditions (I.10) in the case of the second basic problem. Here it is essential that the above conditions be satisfied on the contours of all holes.

[1]The solutions of the problems presented in §8 and 9 are found by A. S. Kosmodamianskiy [1-7].

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We will introduce the function

$$\Psi(z) = \psi_1(z) + z\varphi_1'(z). \tag{II.176}$$

This function will be invariant during translation of the origin of the coordinate system along the x axis just as is the function $\phi(z) = \phi_1(z)$.

Now boundary condition (I.9) can be represented in the form

$$F(z, \bar{z}) = \varphi(z) + (z - \bar{z})\overline{\varphi'(z)} + \psi(z) - (f_{1k} + if_{2k} + C_k) = 0$$
$$(k = 0, 1, \ldots, n-1). \tag{II.177}$$

The functions that are holomorphic in the examined region are /140

$$\varphi(z) = \sum_{k=0}^{n-1}{}' \sum_{m=1}^{\infty}{}' \frac{\alpha_m^{(k)}}{(z - kl)^m},$$
$$\psi(z) = \sum_{k=0}^{n-1}{}' \sum_{m=1}^{\infty}{}' \frac{\beta_m^{(k)}}{(z - kl)^m}. \tag{II.178}$$

Here $\alpha_m^{(k)}$ and $\beta_m^{(k)}$ are arbitrary constants which should be determined from boundary condition (II.177).

The functions $\phi(z)$ and $\psi(z)$ will be found in the N-th approximation in the form

$$\varphi(z) = \sum_{k=0}^{n-1}{}' \sum_{m=1}^{N}{}' \frac{\alpha_m^{(k)}}{(z - kl)^m}, \qquad \psi(z) = \sum_{k=0}^{n-1} \sum_{m=1}^{N+2} \frac{\beta_m^{(k)}}{(z - kl)^m} \tag{II.179}$$

Let us examine the contour with the number ν(ν = 0, 1, ..., n - 1) and substitute expressions (II.179) in the boundary condition on this contour. We will require that the function $F(\sigma) = F_1(z, \bar{z})$, where $\sigma = e^{i\theta}$, a point on the contour of the ν-th hole, is orthogonal to the first N + 4 functions of the total system of functions $\sigma^{\pm k}$ (k = 0, 1, 2, ...). In this case, for the determination of the factors $\alpha_m^{(\nu)}$ and $\beta_m^{(\nu)}$, we obtain the equations

$$\int_{\gamma_\nu} F(\sigma)\sigma^{\pm k} d\sigma = 0. \tag{II.180}$$

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After integration, we find from (II.180) the "final" algebraic equation system in which the factors $\alpha_m^{(\nu)}$ and $\beta_m^{(\nu)}$ are expressed through the factors $\alpha_m^{(k)}$ and $\beta_m^{(k)}$ $(k \neq \nu)$. By carrying out the analogous operation on each contour, we find the algebraic equation system for the determination of all constants $\alpha_m^{(k)}$ and $\beta_m^{(k)}$. By knowing these constants, we find from formulas (II.179), the functions $\phi(z)$ and $\psi(z)$, through which the desired stresses are expressed:

$$\begin{aligned}
\sigma_x &= \sigma_x^0 + \operatorname{Re}\{-(\bar{z}-z)\varphi''(z) + 3\varphi'(z) - \psi'(z)\},\\
\sigma_y &= \sigma_y^0 + \operatorname{Re}\{(\bar{z}-z)\varphi''(z) + \varphi'(z) + \psi'(z)\},\\
\tau_{xy} &= \tau_{xy}^0 + \operatorname{Im}\{(\bar{z}-z)\varphi''(z) - \varphi'(z) + \psi'(z)\}.
\end{aligned} \tag{II.181}$$

Here σ_x^0, σ_y^0, τ_{xy}^0 are stresses in a solid elastic plate under the given forces at infinity.

Two Identical Round Holes. Let us arrange the coordinate system as shown in Figure II.82. The distance between the centers of the holes is assumed to be $2l$. We will assume that at infinity the tension forces p along the line of the centers of the hole and the force q, transverse to the line of the centers[1], are known. /141

In the N-th approximation

$$\varphi(z) = \sum_{k=1}^{N} \alpha_k \left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right], \qquad \psi(z) = \sum_{k=1}^{N+2} \beta_k \left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right]. \tag{II.182}$$

In view of the geometric and force symmetry, we will be required in the given case to satisfy equations (II.180) only on the right hand[2] contour γ. In the expanded form, equation (II.180) on contour γ is written as follows:

$$\begin{aligned}
&\int_\gamma \left[\varphi(\sigma) + \left(\sigma - \frac{1}{\sigma}\right)\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} - \right.\\
&\left. - \frac{p-q}{2}\frac{1}{\sigma} + \frac{p+q}{2}\sigma + c\right]\frac{d\sigma}{\sigma^k} = 0\\
&\qquad (k = 0, \pm 1, \pm 2 \ldots).
\end{aligned} \tag{II.183}$$

Here $\sigma = z - l$ is a point on contour γ.

[1]Other cases of loading are investigated by A. S. Kosmodamianskiy and V. N. Lozhkin [1] and A. S. Kosmodamianskiy [14].

[2]The boundary conditions will be satisfied automatically on the left hand contour.

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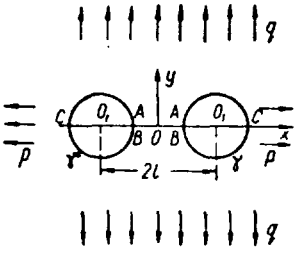


Figure II.82.

In the first approximation (N = 1) the system for the determination of the desired constants α_k and β_k will be found in the form

$$\alpha_1(1+2\varepsilon^2-8\varepsilon^4-3\varepsilon^8)=\frac{p-q}{2}-\varepsilon^2\frac{p+q}{2};$$

$$\beta_1=-\alpha_1(1-2\varepsilon^2)-\frac{p+q}{2},\qquad\text{(II.184)}$$

$$\beta_2=-\alpha_1\varepsilon^3;\qquad \beta_3=\alpha_1(1+\varepsilon^4).$$

In the second approximation (N = 2),

$$\begin{aligned}&\alpha_1(1+2\varepsilon^2-8\varepsilon^4-2\varepsilon^6-3\varepsilon^8-4\varepsilon^{10})+\alpha_2(-6\varepsilon^3+24\varepsilon^5+\\&+6\varepsilon^7+12\varepsilon^9+20\varepsilon^{11})=\frac{p-q}{2}-\varepsilon^2\frac{p+q}{2},\\&\alpha_1(-3\varepsilon^3+12\varepsilon^5+3\varepsilon^7+6\varepsilon^9+10\varepsilon^{11})+\alpha_2(1+12\varepsilon^4-\\&-44\varepsilon^6-9\varepsilon^8-24\varepsilon^{10}-50\varepsilon^{12})=\frac{p+q}{2}\varepsilon^3;\qquad\text{(II.185)}\\&\beta_1=-\alpha_1(1-2\varepsilon^2)-4\alpha_2\varepsilon^3-\frac{p+q}{2},\\&\beta_2=-\alpha_1\varepsilon^3-\alpha_2(2-3\varepsilon^4),\\&\beta_3=\alpha_1(1+\varepsilon^4)-4\alpha_2\varepsilon^5,\\&\beta_4=-\alpha_1\varepsilon^5+\alpha_2(2+5\varepsilon^6).\end{aligned}$$

In systems (II.184) and (II.185) $\varepsilon = 1/2l$.

The results of the calculations of factors α_k and β_k for the various distances between the holes are presented in Table II.32. The factors α_k and β_k in this table are given with an accuracy up to the factor p, while α_k' and β_k' are given with an accuracy up to the factor q. /142

The stresses σ_x and σ_y at points O, A, B and C (see Figure II.82) of the real axis are presented in Table II.33.

Stresses σ_θ on the contour of the right-hand hole are

$$\sigma_\theta=p+q+4\,\mathrm{Re}\,\varphi'(\sigma).\qquad\text{(II.186)}$$

In the fourth approximation,

$$\varphi'(\sigma)=-\sum_{k=1}^{4}k\alpha_k\left[\frac{1}{\sigma^{k+1}}+\frac{(-1)^{k+1}}{(\sigma+2l)^{k+1}}\right].\qquad\text{(II.187)}$$

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TABLE II.32

ε	Constants	First approximation			Second approximation			
		k			k			
		1	2	3	1	2	3	4
$\frac{2}{5}$	α_k	0,380	—	—	0,386	0,050	—	—
	α'_k	—0,525	—	—	—0,525	—0,006	—	—
	β_k	—0,758	—0,024	0,390	—0,775	—0,121	0,394	0,097
	β'_k	—0,143	0,034	—0,538	—0,141	0,045	—0,538	—0,007
$\frac{1}{3}$	α_k	0,397	—	—	0,401	0,039	—	—
	α'_k	—0,496	—	—	—0,497	—0,010	—	—
	β_k	—0,809	—0,015	0,402	—0,818	—0,092	0,405	0,077
	β'_k	—0,114	0,018	—0,502	—0,112	0,039	—0,503	—0,019
$\frac{1}{4}$	α_k	0,429	—	—	0,430	0,022	—	—
	α'_k	—0,486	—	—	—0,487	—0,009	—	—
	β_k	—0,875	—0,007	0,430	—0,878	—0,051	0,423	0,044
	β'_k	—0,075	0,008	—0,488	—0,074	0,025	—0,488	—0,017

Tr. Note: Commas indicate decimal points.

TABLE II.33

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σ	Points from Figure II. 82	First Approximation			Second Approximation		
		1/4	1/3	2/5	1/4	1/3	2/5
σ_x/q	*O*	0,51	0,60	0,44	0,49	0,57	0,40
	A	0,50	0,52	0,36	0,48	0,49	0,31
	B	0,04	0,08	0,05	0,02	0,04	0,00
	C	—0,01	—0,02	—0,01	0,01	0,01	0,02
σ_y/q	*O*	1,47	2,16	3,25	1,47	2,15	3,24
	A	1,67	2,39	3,41	1,67	2,38	3,40
	B	3,12	3,40	3,99	3,07	3,35	3,97
	C	3,03	3,12	3,28	3,09	3,18	3,30
σ_x/p	*O*	0,08	—0,28	—0,42	0,13	—0,15	—0,21
	A	—0,01	—0,29	—0,39	0,04	—0,15	—0,19
	B	—0,08	—0,19	—0,28	—0,03	—0,09	—0,11
	C	0,03	0,06	0,08	—0,01	—0,04	—0,04
σ_y/p	*O*	0,06	—0,13	—0,52	0,06	—0,09	—0,36
	A	—0,02	—0,24	—0,61	—0,01	—0,18	—0,41
	B	—0,83	—0,80	—0,92	—0,70	—0,58	—0,60
	C	—0,82	—0,74	—0,72	—0,95	—0,98	—1,02

Tr. Note: Commas indicate decimal points.

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Stresses σ_θ in the fourth approximation are presented in Table II.34. These results differ little from the (practically accurate) results of D. I. Sherman [5]. For comparison, the results of D. I. Sherman for the very same points on the real axis as found in Table II.33, are presented in Table II.35.

TABLE II.34

θ°	ε					
	σ_θ/p			σ_θ/q		
	2/5	1/3	1/4	2/5	1/3	1/4
0	—0,92	—0,92	—0,92	3,26	3.16	3,07
15	—0,64	—0,64	—0,65	2,98	2,88	2,80
30	0,08	0,08	0,06	2,23	2,12	2,06
45	1,00	1,00	1,00	1,19	1,11	1,06
60	1,88	1,89	1,91	0,15	0,11	0,07
75	2,50	2,51	2,55	—0,63	—0,63	—0,65
90	2,62	2,65	2,71	—0,92	—0,90	—0,89
105	2,10	2,19	2,34	—0,60	—0,55	—0,56
120	1,12	1,28	1,55	0,26	0,32	0,24
135	0,17	0,35	0,64	1,45	1,43	1,26
150	—0,34	—0,22	—0,08	2,67	2,42	2,19
165	—0,42	—0,40	—0,50	3,62	3,07	2,81
180	—0,38	—0,42	—0,62	3,99	3,29	3,03

Tr. Note: Commas indicate decimal points.

TABLE II.35.

ε	σ_x/q				σ_y/q				σ_x/p				σ_y/p			
	O	A	B	C	O	A	B	C	O	A	B	C	O	A	B	C
1/4	0,48	0,46	0,00	0,00	1,50	1,67	3,02	3,07	0,14	0,06	0,00	0,00	0,06	—0,02	—0,61	—0,91
1/3	0,53	0,44	0,00	0,00	2,25	2,37	3,26	3,15	—0,07	—0,06	0,00	0,00	—0,09	—0,16	—0,38	—0,90
2/5	0,41	0,32	0,00	0,00	3,26	3,43	4,03	3,26	—0,06	—0,04	0,00	0,00	—0,30	—0,32	—0,38	—0,90

Tr. Note: Commas indicate decimal points.

The graphs characterizing the distribution of stresses between the holes and near the right-hand hole, constructed on the basis of the data presented in these tables, are shown in Figure II.83. These graphs pertain to the case where $\varepsilon = 0.4$, i.e., when the distance between the holes is one half the radius of one of them.

To achieve great accuracy, it is necessary, when using the Bubnov-Galerkin method, to increase the number of approximations. Then, naturally, the number of equations in system (II.185) is also increased.

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To determine the desired constants α_k and β_k, it is also easy to obtain an infinite equation system[1]:

$$\alpha_k + (-1)^k \sum_{\nu=1}^{\infty} (-1)^\nu \{\nu\alpha_\nu \varepsilon^{k+\nu} (C_{k+\nu}^k - \varepsilon^2 C_{k+\nu+2}^{k+1}) - \beta_\nu^* \varepsilon^{k+\nu} C_{k+\nu-1}^k\} = \omega_k,$$

$$\beta_k^* + q_k - (-1)^k \sum_{\nu=1}^{\infty} (-1)^\nu \alpha_\nu \varepsilon^{k+\nu} C_{k+\nu-1}^k = t_k, \qquad \text{(II.188)} \quad /144$$

where

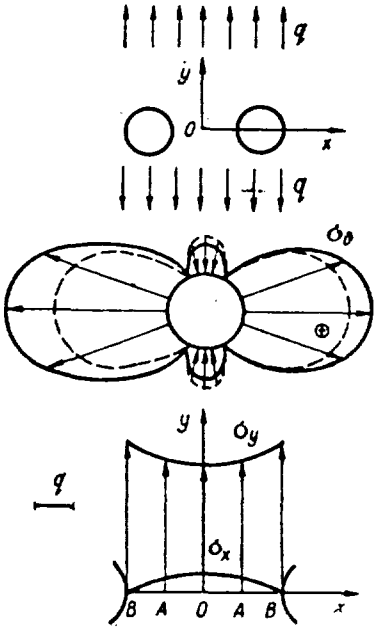


Figure II.83.

$$\varepsilon = \frac{1}{2l}, \qquad q_1 = \sum_{\nu=1}^{\infty} \nu (-1)^\nu \alpha_\nu \varepsilon^{\nu+1},$$

$$\omega_1 = \frac{p-q}{2}, \qquad t_1 = -\frac{p+q}{2},$$

$$\beta_k^* = \beta_k + k\alpha_k - (k-2)\alpha_{k-2}, \qquad \text{(II.189)}$$

$$\omega_k = t_k = q_k = 0 \quad (k \geqslant 2), \quad C_q^r = \frac{q!}{r!(r-q)!}.$$

System (II.188) is written in explicit form, which makes it possible to establish quite easily that it is always quasiregular as long as the given holes are not too close to each other[2]. To prove this fact it is necessary to add up in the infinite sums the factors of the unknown α_k and β_k^* taken with respect to absolute value, and to establish that as the number k increases, the sum of these factors approaches zero. The expressions for the sum of the above factors in the first and second equations (II.188) are:

$$A_{1k} = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k+1} \left[k + 3 - (k+2)\left(\frac{\varepsilon}{1-\varepsilon}\right)^2 - (1-\varepsilon)^{k+1}\right],$$

$$A_{2k} = \delta_k \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 + \left(\frac{1}{1-\varepsilon}\right)^{k+1}. \qquad \text{(II.190)}$$

Here /145

$$\delta_1 = 1, \ \delta_k = 0 \quad (k \geqslant 2).$$

---

[1] V. N. Lozhkin [1] studied the case where the holes are reinforced by absolutely rigid rings, by this method.

[2] See A. S. Kosmodamianskiy [10].

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The value $\varepsilon = 1/2l$ in expressions (II.190) is always less than 1/2, and therefore the values A_{ik} and A_{2k} will approach zero when $k \to \infty$.

It should be noted that if, in system (II.188), we exclude the factors β_k^*, it can be reduced to D. I. Sherman's known system [5].

Three Identical Round Holes. We will place the coordinate system as shown in Figure II.84. The distances between the centers of the holes are l. The loading at infinity is assumed to be the same as in the preceding case.

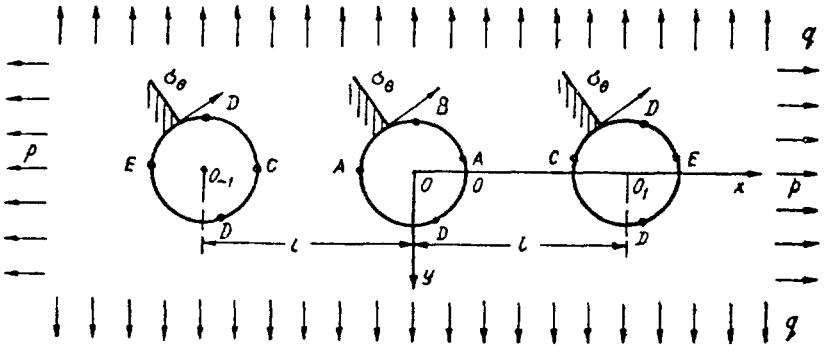


Figure II.84.

Considering geometrical and force symmetry, the functions $\phi(z)$ and $\psi(z)$ are written in the form

$$\varphi(z) = \sum_{k=1}^{\infty}\left\{\alpha_k'\left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right] + \frac{\alpha_k^0}{z^k}\right\},$$
$$\psi(z) = \sum_{k=1}^{\infty}\left\{\beta_k'\left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right] + \frac{\beta_k^0}{z^k}\right\}. \qquad \text{(II.191)}$$

From the boundary conditions on the middle and right hand contours of the holes (on the left hand contours they are automatically satisfied), we obtain the following system for the determination of the factors α_k^0, β_k^0, α_k' and β_k':

$$\alpha_k^0 - [1 + (-1)^{k+1}]\sum_{\nu=1}^{\infty}(-1)^{\nu}\{\nu\alpha_\nu'\varepsilon^{k+\nu}(C_{k+\nu}^k - \varepsilon^2 C_{k+\nu+2}^{k+1}) - \varepsilon^{k+\nu}C_{k+\nu-1}^k\beta_\nu'^*\} = \omega_k,$$

$$\beta_k^0 + 2\delta_k\sum_{\nu=1}^{\infty}(-1)^{\nu}\nu\alpha_\nu'\varepsilon^{\nu+1} + [1 - (-1)^{k+1}]\sum_{\nu=1}^{\infty}(-1)^{\nu}\alpha_\nu'\varepsilon^{k+\nu}C_{k+\nu-1}^k = t_k,$$

$$\alpha_k' + (-1)^k\sum_{\nu=1}^{\infty}\{(-1)^{\nu}\nu\alpha_\nu'\varepsilon_1^{k+\nu}(C_{k+\nu}^k - \varepsilon_1^2 C_{k+\nu+2}^{k+1}) - \nu\alpha_\nu^0\varepsilon^{k+\nu}(C_{k+\nu}^k - \varepsilon^2 C_{k+\nu+2}^{k+1}) - \qquad \text{(II.192)}$$
$$- (-1)^{\nu}\beta_\nu'^*\varepsilon_1^{k+\nu}C_{k+\nu-1}^k + \beta_\nu^{0*}\varepsilon^{k+\nu}C_{k+\nu-1}^k\} = \omega_k,$$

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$$\beta_k'^* + \delta_k \sum_{\nu=1}^{\infty} (-1)^\nu \nu \alpha_\nu' \varepsilon_1^{\nu+1} - \delta_k \sum_{\nu=1}^{\infty} \nu a_\nu^0 \varepsilon^{\nu+1} - (-1)^k \sum_{\nu=1}^{\infty} \{(-1)^\nu \alpha_\nu' \varepsilon_1^{k+\nu} C_{k+\nu-1}^k -$$
$$- \alpha_\nu^0 \varepsilon^{k+\nu} C_{k+\nu-1}^k\} = t_k.$$

Here

$$\beta_k^{0*} = \beta_k^0 + k\alpha_k^0 - (k-2)\alpha_{k-2}^0, \qquad \beta_k'^* = \beta_k' + k\alpha_k' - (k-2)\alpha_{k-2}', \tag{II.193}$$
$$\varepsilon = \frac{1}{l}, \qquad \varepsilon_1 = \frac{1}{2l}.$$

By summing in the infinite sums, the factors of α_k', α_k^0, β_k' and β_k^0, we obtain

$$A_{1k} = 2\left\{\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k+1}\left[k+3-(k+2)\left(\frac{\varepsilon}{1-\varepsilon}\right)^2-(1-\varepsilon)^{k+1}\right]\right\},$$
$$A_{2k} = 2\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k+1} + 2\delta_k\left(\frac{\varepsilon}{1-\varepsilon}\right)^2,$$
$$A_{3k} = \frac{1}{2}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k+1}\left[k+3-(k+2)\left(\frac{\varepsilon}{1-\varepsilon}\right)^2-(1-\varepsilon)^{k+1}\right] +$$
$$+ \frac{1}{2}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{k+1}\left[k+3-(k+2)\left(\frac{\varepsilon}{1+\varepsilon}\right)^2-(1+\varepsilon)^{k+1}\right] + \tag{II.194}$$
$$+ \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{k+1}\left[k+3-(k+2)\left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)^2-(1-\varepsilon_1)^{k+1}\right],$$
$$A_{4k} = \frac{1}{2}\left\{\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k+1} + \left(\frac{\varepsilon}{1+\varepsilon}\right)^{k+1} + \delta_k\left[\left(\frac{\varepsilon}{1+\varepsilon}\right)^2 + \left(\frac{\varepsilon}{1-\varepsilon}\right)^2\right]\right\} +$$
$$+ \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{k+1} + \delta_k\left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)^2.$$

Expressions (II.194) have the same structure as (II.190). They indicate that system (II.192) is quasiregular for any distance between the holes. Stress distribution near the contours of the holes and between the contours is found here in the same manner as during the analysis of a medium with two round holes. However, stress concentration is found to be somewhat different, as seen from the maximum stresses presented below, which occur in an elastic medium with three holes when the medium is under tension, both along the lines of the centers ($p \neq 0$, $q = 0$) and transverse to the lines of the centers ($p = 0$, $q \neq 0$): /147

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ε	1/4	1/3	2/5
$\left(\frac{\sigma_\theta}{p}\right)_{\theta=\pi/2}$	2.42	2.27	2.19
$\left(\frac{\sigma_\theta}{q}\right)_{\theta=0}$	3.10	3.46	4.44

These stresses are related to points B and A, respectively (Figure II.84), of the contour of the inner hole.

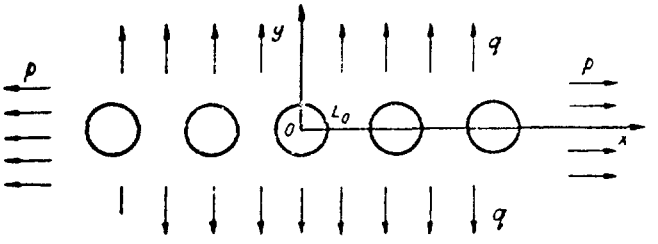


Figure II.85.

Infinite Row of Identical Round Holes. Assume that we have an elastic plane with an infinite row of equal round holes[1], where at infinity, tension is applied in two directions[2]. The origin of the coordinate system is placed at the center of one of the holes, which we will call the basic and the x axis is placed on the center line of the holes (Figure II.85). The distances between the centers of the holes are equal to l.

In the N-th approximation, the functions are

$$\varphi(z) = \sum_{k=1,3,\ldots}^{N} \sum_{n=-\infty}^{\infty} \frac{\alpha_k}{(z-nl)^k},$$
$$\psi(z) = \sum_{k=1,3,\ldots}^{N+2} \sum_{n=-\infty}^{\infty} \frac{\beta_k}{(z-nl)^k}. \qquad \text{(II.195)}$$

Due to the periodicity of the given problem, the factors α_k and β_k are found from the boundary conditions on the contour of the basic hole. They will be satisfied automatically on the other contours.

In the first approximation (N = 1), the system for the determination of /148
the coefficients α_k and β_k will be found in the form

[1]This problem was examined somewhat differently by R. C. J. Howland [2].
[2]The case where the holes are reinforced by elastic rings, or are filled with elastic nuclei, is discussed by A. S. Kosmodamianskiy [13].

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$$\alpha_1(1+4\varepsilon^2P_{11})=\frac{p-q}{2}-\lambda_2\varepsilon^2(p+q),$$

$$\beta_1=-\frac{p+q}{2}-\alpha_1(1-4\lambda_2\varepsilon^2),\quad \beta_3=\alpha_1(1+2\lambda_4\varepsilon^4). \tag{II.196}$$

where

$$\varepsilon=\frac{1}{l},\quad \lambda_{2k}=\sum_{n=1}^{\infty}\frac{1}{n^{2k}},\quad P_{11}=\lambda_2-\varepsilon^2(3\lambda_4+2\lambda_2^2)-3\lambda_4^2\varepsilon^6. \tag{II.197}$$

In the second approximation (N = 2),

$$\begin{aligned}
&\alpha_1(1+4\varepsilon^2P_{12})+6\alpha_3\varepsilon^4P_{32}=\frac{p-q}{2}-\lambda_2\varepsilon^2(p+q),\\
&2\alpha_1\varepsilon^4P_{32}+\alpha_3(1+4\varepsilon^6Q_{32})=-\lambda_4\varepsilon^4(p+q),\\
&\beta_1=-\frac{p+q}{2}-\alpha_1(1-4\lambda_2\varepsilon^2)+12\alpha_3\lambda_4\varepsilon^4,\\
&\beta_3=\alpha_1(1+2\lambda_4\varepsilon^4)-\alpha_3(3-20\lambda_6\varepsilon^6),\\
&\beta_5=2\alpha_1\lambda_6\varepsilon^6+3\alpha_3(1+14\lambda_8\varepsilon^8).
\end{aligned} \tag{II.198}$$

Here

$$\begin{aligned}
P_{32}&=4\lambda_4-\varepsilon^2(15\lambda_6+4\lambda_2\lambda_4)-20\lambda_4\lambda_6\varepsilon^8-70\lambda_6\lambda_8\varepsilon^{10},\\
Q_{32}&=30\lambda_6-\varepsilon^2(105\lambda_8+6\lambda_4^2)-100\lambda_6^2\varepsilon^6-735\lambda_8^2\varepsilon^{10}.
\end{aligned} \tag{II.199}$$

As will be pointed out below, the second approximation insures results that are close to accurate, if the distances between the holes are greater than or equal to one half the radius of one of the holes.

In examining a medium with holes that are located closer together, it is necessary to use the following approximations by obtaining, for the determination of constants α_k and β_k, an abbreviated system from the following infinite system:

$$\begin{aligned}
&\alpha_k+\sum_{m=1}^{\infty}(-1)^m\varepsilon^{k+m}\{\alpha_m[\varepsilon^2\lambda_{k+m+2}((k+2)C_{k+m+1}^{k+2}+\\
&+mC_{k+m+1}^{k})-(k+m)\lambda'_{k+m}C_{k+m-1}^{k}]+B_m^*\lambda_{k+m}C_{k+m-1}^{k}\}=\omega_k,\\
&\beta_k^*+\sum_{m=1}^{\infty}(-1)^m\alpha_m\varepsilon^{m+1}(\varepsilon^{k-1}\lambda'_{k+m}C_{k+m-1}^{k}+\delta_k\alpha_{m+1}m)\}=t_k.
\end{aligned} \tag{II.200}$$

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Here $\lambda_k' = 2\lambda_k$.

System (II.200), as were the infinite systems (II.188) and (II.192) examined earlier, is quasiregular for any distance between the holes[1]. /149

The character of stress distribution in the elastic plane with an infinite number of identical round holes, between the holes and near them, is the same as in plates with two and three holes. Therefore, only the maximum stresses that occur near the basic hole are presented in Table II.36.

TABLE II.36

s	$(\sigma_\theta/p)_{\theta=\pi/2}$ for various approximations			$(\sigma_\theta/q)_{\theta=0}$ for various approximations		
	I	II	III	I	II	III
1/2,2	1,45	1,76	1,75	8,69	10,39	11,15
1/2,5	1,52	1,85	1,81	5,47	5,64	5,71
1/3	1,70	1,96	1,92	4,03	3,95	3,93
1/4	2,09	2,15	2,14	3,33	3,26	3,24

Tr. Note: Commas indicate decimal points.

Concluding Comments. By comparing the results obtained for planes with two, three, and an infinite row of round holes, it is possible to determine the effect of the number of holes and the distance between them on stress concentration.

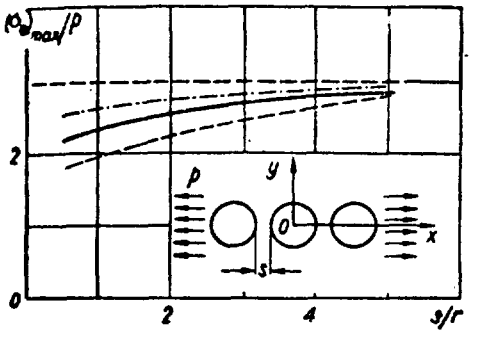


Figure II.86.

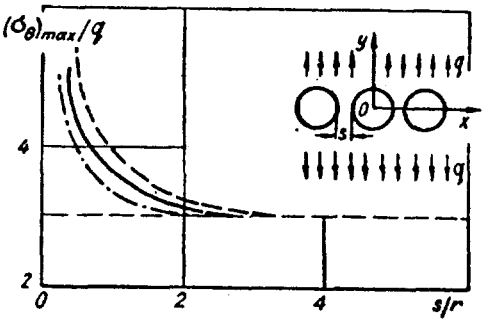


Figure II.87.

Figures II.86 and II.87 represent the graphs that characterize the change of the maximum stress σ_θ on the contour of one of the holes as the distance between the holes changes, where, at infinity, tension forces act along and transverse to the central line: the dot-dash lines correspond to a plane

[1]See A. S. Kosmodamianskiy [10].

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weakened by two holes; the solid lines correspond to three (σ_θ is examined on the inner holes); the broken lines represent planes with an infinite row of equal holes. The curves on these graphs show that the holes have a considerable effect on each other when the distance between them is less than the diameter of one of them. When an elastic plane is under tension along the center line of the holes (Figure II.86), the stress concentration slowly diminishes as the holes become closer to each other, whereupon the degree of decrease increases as the number of holes increases. If, on the other hand, the elastic plane is under tension transverse to the central line of the holes (Figure II.87), the opposite pattern occurs: as the holes become closer to each other, the stress concentration increases, whereupon it increases more rapidly as the number of holes increases. /150

§9. Stresses in an Elastic Plane Weakened by Two Unequal Holes.

In analyzing the stress state in an elastic isotropic plane weakened by two unequal round holes, it is possible to use bipolar coordinates, which method yields an accurate solution of the stated problem. Such solutions for various partial problems were found by Ya. S. Podstrigach [2, 3], M. A. Savruk [1-3], Mori Kyohei [1] and others.

D. I. Sherman [5], in solving problems for multiply-connected regions, used the approximation method and reduced them to the solution of an infinite algebraic system, the equations of which were quasiregular for any distance between the holes. This method was also used by D. I. Sherman [6] for the problem where one of the holes is round and the other is elliptic. Many interesting problems have been analyzed by L. N. Kisler [1, 2], N. P. Moshkin [1, 2] and others, by D. I. Sherman's method.

If unequal holes are located close to each other, the solution of the problem of the stress state for such a multiply-connected region poses considerable difficulties. However, as A. S. Kosmodamianskiy pointed out [2], in the case where the distance between the holes exceeds two diameters of the small hole, it is possible to obtain rather simple results by using for the solution of the stated problems, Schwartz method of series approximations.

Two Unequal Round Holes. Let an elastic isotropic plate be weakened by two unequal round holes, the centers of which lie on a straight line coinciding with the x axis (Figure II.88). The distance between the contours of the holes is S, and between the centers, l. The radii of the small and large holes are $r = 1$ and $R(R > 1)$. The contour of the large hole is denoted through L_1, and of the small hole, through L_0. At infinity, tension forces p and q are given as indicated in Figure II.88.

The approximate solution of this problem in the first approximation, using Schwartz method, was found by A. S. Kosmodamianskiy [2]. The Schwartz method for the solution of the given problem consists in the fact that the solution for the infinite plane with one large round hole is first found by the known method of N. I. Muskhelishvili (see §1, Chapter I). This solution yields /151

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non-zero values $\overline{X}_n$ and $\overline{Y}_n$ on the contour of the small hole. Further, by the same method, the solution for an infinite plate with one small hole, upon the contour of which are applied external forces $(-X_n)$ and $(-Y_n)$, is found. The determination of these two stress fields also yields the desired solution of the problem in the first approximation[1].

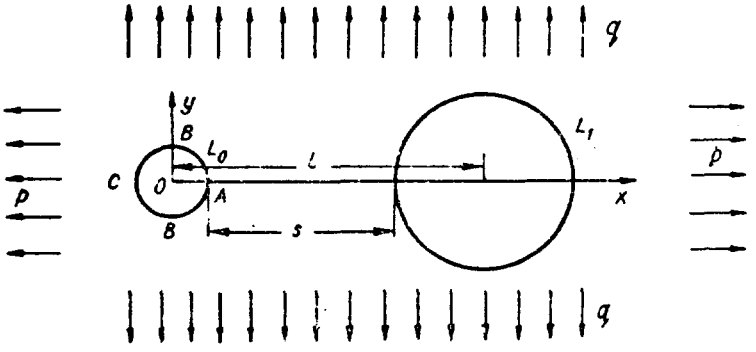


Figure II.88.

For a plate under tension by forces q transverse to the central line, i.e., when $q \neq 0$ and $p = 0$, the stresses near the holes are found from the following equations:

$$\sigma_x + \sigma_y = q + 2q\,\mathrm{Re}\left\{\frac{R^2}{l^2} + \frac{1}{z^2} + \frac{R^2}{(z-l)^2} + \frac{R^2(1+zl-3z^2+lz^3)}{(1-lz)^3} + \frac{3R^4z^2}{(1-lz)^4}\right\},$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = q\left\{1 + \frac{3R^4}{(z-l)^4} + \frac{R^2(z-2\bar{z}+l)}{(z-l)^3} + 2\left(\bar{z} - \frac{1}{z}\right)\left[-\frac{1}{z^3} + \right.\right. \tag{II.201}$$

$$+ \frac{3R^4z(1+lz)}{(1-lz)^5} + \frac{R^2(2l-3z+l^2z)}{(1-lz)^4}\Big] + \frac{1}{z^2}\left[1 + \frac{2R^2}{l^2} + \frac{1}{z^2} + \right.$$

$$\left.\left. + \frac{R^2(1+zl-3z^2+lz^3)}{(1-lz)^3} + \frac{3R^4z^2}{(1-lz)^4}\right] + \frac{R^2}{(1-lz)^2}\right\}.$$

Investigation of the stress state in a plane weakened by two unequal holes has shown that stresses σ_θ, that act on the contour of the small hole L_0 are most important. As concerns stress distribution in the plane near the large hole, it is found in about the same way as in a medium with one hole (a small hole has little effect on the stress state of the medium near a large hole). /152

[1] It is easy to see that in this manner the conditions at infinity and the boundary conditions are satisfied only on the contour of the small round hole L_0. Obviously, deviation in the values of X_n and Y_n from zero on contour L_1 will show the degree of approximation of the solution found.

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The stresses for the points of contour L_0 are

$$\sigma_\theta = q\left(1 + \frac{2R^2}{l^2}\right) + 2q\,\mathrm{Re}\left[\frac{1}{\sigma^2} + \frac{R^2}{(\sigma - l)^2} + \frac{R^2(1 + \sigma l - 3\sigma^2 + l\sigma^3)}{(1 - l\sigma)^4} + \frac{3R^4\sigma^2}{(1-l\sigma)^4}\right], \qquad \text{(II.202)}$$

where

$$\sigma = e^{i\theta} = \cos\theta + i\sin\theta.$$

The stresses at the points A($\theta = 0$), B($\theta = \pi/2$) and C($0 = \pi$) (see Figure II.88) are

$$\begin{aligned}
(\sigma_\theta)_{\theta=0} &= 2q\,[1{,}5 - R^2\varepsilon^2(2\varepsilon + 3\varepsilon^2 + 4\varepsilon^3 + 5\varepsilon^4) + 3R^4\varepsilon^4(1 + 4\varepsilon + 10\varepsilon^2 + 20\varepsilon^3 + 35\varepsilon^4)],\\
(\sigma_\theta)_{\theta=\frac{\pi}{2}} &= 2q\,[-0{,}5 + R^2\varepsilon^2(2 - 15\varepsilon^2 + 45\varepsilon^4) - 3R^4\varepsilon^4(1 - 10\varepsilon^2 + 35\varepsilon^4)],
\end{aligned} \qquad \text{(II.203)}$$

$$(\sigma_\theta)_{\theta=\pi} = 2q\,[1{,}5 + R^2\varepsilon^2(2\varepsilon - 3\varepsilon^2 + 4\varepsilon^3 - 5\varepsilon^4) + 3R^4\varepsilon^4(1 - 4\varepsilon + 10\varepsilon^2 - 20\varepsilon^3 + 35\varepsilon^4)].$$

Here

$$\delta = \frac{1}{l} \ll \frac{1}{2}.$$

Formulas (II.203) are found by keeping the first three components in the expansions of (II.202) into a series for the small parameter ε.

For a plate under tension (Figure II.89) along the center line of the holes ($q = 0$, $p \neq 0$):

$$\begin{aligned}
\sigma_x + \sigma_y &= p - 2p\,\mathrm{Re}\left\{\frac{R^2}{l^2} + \frac{1}{z^2} + \frac{R^2}{(z-l)^2} - \frac{R^2(1 - 3lz + 3z^2 - lz^3)}{(1-lz)^3} + \frac{3R^4z^2}{(1-lz)^4}\right\},\\
\sigma_y - \sigma_x + 2i\tau_{xy} &= p\left\{-1 - \frac{3R^4}{(z-l)^4} + \frac{R^2(z - 2\bar{z} - 3l)}{(z-l)^3} + \right.\\
&+ 2\left(\bar{z} - \frac{1}{z}\right)\left[\frac{1}{z^3} + \frac{3R^2z(1-l^2)}{(1-lz)^4} - \frac{3zR^4(1+lz)}{(1-lz)^5}\right] + \frac{1}{z^2}\left[1 - \frac{1}{z^2} - \right.\\
&\left.\left. - \frac{2R^2}{l^2} + \frac{R^2(1 - 3lz + 3z^2 - lz^3)}{(1-lz)^3} - \frac{3R^4z^2}{(1-lz)^4}\right] - \frac{R^2}{(1-lz)^2}\right\}.
\end{aligned} \qquad \text{(II.204)}$$

At the points of contour L_0 (Figure II.88),

$$\sigma_\theta = p\left(1 - \frac{2R^2}{l^2}\right) - 2p\,\mathrm{Re}\left[\frac{1}{\sigma^2} + \frac{R^2}{(\sigma - l)^2} - \frac{R^2(1 - 3l\sigma^2 + 3\sigma^2 - l\sigma^3)}{(1 - l\sigma)^3} + \frac{3R^4\sigma^2}{(1-l\sigma)^4}\right]. \qquad \text{(II.205)}$$

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By expanding σ_θ (II.205) into a series for the small parameter ε and maintaining the same accuracy as before, we find stresses σ_θ at the points A, B and C: /153

$$(\sigma_\theta)_{\theta=0} = -2p[0{,}5 - R^2\varepsilon^2(2 + 6\varepsilon + 9\varepsilon^2 + 12\varepsilon^3 + 15\varepsilon^4) + 3R^4\varepsilon^4(1 + 4\varepsilon + 10\varepsilon^2 + 20\varepsilon^3 + 35\varepsilon^4)],$$

$$(\sigma_\theta)_{\theta=\frac{\pi}{2}} = p\{1 + 2[1 - R^2\varepsilon^2(4 - 21\varepsilon^2 + 55\varepsilon^4) + 3R^4\varepsilon^4(1 - 10\varepsilon^2 + 35\varepsilon^4)]\}, \quad \text{(II.206)}$$

$$(\sigma_\theta)_{\theta=\pi} = -2p[0{,}5 - R^2\varepsilon^2(2 - 6\varepsilon + 9\varepsilon^2 - 12\varepsilon^3 + 15\varepsilon^4) + 3R^4\varepsilon^4(1 - 4\varepsilon + 10\varepsilon^2 - 20\varepsilon^3 + 35\varepsilon^4)].$$

It should be mentioned that in the case of universal stress (p = q), the formula for the determination of the stresses at the points of contour L_0 is found to be much simpler:

$$\sigma_\theta = 2q\left[1 + 2\,\mathrm{Re}\,\frac{R^2}{(1 - l\sigma)^2}\right]. \quad \text{(II.207)}$$

The stresses at the points A, B and C (see Figure II.88) are

$$(\sigma_\theta)_{\theta=0} = 2q\left[1 + \frac{2R^2}{(1-l)^2}\right], \qquad (\sigma_\theta)_{\theta=\frac{\pi}{2}} = 2q\left[1 + \frac{2R^2(1-l^2)}{(1+l^2)^2}\right],$$

$$(\sigma_\theta)_{\theta=\pi} = 2q\left[1 + \frac{2R^2}{(1+l)^2}\right]. \quad \text{(II.208)}$$

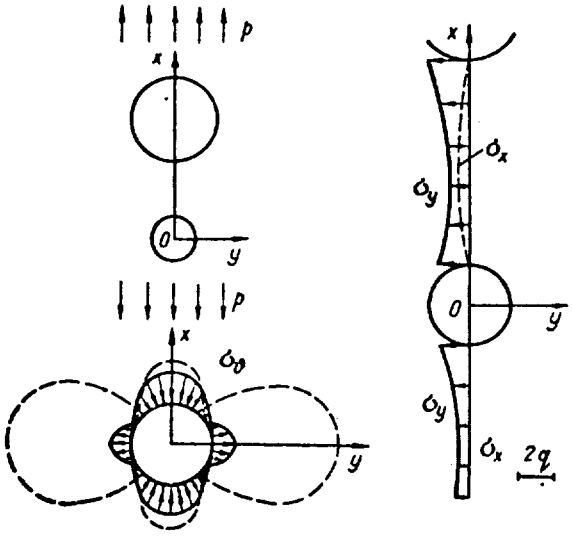


Figure II.89.

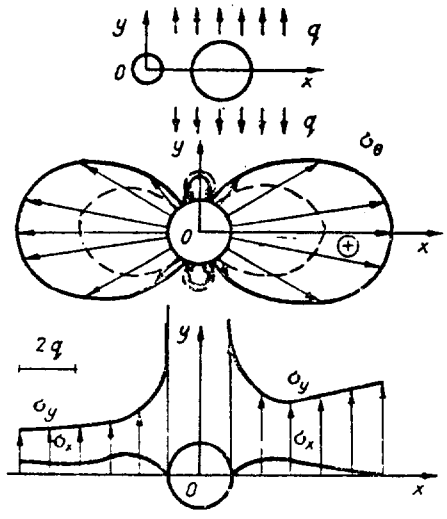


Figure II.90.

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The stresses σ_θ at the points A, B and C for the case of universal tension (p = q) are presented in Tables II.37 and II.38). The values of σ_θ/q at the above three points (see Figure II.88) are presented in Table II.37 as functions of s/r, where the radius of the large hole is 20 times greater than the radius of the small hole. The same values of σ_θ/q are presented in Table II.38 as functions of change (increase) of the radius of the large hole R, where the distance between the holes remains constant and equal to 5r. /154

TABLE II.37

z	s/r						
	5	10	15	20	30	50	100
1	4,56	3,78	3,31	3,00	2,64	2,32	2,01
$\pm i$	—0,36	0,34	0,77	1,05	1,39	1,68	1,89
—1	4,19	3,56	3,17	2,91	2,59	2,31	2,01

Tr. Note: Commas indicate decimal points.

TABLE II.38

z	R/r							
	1	2	3	5	10	20	30	100
1	2,11	2,33	2,56	3,00	3,78	4,56	4,94	5,63
$\pm i$	1,92	1,76	1,54	1,19	0,46	—0,36	—0,77	—1,56
—1	2,06	2,20	2,36	2,69	3,38	4,19	4,63	5,49

Tr. Note: Commas indicate decimal points.

The graphs characterizing the stress distribution σ_θ on the contour L_0 and through cross section y = 0 between the holes for R/r = 20 and s/r = 5 are presented in Figures II.89 and II.90. The broken lines pertain to a medium weakened by one hole L_0, and are given for comparison. These graphs show the effect of the large hole on the stress state near the small hole.

Elliptic and Square Holes. Consider an elastic isotropic plane weakened by two unequal curvilinear holes, one of which is elliptic and the other is square with rounded corners. We will assume that the large axis of the elliptic hole is much larger than the distance between the points of the contour of the square hole that are farthest apart. We will assume that the arrangement of the holes is the same as illustrated in Figure II.91. The contour of the elliptic hole is denoted through L_0, and the contour of the square hole, through L_1. At infinity, as before, we will assume that the elastic plate is

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under tension in two directions by forces p and q. In solving the stated problem by Schwartz approximation method, as was done above, we will confine ourselves to the first approximation[1].

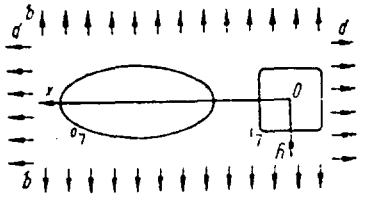


Figure II.91.

The functions that conformally map the exterior of unit circle γ on the exterior of the elliptic and square holes, respectively, are taken in the form /155

$$z - l = R\left(\zeta + \frac{m}{\zeta}\right). \qquad \text{(II.209)}$$

$$z = \omega(\zeta) = R_1\left(\zeta + \frac{m_3}{\zeta^3}\right). \qquad \text{(II.210)}$$

Here R = a + b/2, m = a - b/a + b (a and b are the semi-axes of the ellipse), $m_3 = -1/9$, and R_1 is a constant that governs the dimensions of the square hole.

As in the problem with two unequal round holes, the most important stresses are those near the square hole L_1. In the case of tension (Figure II.91) transverse to the center line of the holes ($q \neq 0$, $p = 0$), the stresses on the contour L_1 are

$$\sigma_\vartheta = q - 2qR(1+m)\{A_1 + 2A_2R_1(\cos\vartheta + m_3\cos 3\vartheta) + 3A_3R_1^2[(1+2m_3)\cos 2\vartheta + m_3^2\cos 6\vartheta]\} - \frac{2q}{R_1C^2}\left\{A\sum_{k=1}^{9}kM_k\cos k\vartheta - B\sum_{k=1}^{9}kM_k\sin k\vartheta\right\}, \qquad \text{(II.211)}$$

where

$$M_1 = \frac{1}{1-m_3}\{-R_1 + R(1+m)[3A_3R_1^3(1+m_3+2m_3^2) + R_1(B_1 - A_1)]\},$$

$$M_2 = RR_1^2(1+m)[B_2 + 2A_2(m_3 - 1)],$$

$$M_3 = -R_1m_3 + RR_1(1+m)[2A_1m_3 + (B_3 - 3A_3)R_1^2],$$

$$M_4 = 2RR_1^2(1+m)A_2m_3, \qquad M_5 = 3RR_1^2(1+m)(1+m_3)m_3A_3, \qquad \text{(II.212)}$$

$$M_6 = RR_1^2(1+m)A_2m_3^2, \qquad M_7 = M_8 = 0, \qquad M_9 = RR_1^3A_3m_3^3(1+m),$$

$$A = \cos\vartheta - 3m_3\cos 3\vartheta, \qquad B = \sin\vartheta + 3m_3\sin 3\vartheta, \qquad C^2 = A^2 + B^2.$$

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$$A_1 = -R^{-1}(q_0^2 - m)^{-1}, \qquad A_2 = q_0^3R^{-2}(q_0^2 - m)^{-3},$$

$$A_3 = -q_0^4(q_0^2 + m)(q_0^2 - m)^{-5}R^{-3},$$

[1]The solution of the problem is given by A. S. Kosmodamianskiy [9].

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$$B_1 = -[mq_0^4 + (3 - 6m + m^2)\, q_0^2 + m](q_0^2 - m^{-3})\, R^{-1},$$

$$B_2 = q_0^3[mq_0^4 + (6 - 12m + 4m^2)\, q_0^2 + m^3](q_0^2 - m)^{-5}R^{-2},$$

$$B_3 = -q_0^4[mq_0^6 + (10 - 20m + 9m^2)\, q_0^2(q_0^2 + m) + m^4](q_0^2 - m)^{-7}R^{-3}, \qquad \text{(II.213)}$$

$$q_0 = -\frac{l}{2R}(1 + \sqrt{1 - 4mR^2l^{-2}}).$$

If, however, the elastic plane is under tension along the center line ($p \neq 0$, $q = 0$), then

$$\sigma_\vartheta = p + 2pR(1 - m)\{A_1 + 2A_2R_1(\cos\vartheta + m_3\cos 3\vartheta) +$$
$$+ 3A_3R_1^2[(1 + 2m_3)\cos 2\vartheta + m_3^2\cos 6\vartheta]\} -$$
$$- \frac{2p}{R_1C^2}\left\{A\sum_{k=1}^{9} kM_k^*\cos k\vartheta - B\sum_{k=1}^{9} kM_k^*\sin k\vartheta\right\}, \qquad \text{(II.214)}$$

where

$$M_1^* = \frac{R_1}{1 - m_3}\{1 - R(1 - m)[3A_3R_1^2(1 + m_3 + 2m_3^2) - A_1 - B_1^*]\},$$

$$M_2^* = -RR_1^2(1 - m)[2A_2(m_3 - 1) - B_2^*],$$

$$M_3^* = -R_1m_3 - R(1 - m)[2A_1m_3 - R_1^2(B_3^* + 3A_3)]R_1,$$

$$M_4^* = -2RR_1^2m_3A_2(1 - m), \qquad \text{(II.215)}$$

$$M_5^* = -3RR_1^3m_3A_3(1 - m)(1 + m_3),$$

$$M_6^* = -RR_1^2m_3^2A_2(1 - m),$$

$$M_7^* = M_8^* = 0,$$

$$M_9^* = -RR_1^3m_3^3A_3(1 - m).$$

Here

$$B_1^* = [q_0^4(m - 2) + q_0^2(3 - 2m + m^2) - m](q_0^2 - m)^{-3}R^{-1},$$

$$B_2^* = [q_0^7(2 - m) + 2q_0^5(3 - 4m + 2m^2) + q_0^4m^2(2 - m)](q_0^2 - m)^{-5}R^{-2}, \qquad \text{(II.216)}$$

$$B_3^* = [q_0^{10}(m - 2) + (q_0^8 + mq_0^6)(10 - 18m + 9m^2) + q_0^4m^3(m - 2)](q_0^2 - m)^{-7}R^{-3}.$$

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Stresses σ_ϑ on the contour of the square hole are represented in Figure II.92 for the case where a/b = 20, R/b - 10.5, m = 19/21, R_1/b = 1.125, m_3 = = -1/9, l/b = 26. As before, the broken lines denote σ_ϑ for the case where the plane is weakened by only one square hole.

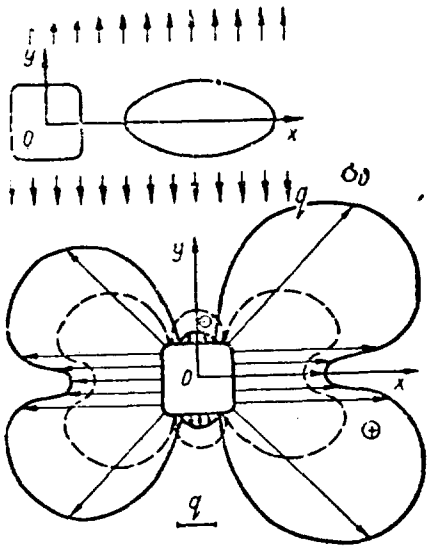


Figure II.92.

Concluding Comments. The above analyses show that in an elastic plane weakened by two unequal holes, the small hole has little effect on the stress state near the large hole. The latter can have a considerable effect on the stress state near the small hole if the distance between the holes is less than half the diameter of the large hole.

When the plate is under tension transverse to the center line (Figure II.92), the stress concentration near the small hole increases sharply. When the plane is under tension along the center line in the case of two round holes (Figure II.89), the stress concentration near the small hole decreases sharply; however, in the case where the large hole is elliptic, the stress state of the medium near the small hole changes insignificantly in comparison with the case where the plane is weakened by one small hole.

The stress state in an infinite plate weakened by two equal round holes under different conditions at infinity and on the contour of the hole has been examined in the works of V. V. Matviyenko [1], N. N. Penin [1], M. A. Savruk [1], A. Atsumi [1-3], Miyao Kadzyu [1, 2], A. M. Sen Gupta [2], Z. Khokao [1], V. V. Yeganyan [1]. The stress state in an infinite plate weakened by two unequal round holes has been analyzed by V. M. Zabludovskiy [1], Ya. S. Podstrigach [2, 3], M. A. Savruk [2-5], M. Z. Narodetskiy [1-3], A. S. Lokshin [2], G. A. Davies, I. R. Hoddinott [1], J. N. Sckhri [1], and Yu. A. Ustinov [1]. The stress state in a plate with several round holes is analyzed in the works of V. N. Buyvol [1, 2], Isida Makoto [1, 2], H. Kraus [1], Okabayasi [1], M. Z. Narodetskiy [5], Peter P. Radkowski [1], Saito Hideo [1], Tan-Li-Min [1, 2], Saito [1], in the monograph by L. E. Hulbert [1], and L. E. Hulbert, F. W. Niedenfuhr [1]. An infinite plate, under tension, containing an infinite /158 number of elliptic holes, was analyzed by Nisitani Hironobu [1, 2]. The problem of stress concentration in a plate under tension, weakened by several rectangular holes, was analyzed by S. K. Roy [1].

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§10. Stresses in Elastic Plate Weakened by Curvilinear Holes

Infinite Row of Identical Curvilinear Holes[1]. We will assume that an elastic medium occupies plane Oxy, in which there is an infinite row of congruent notches of arbitrary shape[2], arranged such that each sequential notch is made by translating the preceding by 2π along the Ox axis. External forces X_n and Y_n, statistically equivalent to zero on each contour, and which are equal at the corresponding points, are applied to the contour of each of the notches. We will place the origin of the coordinate system within one of the contours and mentally break the entire Oxy plane into an infinite row of bands of width 2π by straight lines $y = -3\pi$, $y = -\pi$; $y = -\pi$, $y = \pi$; $y = \pi$, $y = 3\pi$, etc., considering that each notch lies completely within the corresponding band. The stress components and deformation components will be periodic functions with period 2π. It is therefore sufficient to examine them in the band $\pi \leqslant \operatorname{Re} z \leqslant +\pi$ with the given notch[3].

Thus, the solution of the stated problem has been reduced to the determination of the stress and deformation components in the band with the given hole, along the contour l of which are given external forces X_n and Y_n. Through S we will denote the region bounded by straight lines $y = -\pi$, $y = \pi$ and contour l of the notch, and through S*, the region within l.

The solution of the plane problem of elasticity theory for the given external forces X_n and Y_n on the contour reduces to the determination of two analytical functions of complex variable $\phi(z)$ and $\psi(z)$, which satisfy contour conditions (I.9):

$$\overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) = -i\int_0^s (X_n - iY_n)\,ds + \overline{C} = f_1 - if_2, \qquad \text{(II.217)}$$

where C is an arbitrary complex constant.

The stress components σ_x, σ_y and τ_{xy} and deformation components u and v are defined by the functions $\phi(z)$ and $\psi(z)$ from formulas (I.10) and (I.13):

$$\sigma_x + \sigma_y = 4\operatorname{Re}\{\varphi'(z)\},$$

---

[1] See G. N. Savin [6]; the same problem has been solved by somewhat different methods by D. I. Sherman [7, 8], I. I. Vorovich and A. S. Kosmodamianskiy [1], and A. S. Kosmodamianskiy [7, 8].

[2] The case of round holes in the uniaxial homogeneous stress state at infinity was examined by R. C. J. Howland [2]; the basic results of this article are presented in the monograph of G. N. Savin [13].

[3] This problem has a unique solution under conditions that the stress components σ_x, σ_y, τ_{xy}, for $y \to \infty$, uniformly approach zero, while the deformation components u, v, are finite values.

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$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{II.218}$$ /159

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{II.219}$$

Since the left hand sides of equations (II.218) and (II.219), as follows from the statement itself of the problem, are periodic functions of period 2π, then the functions $\phi(z)$ and $\psi(z)$ must have the form

$$\varphi(z) = \varphi_0(z), \qquad \psi(z) = \psi_0(z) - z\varphi_0'(z), \tag{II.220}$$

where $\phi_0(z)$ and $\psi_0(z)$ are periodic functions of period 2π.

Contour condition (II.217) and formulas for stresses and deformations (II.218) and (II.219) in functions $\phi_0(z)$ and $\psi_0(z)$ will have the form

$$\overline{\varphi_0(z)} + (\bar{z} - z)\varphi_0'(z) + \psi_0(z) = f_1 - if_2, \tag{II.221}$$

$$\sigma_x + \sigma_y = 4\mathrm{Re}\{\varphi_0'(z)\},$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[(z - \bar{z})\varphi_0''(z) + \psi_0'(z) - \varphi_0'(z)], \tag{II.222}$$

$$2\mu(u + iv) = \varkappa\varphi_0(z) - (z - \bar{z})\overline{\varphi_0'(z)} - \overline{\psi_0(z)}. \tag{II.223}$$

Functions $\phi_0(z)$ and $\psi_0(z)$ are defined[1] with an accuracy up to expression

$$\varphi_0(z) = C_1^*, \qquad \psi_0(z) = -\bar{C}_1^*, \tag{II.224}$$

where C_1^* is an arbitrary complex constant.

We will map out the band $-\pi \leqslant \mathrm{Re}\ z \leqslant +\pi$ on the plane with the notch on the negative part of the real Ox axis using the function

$$w = e^{iz}. \tag{II.225}$$

The contour of the hole is transformed into some closed contour L which does not contain coordinates within it. Through Σ, we will denote the infinite region beyond contour L of the notch, and through Σ^*, the region within L.

The functions $F_1(w) = \phi_0(-i \ln w)$ and $F_w(w) = \psi_0(-i \ln w)$ are holomorphic in the domain of Σ and satisfy, on contour L, the condition

---

[1]This is clear from the theorem of identity and from the periodicity of the given functions.

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$$\overline{F_1(w)} - w\overline{(\ln w}} + \ln w) F_1'(w) + F_2(w) = \overline{f_1} - i\overline{f_2}. \tag{II.226}$$

To determine the functions $F_1(w)$ and $F_2(w)$, we will use N. I. Muskhelishvili's method for reducing the problem to functional equations[1]. We will multiply both halves of equation (II.226) by 1/2 i dw/w - w_e, where w_e is a point of domain Σ^*, and we will integrate with respect to contour L in the positive direction relative to domain : /160

$$\frac{1}{2\pi i}\int_L \frac{\overline{F_1(w)}}{w - w_e}dw - \frac{1}{2\pi i}\int_L \frac{w(\ln \bar{w} + \ln w)}{w - w_e} F_1'(w)dw = \\ = \frac{1}{2\pi i}\int_L \frac{f_1 - if_2}{w - w_e}dw = A_1 - iA_2 + \bar{C}. \tag{II.227}$$

As $w_e \to w_0$ in (II.227), where w_0 is some point on contour L, by combining (II.227) with equations

$$-\frac{1}{2}\overline{F_1(w)} - \frac{1}{2\pi i}\int_L \frac{\overline{F_1(w)}}{\bar{w} - \bar{w}_e}d\bar{w} = 0,$$

$$-\frac{w_0(\overline{\ln w_0} + \ln w_e)}{2} F_1'(w_0) + \frac{w_0(\overline{\ln w_0} + \ln w_e)}{2\pi i}\int_L \frac{F_1'(w)}{w - w_e}dw = 0, \tag{II.228}$$

we obtain, after simple transformations:

$$\overline{F_1(w_0)} - \frac{1}{2\pi i}\int_L \overline{F_1(w)}d\left\{\ln\frac{w - w_0}{\bar{w} - \bar{w}_0}\right\} + \\ + \frac{1}{2\pi i}\int_L F_1(w)\,d\left\{\frac{w_0 \ln(\bar{w}_0 w_0) - w\ln(\bar{w}w)}{w - w_0}\right\} = -A_1^{(e)} + iA_2^{(e)} - \bar{C}, \tag{II.229}$$

where $A_1^{(e)} - iA_2^{(e)} + \bar{C}$ is the boundary of the right hand side of equation (II.227) as the point w_e approaches point w_0 on contour L outside of it. Equation (II.229) represents Fredholm's integral equation of the second kind with a regular kernel. Generally speaking, however, it is not solvable, since the homogeneous equation corresponding to (II.229) has a nontrivial solution. We

[1]See §1, Chapter I.

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will alter somewhat our equation (II.229) by adding to its left hand side the operator[1]

$$i\overline{\ln w}\,\mathrm{Re}\left\{\frac{b}{2\pi}\int_L \frac{F_1(w)\,dw}{(w-b)^2}\right\}, \tag{II.230}$$

where b is some point within Σ^*.

And so, the equation that solves the problem stated by us will have the form

$$\begin{aligned}&\overline{F_1(w)} - \frac{1}{2\pi i}\int_L \overline{F_1(w)}\,d\left\{\ln\frac{w-w_0}{\bar{w}-\bar{w}_0}\right\} + \\ &+\frac{1}{2\pi i}\int_L F_1(w)\,d\left\{\frac{w_0\ln(\bar{w}_0w_0) - w\ln(\bar{w}\,w)}{w-w_0}\right\} + \\ &+ i\overline{\ln w}\,\mathrm{Re}\left\{\frac{b}{2\pi}\int_L \frac{F_1(w)\,dw}{(w-b)^2}\right\} = -A_1^{(e)} + iA_2^{(e)} - \bar{C}_1.\end{aligned} \tag{II.231}$$

Analysis of equation (II.231) can be carried out in the same manner[2]. We will assume that there exists a solution of equations (II.231); if we introduce into our analysis the functions /161

$$\Phi(w_e) = \frac{1}{2\pi i}\int_L \frac{F_1(w)}{w-w_e}\,dw \tag{II.232}$$

and

$$\Psi(w_e) = -\frac{1}{2\pi i}\int_L \frac{\overline{F_1(w)}}{w-w_e}\,dw + \frac{1}{2\pi i}\int_L \frac{w\ln(\bar{w}w)}{w-w_e}F_1'(w)\,dw + A_1 - iA_2 + \bar{C}, \tag{II.233}$$

we arrive at the equation

$$\overline{\Phi(w)} + w_0(\overline{\ln w_0} + \ln w_0)\,\Phi'(w) - \Psi(w) + i\overline{\ln w}\,\mathrm{Re}\left\{\frac{b}{2\pi}\int_L \frac{F_1(w)}{(w-b)^2}dw\right\} = 0. \tag{II.234}$$

The functions $\Phi(w)$ and $\Psi(w)$, as follows from (II.232) and (II.233), are holomorphic in domain Σ^*.

Equation (II.234) in variable z has the form

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[1]See D. I. Sherman [8, 11].
[2]See N. I. Muskhelishvili [1].

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$$\overline{F(z)} + (z - \bar{z})F'(z) - G(z) + \bar{z}\,\mathrm{Re}\,\{F'(a)\} = 0, \tag{II.235}$$

where

$$\Phi(w) = F(z), \qquad \Psi(w) = G(z), \qquad a = -i\ln b.$$

By multiplying both sides of equation (II.235) by dz and integrating with respect to contour L, we obtain

$$\mathrm{Re}\,\{F'(a)\} = 0, \tag{II.236}$$

whence

$$\overline{F(z)} + (z - \bar{z})F'(z) - G(z) = 0. \tag{II.237}$$

Introducing the definitions

$$F(z) = iF^*(z), \qquad G(z) - zF'(z) = iH^*,$$

we represent equation (II.237) in the form

$$\overline{F^*(z)} + \bar{z}F'^*(z) + H^*(z) = 0. \tag{II.238}$$

Equation (II.238) corresponds to the contour problem of the theory of elasticity with zero external forces on the contour. According to the theorem of identity[1] of the functions

$$F^*(z) = iC_1 z + C_2, \qquad H^*(z) = -\bar{C}_2, \tag{II.239}$$

where C_1 is a real constant; C_2 is complex. Consequently,

$$F(z) = -C_1 z + iC_2, \qquad G(z) = -C_1 z - i\bar{C}_2. \tag{II.240}$$

From (II.236) and (II.240), C_1 is identically equal to zero, i.e., /162

$$F(z) = iC_2, \qquad G(z) = -i\bar{C}_2. \tag{II.241}$$

From equations (II.232), (II.233) and (II.241)

[1] See N. I. Muskhelishvili [1], p. 118.

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$$\Phi(w_e) = \frac{1}{2\pi i}\int_L \frac{F_1(w)}{w - w_e}\,dw = iC_2,$$

$$\Psi(w_e) = -\frac{1}{2\pi i}\int_L \frac{\overline{F_1(w)}}{w - w_e}\,dw + \frac{1}{2\pi i}\int_L \frac{w\ln(\bar{w}w)}{w - w_e}F_1'(w)dw + \tag{II.242}$$

$$+A_1 - iA_2 + \bar{C} = -i\bar{C}_2.$$

By introducing a new function $F_1^* = F_1 + iC_2$ and substituting it in equations (II.242), we find that the function F_1^* is holomorphic in domain Σ and satisfies equation (II.227), in which the right hand side is changed to some constant value $2i\bar{C}_2$:

$$\frac{1}{2\pi i}\int_L \frac{F_1^*(w)}{w - w_e}\,dw = 0,$$

$$\frac{1}{2\pi i}\int_L \frac{\overline{F_1^*(w)}}{w - w_e}\,dw - \frac{1}{2\pi i}\int_L \frac{w\ln(\bar{w}w)}{w - w_e}F_1'^*(w)\,dw = A_1 - iA_2 + \bar{C} + 2i\bar{C}_2. \tag{II.243}$$

It should be noted that the constant C_2 is eliminated when $F_1^*(w)$ is restored with Cauchy's integral in domain Σ, such that it will not be necessary to solve it during the actual solution of the problem.

In this manner, the solution of the stated problem for a hole of any shape can be found from the integral equation (II.231), which has a unique solution. This equation must first be replaced by a system of linear algebraic equations that is conveniently solved with the aid of a computer.

Infinite Row of Equal Square Holes[1]. Let an elastic isotropic plane, weakened by an infinite row of equal square holes with rounded corners, whose contour equations are given by the mapping functions $\omega(\zeta)$ (II.210) for $\rho = 1$. The centers of gravity of the examined holes are placed on the Ox axis at an equal distance l from each other. The origin of the coordinate system Oxy is placed at the center of gravity of one of the holes (it can be any one of the holes), which we will call the basic hole.

As in the preceding section, we will assume that the basic stress state of the elastic plane is homogeneous, i.e.,

$$\sigma_x^0 = p, \qquad \sigma_y^0 = q, \qquad \tau_{xy}^0 = 0. \tag{II.244}$$

[1]The solution of this problem is given by A. S. Kosmodamianskiy [6]. The case where the holes are reinforced with absolutely rigid rings is examined by V. A. Shvetsov [1].

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The mapping function of the exterior of the unit circle on the interior of the basic hole is written in the form of (II.210). /163

Contour condition (II.221) for the given case is

$$\varphi_0(t)+(t-\bar{t})\overline{\varphi_0'(t)}+\overline{\psi_0(t)}=\frac{p-q}{2}\bar{t}-\frac{p+q}{2}t, \tag{II.245}$$

where t is the point on the contour L of the basic hole.

The periodic functions $\phi_0(z)$ and $\psi_0(z)$ (with period $2l$) can be represented in the form[1]

$$\varphi_0(z)=\sum_{k=1}^{\infty}\frac{\alpha_k}{[\zeta(z)]^k}+\sum_{n=-\infty}^{\infty}{}^{*}\sum_{k=1}^{\infty}\frac{\alpha_k}{[\zeta(z+nl)]^k},$$
$$\psi_0(z)=\sum_{k=1}^{\infty}\frac{\beta_k}{[\zeta(z)]^k}+\sum_{n=-\infty}^{\infty}{}^{*}\sum_{k=1}^{\infty}\frac{\beta_k}{[\zeta(z+nl)]^k}, \tag{II.246}$$

where α_k and β_k are constant coefficients, and the variable ζ is related to the variable z by the relation

$$z=\zeta+\frac{m_3}{\zeta^3}+nl \qquad \left(m_3=-\frac{1}{9}\right). \tag{II.247}$$

The first sums in (II.246) represent functions that are holomorphic in the region outside the basic hole, and the second sums represent functions that are holomorphic within the basic hole. The latter can be expanded into converging series and a given (finite) number of terms can be retained within them.

Such expansions are most conveniently performed for the small parameter $\varepsilon = 1/2l$. If, in the above expansions, we retain the terms that contain multiples of ε in powers not exceeding four, we obtain

$$\varphi(z)=\varphi^*(\zeta)-2\lambda_2\varepsilon^2\alpha_1 z-2\lambda_4\varepsilon^4(\alpha_1 z^3+3\alpha_3 z),$$
$$\psi(z)=\psi^*(\zeta)-2\lambda_2\varepsilon^2\beta_1 z-2\lambda_4\varepsilon^4(\beta_1 z^3+3\beta_3 z), \tag{II.248}$$

[1]See S. G. Mikhlin [5].

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where

$$\varphi^*(\zeta)=\sum_{k=1}^{\infty}\frac{\alpha_k}{\zeta^k},\qquad \psi^*(\zeta)=\sum_{k=1}^{\infty}\frac{\beta_k}{\zeta^k},\qquad \lambda_{2k}=\sum_{n=1}^{\infty}n^{-2k}. \tag{II.249}$$

Recalling the form of the mapping function $z=\omega(\zeta)$ (II.210) and expansion (II.248), the boundary condition (II.245) can be reduced to the form

$$\begin{aligned}&\varphi^*(\sigma)+\frac{\omega(\sigma)-\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi^{*'}(\sigma)}+\overline{\psi^*(\sigma)}=\\&=\frac{p-q}{2}\overline{\omega(\sigma)}-\frac{p+q}{2}\omega(\sigma)+4\omega(\sigma)(\alpha_1\lambda_2\varepsilon^2+3\lambda_4\alpha_3\varepsilon^4)+\\&+2\lambda_4\varepsilon^4\alpha_1[\omega(\sigma)]^3+6\lambda_4\alpha_1\varepsilon^4\omega(\sigma)[\overline{\omega(\sigma)}]^2+2\overline{\omega(\sigma)}[\lambda_2\varepsilon^2(\beta_1-\alpha_1)+\\&+3\lambda_4\varepsilon^4(\beta_3-\alpha_3)]+2\lambda_4\varepsilon^4(\beta_1-\alpha_1)[\overline{\omega(\sigma)}]^3.\end{aligned} \tag{II.250}$$

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Hence, by the method described in §8, we find

$$\begin{aligned}\varphi^*(\zeta)&=\frac{1}{\zeta}\left\{\frac{p-q}{2}+\alpha_1[m-2\lambda_2\varepsilon^2+6\lambda_4\varepsilon^4(1+m+2m^2)]+\right.\\&+2\lambda_2\varepsilon^2\beta_1+6\lambda_4\varepsilon^4(\beta_3-\alpha_3)]+\frac{1}{\zeta^3}\left[-\frac{p+q}{2}m+2\varepsilon^2\alpha_1(2\lambda_2m-3\lambda_4\varepsilon^2)+\right.\\&\left.\left.+2\lambda_4\varepsilon^4(\beta_1+6m\alpha_3)\right]+6\alpha_1m\lambda_4\varepsilon^4(1+m)\frac{1}{\zeta^5}+2\alpha_1m^3\lambda_4\varepsilon^4\frac{1}{\zeta^9};\right.\\\psi^*(\zeta)&=-\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)-\omega(\zeta)}{\omega'(\zeta)}\varphi^{*'}(\zeta)-m\alpha_1\zeta+\frac{1}{\zeta}\left[-\frac{p+q}{2}+\right.\\&\left.+2\varepsilon^2\alpha_1(2\lambda_2-9m\lambda_1\varepsilon^2)+6\lambda_4\varepsilon^4(m\beta_1+2\alpha_3)\right]+\\&+\frac{1}{\zeta^3}\left\{\frac{p-q}{2}m-2\varepsilon^2\alpha_1[\lambda_2m-\lambda_4\varepsilon^2(1+6m+3m^2)]+\right.\\&\left.+2\lambda_2\varepsilon^2m\beta_1+6m\lambda_4\varepsilon^4(\beta_3-\alpha_3)\right\}+6m^2\lambda_4\varepsilon^4(\beta_1-3\alpha_1)\frac{1}{\zeta^5}+\\&+6m^2\lambda_4\varepsilon^4\alpha_1\frac{1}{\zeta^7}+2m^3\lambda_4\varepsilon^4(\beta_1-3\alpha_1)\frac{1}{\zeta^9}.\end{aligned} \tag{II.251}$$

The unknown coefficients α_1, α_3, β_1 and β_3 are determined from the following algebraic system of linear equations

$$\begin{aligned}&\alpha_1[1-m+2\lambda_2\varepsilon^2-6\lambda_4\varepsilon^4(1+m+2m^2)]+6\alpha_3\lambda_4\varepsilon^4-\\&-2\beta_1\lambda_2\varepsilon^2-6\beta_3\lambda_4\varepsilon^4=\frac{p-q}{2},\\&2\alpha_1\varepsilon^2(3\lambda_4\varepsilon^2-2\lambda_2m)+\alpha_3(1-12\lambda_4\varepsilon^4m)-2\beta_1\lambda_4\varepsilon^4=-\frac{p+q}{2}m;\end{aligned} \tag{II.252}$$

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$$\alpha_1(1+18m\lambda_4\varepsilon^4-4\lambda_2\varepsilon^2)-3\alpha_3(m+4\lambda_4\varepsilon^4)+\beta_1(1-6m\lambda_4\varepsilon^4)=-\frac{p+q}{2};$$

$$-\alpha_1[1+3m^2+2\lambda_4\varepsilon^4(1+6m+3m^2)--2\lambda_2\varepsilon^2m]+$$

$$+3\alpha_3(1+2m\lambda_4\varepsilon^4)--2\beta_1\lambda_2\varepsilon^2m+\beta_3(1-6\lambda_4\varepsilon^4m)=\frac{p-q}{2}m.$$

By finding the functions $\phi^*(\zeta)$ and $\psi^*(\zeta)$ (II.251), from (II.248) we obtain the functions $\phi(z)$ and $\psi(z)$. The stress components σ_x, σ_y, and τ_{xy} are found from formulas (II.181).

The stresses on the contour of the basic hole (the stresses will be the same on the contours of the other holes), are

$$\sigma_\vartheta=p+q-\frac{4}{C^2}\left\{\frac{p-q}{2}+\alpha_1[m-2\lambda_2\varepsilon^2+6\lambda_4\varepsilon^4(1+m+2m^2)]+\right.$$

$$+2\lambda_2\varepsilon^2\beta_1+6\lambda_4\varepsilon^4(\beta_3-\alpha_3)](A\cos\vartheta-B\sin\vartheta)+$$

$$+3\left[-\frac{p+q}{2}m+2\varepsilon^2\alpha_1(2\lambda_2m-3\lambda_4\varepsilon^2)+2\lambda_4\varepsilon^4(\beta_1+\right.$$

$$\left.+6m\alpha_3)\right](A\cos3\vartheta-B\sin3\vartheta)+30\alpha_1m\lambda_4\varepsilon^4(1+m)(A\cos5\vartheta-B\sin5\vartheta)+ \quad \text{(II.253)}$$

$$+18\alpha_1m^3\lambda_4\varepsilon^4(A\cos9\vartheta-B\sin9\vartheta)-8\varepsilon^2(\lambda_2\alpha_1+3\lambda_4\varepsilon^2\alpha_3)+$$

$$\left.+3\lambda_4\varepsilon^2\alpha_1[(1+2m)\cos2\vartheta+m^2\cos6\vartheta]\right\},$$

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where

$$A=\cos\vartheta-3m\cos3\vartheta,\qquad B=\sin\vartheta+3m\sin3\vartheta,\qquad C^2=A^2+B^2.$$

For stresses at points of the real axis (in particular, at points between the holes), formulas (II.181) can be simplified:

$$\sigma_x=p+\frac{1}{\omega'(\zeta)}[3\varphi^{*'}(\zeta)-\psi^{*'}(\zeta)]-3\varphi_1'(z)+\psi_1'(z),$$

$$\sigma_y=q+\frac{1}{\omega'(\zeta)}[\varphi^{*'}(\zeta)+\psi^{*'}(\zeta)]-\varphi_1'(z)-\psi_1'(z), \quad \text{(II.254)}$$

$$\tau_{xy}=0.$$

For each point of interest to us, ζ should be determined from relations (II.247), and

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$$\varphi_1' = 2\varepsilon^2[\lambda_2\alpha_1 + 3\lambda_4\varepsilon^2(\alpha_3 + \alpha_1 z^2)],$$
$$\psi_1' = 2\varepsilon^2[\lambda_2\beta_1 + 3\lambda_4\varepsilon^2(\beta_3 + \beta_1 z^2)]. \qquad \text{(II.255)}$$

Formulas (II.255) can be used for calculating only the stresses near the contour of the basic hole, all the way up to the points that lie half way between the basic holes and its neighbor.

If the distances between the holes l are greater than two sides of one of the square holes, all the formulas derived earlier can be simplified considerably, since we assume in them that $\alpha_3 = \beta_3 = \varepsilon^4 = 0$. Formula (II.253) in this case is

$$\sigma_\vartheta = p + q - \frac{4}{C^2}\left\{\left[\frac{p-q}{2} + \alpha_1(m - 2\lambda_2\varepsilon^2) + 2\lambda_2\varepsilon^2\beta_1\right](A\cos\vartheta - B\sin\vartheta) + \right.$$
$$\left. + 3m\left(-\frac{p+q}{2} + 4\varepsilon^2\alpha_1\lambda_2\right)(A\cos 3\vartheta - B\sin 3\vartheta)\right\} - 8\varepsilon^2\lambda_2\alpha_1, \qquad \text{(II.256)}$$

where

$$\alpha_1 = \frac{0{,}5(p-q) - \lambda_2\varepsilon^2(p+q)}{1 - m + 4\lambda_2\varepsilon^2}, \qquad \beta_1 = -\frac{p+q}{2} - \alpha_1(1 - 4\lambda_2\varepsilon^2). \qquad \text{(II.257)}$$

Presented below are the values of σ_x/q and σ_y/q at points 1, 2 and 3 (Figure II.93) through cross section y = 0 between the holes, where a homogeneous stress state $\sigma_x^{(\infty)} = 0$, $\sigma_y^{(\infty)} = q$ and $\tau_{xy}^{(\infty)} = 0$ is given at infinity, i.e., when an elastic plate is under tension by forces q = const, transverse to the center line, and under the condition that the distance between the contours of the holes is measured along the Ox axis, equal in length to the side of one of the holes: /166

Points from Figure II.93	1	2	3
σ_x/q	0,00	0,24	0,45
σ_y/q	2,01	1,94	1,70

The values of σ_ϑ/q on the contour of the basic square hole and the values of σ_ϑ^*/q at the same points of the same square hole, but for the case where the elastic plane is weakened by only one hole, are presented below:

$\vartheta°$	0	10	20	30	40	45	50	60	70	80	90
σ_ϑ/q	2,01	2,14	2,57	3,35	4,26	3,19	2,08	0,07	−0,59	−0,77	−0,81
σ_ϑ^*/q	1,85	1,94	2,22	2,69	2,69	2,00	0,97	−0,40	−0,77	−0,84	−0,85

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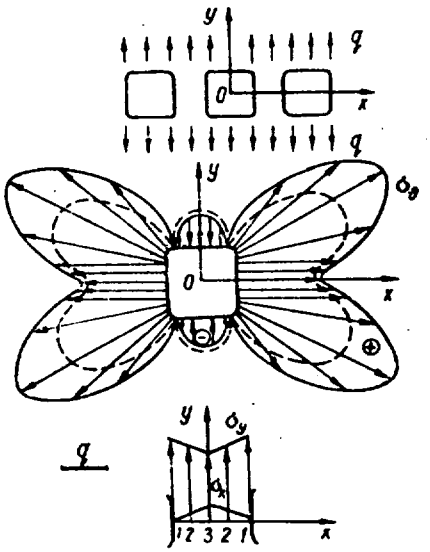


Figure II.93.

Two Identical Square Holes. We will assume that an elastic plane is weakened by two equal square holes of the same shape, and as in the preceding case, the centers of gravity of these holes lie on the Ox axis, separated from each other by distance $2l$ (Figure II.94). We will further assume that a biaxial stress state (II.244) is given at infinity. The desired functions can be represented in the form[1]

$$\varphi(z) = \varphi^*(\zeta) + \varphi_1(z),$$

$$\psi(z) = \psi^*(\zeta) + \psi_1(z), \qquad \text{(II.258)}$$

where

$$\varphi^*(\zeta) = \sum_{m=1}^{\infty} \frac{\alpha_m}{[\zeta(z-l)]^m}, \qquad \psi^*(\zeta) = \sum_{m=1}^{\infty} \frac{\beta_m}{[\zeta(z-l)]^m}.$$

$$\varphi_1(z) = \sum_{m=1}^{\infty} \frac{(-1)^{k+1}\alpha_m}{[\zeta(z+l)]^m}, \qquad \psi_1(z) = \sum_{m=1}^{\infty} \frac{(-1)^{k+1}\beta_m}{[\zeta(z+l)]^m}. \qquad \text{(II.259)}$$

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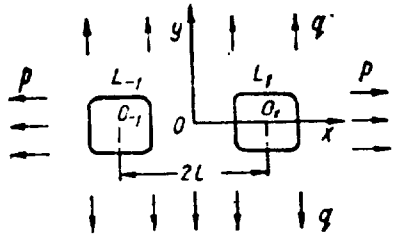


Figure II.94.

To determine the constants α_m and β_m, it is necessary to use boundary conditions (II.245) only on the right hand contour (the boundary condition on the left hand contour will be satisfied automatically).

The functions $\phi_1(z)$ and $\psi_1(z)$ will be holomorphic within the right hand hole. They can be expanded into convergent Fourier series within the circle that inscribes the right hand hole, without coming into contact with the contour of the left hand hole. By limiting ourselves in the state expansion to terms containing, as the factor, the small parameter $\varepsilon = 1/2l$ in a power not exceeding four, we obtain

[1] See A. S. Kosmodamianskiy [4, 7]. This method makes it possible to examine also a finite number of curvilinear holes when the plate is under various stresses, and also thermoelastic problems. See A. S. Kosmodamianskiy [4, 12].

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$$\varphi_1(z) = -(z-l)(\alpha_1\varepsilon^2 - 2\alpha_2\varepsilon^3 + 3\alpha_3\varepsilon^4) + (z-l)^2(\alpha_1\varepsilon^3 - 3\alpha_2\varepsilon^4) - (z-l)^3\alpha_1\varepsilon^4,$$

$$\psi_1(z) = -(z-l)(\beta_1\varepsilon^2 - 2\beta_2\varepsilon^3 + 2\beta_3\varepsilon^4) + (z-l)^2(\beta_1\varepsilon^3 - 3\beta_2\varepsilon^4) - (z-l)^3\beta_1\varepsilon^4. \quad \text{(II.260)}$$

Using the method outlined in §8, we find

$$\varphi^*(\zeta) = \frac{1}{\zeta}\left\{\frac{p-q}{2} + m\alpha_1 + \varepsilon^2(\beta_1 - \alpha_1) - 2\varepsilon^3(\beta_2 - \alpha_2) + 3\varepsilon^4[\alpha_1(1+m^2) + (\beta_3 - \alpha_3)]\right\} + \frac{m}{\zeta^3}\left(-\frac{p+q}{2} + 2\alpha_1\varepsilon^2 - 4\alpha_2\varepsilon^3 + 6\alpha_3\varepsilon^4\right) + m(\alpha_1\varepsilon^3 - 3\alpha_2\varepsilon^4)\left(\frac{2}{\zeta^2} + \frac{m}{\zeta^6}\right) + \alpha_1 m\varepsilon^4\left[\frac{3}{\zeta} + \frac{3(m+1)}{\zeta^5} + \frac{m^2}{\zeta^9}\right] + \frac{2m}{\zeta^4}(\alpha_1\varepsilon^3 - 3\alpha_2\varepsilon^4) - \frac{1}{\zeta^2}[\varepsilon^3(\beta_1 - 2\alpha_1) - 3\varepsilon^4(\beta_2 - 2\alpha_2)] + \frac{\varepsilon^4}{\zeta^3}(\beta_1 - 3\alpha_1);$$

$$\psi^*(\zeta) = -\frac{\overline{\omega}\left(\frac{1}{\zeta}\right) - \omega(\zeta)}{\omega'(\zeta)}\varphi^{*'}(\zeta) - m\alpha_1\zeta + \frac{1}{\zeta}\left[-\frac{p+q}{2} + 2\alpha_1\varepsilon^2 - 4\alpha_2\varepsilon^3 + 6\alpha_3\varepsilon^4\right] + \frac{m}{\zeta^3}\left[\frac{p-q}{2} + \varepsilon^2(\beta_1 - \alpha_1) - 2\varepsilon^3(\beta_2 - \alpha_2) + 3\varepsilon^4(\beta_3 - \alpha_3)\right] - \frac{\varepsilon^3}{\zeta^2}(\alpha_1 - 3\varepsilon\alpha_2) + \frac{\alpha_1\varepsilon^4}{\zeta^3} - \frac{2m\varepsilon^3}{\zeta^4}(\alpha_1 - 3\varepsilon\alpha_2) + 3\alpha_1\varepsilon^4\left[\frac{m(2+m^2)}{\zeta^3} + \frac{m^2}{\zeta^7}\right] - m\varepsilon^3\left(\frac{2}{\zeta^2} + \frac{m}{\zeta^6}\right)[\beta_1 - 2\alpha_1 - 3\varepsilon(\beta_2 - 2\alpha_2)] + m\varepsilon^4\left(\frac{3}{\zeta} + \frac{3m}{\zeta^5} + \frac{m^2}{\zeta^9}\right)(\beta_1 - 3\alpha_1). \quad \text{(II.261)}$$

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The coefficients α_k and β_k (k = 1, 2, 3) in expressions (II.261) are determined from the equation system[1]

$$\alpha_1[1 - m + \varepsilon^2 - 3m\varepsilon^4 - 3\varepsilon^4(1 + 2m^2)] - 2\alpha_2\varepsilon^3 - 3\alpha_3\varepsilon^4 - \varepsilon^2\beta_1 + 2\varepsilon^3\beta_2 - 3\varepsilon^4\beta_3 = \frac{p-q}{2};$$

$$2\alpha_1\varepsilon^3(m-1) + [1 + 6\varepsilon^4(1-m)]\alpha_2 + \varepsilon^3\beta_1 - 3\varepsilon^4\beta_2 = 0;$$

[1]A. S. Kosmodamianskiy [11] proved that for this case an infinite quasiregular algebraic is obtained.

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$$\varepsilon^2\alpha_1(3\varepsilon^2-2m)+4m\varepsilon^2\alpha_2+\alpha_3(1-6m\varepsilon^4)-\beta_1\varepsilon^4=-\frac{p+q}{2}m;$$

$$\beta_1(1-3m\varepsilon^4)+\alpha_1(1-2\varepsilon^2+9m\varepsilon^4)+4\alpha_2\varepsilon^3-3\alpha_3(2\varepsilon^4+m)=-\frac{p+q}{2};$$

$$2m\varepsilon^3\beta_1+\beta_2(1-6m\varepsilon^4)+\varepsilon^3\alpha_1(1-4m)+\alpha_2(2-3\varepsilon^4+12m\varepsilon^4)=0; \qquad \text{(II.262)}$$

$$-\beta_1 m\varepsilon^2+2m\varepsilon^3\beta_2+\beta_3(1-3m\varepsilon^4)-\alpha_1[1+3m^2-m\varepsilon^2+\varepsilon^4+$$

$$+3m\varepsilon^4(2+m^2)]-2m\varepsilon^3\alpha_2+3\alpha_3(1+m\varepsilon^4)=m\frac{p+q}{2}.$$

Finding from (II.262) the coefficients β_k and α_k (k = 1, 2, 3), and placing them in the functions $\phi^*(\zeta)$, $\psi^*(\zeta)$ (II.261), as well as in the functions $\phi_1(z)$ and $\psi_1(z)$ (II.259), and the latter in (II.258), we obtain the final form of the desired functions $\phi(z)$ and $\psi(z)$. The stresses at any point in the vicinity of the holes are found from formulas (II.181).

The stresses on the contour of the right hand hole are

$$\sigma_\vartheta=p+q-\frac{4}{C^2}\sum_{k=1}^{9}kA_k(A\cos k\vartheta-B\sin k\vartheta)-4\{\alpha_1\varepsilon^2-$$

$$-2\alpha_2\varepsilon^3+3\alpha_3\varepsilon^4-2\varepsilon^2(\alpha_1-3\alpha_2\varepsilon)(\cos\vartheta+m\cos 3\vartheta)+ \qquad \text{(II.263)}$$

$$+3\alpha_1\varepsilon^4[(1+2m)\cos 2\vartheta+m^2\cos 6\vartheta]\},$$

where /169

$$A=\cos\vartheta-3m\cos 3\vartheta;\qquad B=\sin\vartheta+3m\sin 3\vartheta;\qquad C^2=A^2+B^2;$$

$$A_1=\frac{p-q}{2}+\varepsilon^2(\beta_1-\alpha_1)-2\varepsilon^3(\beta_2-\alpha_2)+3\varepsilon^4(\beta_3-\alpha_3)+$$

$$+\alpha_1[m(1+3\varepsilon^4)+3\varepsilon^4(1+2m^2)];$$

$$A_2=-2m\varepsilon^3(\alpha_1-3\varepsilon\alpha_2)-\varepsilon^3(\beta_1-2\alpha_1)+3\varepsilon^4(\beta_2-2\alpha_2); \qquad \text{(II.264)}$$

$$A_3=\varepsilon^4(\beta_1-3\alpha_1)+m\left(-\frac{p+q}{2}+2\alpha_1\varepsilon^2-4\alpha_2\varepsilon^3+6\alpha_3\varepsilon^4\right);$$

$$A_4=-2m\varepsilon^3(\alpha_1-3\alpha_2\varepsilon);\qquad A_5=3m\alpha_1\varepsilon^4(1+m);$$

$$A_6=-m^2\varepsilon^3(\alpha_1-3\alpha_2\varepsilon);\qquad A_7=A_8=0;\qquad A_9=m^2\alpha_1\varepsilon^4.$$

The values for stresses on the contour of the right hand hole and through cross section y = 0 between the holes, where the distance between the contours of the holes (Figure II.94) is equal to the length of the side of one of the holes, under the condition that tension forces act at infinity, both along the center line of the holes ($p \neq 0$, $q = 0$), and transverse to the center line of the holes ($p = 0$, $q \neq 0$), are presented in Tables II.39 and II.40. The values

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denoted by the asterisk, as before, pertain to a plane with one square hole of the same shape.

TABLE II.39

Stresses	θ°												
	0	15	30	45	60	75	90	105	120	135	150	165	180
σ_θ/p	—0,77	—0,75	—0,38	1,92	2,61	1,97	1,70	1,74	2,00	1,18	—0,32	—0,45	—0,43
σ^*_θ/p	—0,85	—0,82	—0,40	2,00	2,69	2,05	1,85	2,05	2,69	2,00	—0,40	—0,82	—0,85
σ_θ/q	1,91	2,14	2,83	2,14	—0,42	—0,87	—0,86	—0,74	—0,07	2,67	2,92	2,05	1,78
σ^*_θ/q	1,85	2,05	2,69	2,00	—0,40	—0,82	—0,85	—0,82	—0,40	2,00	2,69	2,05	1,85

Tr. Note: Commas indicate decimal points.

TABLE II.40

$\frac{y}{a}$ z	σ_x/p	σ_y/p	σ^*_x/p	σ^*_y/p	σ_x/q	σ_y/q	σ^*_x/q	σ^*_y/q
1,00	—0,02	—0,46	0,00	—0,85	0,01	1,87	0	1,85
0,75	—0,05	—0,27	0,03	—0,37	0,19	1,80	0,03	1,84
0,50	—0,05	—0,06	0,15	—0,19	0,22	1,74	0,06	1,64
0,25	—0,05	0,02	0,26	—0,04	0,25	1,66	0,13	1,45
0	—0,05	0,05	0,37	0,05	0,27	1,62	0,16	1,32

Tr. Note: Commas indicate decimal points.

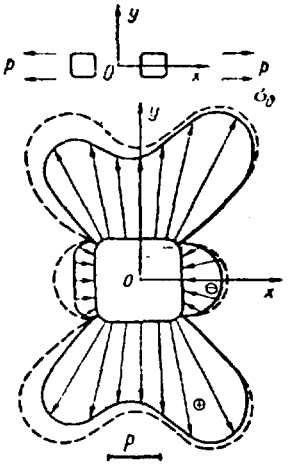


Figure II.95.

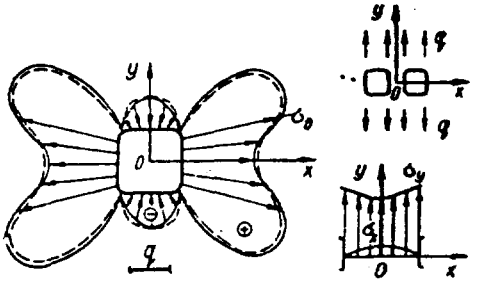


Figure II.96.

The graphs constructed from the data in Tables II.39 and II.40, characterizing the stress distribution near the right hand hole and through cross section y = 0 between the holes, are shown in Figures II.95 and II.96. The broken lines correspond to the case where the plane is weakened by one square hole.

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Concluding Comments. In an elastic plane weakened by square holes, where /170
forces (II.244) are applied at infinity, the same principles hold true as in the case of circular holes.

When the plane is under tension by forces p (at infinity) along the center line, the stress concentration near the corners of the holes decreases as the number of holes increases in comparison with the case where the plane is weakened by only one hole. If, however, the plane (at infinity) is under tension by forces q, transverse to the center line, as the number of holes increases, the stress concentration near them also increases. In the case of two holes, the above decrease or increase in stress concentration is found to be greater from the inside of the holes, and smaller from the outside.

§11. Stress Concentration Near a Double Periodic System of Holes

Statement of Problem. Perforated plates are encountered quite frequently in the construction of modern machinery, and therefore the solution of the problem of stress distribution in an infinite plate weakened by a double periodic system of round holes can considerably simplify the analysis of operation of perforated plates. In the given case, the boundaries at "infinity," which make up functions $\Phi(z)$ and $\Psi(z)$, describing the perturbing stress state, should be replaced by the conditions of double periodicity of these functions. Obviously, the periods of the function will be the distances between the centers of the holes. The general solution of the stated problem is found in the work of G. A. Van Fo Fi [1]. For simplicity, we will examine here only a system of equal[1] holes, located symmetrically (Figures II.97, II.98) with respect to the axes. If the conditions of double periodicity are satisfied, arbitrary constants in the solution should be found from the boundary conditions on the contour of one arbitrary hole. /171

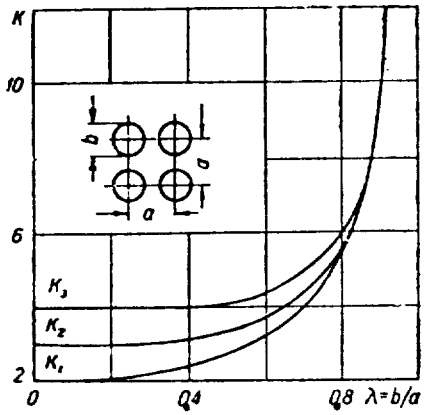


Figure II.97.

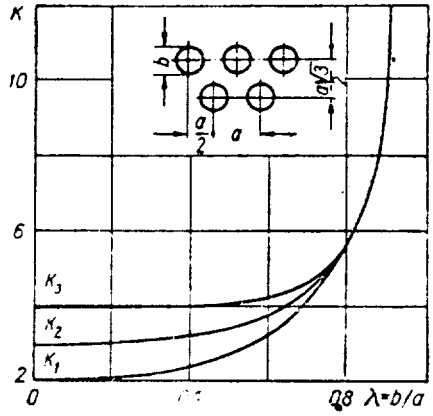


Figure II.98.

[1]See L. A. Fil'shtinskiy [1]. More detailed data concerning perforated plates can be found in the review by E. I. Grigolyuk and L. A. Fil'shtinskiy [1].

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Double Periodic System of Identical Round Holes. Let $\omega_1 = 2$ and $\omega_2 = 2le^{i\alpha}$ ($l > 0$, $\alpha \neq 0$) be the basic periods of a double periodic system, λ is the dimensionless radius of the holes, and $p = m\omega_1 + n\omega_2$ are the coordinates of the centers of the holes ($m, n = \pm 1, \pm i$). From the condition of double periodicity of the stress state of the perforated plate,

$$\sigma_x + \sigma_y = 3\mathrm{Re}\,\Phi(z),$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\{\bar{z}\Phi'(z) + \Psi(z)\} \qquad (z = x + iy) \qquad \text{(II.265)}$$

we have the limitations imposed on complex potentials Φ and Ψ:

$$\Phi(z) = \Phi(z + \omega_j), \quad \Psi(z + \omega_j) = \Psi(z) - \omega_j\Phi'(z) \quad (j = 1, 2). \qquad \text{(II.266)}$$

The conditions of symmetry of the problem impose on the desired functions the following additional relations:

$$\Phi(\bar{z}) = \overline{\Phi(z)}, \quad \Phi(-z) = \Phi(z); \quad \Psi(\bar{z}) = \overline{\Psi(z)}, \quad \Psi(-z) = \Psi(z). \qquad \text{(II.267)}$$

The functions $\Phi(z)$ and $\Psi(z)$ can be constructed either on the basis of Appel's representation, or with the aid of elliptic functions. In the latter case, various representations of the solutions are possible, but the most convenient, /172
obviously, are Weierstrass' functions $\wp(z)$ and the theorem of the representation of any double periodic function through known elliptic functions and its derivatives. In particular, the following functions were used in the work of L. A. Fil'shtinskiy [1]:

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-p)^2} - \frac{1}{p^2} \right\}; \qquad \text{(II.268)}$$

$$Q(z) = \sum_{m,n}{}' \left\{ \frac{\bar{p}}{(z-p)^2} - 2z\frac{\bar{p}}{p^3} - \frac{\bar{p}}{p^2} \right\}. \qquad \text{(II.269)}$$

The functions $Q(z)$ are constructed such that there exists between them the relations

$$Q^{(k)}(z + \omega_j) = Q^{(k)}(z) + \bar{\omega}_j\wp^{(k)}(z),$$
$$Q(z + \omega_j) = Q(z) + \bar{\omega}_j\wp(z) + \gamma_j,$$
$$\gamma_l = 2Q\left(\frac{\omega_l}{2}\right) - \bar{\omega}_j\wp\left(\frac{\omega_l}{2}\right).$$

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Between the parameters γ_i and ω_j exist the relationships

$$\gamma_2\omega_1 - \omega_2\gamma_1 = \delta_1\bar{\omega}_2 - \delta_2\bar{\omega}_1, \quad \delta_1 = 2\zeta\left(\frac{\omega_1}{2}\right), \qquad \delta_2 = 2\zeta\left(\frac{\omega_2}{2}\right),$$

and also

$$\delta_1\omega_2 - \delta_2\omega_1 = 2\pi i.$$

Here $\zeta(z)$ is the known Weierstrass zeta-function. The relationship between the complex potentials and functions $\wp(z)$ and $Q(z)$ is established on the basis of the aforementioned theorem, according to which any double periodic function can be determined through known elliptic functions and its derivatives. In particular, for the functions Φ and Ψ, which satisfy the conditions of periodicity (II.265) and (II.266), the following representations are valid:

$$\Phi(z) = c_0 + \sum_{k=0}^{\infty} c_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)!};$$

$$\Psi(z) = d_0 + \sum_{k=0}^{\infty} d_{2k+2}\frac{\lambda^{2k+2}\wp^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} c_{2k+2}\frac{\lambda^{2k+2}Q^{(2k+1)}(z)}{(2k+1)!}. \tag{II.270}$$

The unknown coefficients c_{2k}, d_{2k} in expansions (II.270) are found from the conditions on the contour of one of the holes and from the static conditions -- equality of the main vector of all forces acting on the contour that circumscribes the hole, to zero.

The latter condition is equivalent to the relations /173

$$g(z+\omega_j) - g(z) = 0,$$

where

$$\chi(z) = \varphi(z) + z\overline{\Phi(z)} + \overline{\Psi(z)}, \qquad \varphi(z) = \int \Phi(z)\,dz, \qquad \psi(z) = \int \Psi(z)\,dz.$$

When this condition is satisfied, we have the formulas

$$c_0 = a_0c_2\lambda^2 + a_1d_2\lambda^2, \qquad d_0 = a_2c_2\lambda^2 + a_3d_2\lambda^2,$$

where, for brevity, we introduce the definitions

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$$a_0 = \frac{\delta_1}{2\omega_1} - a_1, \qquad a_2 = \frac{\overline{\gamma}_1}{\omega_1} + 2a_1, \qquad a_3 = 2a_0, \qquad a_1 = \frac{\pi i}{\overline{\omega}_1\omega_2 - \omega_1\overline{\omega}_2}.$$

To determine the coefficients c_{2k}, d_{2k}, it is convenient to expand the functions $\wp^{(k)}(z)$ and $Q^{(k)}(z)$ into Laurent series in the vicinity of point $z = 0$:

$$\wp^{(2k)}(z) = \frac{1}{z^{2k+2}} + \sum_{n=0}^{\infty} \alpha_{n,k} z^{2k};$$

$$Q^{(2k+1)}(z) = \sum_{n=0}^{\infty} \beta_{n,k} z^{2k},$$

$$\alpha_{n,k} = \frac{(2n+2k+1)!\, q_{n+k+1}}{(2n)!\, 2^{2n+2k+2}},$$

$$\beta_{n,k} = \frac{(2n+2k+2)!}{(2n)!\, 2^{2n+2k+2}} p_{n+k+1},$$

where

$$q_k = \sum_{m,n}{}' \frac{1}{T^{2k}}, \qquad p_k = \sum_{m,n}{}' \frac{\overline{T}}{T^{2k+1}}, \qquad \left(\alpha_{00} = 0, \quad \beta_{00} = 0, \quad T = \frac{1}{2} p\right).$$

If on the contour of the hole we have external forces $N(\theta) - iT(\theta)$, the coefficients of expansion of functions Φ and Ψ are found from the boundary condition

$$\overline{\Phi(\tau)} + \Phi(\tau) - e^{2i\theta}\{\overline{\tau}\Phi'(\tau) + \Psi(\tau)\} = N(\theta) - iT(\theta) \tag{II.271}$$

for $z = \tau = \lambda e^{i\theta}$

$$N(\theta) - iT(\theta) = \sum_{-\infty}^{\infty} A_{2k} e^{2ki\theta},$$

where N, T are the normal and tangential components, respectively, of the force on the contour of the hole.

By introducing into functional equations (II.271) the expansions of functions Φ and Ψ from (II.270), we obtain, relative to c_{2k}, an infinite equation system /174

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$$c_{2j+2} = \sum_{k=0}^{\infty} \beta_{j,k} c_{2k+2} + b_j,$$

where

$$\beta_{jk} = (2j+1)\,\gamma_{j,k}\lambda^{2j+2k+2};$$

$$\gamma_{0,0} = \frac{3}{8} g_2\lambda^2 + \sum_{i=1}^{\infty} \frac{(2i+1)\,g_{i+1}^2\lambda^{4i+2}}{2^{4i+4}} + a_2 + \frac{2a_0a_3\lambda^2}{1-2a_1\lambda^2};$$

$$\gamma_{0,k} = -\frac{(2k+2)\,p_{k+1}}{2^{2k+2}} + \frac{(2k+4)!\,g_{k+2}\lambda^2}{2!\,(2k+2)!\,2^{2k+4}} + \frac{2a_3\lambda^2 g_{k+1}}{(1-2a_1\lambda^2)\,2^{2k+2}} +$$

$$+ \sum_{i=1}^{\infty} \frac{(2k+2j+1)!\,g_{i+1}g_{k+i+1}\lambda^{4i+2}}{(2k+1)!\,(2i)!\,2^{2k+4i+4}};$$

$$\gamma_{i,0} = -\frac{(2j+2)\,p_{j+1}}{2^{2j+2}} + \frac{(2j+4)!\,g_{k+2}\lambda^2}{2!\,(2j+2)!\,2^{2j+4}} + \frac{2a_0\lambda^2 g_{j+1}}{(1-2a_1\lambda^2)\,2^{2j+2}} +$$

$$+ \sum_{i=1}^{\infty} \frac{(2j+2i+1)!\,g_{i+1}g_{i+j+1}\lambda^{4i+2}}{(2j+1)!\,(2i)!\,2^{2j+4i+4}}; \qquad \text{(II.272)}$$

$$\gamma_{j,k} = \gamma_{k,j} = -\frac{(2j+2k+2)!\,p_{j+k+1}}{(2j+1)!\,(2k+1)!\,2^{2j+2k+2}} + \frac{(2j+2k+4)!\,g_{j+k+2}\lambda^2}{(2j+2)!\,(2k+2)!\,2^{2j+2k+4}} +$$

$$+ \sum_{i=0}^{\infty} \frac{(2j+2i+1)!\,(2k+2i+1)\,g_{j+i+1}g_{k+i+1}\lambda^{4i+2}}{(2j+1)!\,(2k+1)!\,(2i+1)!\,(2i)!\,2^{2j+2k+4i+4}} +$$

$$+ \frac{g_{i+1}g_{k+1}}{2^{2j+2k+4}}\left\{1 + \frac{4a_1\lambda^2}{1-2a_1\lambda^2}\right\} \quad (j,k = 1,2,\ldots);$$

$$b_0 = A_2 - \frac{A_0a_3\lambda^2}{1-2a_1\lambda^2} = \sum_{k=0}^{\infty} \frac{g_{k+2}\lambda^{2k+4}}{2^{2k+4}} A_{-2k-2};$$

$$b_j = A_{2j+2} - \frac{(2j+1)\,A_0\lambda^{2j+2}g_{j+1}}{(1-2a_1\lambda^2)\,2^{2j+2}} - \sum_{k=0}^{\infty} \frac{(2j+2k+3)!\,g_{j+k+2}\lambda^{2k+2j+4}}{(2j)!\,(2k+3)!\,2^{2j+2k+4}} A_{-2k-2}.$$

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The constants d_{2k} are found from condition (II.271) on the contour of the hole: /175

$$d_2 = \frac{1}{1-2a_1\lambda^2}\left\{-A_0 + 2a_0\lambda^2 c_2 + 2\sum_{k=1}^{\infty}\frac{g_{k+1}\lambda^{2k+2}}{2^{2k+2}}c_{2k+2}\right.,$$

$$d_{2j+4} = (2j+3)c_{2j+2} + \sum_{k=0}^{\infty}\frac{(2j+2k+3)!\,g_{j+k+2}\lambda^{2j+2k+4}}{(2j+2)!\,(2k+1)!\,2^{2j+2k+4}}c_{2k+2} - A_{-2j-2}.$$

The coefficients of stress concentration on the contours of the holes for square and equilateral triangular systems are represented in Figures II.97 and II.98 by the curves (k_1), (k_2) and (k_3). The curve (k_1) in these figures corresponds to the case of universal tension on the plate; (k_2), to the case of uniaxial tension, and (k_3), to pure displacement. These curves show that the coefficients of stress concentration for triangular and square systems differ only slightly from each other. The mutual arrangement of the holes has a basic effect on stress distribution.

The stress state of a plate perforated by equilateral triangular and square systems of identical reinforced round holes is examined in the work of R. W. Bailey, R. Fidler [1]. Stresses in a perforated plate are discussed in the works of J. W. Dally and A. J. Durelli [1], and also in the article of C. K. Wang [1], which is similar in content.

§12. Elastic Plate with Curvilinear Hole, Reinforced by Elastic Braces, under Tension

Statement of Problem. We will consider an infinite isotropic plate with a bounded single curvilinear hole in the form of an ellipse, square, or rectangle, with rounded corners. We will assume that rigid arc-shaped plates are welded to the material of the plate, located in a state of stress, at arcs L_k of the contour of hole L. Elastic braces of some other material are installed, with some predetermined tightness, between the arc-shaped plates. The problem is to determine the total axial pressure (tension) on each of the braces, depending on their rigidity, preliminary tightness, and stress state of the plane at infinity[1].

To simplify the solution of the problem, we will examine only those cases where the elastic state in the plane is symmetrical relative to the geometrical /176 axes of symmetry of the hole, and the arc-shaped plates, during deformation of the plane and braces, are displaced gradually without encountering rotations.

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[1]The solution in quadratures is given in the article by M. P. Sheremet'yev and I. A. Prusov [1]. The solution of the other problems found in this section (including the approximate solutions) were found by I. A. Prusov [1].

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Furthermore, as regards the braces, we will use the ordinary hypothesis of plane cross sections of rods, considering only their rigidity to tension EF, ignoring in some cases, their rigidity to deflection. We will place the origin of the coordinate system of the complex plane $z = x + iy$ within contour L and introduce to the examination the function $z = \omega(\zeta)$, which conformally maps the exterior of unit circle γ of complex plane $\zeta = \rho e^{i\theta}$ on the exterior of contour L. Then, as is known[1], in the polar coordinates of complex plane ζ in domain $|\zeta| > 1$, we have:

$$\sigma_\rho + \sigma_\theta = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}]; \tag{II.273}$$

$$\sigma_\rho + i\tau_{\rho\theta} = \Phi(\zeta) + \overline{\Phi(\zeta)} - \frac{\overline{\zeta}}{\zeta\omega'(\zeta)}[\omega(\zeta)\overline{\Phi'(\zeta)} + \overline{\omega'(\zeta)\Psi(\zeta)}]; \tag{II.274}$$

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \omega(\zeta)\overline{\Phi(\zeta)} - \overline{\psi(\zeta)} + \text{const}; \tag{II.275}$$

$$X' + iY' = -i[\varphi(\zeta) + \omega(\zeta)\overline{\Phi(\zeta)} + \overline{\psi(\zeta)}]_A^B. \tag{II.276}$$

where $\phi(\zeta)$ and $\psi(\zeta)$ are functions related to $\Phi(\zeta)$ and $\Psi(\zeta)$ by relations $\phi'(\zeta) = \omega'(\zeta)\Phi(\zeta)$ and $\psi'(\zeta) = \omega'(\zeta)\Psi(\zeta)$; u and v are components of displacement in Cartesian coordinates (xOy); X' + iY' is the resultant vector of external forces acting on arc AB from the right during motion from point A to point B.

We will apply the determination of function $\Phi(\zeta)$ to domain $|\zeta| < 1$, assuming that

$$\omega'(\zeta)\Phi(\zeta) = -\omega'(\zeta)\overline{\Phi}\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta^2}\omega(\zeta)\overline{\Phi'}(\zeta) + \\ + \frac{1}{\zeta^2}\overline{\omega'}\left(\frac{1}{\zeta}\right)\overline{\Psi}\left(\frac{1}{\zeta}\right) \quad \text{for} \quad |\zeta| < 1. \tag{II.277}$$

Considering (II.277), on the basis of (II.274)-(II.276), we find the formulas attributed to I. N. Kartsivadze [1]:

$$\sigma_\rho + i\tau_{\rho\theta} = \Phi(\zeta) - \Phi\left(\frac{1}{\zeta}\right) + \overline{\zeta}^2\left[\frac{\omega(\overline{\zeta}^{-1})}{\omega'(\zeta^{-1})} - \frac{\omega(\zeta)}{\rho^2\omega'(\zeta)}\right]\overline{\Phi'(\zeta)} + \\ + \overline{\zeta}^2\,\overline{\omega'(\zeta)}\left[\frac{1}{\omega'(\overline{\zeta}^{-1})} - \frac{1}{\rho^2\omega'(\zeta)}\right]\overline{\Psi(\zeta)}; \tag{II.278}$$

$$2\mu\frac{\partial}{\partial\theta}(u + iv) = i\zeta\omega'(\zeta)\left[\varkappa\Phi(\zeta) + \Phi\left(\frac{1}{\zeta}\right)\right] - \\ - i\rho^2\omega'(\zeta)\left\{\left[\frac{\overline{\zeta}\omega(\overline{\zeta}^{-1})}{\omega'(\overline{\zeta}^{-1})} - \frac{\omega(\zeta)}{\zeta\omega'(\zeta)}\right]\Phi'(\zeta) + \overline{\omega'(\zeta)}\left[\frac{\overline{\zeta}}{\omega'(\overline{\zeta}^{-1})} - \frac{1}{\zeta\omega'(\zeta)}\right]\overline{\Psi(\zeta)}\right\}; \tag{II.279}$$

[1] See formulas (I.10) and (I.27).

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$$2\mu(u+iv)=\varkappa\varphi(\zeta)+\varphi\left(\frac{1}{\zeta}\right)-\left[\omega(\zeta)-\omega\left(\frac{1}{\zeta}\right)\right]\overline{\Phi(\zeta)}+\text{const}; \qquad \text{(II.280)}$$

/17

$$X'+iY'=-i\left\{\varphi(\zeta)-\varphi\left(\frac{1}{\zeta}\right)+\left[\omega(\zeta)-\omega\left(\frac{1}{\zeta}\right)\right]\overline{\Phi(\zeta)}\right\}_A^B. \qquad \text{(II.281)}$$

Henceforth we will consider that function $\Phi(\zeta)$ passes continuously to all points $\zeta = \sigma$ of circle γ from side $|\zeta| > 1$ and $|\zeta| < 1$, with the exception, possibly, of a finite number of points $\sigma_k = e^{i\theta_k}$ close to which

$$|\Phi(\zeta)|<\frac{\text{const}}{|\zeta-\sigma_k|^{\alpha}} \qquad (0\leqslant\alpha<1).$$

We will also assume that for all points σ of circle γ, with the exception, possibly, of the points $\sigma_k = e^{i\theta_k}$,

$$\lim_{\zeta\to\sigma}\Phi(\zeta)(1-\varrho)=\lim_{\zeta\to\sigma}\Phi'(\zeta)\left[\omega(\zeta)-\omega\left(\frac{1}{\zeta}\right)\right]=0. \qquad \text{(II.282)}$$

We will assume that a mapping function $z = \omega(\zeta)$ is of the form

$$\omega(\zeta)=R\left(\zeta+\frac{m}{\zeta}+\frac{n}{\zeta^3}+\frac{l}{\zeta^5}\right), \qquad \text{(II.283)}$$

where R, m, n and l are real parameters.

Then functions $\omega'(\zeta)\Phi(\zeta)$ and $\omega'(\zeta)\Psi(\zeta)$ for $|\zeta| \to \infty$ have the form

$$\begin{aligned}\omega'(\zeta)\Phi(\zeta)&=\Gamma R-\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{\zeta}+0\left(\frac{1}{\zeta^2}\right),\\ \omega'(\zeta)\Psi(\zeta)&=\Gamma' R+\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{\zeta}+0\left(\frac{1}{\zeta^2}\right),\end{aligned} \qquad \text{(II.284)}$$

where X + iY is the resultant vector of external load applied to contour L.

The functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are holomorphic in domain $|\zeta| > 1$. Considering this, and the fact that $\omega'(\zeta) \neq 0$ in domain $|\zeta| \geqslant 1$, on the basis of (II.277) and (II.284), we know that within circle γ the function $\omega'(\zeta)\Phi(\zeta)$ is holomorphic everywhere except for point $\zeta = 0$, where it has, as is readily seen, a pole not exceeding the sixth power:

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$$\omega'(\zeta)\Phi(\zeta)=\frac{1}{\zeta^2}\left[\Gamma R(m+mn+m^2l+3nl)+\bar{A}_2(n+ml)+\bar{A}_4l\right]+$$
$$+\frac{2}{\zeta^3}\left[-\bar{P}'(n+ml)+\bar{A}_3l\right]+\frac{3}{\zeta^4}\left[\Gamma R(n+ml)+\bar{A}_2l\right]-\frac{4}{\zeta^5}\bar{P}'l+ \qquad \text{(II.285)}$$
$$+\frac{5}{\zeta^6}\Gamma Rl+\frac{\bar{\Gamma}'R}{\zeta^3}+\frac{\varkappa P'}{\zeta}+\Phi_0(\zeta).$$

Here $\Phi_0(\zeta)$ is holomorphic in the domain outside the circle of sufficiently large radius: /178

$$P'=\frac{X+iY}{2\pi(1+\varkappa)},\qquad \bar{P}'=\frac{X-iY}{2\pi(1+\varkappa)}; \qquad \text{(II.286)}$$

A_2, A_3, A_4 are coefficients of expansion of function $\omega'(\zeta)\Phi(\zeta)$ into the series

$$\omega'(\zeta)\Phi(\zeta)=\Gamma R-\frac{P}{\zeta}+\frac{A_2}{\zeta^2}+\frac{A_3}{\zeta^3}+\frac{A_4}{\zeta^4}+\ldots \qquad \text{(II.287)}$$

We will assume, in the following, that the axes of the coordinates of the complex plane z = x + iy are directed along the axes of symmetry of hole L, and the stress components at infinity are given in the form

$$X_x^{(\infty)}=p,\qquad Y_y^{(\infty)}=q,\qquad X_y^{(\infty)}=0. \qquad \text{(II.288)}$$

Moreover, we will assume that on the segments of the contour of the hole on which the materials of the plate and arc-shaped plates do not come into contact, the external load is zero and the resultant vector of external load on contour L is also equal to zero:

$$P'=X+iY=0. \qquad \text{(II.289)}$$

Elliptic Hole with Two Braces that Transmit Pressure to Two Rigid Arc-Shaped Plates. Let L be the contour of an elliptic hole with semiaxes a and b, within which are installed arc-shaped plates and braces symmetrical to the coordinate axes (Figure II.99); δ is preliminary tension of each of the braces; l is length, and EF is rigidity to tension of each one of them; L_k are arcs on which the plates are welded to the material of the surface; L_k' are the other parts of contour L; a_k and b_k are the ends of the arcs L_k (k = 1, 2); γ_k, γ_k', α_k and β_k are arcs and points on circle γ corresponding to L_k, L_k' and a_k, b_k.

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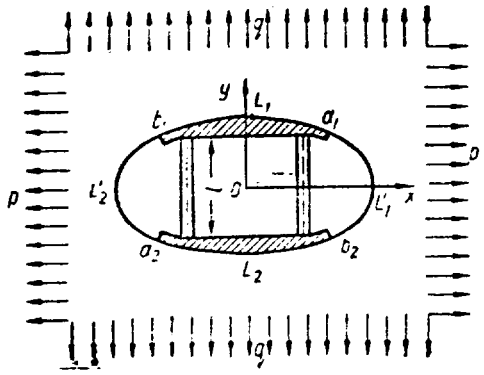


Figure II.99.

In the case at point,

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \quad R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}. \tag{II.290}$$

On the basis of (II.284) and (II.285), function $\omega'(\zeta)\Phi(\zeta)$ in the vicinity $\zeta = 0$ and $|\zeta| = \infty$ acquires the form

$$\omega'(\zeta)\Phi(\zeta) = \Gamma R + 0\left(\frac{1}{\zeta^2}\right) \qquad (|\zeta| \to \infty),$$

$$\omega'(\zeta)\Phi(\zeta) = \frac{R}{\zeta^2}(m\Gamma + \Gamma') + 0(1) \qquad (\zeta \to 0), \tag{II.291}$$

where /179

$$\Gamma = \frac{1}{4}(P+Q); \qquad \Gamma' = \frac{1}{2}(Q-P). \tag{II.292}$$

Considering (II.278) and (II.279) and also the conditions assumed concerning displacements of the arc-shaped plates, we find for function $\omega'(\zeta)\Phi(\zeta)$, the boundary condition

$$[\omega'(\sigma)\Phi(\sigma)]^+ + \varkappa[\omega'(\sigma)\Phi(\sigma)]^- = 0 \text{ on } \gamma_k,$$

$$[\omega'(\sigma)\Phi(\sigma)]^+ - [\omega'(\sigma)\Phi(\sigma)]^- = 0 \text{ on } \gamma_k'. \tag{II.293}$$

Satisfying this condition, we write the general solution of the problem:

$$\omega'(\zeta)\Phi(\zeta) = R\left(\frac{D_1}{\zeta} + \frac{D_2}{\zeta^2} + C_0\zeta^2 + C_1\zeta + C_2\right)X(\zeta), \tag{II.294}$$

where

$$X(\zeta) = (\zeta^2 - \alpha_1^2)^{\lambda}(\zeta^2 - \beta_1^2)^{\bar{\lambda}};$$

$$\lambda = -\frac{1}{2} + i\beta, \qquad \bar{\lambda} = -\frac{1}{2} - i\beta, \qquad \beta = \frac{\ln \varkappa}{2\pi}; \tag{II.295}$$

D_k and C_k are arbitrary constants.

We agree here to take that branch of the ambiguous function $X(\zeta)$ for which, when $|\zeta| \to \infty$, $\zeta^2 X(\zeta) \to 1$.

Keying the solution of (II.294) to conditions (II.291), we find

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$$D_1 = C_1 = 0, \qquad C_0 = \Gamma, \qquad D_2 = (m\Gamma + \Gamma')\, e^{-2\beta\omega}, \tag{II.296}$$

where ω is the central angle corresponding to the arc γ_k ($k = 1, 2$).

By denoting through $2iv_0$ the difference of displacements of the points by L_1 and L_2, and through $2P_0$, the main vector of the forces acting on arc L_1 from any one side, we find, for the determination of v_0, pressure on each of the braces p, and real constant C_2, the following two equations, on the basis of (II.280) and (II.281):

$$(1 + \varkappa) \int_{\gamma_1'} \Phi^-(\sigma)\, \omega'(\sigma)\, d\sigma = 4i\mu v_0,$$

$$(1 + \varkappa) \int_{\gamma_2} \Phi^-(\sigma)\, \omega'(\sigma)\, d\sigma = -2P_0. \tag{II.297}$$

After combining them with equation

$$2v_0 = \delta + \frac{P_0 l}{EF}, \tag{II.298}$$

as a consequence of Hooke's law for rods, we find the total equation system for the determination of all unknowns. After solving this system, we find the following formula for the total pressure on each of the rods: /180

$$P_0 = \frac{\frac{1}{4}\left[D_2 J_2 + C_0 J_{-2} - \frac{J_0}{T_0}(D_2 T_2 + C_0 T_{-2})\right](1 + \varkappa) R e^{\beta\omega} - \mu\delta}{\frac{J_0}{T_0}\sqrt{\varkappa} + \frac{\mu l}{EF}}, \tag{II.299}$$

where

$$J_k = 2\int_0^{\alpha} \frac{\cos(A + k\theta)\, d\theta}{R(\theta)}, \qquad T_k = 2\int_{\alpha}^{\frac{\pi}{2}} \frac{\cos(A + k\theta)\, d\theta}{R(\theta)},$$

$$R(\theta) = \sqrt{|\sin(\theta + \alpha)\sin(\theta - \alpha)|}, \qquad k = 0, \pm 2, \tag{II.300}$$

$$A = \beta \ln\left|\frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)}\right|, \qquad \alpha = \frac{\pi}{2} - \frac{\omega}{4}.$$

Assuming $\nu = 0.25$, $\kappa = 2$ (plane deformation), we find, for the values of ω equal to 90, 60, and 10°, respectively,

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$$P_0(90°) = \frac{3R\,[0{,}707\,(m\Gamma + \Gamma') + \Gamma] - \mu\delta}{\sqrt{2} + \frac{\mu l}{EF}};$$

$$P_0(60°) = \frac{3R\,[0{,}826\,(m\Gamma + \Gamma') + 1{,}05\Gamma] - \mu\delta}{1{,}83 + \frac{\mu l}{EF}}; \qquad \text{(II.301)}$$

$$P_0(10°) = \frac{3R\,[0{,}972\,(m\Gamma + \Gamma') + 1{,}02\Gamma] - \mu\delta}{3{,}59 + \frac{\mu l}{EF}}.$$

Under the conditions of the problem under examination, we find the approximate expression for P_0 for the following simplified hypotheses:

1) normal stresses N on the points of contact between the plate and the arc-shaped plates (or between the plate and the ends of the rods in the absence of the plates), distributed uniformly;

2) tangential stresses in these areas are negligibly small;

3) the vertical components of displacements v of the points of the plate at the points of contact in plane (v, x) form a parabola that is defined by the values of v of the center and ends of the section of contact;

4) the relationship between the average values of these displacements v_{av} and the axial forces in the rods P_0 is

$$2v_{av} = \delta + \frac{P_0 l}{EF}, \qquad \text{(II.302)}$$

where

$$v_{av} = \frac{1}{6}[v(\alpha_1) + 4v(\alpha_0) + v(\beta_1)] \qquad \text{(II.303)}$$

is the height of the rectangle of the same size as the figure bounded by the parabola mentioned above (Figure II.100); α_1 and β_1 are the ends of arc γ_1, α_0 is the mean point on this arc. /181

In the following, we will denote the central arcs that define the positions of points α_k and β_k on circle γ by the same symbols as the points α_k and β_k themselves. Thus, in the given case,

$$\alpha_1 = \frac{\pi}{2} - \frac{\omega}{2}, \qquad \alpha_0 = \frac{\pi}{2}, \qquad \beta_1 = \frac{\pi}{2} + \frac{\omega}{2}. \qquad \text{(II.304)}$$

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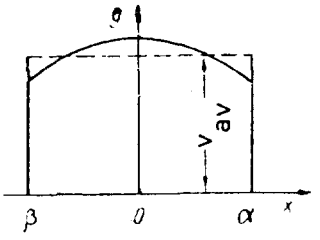


Figure II.100.

On the basis of (II.278), we have the condition on contour γ:

$$[\omega'(\sigma)\Phi(\sigma)]^{+} - [\omega'(\sigma)\Phi(\sigma)]^{-} = f(\sigma), \tag{II.305}$$

where $f(\sigma) = -N\omega'(\sigma)$ on γ_k and $f(\sigma) = 0$ on γ_k'.

Hence, considering (II.291), we find

$$\omega'(\zeta)\Phi(\zeta) = -\frac{N}{2\pi i}\int\limits_{\gamma_1+\gamma_2}\frac{\omega'(\sigma)\,d\sigma}{\sigma-\zeta} + \Gamma R + R(m\Gamma+\Gamma')\frac{1}{\zeta^2}. \tag{II.306}$$

Then, recalling that $\phi(\zeta) = \int\omega'(\zeta)\Phi(\zeta)d\zeta$, we find $\phi(\zeta)$, and along with this, using formulas (II.280) and (II.303), we find v_{av} as a function of N.

Thus, to determine the unknown v_{av}, P_0 and N, we have two equations, namely (II.302) and (II.303). The incomplete third equation

$$P_0 = RN(1+m)\sin\left(\frac{1}{2}\omega\right) \tag{II.307}$$

is found from the condition of equilibrium of one of the plates. As a result of the solution of these equations we find the approximate formula

$$P_0 = \frac{1}{\Delta}\left\{\frac{1}{3}(1+\varkappa)\left(2+\cos\frac{\omega}{2}\right)[(1+m)\Gamma+\Gamma']R - \mu\delta\right\}, \tag{II.308}$$

where

$$\Delta = \frac{1+\varkappa}{6\pi}\left[-2\ln\left(\sin\frac{\omega}{2}\tan^2\frac{\omega}{4}\right) + \frac{\omega}{\sin\frac{\omega}{2}}\left(2+\cos\frac{\omega}{2}\right)\right] -$$

$$-\frac{1}{3}(\varkappa-1)\frac{1-m}{1+m}\tan\frac{\omega}{4} + \frac{\mu l}{EF}.$$

In the practical sense, this formula is much simpler than the analogous formula obtained earlier (II.299), in that it precludes the necessity to calculate the improper integrals J_k and T_k. Assuming in formula (II.308) that $\kappa = 2$, we find /182 for the values of ω, respectively equal to 90, 60 and 10°,

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$$P_0(90°) = \frac{2{,}707R[(1+m)\Gamma+\Gamma'] - \mu\delta}{1{,}490 + \frac{0{,}276m}{1+m} + \frac{\mu l}{EF}};$$

$$P_0(60°) = \frac{2{,}886R[(1+m)\Gamma+\Gamma'] - \mu\delta}{1{,}925 + \frac{0{,}179m}{1+m} + \frac{\mu l}{EF}}; \qquad \text{(II.309)}$$

$$P_0(10°) = \frac{2{,}996R[(1+m)\Gamma+\Gamma'] - \mu\delta}{3{,}710 + \frac{0{,}029m}{1+m} + \frac{\mu l}{EF}}.$$

Comparing equations (II.301) and (II.309), we see that the values P_0 obtained from the approximate formula (II.309) differ only slightly from the corresponding values of P_0 obtained by formula (II.301), for all $\omega \leqslant \pi/2$.

Elliptic Hole with Two Braces That Transmit Pressure to Four Rigid Arc-Shaped Plates. We will assume that the arc plates and the braces are installed within the hole symmetrically with respect to the axes of the ellipse (Figure II.101). The ends of arcs L_k are symmetrical with respect to the center of the ellipse. The elastic braces, of identical length l, and of identical rigidity to tension EF and preliminary tension δ, are arranged such that the arc plates, due to deformation of the braces and plate are displaced without rotations. The boundary conditions on contour γ are also represented in the form (II.293), and the general expression for $\omega'(\zeta)\Phi(\zeta)$, with consideration of symmetry of the elastic state, is

$$\omega'(\zeta)\Phi(\zeta) = R\left[\frac{D_2}{\zeta^2} + C_0\zeta^4 + C_2\zeta^2 + C_4\right]X(\zeta), \qquad \text{(II.310)}$$

where

$$X(\zeta) = (\zeta^2 - \alpha_1^2)^{\lambda}(\zeta^2 - \alpha_1^2)^{\lambda}(\zeta^2 - \beta_1^2)^{\overline{\lambda}}(\zeta^2 - \beta_2^2)^{\overline{\lambda}};$$

D_2, C_0, C_2 and C_4 are real arbitrary constants.

Considering (II.291) and (II.310), we find

$$C_0 = \Gamma, \quad D_2 = (m\Gamma + \Gamma')e^{-4\beta\omega}, \qquad \text{(II.311)}$$

where ω is the central angle corresponding to arcs γ_k.

We will denote through $2iv_1$ and $2u_1$ the difference of displacements of points on arcs L_1 and L_4, L_1 and L_2, and through P_0, the axial force in each of the braces (the resultant vector of forces acting on arc L_k from the side of the material of the plane).

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Then, on the basis of (II.280) and (II.281), we have

$$(1+\varkappa)\int_{\gamma_1'}\Phi^-(\sigma)\,\omega'(\sigma)\,d\sigma = 4\mu v_1 i;$$
$$-i(1+\varkappa)\int_{\gamma_1'}\Phi^-(\sigma)\,\omega'(\sigma)\,d\sigma = iP_0. \qquad \text{(II.312)}$$

Considering (II.310), we find from equations (II.312) /183

$$\begin{aligned}
&D_2J_3 + C_0J_{-3} + C_2J_{-1} + C_4J_1 = \frac{4\mu v_1}{R(1+\varkappa)e^{2\beta\omega}};\\
&D_2T_3 + C_0T_{-3} + C_2T_{-1} + C_4T_1 = -\frac{2P_0}{R(1+\varkappa)e^{\beta(2\omega-\pi)}};\\
&D_2T_3' + C_0T_{-3}' + C_2T_{-1}' + C_4T_1' = 0;\\
&D_2S_3' + C_0S_{-3}' + C_2S_{-1}' + C_4S_1' = \frac{4\mu u_1}{R(1+\varkappa)e^{2\beta\omega}}.
\end{aligned} \qquad \text{(II.313)}$$

The following definitions are made in these equations:

$$J_k = \int_0^{\alpha}\frac{\cos(A+k\theta)\,d\theta}{R(\theta)}; \qquad T_k = \int_{\alpha}^{\varepsilon}\frac{\cos(A+k\theta)}{R(\theta)}\,d\theta;$$

$$T_k' = \int_{\alpha}^{\varepsilon}\frac{\sin(A+k\theta)\,d\theta}{R(\theta)}; \qquad S_k' = \int_{\varepsilon}^{\frac{\pi}{2}}\frac{\sin(A+k\theta)\,d\theta}{R(\theta)};$$

$$\alpha_1 = e^{i\alpha}; \qquad \varepsilon = \alpha+\omega; \qquad \beta = \frac{\ln\varkappa}{2\pi};$$

$$A = \beta\ln\left|\frac{\sin(\theta+\alpha)\sin(\theta-\varepsilon)}{\sin(\theta-\alpha)\sin(\theta+\varepsilon)}\right|;$$

$$R(\theta) = 2\sqrt{\left|\sin(\theta-\alpha)\sin(\theta+\alpha)\sin(\theta-\varepsilon)\sin(\theta+\varepsilon)\right|}.$$

Combining equations (II.313) and

$$2v_1 = \delta + \frac{P_0 l}{EF},$$

we find a system of five equations for the determination of C_1, C_4, P_0, v_1 and u_1. In particular, /184

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$$P_0 = \frac{A^* - \mu\delta}{\sqrt{\varkappa}B + \frac{\mu e}{EF}}, \tag{II.314}$$

where

$$A^* = \frac{(1+\varkappa)R}{2e^{-2\beta\omega}}\left\{D_2\left[J_3 - BT_3 + \frac{T_3'}{T_{-1}'}\left(\frac{BT_{-1}}{J_{-1}} - 1\right)\right] + \right.$$
$$\left. + C_0\left[J_{-3} - BT_{-3} + \frac{T_{-3}'}{T_{-1}'}\left(\frac{BT_{-1}}{J_{-1}} - 1\right)\right]\right\};$$
$$B = \frac{J_1T_{-1} - J_{-1}T_1'}{T_1T_{-1}' - T_{-1}T_1'}.$$

All that remains now is to determine the position of the braces in which the arc plates, during deformations of the plate and braces, are displaced gradually. We will assume that this position of the braces is defined as the distance x_0 from the center of the ellipse to the line of action of the main vector P_0 of the tension forces in the cross section of the brace. To find x_0 we will use the formula

$$M = -(1+\varkappa)\,\mathrm{Re}\int_{\gamma_1}\overline{\omega(\sigma)}\,\omega'(\sigma)\,\Phi^-(\sigma)\,d\sigma, \tag{II.315}$$

which defines the main moment M of forces acting on arc L_1 from the direction of the plate (taken relative to the origin of the coordinate system). As the result, assuming $P_0x_0 = M$, and considering (II.310), we find[1] the equation for the determination of x_0:

$$D_2T_4 + C_0T_{-2} + C_2T_0 + C_4T_2 + m(D_2T_2 + C_0T_{-4} + C_2T_{-2} + C_4T_0) = -\frac{2P_0x_0}{R^2(1+\varkappa)e^{\beta(2\omega-\pi)}}. \tag{II.316}$$

Assuming in (II.314), by way of example, $\omega = 10°$; $\alpha = 45°$; $\nu = 0.25$; $\kappa = 2$, we obtain

$$P_0 = \frac{R[2{,}18\Gamma' + (2{,}39 + 2{,}18m)\Gamma] - \mu\delta}{1{,}88 + \frac{\mu l}{EF}}. \tag{II.317}$$

---

[1]We will disregard the rigidity of the braces to deflection.

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We will assume that the tangential and normal stresses on arcs L_k are distributed uniformly and we will use hypotheses 2-4 presented above. Then, proceeding as before, we find for pressure on each of the braces, the approximate formula

$$P_0 = \frac{A^* - \mu\delta}{B^* + \frac{\mu l}{EF}}, \tag{II.318}$$

where

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$$A^* = \frac{1+\varkappa}{6} R[(1+m)\Gamma + \Gamma'](\sin\alpha + \sin\beta + 4\sin\alpha_0);$$

$$B^* = \frac{1+\varkappa}{6\pi(1+k^2)(\cos\alpha - \cos\beta)}[\cos\alpha \ln\varrho_1 - \cos\beta\ln\varrho_2 +$$

$$+ 2\cos\alpha_0 \ln\varrho_3 + \omega(\sin\alpha + \sin\beta + 4\sin\alpha_0)] -$$

$$-\frac{(\varkappa-1)(1-m)(5\sin\alpha - 4\sin\alpha_0 - \sin\beta)}{12(1+m)(1+k^2)(\cos\alpha - \cos\beta)} + \frac{k(\varkappa-1)(5\cos\alpha - 4\cos\alpha_0 - \cos\beta)}{12(1+k^2)(\cos\alpha - \cos\beta)} -$$

$$-\frac{k(1+\varkappa)(1-m)}{6\pi(1+k^2)(1+m)(\cos\alpha - \cos\beta)}(\sin\alpha\ln\varrho_4 - \sin\beta\ln\varrho_5 - 2\sin\alpha_0\ln\varrho_3);$$

$$\varrho_1 = \left|\frac{\sin\alpha_0 \tan^2\left(\frac{\alpha_0+\alpha}{2}\right)}{\sin\frac{\omega}{2}\cos\alpha\tan^2\left(\frac{\omega}{4}\right)}\right|; \quad \varrho_2 = \left|\frac{\sin\alpha_0\tan^2\left(\frac{\beta+\alpha_0}{2}\right)}{\sin\frac{\omega}{2}\cos\beta\,\mathrm{tg}^2\left(\frac{\omega}{4}\right)}\right|;$$

$$\varrho_3 = \left|\frac{\sin(\alpha_0+\beta)}{\sin(\alpha_0+\alpha)}\right|; \qquad \varrho_4 = \left|\frac{\sin\alpha\tan^2\left(\frac{\alpha_0+\alpha}{2}\right)}{\sin\frac{\omega}{2}\cos\alpha_0\tan^2\left(\frac{\omega}{4}\right)}\right|;$$

$$\varrho_5 = \frac{\sin\beta\tan^2\left(\frac{\alpha_0+\beta}{2}\right)}{\sin\frac{\omega}{2}\cos\alpha_0\tan^2\left(\frac{\omega}{4}\right)}; \qquad k = \frac{(1-m)(\sin\beta - \sin\alpha)}{(1+m)(\cos\alpha - \cos\beta)};$$

$\alpha_0 = \frac{\alpha+\beta}{2}$; $\beta = \alpha + \omega$; $\alpha_1 = e^{i\alpha}$ and $\beta_1 = e^{i\beta}$ are the ends of arc γ_1.

Assuming $\alpha = 45°$, $\omega = 10°$, $\varkappa = 2$, we find by formula (II.318):

$$P_0 = \frac{2{,}30\,R[(1+m)\Gamma + \Gamma'] - \mu\delta}{B + \frac{\mu l}{EF}}, \tag{II.319}$$

where

$$B = \frac{1}{1+k^2}\left[1{,}95 + 0{,}244k + \frac{1-m}{1+m}(0{,}217 + 1{,}63k)\right];$$

$$k = 0{,}8397\frac{1-m}{1+m}.$$

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Considering that when $0 \leqslant m \leqslant 1$ the value of B (in the given case) changes within the range $1.95 \leqslant B \leqslant 2.19$, it is easy to see that the divergence between the values of P_0 found by formula (II.317) and by the approximate formula (II.319), is only slight.

Square Hole with Rigid Plates. Assume that within a square hole with reinforced angles there are two rigid arc-shaped plates, welded to the material of the plate at arcs L_1 and L_2. Between them are installed, with tension δ, two identical braces of length l, with rigidity to tension EF. Arcs L_k and the braces are arranged symmetrically with respect to the middle lines of the contour of the hole (Figure II.102). Recalling that the arc-shaped plates, due to deformation of the plate and braces, are displaced gradually, and proceeding as in the preceding case, we obtain

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$$\omega'(\zeta)\Phi(\zeta) = R\left(\frac{D_4}{\zeta^4} + \frac{D_2}{\zeta^2} + C_0\zeta^2 + C_2\right)X(\zeta). \qquad \text{(II.320)}$$

where

$$X(\zeta) = (\zeta^2 - \alpha_1^2)^{\lambda}(\zeta^2 - \beta_1^2)^{\bar{\lambda}};$$

$$\omega(\zeta) = R\left(\zeta + \frac{n}{\zeta^3}\right);$$

$$n = -\frac{1}{9}; \quad C_0 = \Gamma; \quad D_4 = 3n\Gamma e^{-2\beta\omega};$$

$$D_2 = e^{-2\beta\omega}[\Gamma' + 2n\Gamma(\cos\omega + 4\beta\sin\omega) + nC_2];$$

C_2 is an arbitrary constant[1].

We find the real constant C_2 and pressure on each of the rods P_0 from the equation system

$$\begin{aligned} D_4J_4 + D_2'J_2 + C_0J_{-2} + C_2(J_2D_2' + J_0') &= \lambda_1, \\ D_4T_4 + D_2'T_2 + C_0T_{-2} + C_2(T_2D_2' + T_0) &= \lambda_2, \end{aligned} \qquad \text{(II.321)}$$

---

[1]The constants D_4, D_2 and C_0 are found by subordinating function (II.320) to the following conditions, derived from (II.285) and (II.287):

$$\omega'(\zeta)\Phi(\zeta) = 3\Gamma Rn\zeta^{-4} + (\Gamma' R + n\bar{A}_2)\zeta^{-2} + 0(1) \text{ for } \zeta \to 0;$$

$$\omega'(\zeta)\Phi(\zeta) = \Gamma R + A_2\zeta^{-2} + 0\left(\frac{1}{\zeta^4}\right) \quad \text{for } |\zeta| \to \infty.$$

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where

$$\lambda_1 = \frac{2\mu e^{-\beta\omega}}{R(1+\varkappa)}\left(\delta + \frac{P_0 l}{EF}\right); \quad \lambda_2 = -\frac{2P_0 e^{\beta(\pi-\omega)}}{R(1+\varkappa)};$$

$$D_2' = ne^{-2\beta\omega};$$

$$D_2^* = [\Gamma' + 2n\Gamma(\cos\omega + 4\beta\sin\omega)]\, e^{-2\beta\omega};$$

$$J_k = \int_0^{\alpha} \frac{\cos(A + k\theta)\, d\theta}{R\theta}, \qquad T_k = \int_{\alpha}^{\frac{\pi}{2}} \frac{\cos(A + k\theta)\, d\theta}{R\theta};$$

$$\alpha = \frac{\pi}{2} - \frac{\omega}{2}.$$

Finally, /187

$$P_0 = \frac{A - \mu\delta}{\sqrt{\varkappa}B + \frac{\mu l}{EF}}, \tag{II.322}$$

where

$$A = \frac{(1+\varkappa)R}{2e^{-\beta\omega}}[D_4 J_4 + D_2^* J_2 + C_0 J_{-2} - B(D_4 T_4 + D_2^* T_2 + C_0 T_{-2})];$$

$$B = \frac{J_2 D_2' + J_0}{T_2 D_2' + T_0}.$$

In particular, for values of ω equal to 90 and 10°, we find

$$P_0(90^\circ) = \frac{(2{,}85\Gamma + 2{,}01\Gamma')R - \mu\delta}{1{,}31 + \frac{\mu l}{EF}};$$

$$P_0(10^\circ) = \frac{(3{,}37\Gamma + 2{,}63\Gamma')R - \mu\delta}{3{,}41 + \frac{\mu l}{EF}}. \tag{II.323}$$

It is possible to find the approximate solution of the given problem by using the hypotheses formulated earlier. Considering the properties of the function $\omega'(\zeta)\Phi(\zeta)$, as stated above, we find for pressure on each of the rods, the approximate formula

$$P_0 = \frac{A - \mu\delta}{B + \frac{\mu l}{EF}}, \tag{II.324}$$

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where

$$A = \frac{(1+\varkappa)R}{3}\left[\left(\Gamma + \frac{\Gamma'}{1-n}\right)\left(2+\cos\frac{\omega}{2}\right) - n\Gamma\left(2+\cos\frac{3\omega}{2}\right)\right];$$

$$B = \frac{1}{\sin\frac{\omega}{2} - n\sin\frac{3\omega}{2}}\left\{\frac{1+\varkappa}{6\pi}\left[-2\left(\sin\frac{\omega}{2} - n\sin\frac{3\omega}{2}\right)\ln\left(\sin\frac{\omega}{2}\tan^2\left(\frac{\omega}{4}\right)\right) + \right.\right.$$

$$+ \omega\left(2+\cos\frac{\omega}{2} + n\left(2+\cos\frac{3\omega}{2}\right) - 4n\sin^2\omega\right] +$$

$$\left.\left. + \frac{1+\varkappa}{3\pi}\frac{n\sin\omega}{1-n}\left(2+\cos\frac{\omega}{2}\right) - \frac{\varkappa-1}{3}\left[1-\cos\frac{\omega}{2} + n\left(1-\cos\frac{3\omega}{2}\right)\right]\right\}.$$

Assuming that $\varkappa = 2$ and $n = -1/9$ in the last formula, we find for values of ω equal to 90 and 10°,

$$P_0(90^\circ) = \frac{(2{,}85\Gamma + 2{,}44\Gamma')R - \mu\delta}{1{,}42 + \frac{\mu l}{EF}};$$

$$P_0(10^\circ) = \frac{(3{,}33\Gamma + 2{,}70\Gamma')R - \mu\delta}{3{,}45 + \frac{\mu l}{EF}}. \qquad \text{(II.325)}$$

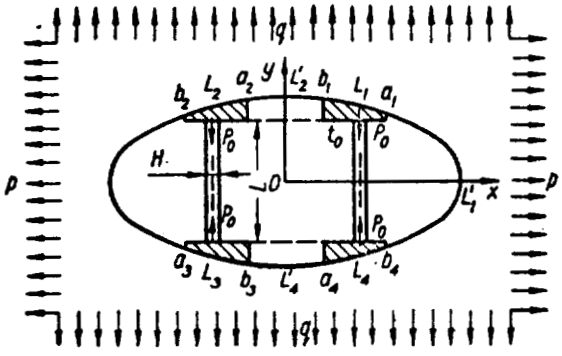


Figure II.101.

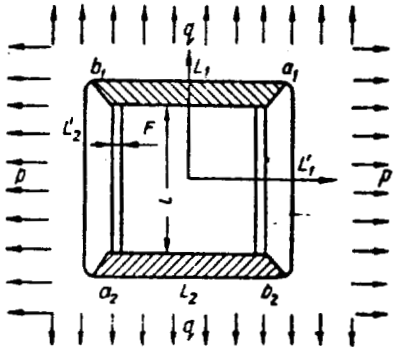


Figure II.102.

As we will see, the values P_0 from (II.323) and (II.325) differ only slightly from each other. /188

Rectangular Hole with Three Supporting Braces. Assume that within a rectangular hole there are three braces 1, 2 and 3 (Figure II.103), with tensions δ_k, rigidity to tension E_kF_k and length l_k (k = 1, 2, 3). We will assume that the axes of the coordinate plane xOy are directed along the axes of

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symmetry of the holes; the axes of braces 1 and 3 are parallel and equidistant from axis Oy, and the axis of brace 2 is directed along the Oy axis. Furthermore, we will assume that $l_3 = l_1$, $E_3F_3 = E_1F_1$, $\delta_3 = \delta_1$. The arcs on which the braces come into contact with the material of the plate without a hole are denoted through L_k (k = 1, 2, ..., 6), and the other parts of contour L of the hole are denoted through L_k'.

We will find the total axial force[1] P_k transmitted to each brace from the direction of the material of the plate. The boundary condition is

$$\omega'(\sigma)\Phi^+(\sigma) - \omega'(\sigma)\Phi^-(\sigma) = f(\sigma) \text{ on } \gamma, \tag{II.326}$$

where

$$f(\sigma) = -N_k\omega'(\sigma) \text{ on } \gamma_k; \quad f(\sigma) = 0 \text{ on } \gamma_k';$$

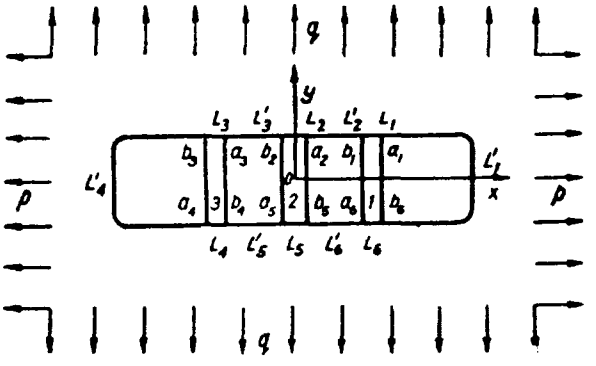


Figure II.103.

N_k is normal stress on L_k (k = 1, 2, ..., 6);

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta} + \frac{n}{\zeta^3} + \frac{l}{\zeta^5}\right).$$

We will note here that due to symmetry of the stress state,

$$N_1 = N_3 = N_4 = N_6; \quad N_2 = N_5.$$

On the basis of (II.326), recalling the form of function $\omega'(\zeta)\Phi(\zeta)$ when $\zeta \to 0$ and $|\zeta| \to \infty$, defined by formulas (II.285) and (II.287), we find /189

$$\omega'(\zeta)\Phi(\zeta) = \Gamma R - \frac{\omega'(\zeta)}{2\pi i}\sum_{k=1}^{6} N_k \ln\frac{\zeta-\beta_k}{\zeta-\alpha_k} + \frac{R}{\zeta^2}(B_0 + B_1N_1 + B_2N_2) + \tag{II.327}$$

$$+ \frac{R^4}{\zeta^4}(C_0 + C_1N_1 + C_2N_2) + \frac{R}{\zeta^6}(D_0 + D_1N_1 + D_2N_2),$$

where

$$B_0 = \frac{1}{\Delta}[\Gamma' + \Gamma(m + mn + 6nl + m^2l + 3ml^2)];$$

$$\Delta = 1 - n - ml - 3l^2;$$

[1]Obviously, due to symmetry, $P_3 = P_1$.

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$$B_1 = -\frac{2}{\pi\Delta}[(m+3nl)\omega_1 + (n+l^2)(\sin 2\beta - \sin 2\alpha) + l(\sin 4\beta - \sin 4\alpha)];$$

$$B_2 = -\frac{1}{\pi\Delta}[(m+3nl)\omega_2 - 2(n+l^2)\sin\omega_2 + 2l\sin 2\omega_2];$$

$$C_0 = \frac{3}{\Delta}[l\Gamma' + \Gamma(n+2ml-n^2-mnl+3nl^2)];$$

$$C_1 = -\frac{2}{\pi\Delta}[3(1-n)(n+ml)\omega_1 + l(1+2n-ml)(\sin 2\beta - \sin 2\alpha) + \\ + 3l^2(\sin 4\beta - \sin 4\alpha)]; \qquad \text{(II.328)}$$

$$C_2 = -\frac{1}{\pi\Delta}[3(1-n)(n+ml)\omega_2 - 2l(1+2n-ml)\sin\omega_2 + 6l^2\sin 2\omega_2];$$

$$D_0 = sl\Gamma; \qquad D_1 = -\frac{10}{\pi}l\omega_1; \qquad D_2 = -\frac{5}{\pi}l\omega_2;$$

$\beta = \alpha + \omega_1$; α_k and β_k are the ends of arcs $\gamma_k(L_k)$; ω_1 is the central angle corresponding to arcs γ_1, γ_3, γ_4, γ_6; ω_2 is the central angle for arcs γ_2 and γ_5; α is a polar angle that defines the position of a point on circle $\gamma(L)$.

Then, using formulas (II.280) and (II.327) and hypothesis 3, we find $v_{av}^{(1)}$ and $v_{av}^{(2)}$ on arcs $L_1(\gamma_1)$ and $L_2(\gamma_2)$:

$$2\mu v_{a\gamma}^{(1)} = 2\mu\frac{v(\alpha_1)+v(\beta_1)+4v(\alpha_0)}{6} = Rn_0 + N_1n_1 + N_2n_2,$$

$$2\mu v_{av}^{(2)} = 2\mu\frac{v(\alpha_2)+v(\beta_2)+4v\left(\frac{\pi}{2}\right)}{6}, \qquad \text{(II.329)}$$

where

$$n_0 = \frac{1+\varkappa}{6}\Big[(\Gamma + B_0)(\sin\alpha + \sin\beta + 4\sin\alpha_0) + \\ + \frac{C_0}{3}(\sin 3\alpha + \sin 3\beta + 4\sin 3\alpha_0) + \frac{D_0}{5}(\sin 5\alpha + \sin 5\beta + 4\sin 5\alpha_0)\Big];$$

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$$n_1 = -\frac{1+\varkappa}{12\pi}\{T(\alpha)\ln\varrho_1 + T(\beta)\ln\varrho_2 + 4T(\alpha_0)\ln\varrho_3 + \\ + 2\omega_1[S(\alpha) + S(\beta) + 4S(\alpha_0)]\} - \frac{1+\varkappa}{6\pi}R\{2n(\sin 2\alpha - \sin 2\beta) + \\ + [2l(\sin 4\alpha - \sin 4\beta) - \pi B_1](\sin\alpha + \sin\beta + 4\sin\alpha_0) + \\ + \frac{1}{3}[2l(\sin 2\alpha - \sin 2\beta) - \pi C_1](\sin 3\alpha + \sin 3\beta + 4\sin 3\alpha_0)\} + \\ + \frac{1+\varkappa}{30}RD_1(\sin 5\alpha + \sin 5\beta + 4\sin 5\alpha_0) + \frac{1-\varkappa}{12}[5S(\alpha) - 4S(\alpha_0) - S(\beta)];$$

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$$n_2 = -\frac{1+\varkappa}{12\pi}\{T(\alpha)\ln\varrho_4 + T(\beta)\ln\varrho_5 - T(\alpha_2)\ln\varrho_6 + 4T(\alpha_0)\ln\varrho_7 +$$

$$+ \omega_2[S(\alpha) + S(\beta) + 4S(\alpha_0)]\} - \frac{1+\varkappa}{6\pi}R\Big[(2n\sin\omega_2 - 2l\sin 2\omega_2 -$$

$$- \pi B_2)(\sin\alpha + \sin\beta + 4\sin\alpha_0) + \left(\frac{2}{3}l\sin\omega_2 - \pi C_2\right)(\sin 3\alpha +$$

$$+ \sin 3\beta + 4\sin 3\alpha_0)] + \frac{1+\varkappa}{30}RD_2(\sin 5\alpha + \sin 5\beta + 4\sin 5\alpha_0);$$

$$m_0 = \frac{1+\varkappa}{3}\left[(\Gamma + B_0)\left(2 + \cos\frac{\omega_2}{2}\right) - \frac{1}{3}C_0\left(2 + \cos\frac{3\omega_2}{2}\right) + \frac{D_0}{2}\left(2 + \cos\frac{5\omega_2}{2}\right)\right];$$

$$m_1 = -\frac{1+\varkappa}{6\pi}\left\{T(\alpha_2)\ln\varrho_8 + T(\beta)\ln\varrho_9 + T(\alpha)\ln\varrho_{10} + 2\omega_1\left[2S\left(\frac{\pi}{2}\right) + S(\alpha_2)\right]\right\} -$$

$$- \frac{1+\varkappa}{3\pi}R\Big\{2n(\sin 2\alpha - \sin 2\beta) + [2l(\sin 4\alpha - \sin 4\beta) - \pi B_1](2 + \sin\alpha_2) +$$

$$+ \frac{1}{3}[2l(\sin 2\alpha - \sin 2\beta) - \pi C_1](-2 + \sin 3\alpha_2)\Big\} +$$

$$+ \frac{1+\varkappa}{15}D_1(2 + \sin 5\alpha_2) + \frac{1-\varkappa}{2}[S(\alpha) - S(\beta)];$$

$$m_2 = -\frac{1+\varkappa}{6\pi}\left\{2T(\alpha_2)\ln\varrho_{11} + \omega_2\left[2S\left(\frac{\pi}{2}\right) + S(\alpha_2)\right]\right\} -$$

$$- \frac{1+\varkappa}{3\pi}R\Big[(2n\sin 2\alpha_2 + 2l\sin 4\alpha_2 - \pi B_2)(2 + \sin\alpha_2) + \qquad \text{(II.330)}$$

$$+ \frac{1}{3}(2l\sin 2\alpha_2 - \pi C_2)(-2 + \sin 3\alpha_2)\Big] +$$

$$+ \frac{1+\varkappa}{15}RD_2(2 + \sin 5\alpha_2) + \frac{\varkappa-1}{3}\left[S\left(\frac{\pi}{2}\right) - S(\alpha_2)\right];$$

$$T(\theta) + iS(\theta) = \omega(\sigma); \qquad \sigma = e^{i\theta};$$

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$$\alpha_0 = \frac{1}{2}(\alpha + \beta); \qquad \alpha_2 = \frac{\pi}{2} - \frac{\omega_2}{2}; \qquad \beta_2 = \frac{\pi}{2} + \frac{\omega_2}{2};$$

$$\varrho_1 = \frac{\sin^2\left(\frac{\alpha+\beta}{2}\right)}{\sin^2\left(\frac{\omega_1}{2}\right)\cos^2\alpha}\left(\frac{\tan\frac{\alpha_0+\alpha}{2}}{\tan\frac{\alpha_0-\alpha}{2}}\right)^4; \qquad \varrho_2 = \frac{\sin^2\left(\frac{\omega_1}{2}\right)\cos^2\beta}{\sin^2\left(\frac{\alpha+\beta}{2}\right)}\left(\frac{\tan\frac{\alpha_0-\beta_1}{2}}{\tan\frac{\alpha_0+\beta_1}{2}}\right)^4;$$

$$\varrho_3 = \left|\frac{\sin(\alpha_0+\beta)}{\sin(\alpha_0+\alpha)}\right|; \qquad \varrho_4 = \left|\frac{\sin(\alpha-\alpha_2)}{\sin(\alpha+\alpha_2)}\right|;$$

$$\varrho_5 = \left|\frac{\sin(\beta-\alpha_2)}{\sin(\beta+\alpha_2)}\right|; \qquad \varrho_6 = \left|\frac{\tan\frac{\alpha-\alpha_2}{2}\operatorname{tg}\frac{\beta-\alpha_2}{2}}{\tan\frac{\alpha+\alpha_2}{2}\operatorname{tg}\frac{\beta+\alpha_2}{2}}\right|\left(\frac{\tan\frac{\alpha_0-\alpha_2}{2}}{\tan\frac{\alpha_0+\alpha_2}{2}}\right)^4;$$

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$$\varrho_7 = \left|\frac{\sin(\alpha_0-\alpha_2)}{\sin(\alpha_0-\beta_2)}\right|; \qquad \varrho_8 = \left[\frac{\cos\left(\alpha+\frac{\omega_2}{2}\right)\cos\left(\beta-\frac{\omega_2}{2}\right)}{\cos\left(\alpha-\frac{\omega^2}{2}\right)\cos\left(\beta+\frac{\omega^2}{2}\right)}\right]^2;$$

$$\varrho_9 = \left|\frac{\tan\frac{\alpha_2-\beta}{2}}{\tan\frac{\alpha_2+\beta}{2}}\right|\tan^4\left(\frac{\pi}{4}-\frac{\beta}{2}\right); \qquad \varrho_{10} = \left|\frac{\tan\frac{\alpha_2+\alpha}{2}}{\operatorname{tg}\frac{\alpha_2-\alpha}{2}}\right|\tan^4\left(\frac{\pi}{4}+\frac{\alpha}{2}\right);$$

$$\varrho_{11} = \frac{\cot^2\left(\frac{\omega_2}{4}\right)}{\sin\left(\frac{\omega_2}{2}\right)}.$$

In formulas (II.330) α and β denote the same as in (II.328); the polar angles that define the positions of the ends of arc $\gamma_2 = \alpha_2\beta_2$ are denoted through α_2 and β_2 (by the same symbols as points α_2 and β_2 themselves).

On the basis of hypothesis 4,

$$2v_{av}^{(1)} = \delta_1 + \frac{P_1 l_1}{E_1 F_1},$$
$$2v_{av}^{(2)} = \delta_2 + \frac{P_2 l_2}{E_2 F_2}. \qquad \text{(II.331)}$$

On the other hand, using formulas (II.281) and (II.327), we find

$$P_1 = [T(\alpha) - T(\beta)]N_1 = k_1 N_1,$$
$$P_2 = [T(\alpha_2) - T(\beta_2)]N_2 = k_2 N_2. \qquad \text{(II.332)}$$

From equations (II.331) and (II.332), recalling (II.329), we find the following formulas for the determination of the axial forces in braces 1 and 2 (see Figure II.103): /192

$$P_1 = \frac{(Rn_0 - \mu\delta_1)\left(\frac{\mu l_2}{E_2F_2} - \frac{m_2}{k_2}\right) + (Rm_0 - \mu\delta_2)\frac{n_2}{k_2}}{\left(\frac{\mu l_1}{E_1F_1} - \frac{n_1}{k_1}\right)\left(\frac{\mu l_2}{E_2F_2} - \frac{m_2}{k_2}\right) - \frac{m_1 n_2}{k_1 k_2}}; \qquad \text{(II.333)}$$

$$P_2 = \frac{(Rn_0 - \mu\delta_1)\frac{m_1}{k_1} + (Rm_0 - \mu\delta_2)\left(\frac{\mu l_1}{E_1F_1} - \frac{n_1}{k_1}\right)}{\left(\frac{\mu l_1}{E_1F_1} - \frac{n_1}{k_1}\right)\left(\frac{\mu l_2}{E_2F_2} - \frac{m_2}{k_2}\right) - \frac{m_1 n_1}{k_1 k_2}}. \qquad \text{(II.334)}$$

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In particular, by assuming $E_1 \to \infty$, we find

$$P_2 = \frac{Rm_0 - \mu\delta_2}{\frac{\mu l_2}{E_2 F_2} - \frac{m_2}{k_2}}; \tag{II.335}$$

assuming $E_2 \to \infty$, we find

$$P_1 = \frac{Rn_0 - \mu\delta_1}{\frac{\mu l_1}{E_1 F_1} - \frac{n_1}{k_1}}. \tag{II.336}$$

Pressure on Cross Pieces between Holes in Infinite Elastic Isotropic Plane. We will use the formulas derived in this section for the determination of the total pressure on each of the cross pieces between identical holes in the form of squares with rounded corners or ellipses. We will assume that the centers of the holes are located on one straight line, and that this straight line is their common axis of symmetry, while the sides of the squares and the principal diameters of the ellipses coincide with the direction of the principal stresses at infinity.

Pressure on Cross Pieces between Three Square Holes. Let the holes be arranged as indicated in Figure II.104; 2a is the side of the squares, a is the width of each cross piece. By regarding three holes, along with the material that separates them, as one rectangular hole with two prismatic rods within it, we find first the parameters in the expression of the mapping function

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta} + \frac{n}{\zeta^3} + \frac{l}{\zeta^5}\right).$$

Thus, using geometrical considerations[1], we find

$$m = \frac{5}{9}, \quad n = -\frac{1}{9}, \quad l = -\frac{1}{45}, \quad R = \frac{45a}{16}.$$

Then, assuming in the parametric equation that /193

$$x = R[(1 + m)\cos\theta + n\cos 3\theta + l\cos 5\theta]$$

we find angles α and β for the abscissa of points of contour L of hole x equal to 2a and a.

[1]By determining, for instance, R, m, n and l from equations $x = 4a$, $d^2x/d\theta^2 = 0$ when $\theta = 0$, $y = b$; $d^2y/d\theta^2 = 0$, when $\theta = \pi/2$.

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The numerical value for pressure on each of the cross pieces is found by formula (II.336), assuming $\delta_1 = 0$, $E_1 = 2\mu(1 + \nu)$. Then, by using, for instance, $p = \lambda q$, $\lambda = \nu/1 - \nu$ and $\kappa = 3 - 4\nu$ (plane deformation), corresponding to Poisson ratios ν equal to 0.25 and 0.50, we find

$$P_1|_{\nu=\frac{1}{4}} = 2{,}05aq, \qquad P_1|_{\nu=\frac{1}{2}} = 1{,}82aq.$$

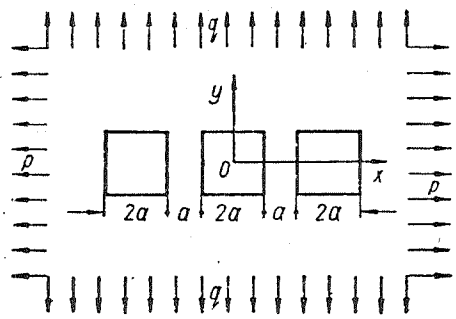


Figure II.104.

If, however, the three holes are circumscribed by an ellipse with semiaxes 4a and a, rather than by a rectangle, then, by using formula (II.318) (assuming $l = 2R(1 - m)\sin\alpha$), we find[1]

$$P_1|_{\nu=\frac{1}{4}} = 2{,}12aq, \qquad P_1|_{\nu=\frac{1}{2}} = 1{,}88aq.$$

Pressure on Cross Pieces between Round and Elliptical Holes. As other examples of the use of the formulas derived in this section, the total pressure P_0 on the cross piece between two identical holes of circular and elliptical form is determined under the conditions of plane deformation for $\nu = 0.3$. The numerical values obtained are presented in Table II.41. The first two rows of this table pertain to the case of round holes of radius r, the next to last, to the case of elliptic holes with semiaxes 10 and 4, and the last, to the case where the ellipses are reduced to slits. The semiaxes of ellipses that "surround" both holes are listed in the first two columns of the table, and the smallest width of the cross piece is listed in the third column (Figure II.105).

TABLE II.41.

a	b	d	P_0	P_0'	P_0''	P_0'''
3,0 r	r	2 r	3,99 qr	3,80 qr	3,77 qr	3,48 qr
2,1 r	r	0,2 r	1,104 qr	1,10 qr	1,10 qr	0,70 qr
22,5	4	5	23,16 q	21,10 q	19,0 q	19,20 q
22,5	0	5	19,30 q	19,30 q	19,6 q	19,30 q

Tr. Note: Commas indicate decimal points.

[1] By assuming $l = 2R(1 - m)\sin\alpha_0$, for the same conditions we obtain, $P_1|_{\nu=\frac{1}{4}} = 2{,}08aq$ and $P_1|_{\nu=\frac{1}{2}} = 1{,}84aq$.

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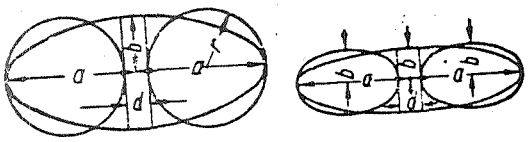


Figure II.105.

We will agree to call the region between the holes, located within the contour surrounding both holes, as the cross piece between the holes (as a whole). The points of intersection of the "surrounding" contour with the contours of the holes are naturally assumed to be the ends of the arcs of contact of the cross piece and the material occupying the exterior of the surrounding contour. /194

By regarding a cross piece as a rod (brace) of varying cross section of some length l, we obtain the average value of 1/EF for the cross piece by two identical elliptic holes:

$$\left(\frac{1}{EF}\right)_{av} = \frac{1}{EF}\left[\int_0^{\gamma} \frac{e+E}{e-\cos\psi}\,d\psi - 2\sqrt{1-\varepsilon^2}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{1}{2}\gamma\right)\right], \qquad \text{(II.337)}$$

where ψ is a parameter in the polar equation of the contour of the elliptic hole;

$$r = \frac{p^*}{1+\varepsilon\cos\psi};$$

γ is the value of the parameter ψ that defines the points of intersection of the surrounding contour with the contour of the holes;

$$e = \frac{k}{p^*-\varepsilon k};$$

k is the distance between the average point of the smallest cross section of the cross piece and the foci of the elliptic holes nearest to it.

For such a determination of the rigidity of a cross piece, the values of P_0 calculated by formula (II.308) are presented in the fourth column of Table II.41. The values of the same P_0'' obtained by D. I. Sherman [3] by a more rigorous, but much more complicated method, are given in the next to last column of this table.

Comparison of P_0 and P_0'' indicates that the values of P_0 are somewhat overstated, particularly in the case of holes that are located farther apart. If, on the other hand, the rigidity of a prismatic rod of area F = d is used as the rigidity of a rod of varying cross section, then, for the very same region of contact between the cross piece and the rest of the plate, we find somewhat understated values of P_0, denoted in Table II.41 through P_0''' (last column).

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It is clear that the actual rigidity 1/EF falls somewhere between these two extreme cases, and there is no doubt that comparison with precise solutions /195 will permit the selection of the average rigidity 1/EF, between these extreme cases, which will yield, in the practical sense, suitable results. Thus, if we denote through Δ' the value $\mu l/EF$ in the first case, assuming that 1/EF is defined by formula (II.337), and through Δ'_0 the value $\mu l/EF$ in the second case (F = d), and assume, for instance,

$$\frac{\mu l}{EF} = \Delta' + \frac{1}{3}(1 + 2m)(\Delta'_0 - \Delta')\frac{d^2}{d^2 - (0{,}5l)^2}, \qquad \text{(II.338)}$$

then, by formula (II.308) or by (II.335), we will find the values for P_0 that are denoted in Table II.41 (fifth column) by P'_0, extremely close to the values of P''_0 obtained by D. I. Sherman [3]. Comparison of the numerical values in Table II.41 shows that the approximate formulas derived for the resultant pressure on the cross pieces between the holes are more precise when the holes are located closer together.

Thus, by analyzing the numerical examples cited in this section, it is possible to derive the following conclusions:

the pressure on braces with rigid arc-shaped plates on the ends, occurring due to landing stresses, is somewhat greater than the pressure on braces without the plates if the corresponding sections of contact of the plates with the main plate are larger than the areas of contact of the braces with the main plate;

the pressure on the braces within the elliptic holes is somewhat greater than that on braces with the same rigidity within a rectangular hole with sides equal to the principal axes of the elliptic hole;

the most accurate total pressure on the cross pieces between round and elliptic holes located close to each other can be found by the same method as used for finding P_0, listed in the fifth column of Table II.41;

pressure on the cross piece between round or elliptic (differing only slightly from round) holes can be found with sufficient accuracy by example of P'_0 in Table II.41, as long as the holes are not located too near or far from each other;

if elliptic holes differ considerably from round, and if their mutual arrangement cannot be regarded as close, the most accurate pressure on the cross pieces between such holes can be found by the same method as for P'''_0 listed in the last column of Table II.41.

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Concluding Comments. The solution for a plane weakened by any finite number of identical round holes, distributed in any manner, and not having common points, was first found by G. N. Bukharinov [1]. W. T. Koiter [1], who reduced the solution of the first basic problem for a plane weakened by a double periodic system of identical holes to the solution of Fredholm's integral equation of the second kind, apparently was the first to investigate double periodic problems. The case of double periodic systems with a regular distribution of holes, under tension, was analyzed by Chen-Lin-Si [1], Chou-Chen-Ti [1], D. I. Sherman [3, 7], R. Bailey, R. Hicke [1], Lin-Chi-Bin [1], Saito Hideo [1]. These authors solved the problems on the basis of Appel's theory (see E. Gursa [1]) for multiply-connected regions. L. A. Fil'shtinskiy [1], G. A. Van Fo Fy [1] and Jan Dvorak [1] solved their problems on the basis of the theory of elliptic functions. /196

Also noteworthy is the article of J. W. Dally and A. I. Durelli [1], a review, in which the results of numerous studies (over the last 10 years) on stress concentration near holes in plates perforated by various systems of identical round holes, are presented (with sources indicated) in compact form as simple graphs and tables. Also noteworthy are the two articles by William Criffel [1-3], in review form, in which data of the concentration factors near a single hole of various form (round, elliptic, square, triangular, rectangular with rounded corners) under various conditions at infinity (uniaxial or biaxial tension, compression, pure displacement, pure deflection, etc) are presented in convenient and compact form (with sources indicated). Presented are data concerning the effect of two round holes on each other; data for an infinite row of equal round holes, and also for plates, perforated with various forms of systems of equal round holes (square, rectangular, rhombic, triangular, etc).

The solutions given above for stress concentration near the various holes indicate that in the uniaxial stress state, zones of the opposite sign, in relation to stresses in infinitely remote points, are formed near the hole, i.e., when the plate is under tension in the region of compressive stresses. In the case of a thin plate, the stresses that occur in these zones can be of such a magnitude that the plate loses its stability and collapses. The stress state in the collapsed zones will differ from that found from the solution of the plane problem. Such problems are analyzed in the work of G. P. Cherepanov [1].

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CHAPTER III. EFFECT OF ANISOTROPY OF MATERIAL ON STRESS DISTRIBUTION AROUND FREE AND REINFORCED HOLES

Abstract. This chapter deals with the anistropic elastic medium. Methods are given of solving the principal boundary value problems for the medium containing one or several elliptical holes, and for curvilinear holes discussed in Chapter II, the method of small parameter was used. A great number of particular problems are given with the numerical results. All results are presented in the forms of tables and diagrams, thus shown the effect of anisotropy on stress concentration around holes under different kinds of medium loading.

§1. Solution of First Basic Problem for Region with Elliptic Hole[1]

We will assume that an elliptic hole, around the contour of which are given the external forces X_n, Y_n (the first basic problem) or components u and v of displacement of its points (second basic problem), is made in an infinite anisotropic elastic plate. We are required to determine the stress state in the examined plane near the hole. We direct the Ox and Oy axes along the corresponding axes of the ellipse. We will denote the semiaxes of the ellipse through a and b. /209

Along with the given plane $z = x + iy$ we will examine the planes z_1 and z_2 which are taken from it by the affine transformation:

$$z_1 = x + s_1 y = x_1 + iy_1,$$
$$z_2 = x + s_2 y = x_2 + iy_2, \qquad \text{(III.1)}$$

where $s_1 = \alpha_1 + i\beta_1$, $s_2 = \alpha_2 + i\beta_2$ are the roots of equation (I.84); $x_1 = x + \alpha_1 y$; $y_1 = \beta_1 y$; $x_2 = x + \alpha_2 y$; $y_2 = \beta_2 y$. By this transformation the given ellipse on plane z is converted into ellipses on planes z_1 and z_2 (Figure III.1).

Through S, $S^{(1)}$ and $S^{(2)}$ we will denote the corresponding regions outside of these ellipses. Then we will determine the functions that conformally map regions S, $S^{(1)}$ and $S^{(2)}$ on the interior of unit circle γ. The function

[1]The solution of this problem was found by S. G. Lekhnitskiy [5] with the aid of series and later by G. N. Savin, using another method [2]; see also H. D. Conway [1], S. G. Lekhnitskiy [7], S. G. Lekhnitskiy, V. V. Soldatov [1] and V. V. Soldatov [1].

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$$z = \omega(\zeta) = \frac{a-b}{2}\zeta + \frac{a+b}{2}\cdot\frac{1}{\zeta}, \qquad \text{(III.2)}$$

as we know (see (I.42)) conformally maps region S on the interior of unit circle γ.

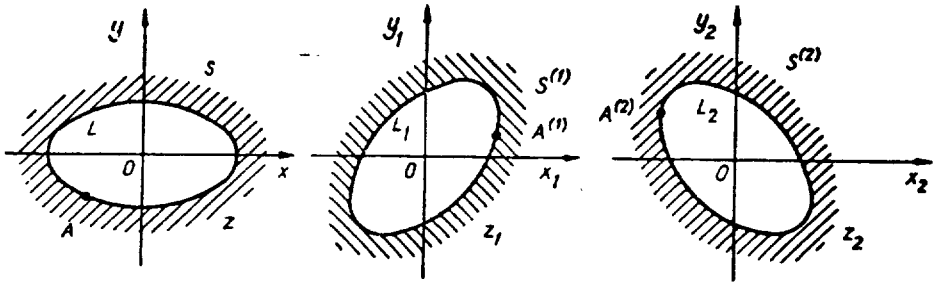


Figure III.1.

The coordinates of the points of the contour of the ellipse of plane S, according to (III.2), are

$$x = a\cos\vartheta, \quad y = -b\sin\vartheta. \qquad \text{(III.3)}$$

After transformation (III.1), the points of contour L of plane S are transformed into points of the contour L_1 of plane $S^{(1)}$, i.e.,

$$z_1 = x + s_1 y = a\cos\vartheta - s_1 b\sin\vartheta = \frac{1}{2}\left[a\left(\sigma + \frac{1}{\sigma}\right) + is_1 b\left(\sigma - \frac{1}{\sigma}\right)\right] =$$

$$= \frac{a+is_1 b}{2}\sigma + \frac{a-is_1 b}{2}\cdot\frac{1}{\sigma}.$$

Thus, the function that conformally maps region $S^{(1)}$ on the interior of unit circle γ is /210

$$z_1 = \omega_1(\zeta) = \frac{a+is_1 b}{2}\zeta + \frac{a-is_1 b}{2}\cdot\frac{1}{\zeta}. \qquad \text{(III.4)}$$

Analogously, we find the function that conformally maps region $S^{(2)}$ on the interior of unit circle γ:

$$z_2 = \omega_2(\zeta) = \frac{a+is_2 b}{2}\zeta + \frac{a-is_2 b}{2}\cdot\frac{1}{\zeta}. \qquad \text{(III.5)}$$

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As follows from the construction of the mapping functions $\omega_1(\zeta)$ and $\omega_2(\zeta)$ (III.4) and (III.5), to points A, $A^{(1)}$ and $A^{(2)}$ of the contours of regions S, $S^{(1)}$, $S^{(2)}$, which are located in affine correspondence (see Figure III.1), corresponds one point on the contour of the unit circle.

We will assume that to contour L of region S (plane with elliptical hole) are applied external forces X_n, Y_n, the resultant vector of which is equal to 0. We will further assume that the stresses at infinity are also equal to zero[1]. In this case, according to §3, Chapter I, the functions $\phi(z_1)$ and $\psi(z_2)$, which are holomorphous in their regions $S^{(1)}$ and $S^{(2)}$ and in contour L of region S, satisfy conditions (I.96):

$$\begin{aligned} 2\,\mathrm{Re}\,[\varphi(z_1)+\psi(z_2)] &= -\int_0^s Y_n ds + C_1 = f_1, \\ 2\,\mathrm{Re}\,[s_1\varphi(z_1)+s_2\psi(z_2)] &= \int_0^s X_n ds + C_2 = f_2. \end{aligned} \tag{III.6}$$

Instead of z_1 in equation (III.6), we will substitute the value $\omega_1(\zeta)$ of (III.4), and instead of z_2, the value $\omega_2(\zeta)$ (III.5). By denoting $\Phi(\zeta) = \phi[\omega_1(\zeta)]$ and $\Psi(\zeta) = \psi[\omega_2(\zeta)]$, we find the boundary conditions for functions $\Phi(\zeta)$ and $\Psi(\zeta)$: /211

$$\begin{aligned} 2\,\mathrm{Re}\,[\Phi(\sigma)+\Psi(\sigma)] &= f_1(\vartheta), \\ 2\,\mathrm{Re}\,[s_1\Phi(\sigma)+s_2\Psi(\sigma)] &= f_2(\vartheta), \end{aligned} \tag{III.7}$$

where $f_1(\vartheta)$ and $f_2(\vartheta)$ are the values of the right hand sides of (III.6), where instead of x and y, are substituted their values from (III.3); $\sigma = e^{i\vartheta}$ is the value of ζ on the contour of unit circle γ.

To define functions $\Phi(\zeta)$ and $\Psi(\zeta)$, which are holomorphous within unit circle γ and satisfy condition (III.7) on its contour, we will use Schwartz formula[2]

$$F(\zeta) = \frac{1}{2\pi i}\int_\gamma U(\vartheta)\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + i\alpha_0, \tag{III.8}$$

where $U(\vartheta)$ is the real part of the function $F(\zeta)$ on the contour of the unit circle; α_0 is some real constant.

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[1]A more general case will be considered below.
[2]See N. I. Muskhelishvili [1], p. 284.

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We will multiply both parts of equations (III.7) by $\frac{1}{2\pi i}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}$ and integrate with respect to contour γ. According to (III.8) we have[1]

$$\Phi(\zeta)+\Psi(\zeta)=\frac{1}{4\pi i}\int_{\gamma} f_1(\vartheta)\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+i\alpha_0,$$
$$s_1\Phi(\zeta)+s_2\Psi(\zeta)=\frac{1}{4\pi i}\int_{\gamma} f_2(\vartheta)\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+i\beta_0. \tag{III.9}$$

By solving equations (III.9) relative to functions $\Phi(\zeta)$ and $\Psi(\zeta)$, we obtain

$$\Phi(\zeta)=\frac{i}{4\pi(s_1-s_2)}\int_{\gamma}[s_2f_1(\vartheta)-f_2(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+\lambda_1,$$
$$\Psi(\zeta)=-\frac{i}{4\pi(s_1-s_2)}\int_{\gamma}[s_1f_1(\vartheta)-f_2(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+\lambda_2, \tag{III.10}$$

where, for brevity, we write

$$\lambda_1=i\frac{\beta_0-\alpha_0s_2}{s_1-s_2};\quad \lambda=-i\frac{\beta_0-\alpha_0s_1}{s_1-s_2}. \tag{III.11}$$

Returning now to variables z_1 and z_2, i.e., by substituting into function $\Phi(\zeta)$, instead of ζ, its value[2]

$$\zeta_1=\frac{z_1-\sqrt{z_1^2-(a^2+s_1^2b^2)}}{a+is_1b}=\frac{a-is_1b}{z_1+\sqrt{z_1^2-(a^2+s_1^2b^2)}}, \tag{III.12}$$

and in function $\Psi(\zeta)$, instead of ζ, its value /212

$$\zeta_2=\frac{z_2-\sqrt{z_2^2-(a^2+s_2^2b^2)}}{a+is_2b}=\frac{a-is_2b}{z_2+\sqrt{z_2^2-(a^2+s_2^2b^2)}}, \tag{III.13}$$

---

[1] In formulas (III.9) the variable ζ in function Φ should be regarded as ζ_1 in (III.12), and in function Ψ, as ζ_2 of (III.13).

[2] Functions (III.12) and (III.13) are found by inverting formulas (III.4) and (III.5).

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we find, finally, the functions $\phi(z_1)$ and $\psi(z_2)$. Since the constants λ_1 and λ_2 of (III.10) have no effect on the stress state, they can be disregarded.

Consider the general case where the main vector of the external forces applied to the contour is not equal to zero and a homogeneous stress state is given at infinity[1]. The functions $\phi(z_1)$ and $\psi(z_2)$ have the form (I.124), i.e.,

$$\begin{aligned} \varphi(z_1) &= A \ln z_1 + B^* z_1 + \varphi_0(z_1), \\ \psi(z_2) &= B \ln z_2 + (B'^* + iC'^*) z_2 + \psi_0(z_2), \end{aligned} \tag{III.14}$$

where $C^* = 0$. The constants B^*, C'^*, B'^* (in the given directions at infinity) are defined by formulas (I.128), and the constants A and B are found from system (I.116).

By substituting functions (III.14) in the boundary conditions (III.6), we find

$$\begin{aligned} 2\,\mathrm{Re}\,[\varphi_0(z_1) + \psi_0(z_2)] &= f_1 - 2\,\mathrm{Re}\,[A \ln z_1 + B^* z_1 + B \ln z_2 + (B'^* + iC'^*) z_2], \\ 2\,\mathrm{Re}\,[s_1\varphi_0(z_1) + s_2\psi_0(z_2)] &= f_2 - 2\,\mathrm{Re}\,[As_1 \ln z_1 + B^* s_1 z_1 + \\ &\quad + Bs_2 \ln z_2 + s_2(B'^* + iC'^*) z_2]. \end{aligned} \tag{III.15}$$

Obviously, functions $\phi_0(z_1)$ and $\psi_0(z_2)$ satisfy the same boundary conditions (III.6), but with different right hand sides. We will denote the right hand sides of (III.15) through

$$\begin{aligned} f_1^0 &= f_1 - 2\,\mathrm{Re}\,[A \ln z_1 + B^* z_1 + B \ln z_2 + (B'^* + iC'^*) z_2], \\ f_2^0 &= f_2 - 2\,\mathrm{Re}\,[As_1 \ln z_1 + B^* s_1 z_1 + Bs_2 \ln z_2 + s_2(B'^* + iC'^*) z_2] \end{aligned} \tag{III.16}$$

and call them the adduced boundary conditions. We will recall that in (III.16) z_1 and z_2 are points of contours $L^{(1)}$ and $L^{(2)}$. Thus, the boundary conditions for functions $\phi_0(z_1)$ and $\psi_0(z_2)$ are

$$\begin{aligned} 2\,\mathrm{Re}\,[\varphi_0(z_1) + \psi_0(z_2)] &= f_1^0, \\ 2\,\mathrm{Re}\,[s_1\varphi_0(z_1) + s_2\psi_0(z_2)] &= f_2^0. \end{aligned} \tag{III.17}$$

---

[1] Cases of heterogeneous stress state at infinity are considered later.

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Comparison of the boundary conditions (III.17) and (III.6) show that functions $\phi_0(z_1)$ and $\psi_0(z_2)$ are found from the very same formulas (III.10) by substituting f_1 and f_2 in their right hand sides by f_1^0 and f_2^0 of (III.16)

Thus, /213

$$\Phi_0(\zeta) = \frac{i}{4\pi(s_1 - s_2)} \int_\gamma [s_2 f_1^0 - f_2^0] \frac{\sigma + \zeta}{\sigma - \zeta} \cdot \frac{d\sigma}{\sigma} + \lambda_1,$$

$$\Psi_0(\zeta) = -\frac{i}{4\pi(s_1 - s_2)} \int_\gamma [s_1 f_1^0 - f_2^0] \frac{\sigma + \zeta}{\sigma - \zeta} \cdot \frac{d\sigma}{\sigma} + \lambda_2. \qquad \text{(III.18)}$$

By knowing functions $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$, and by converting to the variables z_1 of (III.12) and z^2 of (III.13), we find the functions $\phi_0(z_1)$ and $\psi_0(z_2)$. By substituting the values found for these functions in (III.14), we find the final form of the desired functions $\phi(z_1)$ and $\psi(z_2)$. The stress components σ_x, σ_y, τ_{xy} for the functions $\phi(z_1)$ and $\psi(z_2)$ that we have found, are determined[1] from (I.90).

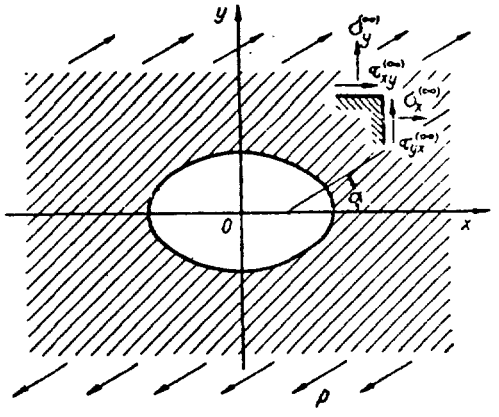


Figure III.2.

Tension of Anisotropic Plate with Elliptic Hole Whose Contour is Free of External Forces. Let the stress state at infinity represent tension by forces p that constitute angle α with the Ox axis (Figure III.2). Consequently, the stress state in infinitely remote parts of the plate are

$$\sigma_x^{(\infty)} = p\cos^2\alpha;\ \sigma_y^{(\infty)} = p\sin^2\alpha;\ \tau_{xy}^{(\infty)} = p\sin\alpha\cos\alpha.$$

We will find the right hand sides of the adduced contour conditions (III.17). Since the contour of the hole is free of external forces, then x = y = 0, and consequently, we may assume that $f_1 = f_2 = 0$ (see (I.64) and (I.116)). By substituting, instead of $\sigma_x^{(\infty)}$, $\sigma_y^{(\infty)}$ and $\tau_{xy}^{(\infty)}$, their values into formulas (I.128), we find

$$B^* = p\frac{\cos^2\alpha + (\alpha_2^2 + \beta_2^2)\sin^2\alpha + \alpha_2\sin 2\alpha}{2[(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2]}.$$

[1] See M. M. Fridman [3], where a detailed analysis of works on elasticity theory of anisotropic media is given.

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$$B'^* = p\frac{[(\alpha_1^2-\beta_1^2)-2\alpha_1\alpha_2]\sin^2\alpha-\cos^2\alpha-\alpha_2\sin 2\alpha}{2[(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)]},$$

$$C'^* = p\left\{\frac{(\alpha_1-\alpha_2)\cos^2\alpha+[\alpha_2(\alpha_1^2-\beta_1^2)-\alpha_1(\alpha_2^2-\beta_2^2)]\sin^2\alpha}{2\beta_2[(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)]}+\right. \tag{III.19}$$

$$\left.+\frac{[(\alpha_1^2-\beta_1^2)-(\alpha_2^2-\beta_2^2)]\sin\alpha\cos\alpha}{2\beta_2[(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)]}\right\}.$$

The adduced contour conditions (III.16) in the given case are /214

$$f_1^0 = -2\,\mathrm{Re}\,[B^*z_1+(B'^*+iC'^*)z_2],$$

$$f_2^0 = -2\,\mathrm{Re}\,[B^*s_1z_1+s_2(B'^*+iC^*)z_2]. \tag{III.20}$$

If we substitute into (III.20), instead of z_1 and z_2, their values through σ from (III.4) and (III.5), and introduce the definitions

$$K_1 = \frac{B^*(a-is_1b)+(B'^*+iC'^*)(a+is_2b)}{2},$$

$$K_2 = \frac{B^*(a-is_1b)-(B'^*+iC'^*)(a-is_2b)}{2},$$

$$K_3 = \frac{B^*s_1(a-is_1b)-s_2(B'^*+iC'^*)(a+is_2b)}{2},$$

$$K_4 = \frac{B^*s_1(a-is_1b)-s_2(B'^*+iC'^*)(a-is_2b)}{2},$$

then the adduced contour conditions (III.20) are

$$f_1^0 = -2\,\mathrm{Re}\left[K_1\sigma+K_2\frac{1}{\sigma}\right],$$

$$f_2^0 = -2\,\mathrm{Re}\left[K_3\sigma+K_4\frac{1}{\sigma}\right].$$

By substituting the values found for f_1^0 and f_2^0 into (III.18), stipulating that

$$\int_\gamma \sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 4\pi i\zeta \text{ and } \int_\gamma \frac{1}{\sigma}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 0,$$

we find

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$$\Phi_0(\zeta) = -\frac{(K_2 - \overline{K}_1) - s_2(K_1 + \overline{K}_2)}{s_1 - s_2}\zeta + \lambda_1,$$
$$\Psi_0(\zeta) = \frac{(K_2 - \overline{K}_1) - s_1(K_1 + \overline{K}_2)}{s_1 - s_2}\zeta + \lambda_2. \qquad \text{(III.21)}$$

After simplifying the expressions obtained, we have

$$\Phi_0(\zeta) = -\frac{p\zeta}{4(s_1 - s_2)}[ib(s_2 \sin 2\alpha + 2\cos^2\alpha) - a(2s_2 \sin^2\alpha + \sin 2\alpha)] + \lambda_1.$$
$$\Psi_0(\zeta) = \frac{p\zeta}{4(s_1 - s_2)}[ib(s_1 \sin 2\alpha + 2\cos^2\alpha) - a(2s_1 \sin^2\alpha + \sin 2\alpha)] + \lambda_2. \qquad \text{(III.22)}$$

By substituting into the function $\Phi_0(\zeta)$ of (III.22), instead of ζ, its value from (III.12), and by substituting into the function $\Psi_0(\zeta)$ its value from (III.13), we find /215

$$\varphi_0(z_1) = -\frac{ip(a - is_1 b)}{4(s_1 - s_2)}\left\{\frac{b(s_2 \sin 2\alpha + 2\cos^2\alpha)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2 b^2)}} + \frac{ia(2s_2 \sin^2\alpha + \sin 2\alpha)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2 b^2)}}\right\},$$
$$\psi_0(z_2) = \frac{ip(a - is_2 b)}{4(s_1 - s_2)}\left\{\frac{b(s_1 \sin 2\alpha + 2\cos^2\alpha)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2 b^2)}} + \frac{ia(2s_1 \sin^2\alpha + \sin 2\alpha)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2 b^2)}}\right\}. \qquad \text{(III.23)}$$

In formulas (III.23) the nonessential constants λ_1 and λ_2 of (III.11) will be omitted.

In order to find $\phi(z_1)$ and $\psi(z_2)$, it is necessary to substitute the values found for the functions $\phi_0(z_1)$ and $\psi_0(z_2)$ from (III.23) into (III.14), assuming in the latter that A = B = 0 (since the contour of the hole is free of external forces).

Hence, finally:

---

[1] The solution was found by G. N. Savin [2]. For the biaxial stress state, where N_1 and N_2 are the main stresses at infinity and α_1 is the angle between the main axis corresponding to N_1 and the Ox axis, it is necessary to find two solutions for (III.23): one, for $p = N_1$ and $\alpha = \alpha_1$, and the other, for $p = N_2$ and $\alpha = \alpha_1 + 90°$. The same problem is analyzed from a different point of view by S. G. Lekhnitskiy [5, 16].

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$$\varphi(z_1) = B^* z_1 + \varphi_0(z_1),$$
$$\psi(z_2) = (B'^* + iC'^*) z_2 + \psi_0(z_2), \tag{III.24}$$

where the values B*, B'* and C'* are given by formulas (III.19). By substituting the functions $\phi(z_1)$ and $\psi(z_2)$ thus found into formulas (I.90) and simplifying, we find the formulas for the stress components

$$\sigma_x = p\cos^2\alpha + 2\,\mathrm{Re}\,[s_1^2\varphi_0'(z_1) + s_2^2\psi_0'(z_2)],$$
$$\sigma_y = p\sin^2\alpha + 2\,\mathrm{Re}\,[\varphi_0'(z_1) + \psi_0'(z_2)], \tag{III.25}$$
$$\tau_{xy} = p\sin\alpha\cos\alpha - 2\,\mathrm{Re}\,[s_1\varphi_0'(z_1) + s_2\psi_0'(z_2)],$$

where $\phi_0(z_1)$ and $\psi_0(z_2)$ are functions (III.23). By substituting in expressions (III.23) and (III.25) p by -p, we find the solution of the same problem for the case where the plate is compressed by forces p at infinity.

Let us examine a few partial cases.

Plate under Tension along Ox Axis (see Figure III.2). By assuming $\alpha = 0$ in formulas (III.23), we find

$$\varphi_0(z_1) = -\frac{ipb}{2(s_1 - s_2)} \cdot \frac{a - is_1 b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2 b^2)}},$$
$$\psi_0(z_2) = \frac{ipb}{2(s_1 - s_2)} \cdot \frac{a - is_2 b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2 b^2)}}. \tag{III.26}$$

The stress components are found from formulas (III.25), assuming that $\alpha = 0$ in them.

Plate under Stress along Oy Axis (see Figure III.2). Assuming that $\alpha = \pi/2$ in formulas (III.23), we obtain /216

$$\varphi_0(z_1) = \frac{aps_2}{2(s_1 - s_2)} \cdot \frac{a - is_1 b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2 b^2)}},$$
$$\psi_0(z_2) = -\frac{aps_1}{2(s_1 - s_2)} \cdot \frac{a - is_2 b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2 b^2)}}. \tag{III.27}$$

The formulas for σ_x, σ_y and τ_{xy} will not be written, since they are found very readily from formulas (III.25), assuming that $\alpha = \pi/2$ in them. By superposing

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the solutions of (III.26) and (III.27) one on the other, we find the solution for the case of three dimensional tension (compression) of a plate at infinity with intensity p of forces.

Slot of Length 2a (or 2b) along Ox Axis (or along Oy Axis). Assuming that b = 0 in (III.26), we obtain

$$\varphi_0(z_1) = 0, \quad \psi_0(z_2) = 0, \tag{III.28}$$

i.e., a rectinlinear slot in an anisotropic medium, made in the direction of tension, does not affect stresses in this medium. The same may be concluded from (III.27), assuming a = 0.

Consider a slot of length 2a (or 2b) located perpendicular to the forces of tension at infinity.

Assuming that a = 0 in functions (III.27), we find the functions for the case of a plate under stress along the Oy axis with a slot of length 2a located on the Ox axis:

$$\varphi_0(z_1) = \frac{a^2 p s_2}{2(s_1 - s_2)} \cdot \frac{1}{z_1 + \sqrt{z_1^2 - a^2}},$$

$$\psi_0(z_2) = -\frac{a^2 p s_1}{2(s_1 - s_2)} \cdot \frac{1}{z_2 + \sqrt{z_2^2 - a^2}}. \tag{III.29}$$

Assuming that $\alpha = \pi/2$ in formulas (III.25), and substituting in them the values found for the functions $\phi_0(z)$ and $\psi_0(z)$ of (III.29), we obtain the stress components for the cross section y = 0 when $|x| > a$:

$$\sigma_x = -\frac{a^2 p}{\sqrt{x^2 - a^2}(x + \sqrt{x^2 - a^2})} \operatorname{Re}[s_1 s_2],$$

$$\sigma_y = \frac{a^2 p}{\sqrt{x^2 - a^2}(x + \sqrt{x^2 - a^2})} + p, \tag{III.30}$$

$$\tau_{xy} = 0.$$

It is interesting to note that σ_y, as seen in (III.30), does not depend on the elastic constants of the material.

If, however, a slot of length 2b is made in the same plate along the Oy axis and it is under tension along the Ox axis, then the stress components for $|y| > b$ are /217

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$$\sigma_x = p + \frac{pb^2}{\sqrt{y^2 - b^2}\,(y + \sqrt{y^2 - b^2})},$$
$$\sigma_y = -\frac{pb^2}{\sqrt{y^2 - b^2}\,(y + \sqrt{y^2 - b^2})}\,\mathrm{Re}\left[\frac{1}{s_1 s_2}\right], \qquad \text{(III.31)}$$
$$\tau_{xy} = 0.$$

From (III.31) we see that stresses σ_x are independent of the elastic constants of the material from which the plate is made.

However, stresses σ_y in (III.31) are not equal to the corresponding stresses σ_x in (III.30). For comparison, it is necessary to assume that a = b in both formulas. These stresses will be equal only in the case of an isotropic medium, where equation (I.82) is converted to a biharmonic equation, the roots of the characteristic equation of which, namely s_1 and s_2, are equal to i:

$$\mathrm{Re}\,[s_1 s_2] = \mathrm{Re}\left[\frac{1}{s_1 s_2}\right] = -1.$$

To understand the changes that occur in the distribution of stresses around the hole, which were caused by the anistropy of the medium, it is sufficient to analyze the stress state along the contour of the hole and near it for various anisotropic materials through a few cross sections.

Consider a plate under tension along the Ox axis. By differentiating functions $\phi_0(z_1)$ and $\psi_0(z_2)$ in (III.26), it is easy to find[1] that

$$\varphi_0'(z_1) = -\frac{i}{2(s_1 - s_2)} \cdot \frac{pb}{a + is_1 b}\left[1 - \frac{z_1}{\sqrt{z_1^2 - (a^2 + s_1^2 b^2)}}\right],$$
$$\psi_0'(z_2) = \frac{i}{2(s_1 - s_2)} \cdot \frac{pb}{a + is_2 b}\left[1 - \frac{z_2}{\sqrt{z_2^2 - (a^2 + s_2^2 b^2)}}\right]. \qquad \text{(III.32)}$$

We will calculate the stresses around an elliptical hole through certain cross sections, considering that the coordinate axes xOy coincide with the straight lines of intersection by the plane of elastic symmetry of the material. By substituting functions (III.32) into the first formula of (III.27), we find[2] the stresses through the cross section x = 0

---

[1] See S. G. Lekhnitskiy [6].

[2] As will be pointed out below, $s_1 = i\beta_1$ and $s_2 = i\beta_2$ for all materials examined. These values are also used in the derivation of formula (III.33).

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$$\sigma_x = p + \frac{pb}{\beta_1 - \beta_2}\left\{\frac{\beta_1^2}{a - \beta_1 b}\left(1 - \frac{\beta_1 y}{\sqrt{a^2 + \beta_1^2 (y^2 - b^2)}}\right) - \right.$$
$$\left. - \frac{\beta_2^2}{a - \beta_2 b}\left(1 - \frac{\beta_2 y}{\sqrt{a^2 + \beta_2^2 (y^2 - b^2)}}\right)\right\}. \qquad \text{(III.33)}$$

Assuming that y = b in formula (III.33), we obtain /218

$$\sigma_x = p\left[1 + (\beta_1 + \beta_2)\frac{b}{a}\right]. \qquad \text{(III.34)}$$

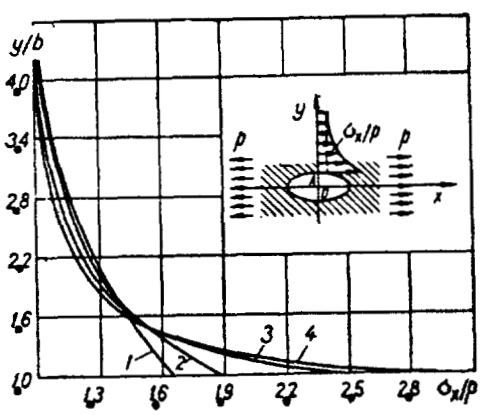


Figure III.3.

If the medium is isotropic, then $\beta_1 = \beta_2 = 1$, and from formula (III.34) we find formula (II.65)

$$\sigma_x = p\left(1 + 2\frac{b}{a}\right).$$

We will determine σ_y through cross section y = 0. By substituting in functions $\phi_0'(z_1)$ and $\psi_0'(z_2)$ of (III.32), $z_1 = z_2 = x$ from the second formula of (I.53), we find

$$\sigma_y = -\frac{pb}{\beta_1 - \beta_2}\left\{\frac{1}{a - \beta_1 b} \times \left(1 - \frac{x}{\sqrt{x^2 - a^2 + \beta_1^2 b^2}}\right) - \frac{1}{a - \beta_2 b} \times \left(1 - \frac{x}{\sqrt{x^2 - a^2 + \beta_2^2 b^2}}\right)\right\}; \qquad \text{(III.35)}$$

for x = a,

$$\sigma_y = -p\frac{1}{\beta_1 \beta_2}. \qquad \text{(III.36)}$$

Assuming that $\beta_1 = \beta_2 = 1$ in (III.36), we find, in the case of an isotropic medium (see formula (II.66)), $\sigma_y = -p$.

Figures III.3 and III.4 represent the graphs of stresses σ_x through cross section x = 0 calculated by formula (III.33) for a/b = 3 and a/b = 1/3, respectively for various anisotropic materials:

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1) for oak, the elastic constants of which are[1] /219

$$a_{11} = \frac{1}{E_x} = \frac{1}{582} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{22} = \frac{1}{E_y} = \frac{1}{219} \cdot \frac{10^{-6}}{9{,}81},$$
$$a_{12} = -\frac{\nu_x}{E_x} = -\frac{1}{1800} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{66} = \frac{1}{G_{xy}} = \frac{1}{132} \cdot \frac{10^{-6}}{9{,}81},$$

according to which, from equation (I.84), we find

$$s_1 = 1{,}679i, \quad s_2 = 0{,}971i, \quad s_3 = \bar{s}_1, \quad s_4 = \bar{s}_2;$$

2) for birch

$$a_{11} = \frac{1}{1670} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{12} = -\frac{1}{3400} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{22} = \frac{1}{113} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{66} = \frac{1}{120} \cdot \frac{10^{-6}}{9{,}81},$$
$$s_1 = 3{,}416i, \quad s_2 = 1{,}126i;$$

3) for spruce

$$a_{11} = \frac{1}{1700} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{12} = -\frac{1}{4600} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{22} = \frac{1}{83} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{66} = \frac{1}{64} \cdot \frac{10^{-6}}{0{,}81},$$
$$s_1 = 5{,}0007i, \quad s_2 = 0{,}905i.$$

Curve 1 in Figures III.3-III.10 correspond to an isotropic plate, curve 2 to an oak plate, curve 3, to a birch plate and curve 4, to spruce.

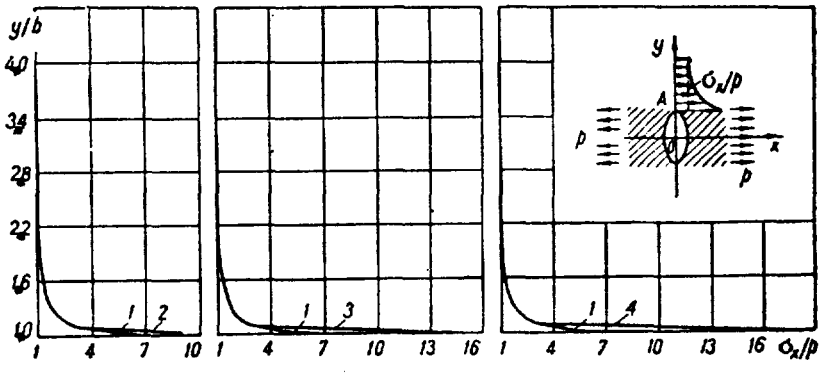


Figure III.4.

Figures III.5 and III.6 represent the graphs of stresses σ_x through cross section x = 0, calculated by formula (III.33) for the same anisotropic materials as represented in Figures III.3 and III.4, but for the case where the Ox axis

[1] The adduced constants have the dimension m^2/n.

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coincides with the direction corresponding to the least elasticity modulus. The roots of equation (I.83) for this case are different, since some elastic constants are replaced by others. For instance:

for oak

$$a_{22} = \frac{1}{E_y} = \frac{1}{582} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{12} = -\frac{\nu_y}{E_y} = -\frac{1}{1800} \cdot \frac{10^{-6}}{9{,}81}, \quad a_{11} = \frac{1}{E_x} = \frac{1}{219} \cdot \frac{10^{-6}}{9{,}81};$$

$$a_{66} = \frac{1}{G_{xy}} = \frac{1}{132} \cdot \frac{10^{-6}}{9{,}81}, \quad s_1 = 1{,}030i, \quad s_2 = 0{,}595i;$$

for birch /221

$$s_1 = 0{,}889i, \quad s_2 = 0{,}293i;$$

for spruce

$$s_1 = 1{,}105i, \quad s_2 = 0{,}200i.$$

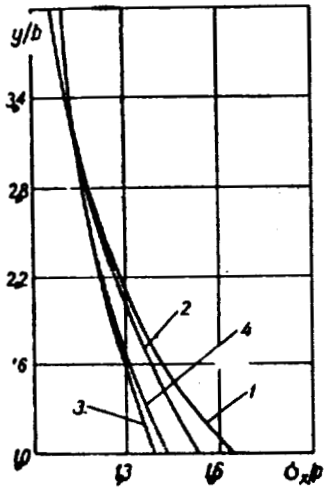


Figure III.5.

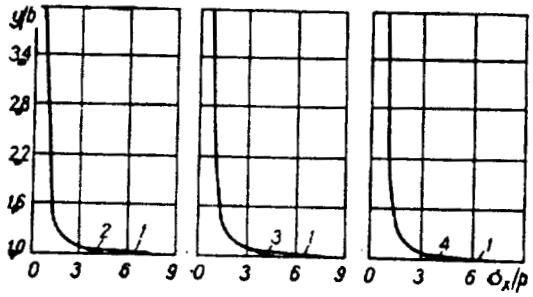


Figure III.6.

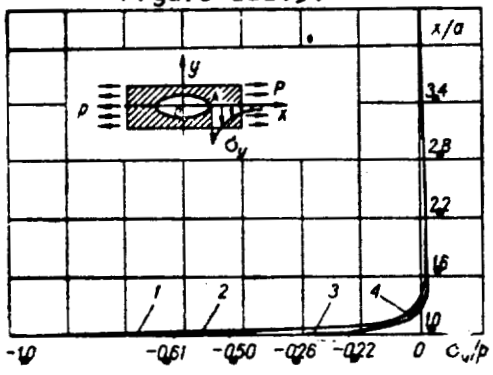


Figure III.7.

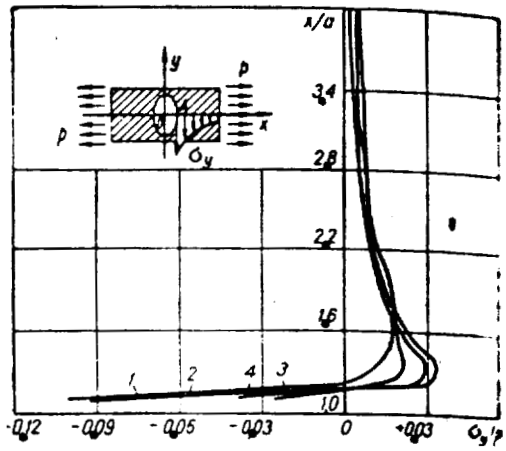


Figure III.8.

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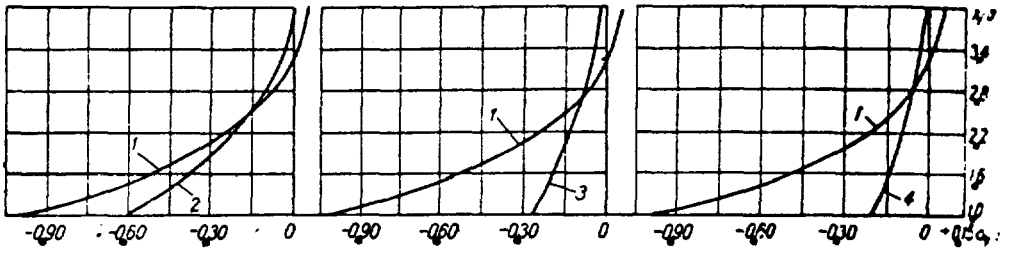


Figure III.9.

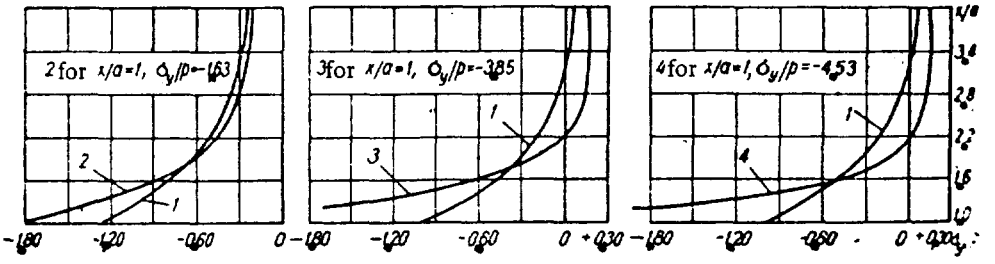


Figure III.10.

Figures III.7 and III.8 represent the graphs of stresses σ_y through cross section y = 0 calculated by formula (III.35) for a/b = 3 and a/b = 1/3, respectively, for the same anisotropic materials.

Figures III.9 and III.10 show the graphs of the same stresses σ_y as Figures III.7 and III.8, but for axes rotated by 90° relative to the position in the first case.

We see from Figures III.3-III.10 that the stress concentration around the elliptic hole, as in the case of an isotropic medium, is of a local character. The anisotropy of the medium involves substantial corrections in the stresses in an anisotropic material only in a small area around the hole. The pattern of perturbation of the stress state vanishes by measure of distance from the hole[1].

The values of $(\sigma_x/p)_A$ at point A (see Figures III.3 and III.4) around an elliptic hole for a/b = 3 and a/b = 1/3 are presented in Table III.1 for various anisotropic materials. It is convenient to calculate the stresses around an elliptic hole according to confocal ellipses

$$x = a\varrho\cos\vartheta,\quad y = b\varrho\sin\vartheta\quad (\varrho e^{i\vartheta} = \zeta,\ \varrho \geqslant 1), \tag{III.37}$$

by converting[2] to a new system of stress components: σ_ϑ, normal stress on

[1] See A. V. Stepanov [1].
[2] See S. G. Lekhnitskiy [1].

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surface perpendicular to the ellipses of interest; σ_ρ, normal stress on surface tangent to ellipse, and $\tau_{\rho\vartheta}$, tangential stress on these surfaces. The formulas of stress components for tension along the Ox axis around the contour of an elliptic hole ($\rho = 1$) are

$$\sigma_\vartheta = p\left\{\frac{\sin^2\vartheta}{\sin^2\vartheta + k^2\cos^2\vartheta} + \frac{k}{\sin^2\vartheta + k^2\cos^2\vartheta}\operatorname{Re}\left[\frac{e^{-i\vartheta}}{s_1 - s_2}\times\right.\right.$$
$$\left.\left.\times\left(\frac{(s_1\sin\vartheta + k\cos\vartheta)^2}{\sin\vartheta - s_1 k\cos\vartheta} - \frac{(s_2\sin\vartheta + k\cos\vartheta)^2}{\sin\vartheta - s_2 k\cos\vartheta}\right)\right]\right\}, \qquad \text{(III.38)}$$

where $k = b/a$. Assuming that $s_1 = \alpha_1 + i\beta_1$, $s_2 = \alpha_2 + i\beta_2$ in (III.38), and by removing the real part, we find the formula for σ_ϑ around the contour of the elliptic hole. When $k = b/a = 1$, we find from (III.38) the stresses around the contour of the round hole.

TABLE III.1

Material	Tension			
	Along major modulus		Along minor modulus	
	$a/b=3$	$a/b=1/3$	$a/b=3$	$a/b=1/3$
Oak	1,883	8,951	1,542	5,875
Birch	2,514	14,626	1,394	4,546
Pine	2,522	14,698	1,395	4,558
Spruce	2,969	18,718	1,435	4,916
Isotropic medium	1,667	7,000	1,667	7,000

Tr. Note: Commas indicate decimal points.

Assuming that $s_1 = s_2 = 1$ in (III.38), we find, after limit transition, the known formula (II.64) for σ_ϑ in the case of an isotropic medium:

$$\sigma_\vartheta = p\frac{\sin^2\vartheta + 2k\sin^2\vartheta - k^2\cos^2\vartheta}{\sin^2\vartheta + k^2\cos^2\vartheta}. \qquad \text{(III.39)}$$

Since in the general case of anisotropy, i.e., when $s_1 = \alpha_1 + i\beta_1$ and $s_2 = \alpha_2 + i\beta_2$, formula (III.38) is extremely awkward, we will introduce formulas σ_ϑ for the most characteristic points of the contour. Stresses σ_ϑ at the points $x = \pm a$ and $y = 0$, i.e., $\vartheta = 0$ and $\vartheta = \pi$ are found from (III.38) in the form /222

$$\sigma_\vartheta = p\operatorname{Re}\left[\frac{1}{s_1 s_2}\right] = p\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)}.$$

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The stress in these points clearly does not depend on k = b/a, i.e. it is identical for all ellipses and depends only on the elastic characteristics of the anisotropic material.

If the medium is isotropic, then $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$, and we find the known formula $\sigma_\vartheta = -p$ (see (II.66)). The stresses at the points y = b, x = 0, i.e. when $\vartheta = \pm\pi/2$, are found from (III.38):

$$\sigma_\vartheta = p\left[1 + \frac{b}{a}(\beta_1 + \beta_2)\right].$$

In the case of an isotropic medium $\beta_1 = \beta_2 = 1$, and we find the known formula $\sigma_\vartheta = p\ (1 + 2\ b/a)$ (see (II.65)).

When an isotropic plate is under tension along the Ox axis, the greatest stresses σ_ϑ on the contour of the elliptical hole occur at the points $\vartheta = \pm\pi/2$. If the medium is anisotropic, then this conclusion, generally speaking, is not true. The fact is that even in the simplest case of an anisotropic material, particularly in the presence of three planes of elastic symmetry within the material and under the condition that the axes are coincident with the straight lines of intersection of these planes of elastic symmetry, stresses σ_ϑ around an elliptic (in particular, round) hole for $\vartheta = 0$ and $\vartheta = \pi$ can greatly exceed in absolute value the stresses σ_ϑ for $\vartheta = \pm\pi/2$. This will occur, for instance, in the case of tension in the direction of the minor modulus E_1 of a spruce plate weakened by a round hole, for which $s_1 = 1.105i$ and $s_2 = 0.200i$. In this case $(\sigma_\vartheta)_{\vartheta=0} \simeq -4.5p$, whereas $(\sigma_\vartheta)_{\vartheta=\pm\pi/2} = 2.3p$. The same will be true in the case of a birch plate under tension and weakened by a round hole, specifically $(\sigma_\vartheta)_{\vartheta=0} = -3.85p$, whereas $(\sigma_\vartheta)_{\vartheta=\pm\pi/2} = 2.18p$.

For certain anisotropic materials, stresses σ_ϑ are determined along the contour of an elliptic hole for a/b = 3 and a/b = 1/3 (the semiaxis a lies on the Ox axis, and the semiaxis b lies on the Oy axis) for two cases:

1) when the straight lines of intersection of the planes of elastic symmetry of an anisotropic material are used as the Ox and Oy axes;

2) when the Ox and Oy axes are rotated in relation to their position in the first case by angle $\delta = 30°$.

The values of stresses σ_ϑ/p along the contour of an elliptic hole are presented in Tables III.2 and III.3 for a/b = 3 and a/b = 1/3, respectively, for the cases of plywood and spruce.

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TABLE III.2

$\vartheta°$	Isotropic medium	Anisotropic medium					
		Plywood I ($E_1/E_3=12$)		Plywood II ($E_1/E_3\simeq2,1$)		Spruce ($E_1/E_3\simeq20,5$)	
		$\delta=0$	$\delta=30°$	$\delta=0$	$\delta=30°$	$\delta=0$	$\delta=30°$
0	—1,000	—0,290	—0,344	—0,689	—0,919	—0,221	—0,327
15	—0,077	—0,162	—1,132	0,018	—0,763	—0,114	—1,287
30	1,000	0,282	0,933	0,523	1,482	0,163	1,098
45	1,400	0,957	2,215	0,899	2,159	0,664	2,519
60	1,571	1,667	2,148	1,553	1,835	1,464	2,148
75	1,646	2,200	—	2,184	—	2,447	—
90	1,667	2,400	1,680	2,463	1,432	2,969	1,556
120	1,571	1,667	1,230	1,553	1,214	1,464	1,166
150	1,000	—0,282	0,672	0,523	1,077	0,163	0,720
180	—1,000	0,290	—0,344	—0,689	—0,919	—0,221	—0,327

Tr. Note: Commas indicate decimal points.

TABLE III.3

$\vartheta°$	Isotropic medium	Anisotropic medium					
		Plywood I ($E_1/E_3=12$)		Plywood II ($E_1/E_3\simeq2,1$)		Spruce ($E_1/E_3\simeq20,5$)	
		$\delta=0$	$\delta=30°$	$\delta=0$	$\delta=30°$	$\delta=0$	$\delta=30°$
0	—1,000	—0,290	—0,344	—0,689	—0,919	—0,221	—0,327
30	—0,714	—0,248	—1,493	—0,354	—0,833	—0,175	—1,200
60	1,000	—0,093	—1,214	—0,569	—0,224	0,078	—1,455
70	—	—	—	—	1,533	—	—
75	3,859	1,449	3,999	1,749	5,080	0,991	2,609
80	5,251	3,280	10,429	3,068	10,835	2,274	14,499
85	6,484	8,525	10,909	7,033	8,007	6,836	10,679
90	7,000	13,601	7,120	14,171	4,888	18,720	6,001
120	1,000	0,093	0,945	0,569	2,025	0,078	1,062
150	—0,714	—0,248	0,114	—0,354	—0,644	—0,175	0,147
180	—1,000	—0,290	—0,344	—0,689	—0,919	—0,221	—0,327

Tr. Note: Commas indicate decimal points.

The elastic constants of plywood I in the first case ($\delta = 0$) are: /223

$$a_{11}=\frac{1}{1,4}\cdot\frac{10^{-9}}{9,81},\qquad a_{12}=-\frac{0,46}{1,4}\cdot\frac{10^{-9}}{9,81},$$

$$a_{22}=\frac{12}{1,4}\cdot\frac{10^{-9}}{9,81}\qquad a_{66}=\frac{11.667}{1,4}\cdot\frac{10^{-9}}{9,81},\qquad a_{16}=a_{26}=0;$$

the roots of equation (I.84) are

$$s_1=3,08i,\qquad s_2=1,12i,\qquad s_3=\bar{s}_1,\qquad s_4=\bar{s}_2;$$

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the Ox axis is directed along the major modulus $E_1:E_2 = 12$. When the axes are rotated by some angle, the values of the elastic constants a_{ik} are changed. In the new system of coordinates, the values of a'_{ik} are defined by the known formulas of transformation of the elastic constants of anisotropic bodies[1]. /224
The elastic constants for plywood I in the second case ($\delta = 30°$) are

$$a'_{11} = 2{,}376\,\frac{10^{-9}}{9{,}81}, \qquad a'_{22} = 6{,}305\,\frac{10^{-9}}{9{,}81},$$

$$a'_{12} = 0{,}0267\,\frac{10^{-9}}{9{,}81}, \qquad a'_{16} = 3{,}050\,\frac{10^{-9}}{9{,}81},$$

$$a'_{26} = 3{,}746\,\frac{10^{-9}}{9{,}81}, \qquad a'_{66} = 9{,}541\,\frac{10^{-9}}{9{,}81},$$

$$s_1 = 1{,}1775 + i0{,}9875, \qquad s_3 = \bar{s}_1,$$

$$s_2 = 0{,}1075 + i1{,}0519, \qquad s_4 = \bar{s}_2.$$

The elastic constants of plywood II in the first case are:

$$a_{11} = \frac{1}{1{,}6}\cdot\frac{10^{-9}}{9{,}81}, \qquad a_{12} = -\frac{0{,}05}{1{,}6}\,\frac{10^{-9}}{9{,}81}, \qquad a_{22} = \frac{1}{0{,}765}\cdot\frac{10^{-9}}{9{,}81},$$

$$a_{66} = \frac{1}{0{,}097}\cdot\frac{10^{-9}}{9{,}81}, \qquad a_{26} = a_{16} = 0;$$

$$s_1 = 4{,}03i, \qquad s_2 = 0{,}36i, \qquad s_3 = \bar{s}_1, \qquad s_4 = \bar{s}_2;$$

in the second case:

$$a'_{11} = 2{,}354\,\frac{10^{-9}}{9{,}81}, \qquad a'_{22} = 2{,}695\,\frac{10^{-9}}{9{,}81}, \qquad a'_{66} = 4{,}072\,\frac{10^{-9}}{9{,}81},$$

$$a'_{12} = -1{,}590\,\frac{10^{-9}}{9{,}81}, \qquad a'_{16} = 2{,}092\,\frac{10^{-9}}{9{,}81}, \qquad a'_{26} = -1{,}530\,\frac{10^{-9}}{9{,}81},$$

$$s_1 = 1{,}373 + i0{,}842, \qquad s_3 = \bar{s}_1, \qquad s_4 = \bar{s}_2, \qquad s_2 = 0{,}483 + i0{,}454.$$

The elastic constants of spruce in the first case are:

$$E_1 : E_2 = 20{,}5,$$

$$a_{11} = 0{,}588\,\frac{10^{-7}}{9{,}81}, \qquad a_{22} = 12{,}040\,\frac{10^{-7}}{9{,}81},$$

$$a_{12} = -0{,}217\,\frac{10^{-7}}{9{,}81}, \qquad a_{66} = 15{,}630\,\frac{10^{-7}}{9{,}81},$$

$$a_{16} = a_{26} = 0, \qquad s_2 = 5{,}001i, \qquad s_2 = 0{,}905i, \qquad s_3 = \bar{s}_1, \qquad s_4 = \bar{s}_2;$$

[1] See S. G. Lekhnitskiy [1].

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in the second case:

$$a'_{11} = 3{,}932\,\frac{10^{-7}}{9{,}81}, \qquad a'_{22} = 9{,}658\,\frac{10^{-7}}{9{,}81}, \qquad a'_{66} = 13{,}704\,\frac{10^{-7}}{9{,}81},$$

$$a'_{12} = -\,0{,}697\,\frac{10^{-7}}{9{,}81}, \qquad a'_{16} = 5{,}509\,\frac{10^{-7}}{9{,}81}, \qquad a'_{26} = 4{,}397\,\frac{10^{-7}}{9{,}81},$$

$$s_1 = 1{,}483 + i0{,}718 \qquad s_2 = -\,0{,}083 + i0{,}949, \qquad s_3 = \overline{s_1}, \qquad s_4 = \overline{s_2}.$$

The data in Tables III.2 and III.3 show that when the Ox and Oy axes coincide with the straight lines of intersection of the planes of elastic symmetry of the material, the greatest and smallest stresses for the case of tension along the Ox axis will occur at the very same points $\vartheta = \pi/2$ and $\vartheta = 0$ as for an isotropic medium. However, it cannot be verified that the greatest absolute value of σ_ϑ will occur for $\vartheta = \pi/2$, and that the least will occur for $\vartheta = 0$. It is necessary to calculate and compare the values of σ_ϑ at these two points. /225

Elliptic Hole, the Edge of Which Is under Uniform Tangential Stress. In this case,

$$X_n = T\cos(t, x) = T\frac{dx}{ds}, \qquad Y_n = T\cos(t, y) = T\frac{dy}{ds}.$$

Consequently, from (III.6)

$$f_1 = -\int_0^s Y_n ds + C_1 = -Ty + \text{const},$$

$$f_2 = \int_0^s X_n ds + C_2 = Tx + \text{const}.$$

The right hand sides of boundary conditions (III.7), considering (III.3), are found in the form

$$f_1(\vartheta) = Tb\sin\vartheta = -i\frac{Tb}{2}\left(\sigma - \frac{1}{\sigma}\right),$$

$$f_2(\vartheta) = Ta\cos\vartheta = \frac{Ta}{2}\left(\sigma + \frac{1}{\sigma}\right).$$

From equations (III.10), recalling that

$$\int_\gamma \sigma\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 4\pi i\zeta, \qquad \int_\gamma \frac{1}{\sigma}\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 0,$$

we will find the functions

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$$\Phi(\zeta) = \frac{T(a + is_2 b)}{2(s_1 - s_2)}\zeta,$$

$$\Psi(\zeta) = -\frac{T(a + is_1 b)}{2(s_1 - s_2)}\zeta,$$

or, in variables z_1 and z_2:

$$\varphi(z_1) = \frac{T(a + is_2 b)}{2(s_1 - s_2)} \cdot \frac{a - is_1 b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2 b^2)}},$$

$$\psi(z_2) = -\frac{T(a + is_1 b)}{2(s_1 - s_2)} \cdot \frac{a - is_2 b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2 b^2)}}. \qquad \text{(III.40)}$$

If in formulas (III.40) we assume that a = b = R and specify that M = $= -2\pi R^2 T$, and then proceed to the limit so that the value of the moment M will not be changed, and so that the radius of the hole R approaches zero, then we find the formulas /226

$$\varphi(z_1) = -\frac{M(1 + is_2)(1 - is_1)}{8\pi(s_1 - s_2)} \cdot \frac{1}{z_1},$$

$$\psi(z_2) = \frac{M(1 + is_1)(1 - is_2)}{8\pi(s_1 - s_2)} \cdot \frac{1}{z^2},$$

which express the effect of a concentrated couple with moment M, applied at the origin of the coordinates of an infinite anisotropic plate.

Elliptic Hole, the Edge of Which Is under Uniform Pressure p. In this case

$$X_n = -p\cos(n, x) = -p\frac{dy}{ds}, \qquad Y_n = -p\cos(n, y) = p\frac{dx}{ds}.$$

Consequently, from (III.7) for the given problem, we have

$$f_1(\vartheta) = -pa\cos\vartheta + \text{const} = -\frac{pa}{2}\left(\sigma + \frac{1}{\sigma}\right) + \text{const},$$

$$f_2(\vartheta) = pb\sin\vartheta + \text{const} = -i\frac{pb}{2}\left(\sigma - \frac{1}{\sigma}\right) + \text{const}.$$

By substituting the values found for $f_1(\vartheta)$ and $f_2(\vartheta)$ into equations (III.10), we find

$$\Phi(\zeta) = -\frac{ip(b + ias_2)}{2(s_1 - s_2)}\zeta, \qquad \Psi(\zeta) = \frac{ip(b + ias_1)}{2(s_1 - s_2)}\zeta.$$

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Converting to variables z_1 and z_2, we obtain

$$\varphi(z_1) = -\frac{ip(b+ias_2)}{2(s_1-s_2)} \cdot \frac{a-is_1b}{z_1+\sqrt{z_1^2-(a^2+s_1^2b^2)}},$$

$$\psi(z_2) = \frac{ip(b+ias_1)}{2(s_1-s_2)} \cdot \frac{a-is_2b}{z_2+\sqrt{z_2^2-(a^2+s_2^2b^2)}}. \qquad \text{(III.41)}$$

Rectilinear Slit, the Edge of Which Is under Uniform Pressure p. The solution is found from the preceding problem, assuming that b = 0 or a = 0 in functions $\phi(z_1)$ and $\psi(z_2)$ of (III.41). In the first case we will have a slit of length 2a along the real axis, and in the second, a slit of length 2b along the imaginary axis.

The functions for the slit of length 2a lying on the real axis are

$$\varphi(z_1) = \frac{a^2s_2p}{2(s_1-s_2)} \cdot \frac{1}{z_1+\sqrt{z_1^2-a^2}},$$

$$\psi(z_2) = -\frac{a^2s_2p}{2(s_1-s_2)} \cdot \frac{1}{z_2+\sqrt{z_2^2-a^2}}. \qquad \text{(III.42)}$$

By comparing functions $\phi(z_1)$ and $\psi(z_2)$ of (III.42) with functions $\phi_0(z_1)$ and $\psi_0(z_2)$ of (III.29), we see that they are completely coincident. /227

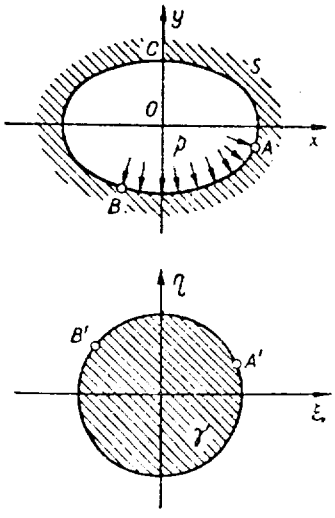


Figure III.11.

The general solution of the problem of elastic equilibrium of an anisotropic plate with cross sections located along a straight line is examined by S. G. Mikhlin [1] and, by a different method, P. A. Zagubizhenko [1], who also analyzed the case of compression of an anisotropic plate with a single rectilinear slit[1].

Elliptic Hole, Part of the Edge of Which Is under Normal Pressure. Case of Concentrated Forces. We will assume that uniform pressure p (Figure III.11) is applied only to segment AB of the contour of the hole, and that the remaining part of the contour BCA is free of external forces. The stresses at infinity, as before, are assumed to be equal to zero.

Since the main vector of external forces applied to the contour of the elliptic hole are not equal to

[1]See also G. P. Cherepanov [1] and I. A. Prusov [1].

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zero, it is necessary to use the adduced contour conditions f_1^0 and f_2^0 of (III.16) in formulas (III.10) instead of f_1 and f_2.

We will first determine f_1 and f_2. We will use point A as the origin and assume that $f_1 = f_2 = 0$ in it. As in the preceding example, on segment AB we have

$$f_1 = -pa\cos\vartheta + C_1 = -\frac{pa}{2}\left(\sigma + \frac{1}{\sigma}\right) + C_1,$$

$$f_2 = pb\sin\vartheta + C_2 = -i\frac{pb}{2}\left(\sigma - \frac{1}{\sigma}\right) + C_2.$$

To points A and B on plane S will correspond points A' and B', i.e. $\sigma_1 = e^{i\vartheta_1}$ and $\sigma_2 = e^{i\vartheta_2}$ on the contour of unit circle .

Relative to constants C_1 and C_2, arbritrary assumptions cannot be made in the given case (i.e. we cannot assume that $C_1 = C_2 = 0$), since the functions f_1 and f_2 must be continuous.

According to definition, at point A $f_1 = f_2 = 0$, i.e., pa cos ϑ_1 + C_1 = 0; pb sin ϑ_1 + C_2 = 0 or C_1 = pa cos ϑ_1; C_2 = -pb sin ϑ_1. Consequently, the functions on segment AB are

$$f_1 = -pa(\cos\vartheta - \cos\vartheta_1) = -\frac{pa}{2}\left[\sigma + \frac{1}{\sigma} - \left(\sigma_1 + \frac{1}{\sigma_1}\right)\right],$$

$$f_2 = pb(\sin\vartheta - \sin\vartheta_1) = -i\frac{pb}{2}\left[\sigma - \frac{1}{\sigma} - \left(\sigma_1 - \frac{1}{\sigma_1}\right)\right];$$

and on segment BCA /228

$$f_1 = -pa(\cos\vartheta_2 - \cos\vartheta_1) = -\frac{pa}{2}\left[\left(\sigma_2 + \frac{1}{\sigma_2}\right) - \left(\sigma_1 + \frac{1}{\sigma_1}\right)\right],$$

$$f_2 = pb(\sin\vartheta_2 - \sin\vartheta_1) = -i\frac{pb}{2}\left[\left(\sigma_2 - \frac{1}{\sigma_2}\right) - \left(\sigma_1 - \frac{1}{\sigma_1}\right)\right].$$

By passing one time around the contour of the hole in the positive direction, we find that the functions f_1 and f_2 acquire the increments

$$[f_1]_L = -pa(\cos\vartheta_2 - \cos\vartheta_1),$$

$$[f_2]_L = pb(\sin\vartheta_2 - \sin\vartheta_1).$$

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But $f_1 = -\int_0^s Y_n ds + C_1$ и $f_2 = \int_0^s X_n ds + C_2$.

Consequently,

$$Y = -[f_1]_L = \frac{pa}{2}\left[\left(\sigma_2 + \frac{1}{\sigma_2}\right) - \left(\sigma_1 + \frac{1}{\sigma_1}\right)\right],$$
$$X = [f_2]_L = -\frac{pb}{2} i\left[\left(\sigma_2 - \frac{1}{\sigma_2}\right) - \left(\sigma_1 - \frac{1}{\sigma_1}\right)\right].$$

These values of the projection of the resultant vector of external forces should be substituted into the constants A_k and B_k defined by system (I.116).

By defining A = A' +iA", B = B' + iB", and by substituting into (III.14), instead of z_1 and z_2, their values through σ from (III.4) and (III.5), we obtain

$$\Phi(\zeta) = -(A' + iA'')\ln\zeta + \Phi_0(\zeta),$$
$$\Psi(\zeta) = -(B' + iB'')\ln\zeta + \Psi_0(\zeta),$$

where $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ are functions that are holomorphic within γ. By substituting the values of these functions into the contour conditions (III.6), we find[1]

$$2\mathrm{Re}[\Phi_0(\sigma) + \Psi_0(\sigma)] = f_1 + 2\mathrm{Re}[(A' + iA'')\ln\sigma + (B' + iB'')\ln\sigma] = f_1^0,$$
$$2\mathrm{Re}[s_1\Phi_0(\sigma) + s_2\Psi_0(\sigma)] = f_2 + 2\mathrm{Re}[s_1(A' + iA')\ln\sigma + s_2(B' + iB')\ln\sigma] = f_2^0,$$

hence

$$f_1^0 = f_1 + 2i(A' + B')\ln\sigma,$$
$$f_2^0 = f_2 + 2i(A'\beta_1 + A'\alpha_1 + B'\beta_2 + B'\alpha_2)\ln\sigma.$$

From formula (III.18) we find the functions

$$\Phi_0(\zeta) = \frac{i}{4\pi(s_1 - s_2)}\int_\gamma [s_2 f_1^0 - f_2^0]\frac{\sigma + \zeta}{\sigma - \zeta}\cdot\frac{d\sigma}{\sigma} =$$
$$= \frac{i}{4\pi(s_1 - s_2)}\left\{M_1\int_\gamma \ln\sigma\frac{\sigma + \zeta}{\sigma - \zeta}\cdot\frac{d\sigma}{\sigma} + \frac{p}{2}\left[as_2\left(\sigma_1 + \frac{1}{\sigma_1}\right) - ib\left(\sigma_1 - \frac{1}{\sigma_1}\right)\right]\right.$$

[1] For ln σ, it is necessary to take some single arm of this function.

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$$\times \int_{\gamma} \frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} - \frac{p}{2}\left[as_2\left(\sigma_2+\frac{1}{\sigma_2}\right) - ib\left(\sigma_2-\frac{1}{\sigma_2}\right)\right]\int_{\sigma_2}^{\sigma_1}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} -$$

$$-\frac{p}{2}(as_2-ib)\int_{\sigma_1}^{\sigma_2}\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} - \frac{p}{2}(as_2+ib)\int_{\sigma_1}^{\sigma_2}\frac{1}{\sigma}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}\Big\};$$

$$\Psi_0(\zeta) = -\frac{i}{4\pi(s_1-s_2)}\int_{\gamma}[s_1f_1^0 - f_2^0]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = \tag{III.43}$$

$$= -\frac{i}{4\pi(s_1-s_2)}\Big\{M_2\int_{\gamma}\ln\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \frac{p}{2}\left[as_1\left(\sigma_1+\frac{1}{\sigma_1}\right) - ib\left(\sigma_1-\frac{1}{\sigma_1}\right)\right]\times$$

$$\times\int_{\gamma}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} - \frac{p}{2}\left[as_1\left(\sigma_2+\frac{1}{\sigma_2}\right) - ib\left(\sigma_2-\frac{1}{\sigma_2}\right)\right]\int_{\sigma_2}^{\sigma_1}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} -$$

$$-\frac{p}{2}(as_1-ib)\int_{\sigma_2}^{\sigma_1}\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} - \frac{p}{2}(as_1+ib)\int_{\sigma_2}^{\sigma_1}\frac{1}{\sigma}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}\Big\},$$

where

$$M_1 = 2i[s_2(A''+B'') - (A'\beta_1 + A''\alpha_1 + B'\beta_2 + B''\alpha_2)],$$
$$M_2 = 2i[s_1(A''+B'') - (A'\beta_1 + A''\alpha_2 + B'\beta_2 + B''\alpha_2)].$$

In expressions (III.43) we encounter the integrals

$$\int_{\gamma}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 2\pi i; \qquad \int_{\gamma}\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 4\pi i\zeta;$$

$$\int_{\gamma}\frac{1}{\sigma}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 0;$$

$$\int\ln\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 2\int\frac{\ln\sigma}{\sigma-\zeta}d\sigma - \int\frac{\ln\sigma}{\sigma}d\sigma =$$

$$= 2\pi^2 + 4\pi i\ln(\sigma-\zeta) + \text{const};$$

$$\int\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = \int\left[\frac{2}{\sigma-\zeta} - \frac{1}{\sigma}\right]d\sigma = 2\ln(\sigma-\zeta) - \ln\sigma + \text{const};$$

$$\int\sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = \int\left[1 + \frac{2\zeta}{\sigma-\zeta}\right]d\sigma = \sigma + 2\zeta\ln(\sigma-\zeta) + \text{const};$$

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$$\int \frac{1}{\sigma}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = \int\left[\frac{2}{\zeta}\cdot\frac{1}{\sigma-\zeta} - \frac{1}{\sigma^2} - \frac{2}{\zeta}\cdot\frac{1}{\sigma}\right]d\sigma =$$

$$= \frac{2}{\zeta}\ln(\sigma-\zeta) + \frac{1}{\sigma} - \frac{2}{\zeta}\ln\sigma + \text{const.}$$

Considering the values of the latter integrals, it is easy to obtain the final expressions for $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$. We will not write them out since they are quite cumbersome, but will simply point out that if we assume that $\sigma_2 = \sigma_1$ in them (the entire contour is under stress), we will find the simple formulas derived above for this case. /230

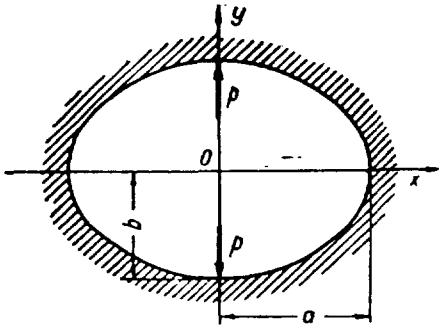


Figure III.12.

By substituting the expressions found for $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ into formulas

$$\Phi(\zeta) = -(A' + iA'')\ln\zeta + \Phi_0(\zeta),$$

$$\Psi(\zeta) = -(B' + iB'')\ln\zeta + \Psi_0(\zeta)$$

and by returning by formulas (III.12) and (III.13) to the variables z_1 and z_2, we obtain the desired functions $\phi(z_1)$ and $\psi(z_2)$. By simple limit transition in the functions obtained, it is easy to find the solution for the case of concentrated force P applied (normally) to the contour at at any of its points. For this purpose it is necessary to shorten arc AB so that $\lim_{\overset{\frown}{AB}\to 0}\{p\cdot\overset{\frown}{AB}\} = P$. In this manner we can find the solution for several individual forces. By combining the solution, obviously, we find the solution for any finite number of arbitrary, but concentrated forces perpendicular to the contour of the hole. For instance, the case of two concentrated forces P applied at points (0, -b) and (0, +b) of the contour of an elliptic hole (Figure III.12) is found as the limit case of a uniformly distributed pressure p applied to the contour of an elliptic hole at certain of its segments, distributed symmetrically relative to the Oy axis.

Through $\sigma_1 = e^{i\vartheta_1}$ and $\sigma_2 = e^{i\vartheta_2}$ we will denote the points on the unit circle that correspond to points A and B of segment AB on the contour of the elliptic hole (see Figure III.11, where the points A and B in the given case should be located symmetrically in relation to the Oy axis). Such a segment of the contour is under stress by uniformly distributed pressure p in the upper part of the contour of the hole as well (Figure III.11). We will denote this through CD. The points C and D, obviously, will coincide on the unit circle to points $\sigma_3 = e^{i\vartheta_3}$ and $\sigma_4 = e^{i\vartheta_4}$.

The functions $\Phi(\zeta)$ and $\Psi(\zeta)$, where, on the above stated segments AB and CD

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of the contour of the elliptic hole, a uniform pressure p is applied, we find from formulas (III.10):

$$\Phi(\zeta) = -\frac{p}{8\pi(s_1 - s_2)i}\left\{(b + ibs_2)\left[\sigma_2 - \sigma_1 + 2\zeta \ln\frac{\sigma_2 - \zeta}{\sigma_1 - \zeta}\right] - \right.$$

$$-(b - ias_2)\times\left[\frac{2}{\zeta}\ln\frac{\sigma_2 - \zeta}{\sigma_1 - \zeta} + \frac{1}{\sigma_2} - \frac{1}{\sigma_1} - \frac{2}{\zeta}\ln\frac{\sigma_2}{\sigma_1}\right] - K_1\left[2\ln\frac{\sigma_2 - \zeta}{\sigma_1 - \zeta} - \ln\frac{\sigma_2}{\sigma_1}\right] \quad /231$$

$$+(K_2 - K_1)\left[2\ln\frac{\sigma_3 - \zeta}{\sigma_2 - \zeta} - \ln\frac{\sigma_3}{\sigma_2}\right] + (b + ias_2)\left[\sigma_4 - \sigma_3 + 2\zeta\ln\frac{\sigma_4 - \zeta}{\sigma_3 - \zeta}\right] -$$

$$-(b - ias_2)\left[\frac{2}{\zeta}\ln\frac{\sigma_4 - \zeta}{\sigma_3 - \zeta} - \frac{1}{\sigma_4} - \frac{1}{\sigma_3} - \frac{2}{\zeta}\ln\frac{\sigma_4}{\sigma_3}\right] + (K_2 - K_1 - K_3)\times$$

$$\left.\times\left[2\ln\frac{\sigma_4 - \zeta}{\sigma_3 - \zeta} - \ln\frac{\sigma_4}{\sigma_3}\right] + (K_4 + K_2 - K_3 - K_1)\left[2\ln\frac{\sigma_1 - \zeta}{\sigma_4 - \zeta} - \ln\frac{\sigma_1}{\sigma_4}\right]\right\},$$

where

$$K_1 = (b + ias_2)\sigma_1 - (b - ias_2)\frac{1}{\sigma_1},$$

$$K_2 = (b + ias_2)\sigma_2 - (b - ias_2)\frac{1}{\sigma_2},$$

$$K_3 = (b + ias_2)\sigma_3 - (b - ias_2)\frac{1}{\sigma_3},$$

$$K_4 = (b + ias_2)\sigma_4 - (b - ias_2)\frac{1}{\sigma_4}.$$

The analogous expression for $\Psi(\zeta)$ is easily found from $\Phi(\zeta)$ by substitution of s_2 by s_1.

Proceeding now to the limit such that $\lim_{\widehat{AB}\to 0}\{p\cdot\widehat{AB}\} = P$ and $\lim_{\widehat{CD}\to 0}\{p\cdot\widehat{CD}\} = P$,

we find functions $\Phi(\zeta)$ and $\Psi(\zeta)$ in the case where two mutually cancelling concentrated forces are applied to the contour of an elliptic hole at the points (0, -b) and (0, +b) (see Figure III.12):

$$\Phi(\zeta) = -\frac{iP}{2\pi(s_1 - s_2)}as_2\ln\frac{\zeta - i}{\zeta + i} + \text{const},$$

$$\Psi(\zeta) = \frac{iP}{2\pi(s_1 - s_2)}as_1\ln\frac{\zeta - i}{\zeta + i} + \text{const}. \qquad \text{(III.44)}$$

By substituting into $\Phi(\zeta)$ of (III.44), instead of ζ, its value in (III.12), and into $\Psi(\zeta)$ of (III.44), the value of ζ in (III.13), we find the functions $\phi(z_1)$ and $\psi(z_2)$. From the functions thus found we determine, by formulas (I.90), the stress components σ_x, σ_y and τ_{xy}.

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Of particular interest are stresses σ_y at the points $(\pm a; 0)$, i.e., at the points of intersection of the contour of the hole with the Ox axis. If we assume that $s_1 = i\beta_1$ and $s_2 = i\beta_2$, then at these points

$$\sigma_y = \frac{P}{a\pi} \cdot \frac{\beta_1 + \beta_2}{\beta_1\beta_2} \cdot \frac{a}{b} = \frac{P}{\pi} \cdot \frac{\beta_1 + \beta_2}{\beta_1\beta_2} \cdot \frac{1}{b}. \qquad \text{(III.45)}$$

If the medium is isotropic, then $\beta_1 = \beta_2 = 1$, and from (III.45) we find[1] the following stresses, which are independent of a: /232

$$\sigma_y = 2\frac{P}{a\pi} \cdot \frac{a}{b} = \frac{2P}{\pi} \cdot \frac{1}{b}.$$

In Figure III.13 are presented the curves of various anisotropic material: oak ($s_1 = 1.679i$; $s_2 = 0.971i$); spruce ($s_1 = 5.0007i$, $s_2 = 0.905i$); birch ($s_1 = 3.416i$, $s_2 = 1.126i$). Curve 1 corresponds to an isotropic plate, curve 2, to oak, curve 3, to spruce and curve 4 to birch.

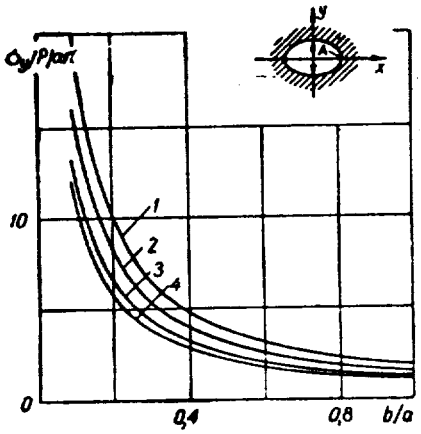


Figure III.13.

Stresses Around Elliptic or Round Hole in Anisotropic Rod (Beam) under Pure Deflection[2]. We will assume that an anisotropic rod (beam) with an elliptic hole (Figure III.14) is in a state of pure deflection. Deflecting moment M, its direction, and the position of the hole are shown in Figure III.14. We will find the stress state around the hole, assuming the contour of the hole to be free of external stresses. The solution of this problem will be found under the same assumptions as for an isotropic medium, i.e., we will assume that the plate is infinite and that the stress state in the remote areas from the hole is independent of the elliptic hole in the deflected beam. The stress state around the hole will be represented in the form

[1] A complex and inconvenient solution in the form of infinite trigonometric series for an isotropic medium was found by P. S. Symonds [1]. These series converge very slowly, and the graph for σ_ϑ represents a rough approximation.

[2] The solutions was found by S. G. Lekhnitskiy [1] and, by a different method, by G. N. Savin [3]. See also V. B. Lipkin [1].

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$$\sigma_x = \sigma_x^0 + \sigma_x^*,$$

$$\sigma_y = \sigma_y^0 + \sigma_y^*,$$

$$\tau_{xy} = \tau_{xy}^0 + \tau_{xy}^*,$$

where σ_x^0, σ_y^0, and τ_{xy}^0 are stresses of the basic state in an anisotropic rod (beam) under pure deflection, not weakened by an elliptic hole, σ_x^*, σ_y^* and τ_{xy}^* are additional stress components caused by the presence of a hole.[1] Here /233

$$\begin{aligned} \sigma_x^0 &= -\frac{M}{J}(x\sin\alpha + y\cos\alpha + d)\cos^2\alpha, \\ \sigma_y^0 &= -\frac{M}{J}(x\sin\alpha + y\cos\alpha + d)\sin^2\alpha, \\ \tau_{xy}^0 &= \frac{M}{J}(x\sin\alpha + y\cos\alpha + d)\sin\alpha\cos\alpha, \end{aligned} \tag{III.46}$$

where J is the moment of inertia of the cross section of the beam; d is the distance between the center of the elliptic hole and the neutral axis of the beam (rod); α is the angle formed by the Ox axis and the neutral line of the beam (rod). The corresponding function of stresses for pure deflection[1] is

$$U_0(x, y) = -\frac{M}{6J}[y^3\cos^3\alpha + 3xy\cos^2\alpha\sin\alpha + 3y^2d\cos^2\alpha + 3x^2y\sin^2\alpha\cos\alpha + \\ + 3xyd\sin 2\alpha + 3x^2d\sin^2\alpha + x^3\sin^3\alpha]; \tag{III.47}$$

$$U(x, y) = U_0(x, y) + U^*(x, y), \tag{III.48}$$

where $U_0(x, y)$ is the known function (III.47); $U^*(x, y)$ is an unknown function of stresses corresponding to the stress state σ_x^*, σ_y^* and τ_{xy}^*.

Due to the linearity of basic equation (I.82), if we substitute in it, instead of the function U(x, y), its value from (III.48), we arrive at the case where the function U*(x, y) will satisfy the very same equation (I.82): /234

$$a_{22}\frac{\partial^4U^*}{\partial x^4} - 2a_{26}\frac{\partial^4U^*}{\partial x^3\partial y} + (2a_{12} + a_{66})\frac{\partial^4U^*}{\partial x^2\partial y^2} - 2a_{16}\frac{\partial^4U^*}{\partial x\partial y^3} + a_{11}\frac{\partial^4U^*}{\partial y^4} = 0. \tag{III.49}$$

The general integral of this equation will have the form (I.85):

---

[1] For more complex forms of external load applied to an anisotropic beam on its upper and lower edges, the Airy function U(x, y) should be found by A. A. Kurdyumov's method [1].

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$$U^*(x, y) = F_1^*(z_1) + F_2^*(z_2) + \overline{F_1^*(z_1)} + \overline{F_2^*(z_2)}. \tag{III.50}$$

By substituting into (I.95), instead of σ_x, σ_y and τ_{xy}, their values

$$\sigma_x = \sigma_x^0 + \frac{\partial^2 U^*}{\partial y^2},$$

$$\sigma_y = \sigma_y^0 + \frac{\partial^2 U^*}{\partial x^2},$$

$$\tau_{xy} = \tau_{xy}^0 - \frac{\partial^2 U^*}{\partial x \partial y}$$

recalling that the contour of the hole is free of external forces, i.e. assuming $X_n = Y_n = 0$, we find the contour conditions for the function U*(x, y):

$$\frac{\partial^2 U^*}{\partial y^2}\cos(n, x) - \frac{\partial^2 U^*}{\partial x \partial y}\cos(n, y) = -[\sigma_x^0 \cos(n, x) + \tau_{xy}^0 \cos(n, y)],$$

$$-\frac{\partial^2 U^*}{\partial x \partial y}\cos(n, x) + \frac{\partial^2 U^*}{\partial x^2}\cos(n, y) = -[\tau_{xy}^0 \cos(n, x) + \sigma_y^0 \cos(n, y)].$$

Recalling that cos(n, x) = dy/ds and cos(n, y) = -dx/ds, we find

$$\frac{\partial U^*}{\partial x} = \int_0^s (\tau_{xy}^0 dy - \sigma_y^0 dx) + C_1,$$

$$\frac{\partial U^*}{\partial x} = -\int_0^s (\sigma_x^0 dy - \tau_{xy}^0 dx) + C_2. \tag{III.51}$$

Introducing, as before, the definitions

$$\frac{dF_1^*}{dz_1} = \varphi_0(z_1), \quad \frac{dF_2^*}{dz_2} = \psi_0(z_2),$$

we find, for these functions, the contour conditions in the form

$$\varphi_0(z_1) + \overline{\varphi_0(z_1)} + \psi_0(z_2) + \overline{\psi_0(z_2)} =$$

$$= \int_0^s (\tau_{xy}^0 dy - \sigma_y^0 dx) + C_1 = f_1,$$

$$s_1\varphi_0(z_1) + \overline{s_1}\cdot\overline{\varphi_0(z_1)} + s_2\cdot\psi_0(z_2) + \overline{s_2}\cdot\overline{\psi_0(z_2)} = \tag{III.52}$$

$$= -\int_0^s (\sigma_x^0 dy - \tau_{xy}^0 dx) + C_2 = f_2,$$

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where C_1, C_2 are arbitrary real constants, and σ_x^0, σ_y^0 and τ_{xy}^0 are known functions of the coordinates of the points of the contour, given by equations (III.46).

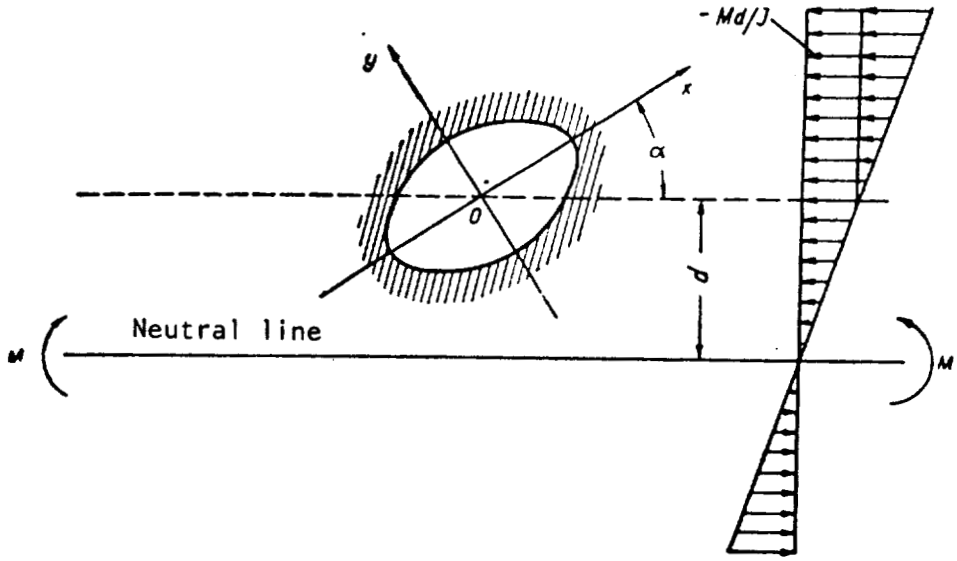


Figure III.14.

The contour conditions for $\phi_0(z_1)$ and $\psi_0(z_2)$ (III.52) have the very same form (III.6), as for functions $\phi(z_1)$ and $\psi(z_2)$. In this manner the solution of the stated problem reduces to a problem that has already been solved.

Let us return to our problem. We will determine the right hand parts of (III.52). By substituting, instead of σ_x^0, σ_y^0, τ_{xy}^0, their values from (III.46), we find

$$f_1 = \frac{M}{J}\left[\frac{y^2}{2}\sin\alpha\cos^2\alpha + \frac{x^2}{2}\sin^3\alpha + xy\sin^2\alpha\cos\alpha + \right.$$
$$\left. + xd\sin^2\alpha + yd\sin\alpha\cos\alpha\right] + \text{const},$$

$$f_2 = \frac{M}{J}\left[\frac{y^2}{2}\cos^3\alpha + \frac{x^2}{2}\sin^2\cos\alpha + xy\sin\alpha\cos^2\alpha + \right.$$
$$\left. + xd\sin\alpha\cos\alpha + yd\cos^2\alpha\right] + \text{const}.$$

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By substituting into the expressions for f_1 and f_2, instead of x and y, their values

$$x = \frac{a}{2}\left(\sigma + \frac{1}{\sigma}\right), \quad y = \frac{ib}{2}\left(\sigma - \frac{1}{\sigma}\right),$$

we obtain

$$f_1 = \frac{M}{J}\left[\frac{\sin\alpha}{8}(a^2\sin^2\alpha - b^2\cos^2\alpha + iab\sin 2\alpha)\sigma^2 + \right.$$
$$+ \frac{d\sin\alpha}{2}(a\sin\alpha + ib\cos\alpha)\sigma + \frac{d\sin\alpha}{2}(a\sin\alpha - ib\cos\alpha)\frac{1}{\sigma} +$$
$$\left. + \frac{\sin\alpha}{8}(a^2\sin^2\alpha - b^2\cos^2\alpha - iab\sin 2\alpha)\frac{1}{\sigma^2}\right] + \text{const},$$

$$f_2 = \frac{M}{J}\left[\frac{\cos\alpha}{8}(a^2\sin^2\alpha - b^2\cos^2\alpha + iba\sin 2\alpha)\sigma^2 + \right.$$
$$+ \frac{d\cos\alpha}{2}(a\sin\alpha + ib\cos\alpha)\sigma + \frac{d\cos\alpha}{2}(a\sin\alpha - ib\cos\alpha)\frac{1}{\sigma} +$$
$$\left. + \frac{\cos\alpha}{8}(a^2\sin^2\alpha - b^2\cos^2\alpha - iab\sin 2\alpha)\frac{1}{\sigma^2}\right] + \text{const}.$$

Then, by substituting the values for f_1 and f_2 into formulas (III.10), /236
noting that

$$\int_\gamma \sigma^2\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 4\pi i\zeta^2, \qquad \int_\gamma \sigma\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 4\pi i\zeta,$$
$$\int_\gamma \frac{1}{\sigma}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 0, \qquad \int_\gamma \frac{1}{\sigma^2}\cdot\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} = 0,$$

we find

$$\Phi(\zeta) = -\frac{M}{8J(s_1 - s_2)}[(s_2\sin\alpha - \cos\alpha)(a^2\sin^2\alpha - b^2\cos^2\alpha + iab\sin 2\alpha)\zeta^2 +$$
$$+ 4d(s_2\sin\alpha - \cos\alpha)(a\sin\alpha + ib\cos\alpha)\zeta] + \text{const},$$
$$\Psi(\zeta) = \frac{M}{8J(s_1 - s_2)}[(s_1\sin\alpha - \cos\alpha)(a^2\sin^2\alpha - b^2\cos^2\alpha + iab\sin 2\alpha)\zeta^2 + \qquad \text{(III.53)}$$
$$+ 4d(s_1\sin\alpha - \cos\alpha)(a\sin\alpha + ib\cos\alpha)\zeta] + \text{const}.$$

Further, by substituting into the functions $\Phi(\zeta)$ and $\Psi(\zeta)$, instead of ζ, their values from (III.12) and (III.13), respectively, we find the final form of

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the desired functions $\phi_0(z_1)$ and $\psi_0(z_2)$. The stress components for the desired functions are found from formulas

$$\begin{aligned}\sigma_x &= \sigma_x^0 + 2\,\mathrm{Re}\,[s_1^2\varphi_0'(z_1) + s_2^2\psi_0'(z_2)],\\ \sigma_y &= \sigma_y^0 + 2\,\mathrm{Re}\,[\varphi_0'(z_1) + \psi_0'(z_2)],\\ \tau_{xy} &= \tau_{xy}^0 - 2\,\mathrm{Re}\,[s_1\varphi_0'(z_1) + s_2\psi_0'(z_2)].\end{aligned} \tag{III.54}$$

We will not write out the functions $\phi_0(z_1)$ and $\psi_0(z_2)$ in the general case, since the statement of this problem is obvious, but will write them out for the case where $\alpha = 0$ (Figure III.15).

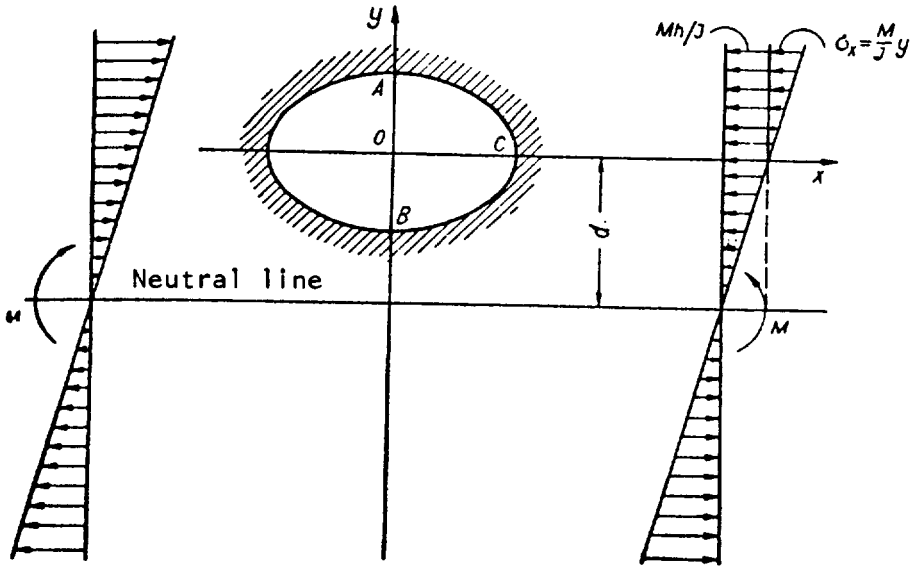


Figure III.15.

For this case

$$\begin{aligned}\varphi_0(z_1) &= \frac{Mdb}{2J}\cdot\frac{i}{s_1 - s_2}\cdot\frac{a - is_1b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}} - \\ &- \frac{Mb^2}{8J}\cdot\frac{1}{s_1 - s_2}\left[\frac{a - is_1b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}}\right]^2,\\ \psi_0(z_2) &= -\frac{Mdb}{2J}\cdot\frac{i}{s_1 - s_2}\cdot\frac{a - is_2b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}} + \\ &+ \frac{Mb^2}{8J}\cdot\frac{1}{s_1 - s_2}\left[\frac{a - is_2b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}}\right]^2.\end{aligned} \tag{III.55}$$

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Comparison of functions (III.55) with (III.26) shows that the first components in (III.55) give the solution of the problem in the case of tension (compression) of an isotropic rod weakened by an elliptic hole, along the Ox axis with intensity of forces p = -Md/J (Figure III.15); the second components in (III.55), however, correspond to pure deflection of a beam (rod), where the neutral line coincides with the Ox axis. /237

Additional stress components σ_x^*, σ_y^*, τ_{xy}^* can be determined from functions (III.55).

It is clear from (III.55) that the additional stress components caused by pure deflection of a rod (second components in (III.55)) will attenuate more rapidly by measure of distance from the hole than the additional stress components caused by tension (compression) of the rod (first components in (III.55)). Hence we may conclude that with pure deflection the zone of perturbation of the stress state around the elliptic hole when d = 0 in an anisotropic rod will be less than the corresponding zone when this rod is under tension.

We will find that the values of stresses σ_ϑ are the points A, B and C (see Figure III.15) of an elliptic hole. We assume that $s_1 = \alpha_1 + i\beta_1$ and $s_2 = \alpha_2 + i\beta_2$, i.e. the general case of anisotropy.

The stress at point A is

$$\sigma_\phi = -\frac{M}{J}\left[(b+d) + k\frac{b+2d}{2}(\beta_1+\beta_2)\right], \qquad \text{(III.56)}$$

where k = b/a is the ratio of the semiaxes of the elliptic hole. If the medium is isotropic, then $\beta_1 = \beta_2 = 1$, and from (III.56) we have

$$\sigma_\phi = -\frac{M}{J}\left[(b+d) + \frac{b}{a}(b+2d)\right]. \qquad \text{(III.57)}$$

If $\alpha = 0$, i.e., the center of the elliptic hole lies on the neutral line, /238 then from (III.57) we find formula (II.102)

$$\sigma_\phi = -\frac{Mb}{J}\left(1+\frac{b}{a}\right).$$

The stress at point B is

$$\sigma_\phi = -\frac{M}{J}\left[(d-b) + \frac{2d-b}{2}k(\beta_1+\beta_2)\right]. \qquad \text{(III.58)}$$

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The stress at point C is

$$\sigma_\phi = \frac{M}{J}\left[\frac{(\beta_1\beta_2 - \alpha_1\alpha_2)\,d}{(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)} + \frac{b}{2}\cdot\frac{\alpha_1\beta_2 + \alpha_2\beta_1)}{(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)}\right]. \qquad \text{(III.59)}$$

In the case of an isotropic medium ($\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$), we obtain

$$\sigma_\phi = \frac{Md}{J}.$$

It follows from (III.59) that when d = 0, in the general case of anisotropy, the stress at point C, lying on the neutral axis of the rod, will not be equal to zero:

$$\sigma_\phi = \frac{M_b}{2J}\cdot\frac{\beta_1\alpha_2 + \alpha_1\beta_2}{(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)}. \qquad \text{(III.60)}$$

Figure III.16 represents the stress-strain diagrams[1] for stresses σ_x through cross section x = 0 in an anisotropic beam weakened by an elliptic hole when a/b = 3. Three types of plywood represent the materials from which the beam is made[2].

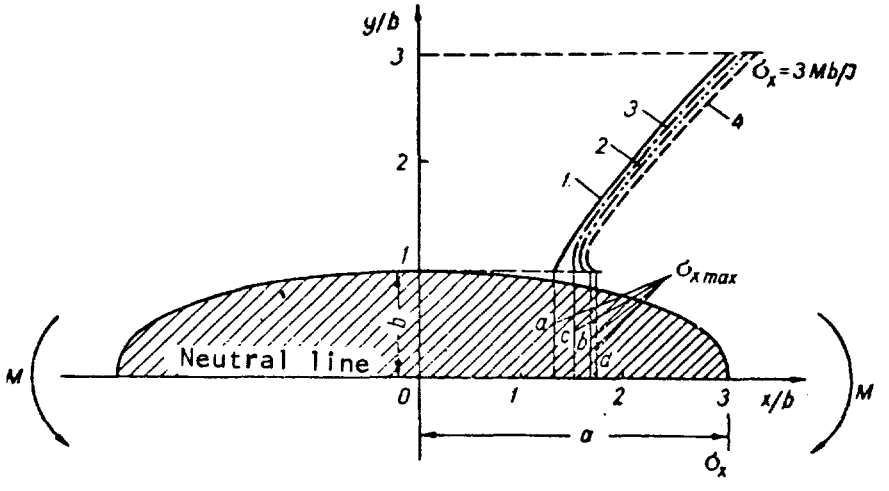


Figure III.16.

[1] The coinciding numerical values are represented (for clarity) in Figure III.16 in the form of parallel curves.
[2] See V. G. Chentsov [1], p. 24 and S. V. Serensen [1], p. 187.

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Curve 1 corresponds to an isotropic plate /239

$$\sigma_{x,\max} = 1{,}33\,\frac{Mb}{J}.$$

Curve 2 represents the material with elastic constants

$$E_x = 1{,}6\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad E_y = 0{,}905\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad G_{xy} = 0{,}097\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2$$
$$\nu_x = 0{,}05;$$
$$\sigma_{x,\max} = 1{,}727\,\frac{Mb}{J}.$$

Curve 3 represents the material with elastic constants

$$E_x = 0{,}5\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad E_y = 1{,}6\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad G_{xy} = 0{,}097\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2$$
$$\nu_y = 0{,}05;$$
$$\sigma_{x,\max} = 1{,}51\,\frac{Mb}{J}.$$

Curve 4 represents the material with elastic constants

$$E_x = 1{,}4\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad E_y = \frac{1{,}4}{12}\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2 \quad G_{xy} = 0{,}12\cdot 9{,}81\cdot 10^9 \text{ n/cm}^2$$
$$\nu_x = 0{,}46.$$
$$\sigma_{x,\max} = 1{,}70\,\frac{Mb}{J}.$$

We conclude from the stress-strain diagrams shown in Figure III.16 that:

1) the zone of perturbation of stresses caused by the hole is small;

2) the perturbations vanish rapidly by measure of distance from the hole, and the stress state in the beam approaches the basic stress state (III.46);

3) anisotropy in the immediate vicinity of the hole has a considerable effect on stresses[1].

A. S. Dorogobed [1] examined the pattern of the stress state in an orthotropic plate with a round hole under pure displacement.

---

[1] The analogous pattern prevails (see G. N. Savin [4]) in the case of an anisotropic beam with an elliptic or round hole under deflection by a constant shear force.

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Elastic Equilibrium of Anisotropic Plate with Elliptical Hole under the Effect of Force or Moment Applied to Some Point of the Plate[1]. We will define the functions $\phi(z_1)$ and $\psi(z_2)$ in an infinite homogeneous anisotropic plate weakened by an elliptic hole and deformed by a force or moment applied at an arbitrary point. The edge of the hole is assumed to be under stress by forces acting in the middle of the plate, the resultant vector of which forces have the components X and Y. The x and y axes are directed along the corresponding axes of the ellipse. We will denote through a and b its semiaxes. The thickness of an anisotropic, but generally not orthotropic, plate is assumed to be unity.

The boundary conditions for the given external forces X_n and Y_n are given /240
by formulas (I.96), which can be represented in the form

$$(1+is_1)\varphi(z_1)+(1+is_2)\psi(z_2)+(1+i\overline{s_1})\overline{\varphi(z_1)}+(1+i\overline{s_2})\overline{\psi(z_2)}= \\ =f_1+if_2=f, \tag{III.61}$$

where

$$(1-s_1i)\varphi(z_1)+(1-is_2)\psi(z_2)+(1-i\overline{s_1})\overline{\varphi(z_1)}+(1-i\overline{s_2})\overline{\psi(z_2)}= \\ =f_1-if_2=\overline{f},$$

$$f=f_1+if_2=i\int_0^s(X_n+iY_n)\,ds+\text{const}. \tag{III.62}$$

The constant in the right hand side of relation (III.62) can be assumed to be equal to zero in the cases examined below.

The results of the given section will be used in §3 and partially in §5 of the present chapter.

The Effect of Force. We will assume that at an arbitrary point of an anisotropic plate with an elliptic hole, a concentrated force $\overrightarrow{P}(P_x, P_y)$ is applied in its plane. We will determine the form of stress functions, assuming that the edge of the hole is also stressed by forces, the resultant vector of which has the components X and Y.

The functions $\phi(z_1)$ and $\psi(z_2)$ from the loading only of the edge of the elliptic hole have the form (I.124), where $B^* = B'^* = C'^* = C^* = 0$, i.e.

---

[1]The solution was found by D. V. Grilitskiy [1]. Along the way we will correct the errors that have crept in there. For an isotropic medium the solution of this problem was found by B. Karunes [1].

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$$\varphi_1(z_1) = A^{(1)} \ln z_1 + \varphi_{10}(z_1),$$
$$\psi_1(z_2) = B^{(1)} \ln z_2 + \psi_{10}(z_2). \qquad \text{(III.63)}$$

Here $\phi_{10}(z_1)$ and $\psi_{10}(z_2)$ are holomorphic functions outside of the elliptic holes in planes z_1 and z_2, respectively. The constants $A^{(1)}$ and $B^{(1)}$ are defined by formulas (I.116).

We will now assume that the edge of the hole is free of forces, and that at some point $z_0 = x_0 + iy_0$ of the plate is applied a concentrated force $\overline{P}$, with components P_x and P_y. We will also establish the form of functions $\phi_1(z_1)$ and $\psi_1(z_2)$ in this case.

If the concentrated force is applied at the origin of the coordinate system of a solid infinite anisotropic plate, then, as we know,

$$\varphi_2(z_1) = A^{(2)} \ln z_1, \qquad \psi_2(z_2) = B^{(2)} \ln z_2, \qquad \text{(III.64)}$$

where $A^{(2)}$ and $B^{(2)}$ are defined by formulas (I.116), hence, instead of X and Y, we must substitute P_x and P_y, respectively[1].

It is easy to show[2] that both functions for an anisotropic medium are invariant in the case of parallel translation of the origin of the coordinate system to a new point. Therefore, if the force is applied at an arbitrary point, defined by coordinates x_0, y_0, then the functions are /241

$$\varphi_2(z_1) = A^{(2)} \ln (z_1 - z_{10}), \qquad \psi_2(z_2) = B^{(2)} \ln (z_2 - z_{20}), \qquad \text{(III.65)}$$

where

$$z_{10} = x_0 + s_1 y_0, \qquad z_{20} = x_0 + s_2 y_0 \qquad \text{(III.66)}$$

are points corresponding to point z_0 of application of the force in the physical plane.

---

[1]Here, for contrast, various definitions are given for the components of the resultant vector of external forces applied to the contour of the hole, and for the components of the concentrated force $\overline{P}$ applied to an internal point of the plate. But if this is not necessary here, then the components of force $\overline{P}$ will also be denoted through X and Y.

[2]See D. V. Grilitskiy [1].

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In the presence of a hole the functions are

$$\varphi_2(z_1) = A^{(2)} \ln(z_1 - z_{10}) + \varphi_{20}(z_1),$$
$$\varphi_2(z_2) = B^{(2)} \ln(z_2 - z_{20}) + \psi_{20}(z_2), \qquad \text{(III.67)}$$

where $\phi_{20}(z_1)$ and $\psi_{20}(z_2)$ are holomorphic functions at infinity.

Thus, in the general case,

$$\varphi(z_1) = A^{(1)} \ln z_1 + A^{(2)} \ln(z_1 - z_{10}) + \varphi_0(z_1),$$
$$\psi(z_2) = B^{(1)} \ln z_2 + B^{(2)} \ln(z_2 - z_{20}) + \psi_0(z_2). \qquad \text{(III.68)}$$

We will map the exterior of the unit circle in plane ζ onto the exterior of elliptic holes in planes z_1 and z_2. The functions that accomplish this mapping have the form (III.4) and (III.5). The opposite transformations are given by formulas (III.12) and (III.13).

By substituting in conditions (III.61), instead of z_1 and z_2, their values from (III.4) and (III.5), and by denoting $\Phi(\zeta) = \phi[\omega_1(\zeta)]$, $\Psi(\zeta) = \Psi[\omega_2(\zeta)]$, we find the contour conditions for the functions $\Phi(\zeta)$ and $\Psi(\zeta)$:

$$(1 + is_1)\Phi(\sigma) + (1 + is_2)\Psi(\sigma) + (1 + i\bar{s}_1)\overline{\Phi(\sigma)} + (1 + i\bar{s}_2)\overline{\Psi(\sigma)} = f(\sigma),$$
$$(1 - is_1)\Phi(\sigma) + (1 - is_2)\Psi(\sigma) + (1 - i\bar{s}_1)\overline{\Phi(\sigma)} + (1 - i\bar{s}_2)\overline{\Psi(\sigma)} = \overline{f(\sigma)}. \qquad \text{(III.69)}$$

From (III.68), recalling formulas of transformation (III.4) and (III.5), we find the functions

$$\Phi(\zeta) = A^{(1)} \ln \zeta + A^{(2)} \ln \frac{(\zeta - \zeta_{10})[(a - is_1 b)\zeta\zeta_{10} - (a + is_1 b)]}{\zeta\zeta_{10}} + \Phi_0(\zeta),$$
$$\Psi(\zeta) = B^{(1)} \ln \zeta + B^{(2)} \ln \frac{(\zeta - \zeta_{10})[(a - is_2 b)]\zeta\zeta_{20} - (a + is_2 b)]}{\zeta\zeta_{20}} + \Psi_0(\zeta), \qquad \text{(III.70)}$$

where $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ are holomorphic functions outside of unit circle γ of plane ζ;

$$\zeta_{10} = \frac{z_{10} + \sqrt{z_{10}^2 - a^2 - s_1^2 b^2}}{a - is_1 b}, \qquad \zeta_{20} = \frac{z_{20} + \sqrt{z_{20}^2 - a^2 - s_2^2 b^2}}{a - is_2 b}. \qquad \text{(III.71)}$$ /242

By substituting functions (III.70) into conditions (III.69) and by making certain transformations, we will have on γ:

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$$\Phi_0(\sigma)+\frac{\bar{s}_1-s_2}{s_1-s_2}\overline{\Phi}_0\left(\frac{1}{\sigma}\right)+\frac{\bar{s}_2-s_2}{s_1-s_2}\overline{\Psi}_0\left(\frac{1}{\sigma}\right)=\frac{1-is_2}{2i(s_1-s_2)}f-\frac{1+is_2}{2i(s_1-s_2)}\bar{f}-$$

$$-\left[A^{(1)}-\frac{\overline{A}^{(1)}(\bar{s}_1-s_2)+\overline{B}^{(1)}(\bar{s}_2-s_2)}{s_1-s_2}\right]\ln\sigma-$$

$$-A^{(2)}\ln\frac{(\sigma-\zeta_{10})[(a-is_1b)\sigma\zeta_{10}-(a+is_1b)]}{\sigma\zeta_{10}}-$$

$$-\frac{\bar{s}_1-s_2}{s_1-s_2}\overline{A}^{(2)}\ln\frac{(1-\sigma\bar{\zeta}_{10})[(a+i\bar{s}_1b)\bar{\zeta}_{10}-(a-i\bar{s}_1b)\sigma]}{\sigma\bar{\zeta}_{10}}-$$

$$-\frac{\bar{s}_2-s_2}{s_1-s_2}\overline{B}^{(2)}\ln\frac{(1-\sigma\bar{\zeta}_{20})[(a+i\bar{s}_2b)\bar{\zeta}_{20}-(a-i\bar{s}_2b)\sigma]}{\sigma\bar{\zeta}_{20}};$$

$$\Psi_0(\sigma)-\frac{\bar{s}_1-s_1}{s_1-s_2}\overline{\Phi}_0\left(\frac{1}{\sigma}\right)-\frac{\bar{s}_2-s_1}{s_1-s_2}\overline{\Psi}_0\left(\frac{1}{\sigma}\right)= \qquad \text{(III.72)}$$

$$=-\frac{1-is_1}{2i(s_1-s_2)}f+\frac{1+is_1}{2i(s_1-s_2)}\bar{f}-\left[B^{(1)}+\frac{\overline{A}^{(1)}(\bar{s}_1-s_1)+\overline{B}^{(1)}(\bar{s}_2-s_1)}{s_1-s_2}\right]\ln\sigma-$$

$$-B^{(2)}\ln\frac{(\sigma-\zeta_{20})[(a-is_2b)\sigma\zeta_{20}-(a+is_2b)]}{\sigma\zeta_{20}}+$$

$$+\frac{\bar{s}_1-s_1}{s_1-s_2}\overline{A}^{(2)}\ln\frac{(1-\sigma\bar{\zeta}_{20})[(a+i\bar{s}_1b)\bar{\zeta}_{10}-(a-i\bar{s}_1b)\sigma]}{\sigma\bar{\zeta}_{10}}+$$

$$+\frac{\bar{s}_2-s_1}{s_1-s_2}\overline{B}^{(2)}\ln\frac{(1-\sigma\bar{\zeta}_{20})[(a+i\bar{s}_2b)\bar{\zeta}_{20}-(a-i\bar{s}_2b)\sigma]}{\sigma\bar{\zeta}_{20}}.$$

Then, by multiplying equations (III.72) by $1/2\pi i \cdot d\sigma/\sigma - \zeta$, where ζ is located outside of circle γ, and by integrating along γ, we find

$$\Phi_0(\zeta)=\frac{l_2}{2\pi i}\int_\gamma\frac{f\,d\sigma}{\sigma-\zeta}-\frac{m_2}{2\pi i}\int_\gamma\frac{\bar{f}\,d\sigma}{\sigma-\zeta}+(A^{(1)}-C_2)\ln\frac{\sigma_1-\zeta}{\zeta}+$$

$$+A^{(2)}\ln\frac{\zeta\zeta_{10}}{(a-is_1b)\zeta\zeta_{10}-(a+is_1b)}+A_2\ln\frac{\zeta\bar{\zeta}_{10}}{\zeta\bar{\zeta}_{10}-1}+B_2\ln\frac{\zeta\bar{\zeta}_{20}}{\zeta\bar{\zeta}_{20}-1};$$

$$\Psi_0(\zeta)=-\frac{l_1}{2\pi i}\int_\gamma\frac{f\,d\sigma}{\sigma-\zeta}+\frac{m_1}{2\pi i}\int_\gamma\frac{\bar{f}\,d\sigma}{\sigma-\zeta}+(B^{(1)}+C_1)\ln\frac{\sigma_1-\zeta}{\zeta}+ \qquad \text{(III.73)} \quad \underline{/243}$$

$$+B^{(2)}\ln\frac{\zeta\zeta_{20}}{(a-is_2b)\zeta\zeta_{20}-(a+is_2b)}-A_1\ln\frac{\zeta\bar{\zeta}_{10}}{\zeta\bar{\zeta}_{10}-1}-B_1\ln\frac{\zeta\bar{\zeta}_{20}}{\zeta\bar{\zeta}_{20}-1},$$

where σ_1 is an arbitrary point on γ, used as the origin:

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$$C_k = \frac{(\bar{s}_1 - s_k)\overline{A}^{(1)} + (\bar{s}_2 - s_k)\overline{B}^{(1)}}{s_1 - s_2}, \qquad l_k = \frac{i(1 - is_k)}{2(s_1 - s_2)},$$

$$m_k = \frac{i(1 + is_k)}{2(s_1 - s_2)}, \qquad A_k = \frac{(\bar{s}_1 - s_k)\overline{A}^{(2)}}{s_1 - s_2}, \qquad B_k = \frac{(\bar{s}_2 - s_k)\overline{B}^{(2)}}{s_1 - s_2} \tag{III.74}$$

$$(k = 1, 2).$$

From (III.70), considering expressions (III.73), we find

$$\Phi(\zeta_1) = \frac{l_2}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma - \zeta_1} - \frac{m_2}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma - \zeta_1} + C_2 \ln \zeta_1 + (A^{(1)} - C_2)\ln(\sigma_1 - \zeta_1) +$$
$$+ A^{(2)} \ln(\zeta_1 - \zeta_{10}) + A_2 \ln \frac{\zeta_1 \bar{\zeta}_{10}}{\zeta_1 \bar{\zeta}_{10} - 1} + B_2 \ln \frac{\zeta_1 \bar{\zeta}_{20}}{\zeta_1 \bar{\zeta}_{20} - 1}; \tag{III.75}$$

$$\Psi(\zeta_2) = -\frac{l_1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma - \zeta_2} + \frac{m_1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma - \zeta_{12}} - C_1 \ln \zeta_2 + (B^{(1)} + C_1)\ln(\sigma_1 - \zeta_2) +$$
$$+ B^{(2)} \ln(\zeta_2 - \zeta_{20}) - A_1 \ln \frac{\zeta_2 \bar{\zeta}_{10}}{\zeta_2 \bar{\zeta}_{10} - 1} - B_1 \ln \frac{\zeta_2 \bar{\zeta}_{20}}{\zeta_2 \bar{\zeta}_{20} - 1}. \tag{III.76}$$

The formulas that we have derived make it possible to find the solution for several partial cases:

1) for a force equal to zero, in formulas (III.74), (III.75) and (III.76), we may assume that $A^{(2)} = B^{(2)} \equiv 0$;

2) for a resultant vector of external forces applied to the contour of an elliptic hole equal to zero; in this case, in formulas (III.75) and (III.76) it is assumed that $A^{(1)} = B^{(1)} = C_1 = C_2 \equiv 0$;

3) if, moreover, the contour of the hole is free of forces, then $f = \bar{f} \equiv 0$ and the integrals in the preceding formulas vanish; in this case we obtain the simple expressions

$$\Phi(\zeta_1) = A^{(2)} \ln(\zeta_1 - \zeta_{10}) + A_2 \ln \frac{\zeta_1 \bar{\zeta}_{10}}{\zeta_1 \bar{\zeta}_{10} - 1} + B_2 \ln \frac{\zeta_1 \bar{\zeta}_{20}}{\zeta_1 \bar{\zeta}_{20} - 1},$$
$$\Psi(\zeta_2) = B^{(2)} \ln(\zeta_2 - \zeta_{20}) - A_1 \ln \frac{\zeta_2 \bar{\zeta}_{10}}{\zeta_2 \bar{\zeta}_{10} - 1} - B_1 \ln \frac{\zeta_2 \bar{\zeta}_{20}}{\zeta_2 \bar{\zeta}_{20} - 1}. \tag{III.77}$$

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In particular, if the force is applied at a point on the contour of the hole, then, assuming that $\zeta_{10} = \zeta_{20} = \sigma_0$ in formulas (III.77), where σ_0 is a point on the unit circle, we find /244

$$\Phi(\zeta_1) = (A^{(2)} - A_2 - B_2)\ln(\zeta_1 - \sigma_0) + (A_2 + B_2)\ln\zeta_1,$$
$$\Psi(\zeta_2) = (B^{(2)} + A_1 + B_1)\ln(\zeta_2 - \sigma_0) - (A_1 + B_1)\ln\zeta_2. \tag{III.78}$$

By knowing the functions $\Phi(\zeta_1)$ and $\Psi(\zeta_2)$ and by returning to the variables z_1 and z_2 in (III.12) and (III.13), we find the functions $\phi(z_1)$ and $\psi(z_2)$, which help us to find the stress components σ_x, σ_y and τ_{xy} by formulas (I.90).

The normal stress along the contour of the hole is

$$\sigma_\vartheta = \frac{2}{a^2\sin^2\vartheta + b^2\cos^2\vartheta}\operatorname{Re}[(b\cos\vartheta + s_1 a\sin\vartheta)^2\varphi'(z_1) + (b\cos\vartheta + s_2 a\sin\vartheta)^2\psi'(z_2)]. \tag{III.79}$$

In the case of a round hole a = b and formula (III.79) acquires the form

$$\sigma_\vartheta = 2\operatorname{Re}[(\cos\vartheta + s_1\sin\vartheta)^2\varphi'(z_1) + (\cos\vartheta + s_2\sin\vartheta)^2\psi'(z_2)]. \tag{III.80}$$

For grade I plywood, by directing the x axis along the grain, i.e., by making it coincide with the direction corresponding to the major modulus of elasticity, we find the elastic constants[1]

$$E_1 = 1{,}4\cdot 9{,}81\cdot 10^9, \quad E_2 = \frac{1{,}4}{12}9{,}81\cdot 10^9; \quad \nu_1 = 0{,}46,$$
$$\nu_2 = \frac{0{,}46}{12}; \quad s_1 = \beta_1 i = 3{,}08i, \quad s_2 = \beta_2 i = 1{,}12i.$$

By formula (III.80) we calculate stresses σ_ϑ along the contour of the round hole, where the force is applied on the contour at the point of intersection of the x axis with the contour. Here we examine two cases of the effect of the force: in the direction of the positive axis x, and in the direction of the negative axis y. For comparison we give the corresponding data for an isotropic plate when $\nu = 0.3$. The results of these calculations for angle ϑ are represented in Table III.4, where the values of stresses σ_ϑ are given in fractions P/R (P is the force, R is the radius of the hole).

[1]See S. G. Lekhnitskiy [1], Chapter II, §11, and also G. N. Savin [4], Chapter III, §1.

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The following conclusions can be derived from the data presented in Table III.4:

1) for a force directed along the positive x axis, the values of σ_ϑ will be equal in absolute value and sign to the corresponding values of σ_ϑ within the range 0-180°;

2) for a force directed along the negative y axis, the values of σ_ϑ will be equal in absolute value, but opposite in sign to the corresponding values of σ_ϑ in the range 0-180°.

TABLE III.4.

ϑ°	force directed along positive x axis		force directed along negative y axis		ϑ°	force directed along positive x axis		force directed along negative y axis	
	plywood plate	isotropic plate	plywood plate	isotropic plate		plywood plate	isotropic plate	plywood plate	isotropic plate
0	0,152	0,748	∞	∞	100	0,371	0,244	—0,636	—0,156
10	0,156	0,742	2,247	3,561	110	0,135	0,171	—0,487	—0,181
20	0,168	0,722	1,166	1,658	120	0,0289	0,103	—0,325	—0,188
30	0,189	0,690	0,822	0,973	130	—0,0135	0,0421	—0,212	—0,181
40	0,225	0,648	0,662	0,598	140	—0,0303	—0,0109	—0,138	—0,160
50	0,282	0,594	0,565	0,353	150	—0,0368	—0,0538	—0,0881	—0,130
60	0,374	0,533	0,477	0,179	160	—0,0392	—0,0855	—0,0524	—0,0908
70	0,516	0,465	0,331	0,0508	170	—0,0399	—0,105	—0,0244	—0,0468
80	0,685	0,393	0,0067	—0,0438	180	—0,0400	—0,111	0	0
90	0,668	0,318	—0,476	—0,111					

Tr. Note: Commas indicate decimal points.

The Effect of the Moment. We will determine the functions $\phi(z_1)$ and $\psi(z_2)$ for an anisotropic plate with an elliptic hole, where a concentrated couple with moment M is applied at an arbitrary point. As before, we will assume that the edge of the hole is loaded by external forces with a resultant vector not equal to zero.

By tracing the preceding path, we establish first the form of functions /245 $\phi(z_1)$ and $\psi(z_2)$.

If constant tangential forces of intensity T are applied to the contour of a round hole of radius R in an infinite anisotropic medium, then, as follows from (III.40),

$$\varphi(z_1) = \frac{TR^2(1+is_2)(1-is_1)}{2(s_1-s_2)\left(z_1+\sqrt{z_1^2-R^2(1+s_1^2)}\right)},$$

$$\psi(z_2) = -\frac{TR^2(1+is_1)(1-is_2)}{2(s_1-s_2)\left(z_2+\sqrt{z_2^2-R^2(1+s_2^2)}\right)}. \qquad \text{(III.81)}$$

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If we assume in (III.81) that

$$T = -\frac{M}{2\pi R^2},$$

where M is the main moment of external forces with respect to the center of the hole, and proceed to the limit, where $R \to 0$, such that M remains constant, then we find

$$\varphi(z_1) = -\frac{M(1+is_2)(1-is_1)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_1},$$
$$\psi(z_2) = \frac{M(1+is_1)(1-is_2)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_2}. \qquad \text{(III.82)}$$

Functions (III.82) represent stress functions for an infinite anisotropic elastic plate in the case where a concentrated couple with moment M is applied at the origin of the coordinates.

On the basis of the preceding formulas it is easy to find the functions for an anisotropic plate with an elliptic hole, at the point (x_0, y_0) of which is applied a concentrated moment M: /246

$$\varphi_2(z_1) = -\frac{M(1+is_2)(1-is_1)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_1-z_{10}} + \varphi_{20}(z_1),$$
$$\psi_2(z_2) = \frac{M(1+is_1)(1-is_2)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_2-z_{20}} + \psi_{20}(z_2), \qquad \text{(III.83)}$$

where $\phi_{20}(z_1)$ and $\psi_{20}(z_2)$ are holomorphic functions outside of the hole.

In the general case, i.e., where the edge of the hole is also loaded by forces, with a resultant vector not equal to zero, the functions $\phi(z_1)$ and $\psi(z_2)$ are represented by the sum of expressions (III.63) and (III.83):

$$\varphi(z_1) = A^{(1)}\ln z_1 - \frac{M(1+is_2)(1-is_1)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_1-z_{10}} + \varphi_0(z_1),$$
$$\psi(z_2) = B^{(1)}\ln z_2 + \frac{M(1+is_1)(1-is_2)}{8\pi(s_1-s_2)}\cdot\frac{1}{z_2-z_{20}} + \psi_0(z_2). \qquad \text{(III.84)}$$

Proceeding in (III.84) to variable ζ, we obtain

$$\Phi(\zeta_1) = A^{(1)}\ln\zeta - \frac{M(1+is_2)(1-is_1)}{4\pi(s_1-s_2)}\cdot\frac{\zeta\zeta_{10}}{(\zeta-\zeta_{10})[(a-is_1b)\zeta\zeta_{10}-(a+is_1b)]} + \Phi_0(\zeta).$$
$$\Psi(\zeta_2) = B^{(1)}\ln\zeta + \frac{M(1+is_1)(1-is_2)}{4\pi(s_1-s_2)}\cdot\frac{\zeta\zeta_{20}}{(\zeta-\zeta_{20})[(a-is_2b)\zeta\zeta_{20}-(a+is_2b)]} + \Psi_0(\zeta). \qquad \text{(III.85)}$$
(III.85)

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By substituting expressions (III.85) into the boundary conditions (III.69) we find on unit circle γ:

$$\Phi_0(\sigma)+\frac{\bar{s}_1-s_2}{s_1-s_2}\overline{\Phi}_0\left(\frac{1}{\sigma}\right)+\frac{\bar{s}_2-s_2}{s_1-s_2}\cdot\overline{\Psi}_0\left(\frac{1}{\sigma}\right)=$$
$$=\frac{1-is_2}{2i(s_1-s_2)}\hat{f}-\frac{1+is_2}{2i(s_1-s_2)}\bar{f}-(A^{(1)}-C_2)\ln\sigma+$$
$$+\frac{M(1+is_2)(1-is_1)}{4\pi(s_1-s_2)}\cdot\frac{\sigma\zeta_{10}}{(\sigma-\zeta_{10})[(a-is_1b)\sigma\zeta_{10}-(a+is_1b)]}+$$
$$+\frac{\bar{s}_1-s_2}{s_1-s_2}\cdot\frac{M(1-i\bar{s}_2)(1+i\bar{s}_1)}{4\pi(\bar{s}_1-\bar{s}_2)}\cdot\frac{\sigma\bar{\zeta}_{10}}{(1-\sigma\bar{\zeta}_{10})[(a+i\bar{s}_1b)\bar{\zeta}_{10}-(a-i\bar{s}_1b)\sigma]}-$$
$$-\frac{\bar{s}_2-s_2}{s_1-s_2}\cdot\frac{M(1-i\bar{s}_1)(1+i\bar{s}_2)}{4\pi(\bar{s}_1-\bar{s}_2)}\cdot\frac{\sigma\bar{\zeta}_{20}}{(1-\sigma\bar{\zeta}_{20})[(a+i\bar{s}_2b)\bar{\zeta}_{20}-(a-i\bar{s}_2b)\sigma]};$$

$$\Psi_0(\sigma)-\frac{\bar{s}_1-s_1}{s_1-s_2}\overline{\Phi}_0\left(\frac{1}{\sigma}\right)-\frac{\bar{s}_2-s_1}{s_1-s_2}\overline{\Psi}_0\left(\frac{1}{\sigma}\right)= \qquad \text{(III.86)}$$
$$=-\frac{1-is_1}{2i(s_1-s_2)}\hat{f}+\frac{1+is_1}{2i(s_1-s_2)}\bar{f}-(B^{(1)}+C_1)\ln\sigma-$$
$$-\frac{M(1+is_1)(1-is_2)}{4\pi(s_1-s_2)}\cdot\frac{\sigma\zeta_{20}}{(\sigma-\zeta_{20})[(a-is_2b)\sigma\zeta_{20}-(a+is_2b)]}- \qquad \text{/247}$$
$$-\frac{\bar{s}_1-s_1}{s_1-s_2}\cdot\frac{M(1-i\bar{s}_2)(1+i\bar{s}_1)}{4\pi(\bar{s}_1-\bar{s}_2)}\cdot\frac{\sigma\bar{\zeta}_{10}}{(1-\sigma\bar{\zeta}_{10})[(a+i\bar{s}_1b)\bar{\zeta}_{10}-(a-i\bar{s}_1b)\sigma]}+$$
$$+\frac{\bar{s}_2-s_1}{s_1-s_2}\cdot\frac{M(1-i\bar{s}_1)(1+i\bar{s}_2)}{4\pi(\bar{s}_1-\bar{s}_2)}\cdot\frac{\sigma\bar{\zeta}_{20}}{(1-\sigma\bar{\zeta}_{20})[(a+i\bar{s}_2b)\bar{\zeta}_{20}-(a-i\bar{s}_2b)\sigma]}.$$

From boundary conditions (III.86), we find the functions

$$\Phi_0(\zeta)=\frac{l_2}{2\pi i}\int_\gamma\frac{f d\sigma}{\sigma-\zeta}-\frac{m_2}{2\pi i}\int_\gamma\frac{\bar{f}d\sigma}{\sigma-\zeta}+(A^{(1)}-C_2)\ln\frac{\sigma_1-\zeta}{\zeta}-$$
$$-\frac{Mn_1(a+is_1b)\zeta_{10}}{4\pi[(a-is_1b)\zeta\zeta_{10}-(a+is_1b)]}-\frac{M\gamma_1\bar{\zeta}_{10}}{4\pi(\zeta\bar{\zeta}_{10}-1)}+\frac{M\gamma_2\bar{\zeta}_{20}}{4\pi(\zeta\bar{\zeta}_{20}-1)}, \qquad \text{(III.87)}$$
$$\Psi_0(\zeta_2)=-\frac{l_1}{2\pi i}\int_\gamma\frac{f d\sigma}{\sigma-\zeta}+\frac{m_1}{2\pi i}\int_\gamma\frac{\bar{f}d\sigma}{\sigma-\zeta}+(B^{(1)}+C_1)\ln\frac{\sigma_1-\zeta}{\zeta}+$$
$$+\frac{Mn_2(a+is_2b)\zeta_{20}}{4\pi[(a-is_2b)\zeta\zeta_{20}-(a+is_2b)]}+\frac{M\delta_1\bar{\zeta}_{10}}{4\pi(\zeta\bar{\zeta}_{10}-1)}-\frac{M\delta_2\bar{\zeta}_{20}}{4\pi(\zeta\bar{\zeta}_{20}-1)},$$

where C_1 and C_2 are defined by formula (III.74).

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Here we introduce the following definitions:

$$n_k = \frac{(1 - is_k)(1 + is_{k+1})}{(s_1 - s_2)[(a - is_k b)\zeta_{k0}^2 - (a + is_k b)]} \qquad (s_3 = s_1);$$

$$\gamma_k = \bar{n}_k \frac{\bar{s}_k - s_2}{s_1 - s_2}; \qquad \delta_k = \bar{n}_k \frac{\bar{s}_k - s_1}{s_1 - s_2} \qquad (k = 1, 2). \tag{III.88}$$

On the basis of (III.85) and (III.87), we finally have

$$\Phi(\zeta_1) = \frac{l_2}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma - \zeta_1} - \frac{m_2}{2\pi i}\int_\gamma \frac{\bar{f} d\sigma}{\sigma - \zeta_1} + C_2 \ln \zeta_1 +$$

$$+ (A^{(1)} - C_2)\ln(\sigma_1 - \zeta_1) - \frac{M}{4\pi}\left(\frac{n_1\zeta_{10}^2}{\zeta_1 - \zeta_{10}} + \frac{\gamma_1\bar{\zeta}_{10}}{\zeta_1\bar{\zeta}_{10} - 1} - \frac{\gamma_2\bar{\zeta}_{20}}{\zeta_1\bar{\zeta}_{20} - 1}\right),$$

$$\Psi(\zeta_2) = -\frac{l_1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma - \zeta_2} + \frac{m_1}{2\pi i}\int_\gamma \frac{\bar{f} d\sigma}{\sigma - \zeta_2} - C_1 \ln \zeta_2 + \tag{III.89}$$

$$+ (B^{(1)} + C_1)\ln(\sigma_1 - \zeta_2) + \frac{M}{4\pi}\left(\frac{n_2\zeta_{20}^2}{\zeta_2 - \zeta_{20}} + \frac{\delta_1\bar{\zeta}_{10}}{\zeta_2\bar{\zeta}_{10} - 1} - \frac{\delta_2\bar{\zeta}_{20}}{\zeta_2\bar{\zeta}_{20} - 1}\right).$$

When the contour of the elliptic hole is free of external forces, formulas (III.89) acquire the form /248

$$\Phi(\zeta_1) = -\frac{M}{4\pi}\left(\frac{n_1\zeta_{10}^2}{\zeta_1 - \zeta_{10}} + \frac{\gamma_1\bar{\zeta}_{10}}{\zeta_1\bar{\zeta}_{10} - 1} - \frac{\gamma_2\bar{\zeta}_{20}}{\zeta_1\bar{\zeta}_{20} - 1}\right),$$

$$\Psi(\zeta) = \frac{M}{4\pi}\left(\frac{n_2\zeta_{20}^2}{\zeta_2 - \zeta_{20}} + \frac{\delta_1\bar{\zeta}_{20}}{\zeta_2\bar{\zeta}_{10} - 1} - \frac{\delta_2\bar{\zeta}_{20}}{\zeta_2\bar{\zeta}_{20} - 1}\right). \tag{III.90}$$

If a moment is applied to a point of the contour of the hole, then, assuming in (III.88) and (III.90) that $\zeta_{10} = \zeta_{20} = \sigma_0$, we obtain

$$\Phi(\zeta_1) = -\frac{M}{4\pi}\cdot\frac{n_1\sigma_0^2 + \gamma_1 - \gamma_2}{\zeta_1 - \sigma_0},$$

$$\Psi(\zeta_2) = \frac{M}{4\pi}\cdot\frac{n_2\sigma_0^2 + \delta_1 - \delta_2}{\zeta_2 - \sigma_0}. \tag{III.91}$$

The values of normal stress σ_ϑ around the contour of a round hole in the same type of plywood plate, for which Table III.4 is compiled for angle ϑ, which changes within the range 0-180°, when the moment is applied at the contour of hole at the point of intersection with the x axis, are presented in Table III.5.

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The values of σ_ϑ within the range of change of angle ϑ from 0 to -180° are equal in absolute value, but opposite in sign, to the corresponding values for change of angle ϑ from 0 to +180°. The values of stresses σ_ϑ are presented in fractions M/R^2.

TABLE III.5.

ϑ°	plywood plate	isotropic plate	ϑ°	plywood plate	isotropic plate
0	—∞	—∞	100	—0,443	—0,267
10	—0,664	—3,638	110	—0,228	—0,223
20	—0,364	—1,805	120	—0,166	—0,184
30	—0,284	—1,188	130	—0,0626	—0,148
40	—0,267	—0,874	140	—0,0354	—0,116
50	—0,288	—0,683	150	—0,0204	—0,0853
60	—0,349	—0,551	160	—0,0113	—0,0561
70	—0,465	—0,455	170	—0,0051	—0,0278
80	—0,630	—0,379	180	0	0
90	—0,668	—0,318	—	—	—

Tr. Note: Commas indicate decimal points.

B. Karunes [1] analyzes the analogous problem for the case of an isotropic plate in which there is a hole with a free edge. Certain partial cases of the problem solved by B. Karunes were analyzed later by F. Szelagowski [1, 2].

§2. Solution of Second Basic Problem for Region with Elliptic Hole[1]

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We will assume that displacements of points of contour L of an elliptic hole are known. The contour conditions for functions $\phi(z_1)$ and $\psi(z_2)$ in this case have the form (I.97):

$$2\,\mathrm{Re}\,[p_1\varphi(z_1)+p_2\psi(z_2)]=g_1(s),$$
$$2\,\mathrm{Re}\,[q_1\varphi(z_1)+q_2\psi(z_2)]=g_2(s). \qquad \text{(III.92)}$$

We will assume that the resultant vector of external forces which cause the given displacements of the points of contour L, and the stresses at infinity, are equal to zero. Then $\phi(z_1)$ and $\psi(z_2)$ are holomorphic functions in regions $S^{(1)}$ and $S^{(2)}$.

By using the same attack as in the case of the first basic problem, we obtain

[1]The solution is given by G. N. Savin [2]; the solution of this problem was found earlier by S. G. Lekhnitskiy [1, 5] with the aid of series.

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$$p_1\Phi(\zeta) + p_2\Psi(\zeta) = \frac{1}{4\pi i}\int_\gamma g_1(\vartheta)\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + i\alpha_0,$$
$$q_1\Phi(\zeta) + q_2\Psi(\zeta) = \frac{1}{4\pi i}\int_\gamma g_2(\vartheta)\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + i\beta_0, \qquad \text{(III.93)}$$

or

$$\Phi(\zeta) = \frac{i}{4\pi(p_1q_2 - p_2q_1)}\int_\gamma [p_2g_2(\vartheta) - q_2g_1(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_1,$$
$$\Psi(\zeta) = -\frac{i}{4\pi(p_1q_2 - p_2q_1)}\int_\gamma [p_1g_2(\vartheta) - q_2g_1(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_2, \qquad \text{(III.94)}$$

where

$$\lambda_1 = i\frac{\alpha_0q_2 - \beta_0p_2}{p_1q_2 - p_2q_1}, \qquad \lambda_2 = -i\frac{\alpha_0q_1 - \beta_0p_1}{p_1q_2 - p_2q_1}. \qquad \text{(III.95)}$$

By substituting ζ in functions (III.94) by its values through z_1 and z_2, respectively, by formulas (III.12) and (III.13), we find the final form of the desired functions $\phi(z_1)$ and $\psi(z_2)$.

Since the contour conditions (III.92) are satisfied by functions $\phi(z_1)$ and $\psi(z_2)$, we will assume henceforth, for all λ_1 and λ_2, that they are equal to zero.

The functions $\phi(z_1)$ and $\psi(z_2)$ are holomorphic in their regions $S^{(1)}$ and $S^{(2)}$ (see Figure III.1). Consequently, displacements u and v will be limited at infinity.

We will return now to the general case where the resultant vector of external forces caused by the given displacements of the points of contour L, and the stresses at infinity, are not equal to zero. In this case the functions (z_1) /250
and (z_2) will have the form (III.14). The constants A, B, B*, B'* and C'* are defined by the same equations (I.116) and (I.127), as in the case of the first basic problem.

We see, therefore, that in order to determine the solution of the second basic problem it is necessary to know the resultant vector of external forces applied to the contour L of the hole. By substituting functions $\phi(z_1)$ and $\psi(z_2)$ from (III.14) into contour conditions (III.92), and by transposing the known functions into the right hand sides, we obtain

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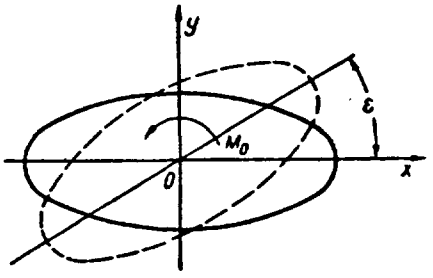


Figure III.17.

$$2\mathrm{Re}\,[p_1\varphi_0(z_1) + p_2\psi_0(z_2)] = g_1^0(s),$$
$$2\mathrm{Re}\,[q_1\varphi_0(z_1) + q_2\psi_0(z_2)] = g_2^0(s), \qquad \text{(III.96)}$$

where

$$g_1^0(s) = g_1(\vartheta) - 2\mathrm{Re}\,[Ap_1 \ln z_1 + B^* p_1 z_1 + Bp_2 \ln z_2 + p_2(B'^* + iC'^*)z_2],$$
$$g_2^0(s) = g_2(\vartheta) - 2\mathrm{Re}\,[Aq_1 \ln z_1 + B^* q_1 z_1 + Bq_2 \ln z_2 + q_2(B'^* + iC'^*)z_2]. \qquad \text{(III.97)}$$

It is clear from (III.96) that if we substitute in formulas (III.94), instead of $g_1(s)$ and $g_2(s)$, the functions $g_1^0(s)$ and $g_2^0(s)$ of (III.97), we obtain the desired functions $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$. Returning to the variables z_1 and z_2 according to formulas (III.12) and (III.13), we find the functions $\phi_0(z_1)$ and $\psi_0(z_2)$, and by placing them into formulas (III.14), we obtain the final form of functions $\phi(z_1)$ and $\psi(z_2)$.

Stresses in Anisotropic Plate Around Absolutely Rigid Elliptic Nucleus (or Ring), Under the Effect of Couple M_0. We will assume that stresses at infinity are equal to zero. Under the effect of couple M_0, the rigid nucleus will rotate only by some angle ε (Figure III.17), which must be determined. Consequently, displacements of points of the contour are

$$u = -\varepsilon y, \quad v = \varepsilon x,$$

i.e.,

$$u = -\varepsilon y, \qquad v = \varepsilon x,$$
$$g_1 = -\varepsilon y = -i\frac{\varepsilon b}{2}\left(\sigma - \frac{1}{\sigma}\right),$$
$$g_2 = \varepsilon x = \frac{\varepsilon a}{2}\left(\sigma + \frac{1}{\sigma}\right). \qquad \text{(III.98)}$$

By substituting the values found for g_1 and g_2 into formula (III.94), we find

$$\Phi(\zeta) = -\frac{\varepsilon(ap_2 + ibq_2)}{2(p_1q_2 - p_2q_1)}\zeta,$$
$$\Psi(\zeta) = \frac{\varepsilon(ap_1 + ibq_1)}{2(p_1q_2 - p_2q_1)}\zeta.$$

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By substituting here, instead of ζ, its values through z_1 and z_2 from (III.12) and (III.13), we obtain /251

$$\varphi(z_1) = -\frac{\varepsilon(ap_2 + ibq_2)}{2(p_1q_2 - p_2q_1)} \cdot \frac{a - is_1b}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}};$$

$$\psi(z_2) = \frac{\varepsilon(ap_1 + ibq_1)}{2(p_1q_2 - p_2q_1)} \cdot \frac{a - is_2b}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}}. \qquad \text{(III.99)}$$

The functions $\phi(z_1)$ and $\psi(z_2)$ (III.99) obviously satisfy the condition $\phi(\infty) = \psi(\infty) = 0$, and consequently, displacements at infinity are equal to zero.

We will determine the value ε from the condition of equality of moment M_0, applied to the nucleus, and the moment of forces transmitted from the nucleus to the contour of the elliptic hole. The expression for the moment is given by formula (I.101). Since functions $\phi(z_1)$ and $\psi(z_2)$ are unique, then in our case the formula acquires the form

$$M_0 = 2\mathrm{Re}[F_1(z_1) + F_2(z_2)]. \qquad \text{(III.100)}$$

Hence, the main moment M of forces acting from the direction of the nucleus on the surrounding medium will be equal to the increment of expression

$$2\mathrm{Re}\left\{\int_{L^{(1)}} \varphi(z_1)\,dz_1 + \int_{L^{(2)}} \psi(z_2)\,dz_2\right\}$$

during rotation around the contour of the ellipse in the clockwise direction. Obviously,

$$\int_{L^{(1)}} \varphi(z_1)\,dz_1 = \int_{\gamma} \Phi(\sigma)\,\omega'(\sigma)\,d\sigma,$$

$$\int_{L^{(2)}} \psi(z_2)\,dz_2 = \int_{\gamma} \Psi(\sigma)\,\omega'(\sigma)\,d\sigma.$$

Moreover,

$$\int_{\gamma} \Phi(\sigma)\,\omega'(\sigma)\,d\sigma = -\frac{\varepsilon(ap_2 + ibq_2)}{2(p_1q_2 - p_2q_1)} \int_{\gamma} \sigma\left[\frac{a + is_1b}{2} - \frac{a - is_1b}{2}\frac{1}{\sigma^2}\right]d\sigma =$$

$$= \left[\frac{\varepsilon(ap_2 + ibq_2)(a - is_1b)}{4(p_1q_2 - p_2q_1)} \ln\sigma\right]_{\gamma} = i\frac{\varepsilon\pi(ap_2 + ibq_2)(a - is_1b)}{2(p_1q_2 - p_2q_1)}.$$

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Analogously, we find

$$\int_{\gamma} \Psi(\sigma)\omega'(\sigma)d\sigma = -i\frac{\varepsilon\pi(ap_1 + ibq_1)(a - is_2b)}{2(p_1q_2 - p_2q_1)}.$$

Consequently,

$$M_0 = \varepsilon \operatorname{Re}\left\{i\frac{\pi[(ap_2 + ibq_2)(a - is_1b) - (ap_1 + ibq_1)(a - is_2b)]}{p_1q_2 - p_2q_1}\right\}. \qquad \text{(III.101)}$$

For brevity, we will denote through N the expression found in the braces of (III.101); then /252

$$\varepsilon = \frac{M_0}{\operatorname{Re} N}. \qquad \text{(III.102)}$$

By substituting the value found for ε from (III.102) into functions (III.99), we find the final form of functions $\phi(z_1)$ and $\psi(z_2)$.

Anisotropic Infinite Plate under Tension with Elliptic, Absolutely Rigid Nucleus (or Ring). We will assume that an anisotropic plate with a welded, absolutely rigid elliptic nucleus is subjected to tension at infinity by forces p, constituting angle α with the Ox axis (Figure III.18). The rigid nucleus in this plate can be translated gradually and it is rotated by some angle ε. We can disregard a gradual translation of the nucleus, since in this case the elastic state of the material surrounding the nucleus is not changed. Consequently, as in the preceding problem, translations of the points of contour are

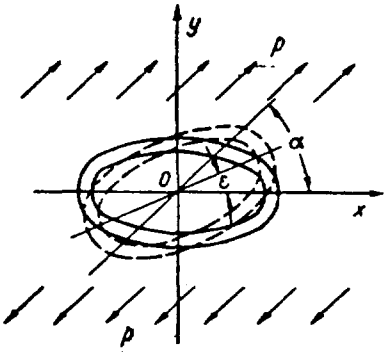


Figure III.18.

$$u = -\varepsilon y, \qquad v = \varepsilon x. \qquad \text{(III.103)}$$

In order to solve the problem as stated, it is necessary to use, rather than g_1 and g_2 from (III.98), g_1^0 and g_2^0 from (III.97), assuming that A = B = 0 in them, since the resultant vector of external forces is equal to zero; the values B*, B'*, C'* are given by formulas (III.19).

Thus,

$$\begin{aligned} g_1^0 &= -\varepsilon y - 2\operatorname{Re}[B^* p_1 z_1 + p_2(B'^* + iC'^*)z_2], \\ g_2^0 &= \varepsilon x - 2\operatorname{Re}[B^* q_1 z_1 + q_2(B'^* + iC'^*)z_2], \end{aligned} \qquad \text{(III.104)}$$

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or

$$g_1^0 = -i\frac{eb}{2}\left(\sigma - \frac{1}{\sigma}\right) - \left\{B^* p_1\left(\frac{a+is_1b}{2}\sigma + \frac{a-is_1b}{2}\frac{1}{\sigma}\right) + \right.$$
$$+ B^*\bar{p}_1\left(\frac{a-i\bar{s}_1b}{2}\cdot\frac{1}{\sigma} + \frac{a+i\bar{s}_1b}{2}\sigma\right) + p_2(B'^* + iC'^*)\left(\frac{a+is_2b}{2}\sigma + \frac{a-is_2b}{2}\frac{1}{\sigma}\right) +$$
$$\left. + \bar{p}_2(B'^* - iC'^*)\left(\frac{a-i\bar{s}_2b}{2}\frac{1}{\sigma} + \frac{a+i\bar{s}_2b}{2}\sigma\right)\right\};$$
$$g_2^0 = \frac{a\varepsilon}{2}\left(\sigma + \frac{1}{\sigma}\right) - \left\{B^* q_1\left(\frac{a+is_1b}{2}\sigma + \frac{a-is_1b}{2}\frac{1}{\sigma}\right) + \right.$$
$$+ B^*\bar{q}_1\left(\frac{a-i\bar{s}_1b}{2}\frac{1}{\sigma} + \frac{a+i\bar{s}_1b}{2}\sigma\right) + q_2(B'^* + iC'^*)\left(\frac{a+is_2b}{2}\sigma + \frac{a-is_2b}{2}\frac{1}{\sigma}\right)$$
$$\left. + \bar{q}_2(B'^* - iC'^*)\left(\frac{a-i\bar{s}_2b}{2}\frac{1}{\sigma} + \frac{a+i\bar{s}_2b}{2}\sigma\right)\right\}$$

By substituting into formulas (III.94), instead of $g_1(\vartheta)$ and $g_2(\vartheta)$ the expressions found for g_1^0 and g_2^0, and noting that /253

$$\int_\gamma \sigma\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} = 4\pi i\zeta; \qquad \int_\gamma \frac{1}{\sigma}\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} = 0,$$

we obtain

$$\Phi_0(\zeta) = -(P_1\varepsilon + Q_1)\zeta,$$
$$\Psi_0(\zeta) = (P_2\varepsilon + Q_2)\zeta, \qquad \text{(III.105)}$$

where

$$P_1 = \frac{ap_2 + ibq_2}{2(p_1q_2 - p_2q_1)}; \qquad P_2 = \frac{ap_1 + ibq_1}{2(p_1q_2 - p_2q_1)};$$
$$Q_1 = \frac{1}{2(p_1q_2 - p_2q_1)}\{B^*[(a+is_1b)(p_1q_2 - q_1p_2) +$$
$$+ (a+is_1\bar{b})(\bar{p}_1q_2 - \bar{q}_1p_2)] + (B'^* - iC'^*)(a+i\bar{s}_2b)(\bar{p}_2q_2 - p_2\bar{q}_2)\}; \qquad \text{(III.106)}$$
$$Q_2 = \frac{1}{2(p_1q_2 - p_2q_1)}\{B^*(a+i\bar{s}_1b)(\bar{p}_1q_1 - \bar{q}_1p_1) +$$
$$+ (B'^* + iC'^*)(a+is_2b)(p_2q_1 - p_1q_2) + (B'^* - iC'^*)(a+i\bar{s}_2b)(\bar{p}_2q_1 - \bar{q}_2p_1)\}.$$

Proceeding in functions $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ of (III.105), to the variables z_1 and z_2, we obtain

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$$\varphi_0(z_1) = -\frac{(P_1\varepsilon + Q_1)(a - is_1b)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}}$$

$$\psi_0(z_2) = \frac{(P_2\varepsilon + Q_2)(a - is_2b)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}}$$

Consequently, the functions are

$$\varphi(z_1) = B^* z_1 - \frac{(P_1\varepsilon + Q_1)(a - is_1b)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}},$$

$$\psi(z_2) = (B'^* + iC'^*)z_2 + \frac{(P_2\varepsilon + Q_2)(a - is_2b)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}} \qquad \text{(III.107)}$$

In order to determine, finally, the functions $\phi(z_1)$ and $\psi(z_2)$ of (III.107), it is necessary also to determine the angle of rotation of the nucleus ε from the condition of equality to zero of the moment of forces acting on the nucleus from the direction of the surrounding material. This moment is found from formula (I.101). Since functions (III.107) are unique, then we must compute the increment of the expression

$$2\mathrm{Re}\,[F_1(z_1) + F_2(z_2)] \qquad \text{(III.108)}$$

during revolution around the contour of the elliptic hole and equate it to zero. /254 Here

$$F_1(z_1) = \int_{L^{(1)}} \varphi(z_1)\,dz_1, \qquad F_2(z_2) = \int_{L^{(2)}} \psi(z_2)\,dz_2.$$

Similar expressions were computed in the preceding section. By setting the increment of expression (III.108) to zero, we obtain

$$2\mathrm{Re}\,\{(P_1\varepsilon + Q_1)(a - is_1b)\,i - (P_2\varepsilon + Q_2)(a - is_2b)\,i\} = 0.$$

Solving the latter expression relative to ε, we find

$$\varepsilon = \frac{\mathrm{Re}\,\{i[Q_2(a - is_2b) - Q_1(a - is_1b)]\}}{\mathrm{Re}\,\{i[P_1(a - is_1b) - P_2(a - is_2b)]\}}. \qquad \text{(III.109)}$$

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By substituting the value found for ε from (III.109) into functions $\phi(z_1)$ and $\psi(z_2)$ of (III.107), we find them in final form.

If the nucleus is prevented from rotating, then $\varepsilon = 0$. Assuming in functions $\phi(z_1)$ and $\psi(z_2)$ of (III.109) that $\varepsilon = 0$, we obtain the solution for the latter case. Then, by differentiating functions $\phi_0(z_1)$ and $\psi_0(z_2)$ with respect to z_1 and z_2, respectively, for $\varepsilon = 0$, and assuming

$$z_1 = a\cos\vartheta + s_1 b\sin\vartheta, \qquad z_2 = a\cos\vartheta + s_2 b\sin\vartheta,$$

we obtain the expressions for the functions along the contour of the weld between the plate and the elliptical nucleus:

$$\varphi_0'(z_1) = -Q_1\frac{\sin\vartheta + i\cos\vartheta}{a\sin\vartheta - s_1 b\cos\vartheta},$$
$$\psi_0'(z_2) = Q_2\frac{\sin\vartheta + i\cos\vartheta}{a\sin\vartheta - s_2 b\cos\vartheta}. \qquad \text{(III.110)}$$

By substituting (III.110) into (III.37) for $s_1 = i\beta_1$ and $s_2 = i\beta_2$, and recalling that Q_1 and Q_2 are real numbers[1], we obtain the formulas for the stress components along the contour of the weld:

$$\sigma_\varrho = p\frac{b^2\cos^2\vartheta}{a^2\sin^2\vartheta + b^2\cos^2\vartheta} + \frac{2[(Q_2 - Q_1)a\sin^2\vartheta + (Q_2\beta_2 - Q_1\beta_1)b\cos^2\vartheta]}{a^2\sin^2\vartheta + b^2\cos^2\vartheta};$$
$$\sigma_\vartheta = \frac{1}{a^2\sin^2\vartheta + b^2\cos^2\vartheta}\left\{pa^2\sin^2\vartheta + 2\mathrm{Re}\left[\frac{\sin\vartheta + i\cos\vartheta}{a\sin\vartheta - s_2 b\cos\vartheta}Q_2(as_2\sin\vartheta + b\cos\vartheta)^2 - \frac{\sin\vartheta + i\cos\vartheta}{a\sin\vartheta - s_1 b\cos\vartheta}Q_1(as_1\sin\vartheta + b\cos\vartheta)^2\right]\right\}; \qquad \text{(III.111)}$$
$$\tau_{\varrho\vartheta} = -\frac{1}{a^2\sin^2\vartheta + b^2\cos^2\vartheta}\{pab\sin\vartheta\cos\vartheta - 2\mathrm{Re}([Q_2(s_2 a\sin\vartheta + b\cos\vartheta) - Q_1(s_1 a\sin\vartheta + b\cos\vartheta)](\sin\vartheta + i\cos\vartheta))\}.$$

If the medium is isotropic, then $\beta_1 = \beta_2 = 1$; $a_{11} = a_{12} = 1/E$, $a_{12} = -\nu/E$, and formula (III.111) will acquire the form /255

[1]The values P_1 and P_2 (III.106) will be purely imaginary here; consequently, ε in (III.109) for these values of Q_1, Q_2, P_1 and P_2 will be equal to zero.

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$$v_0 = \frac{\mu}{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta} \times$$
$$\times \left\{ b^2 \cos^2 \vartheta + \frac{[a(1-\nu) - 2b\nu]\, a \sin^2 \vartheta + [2a - \nu b(1-\nu)]\, b \cos^2 \vartheta}{3 + 2\nu - \nu^2} \right\}. \quad \text{(III.112)}$$

Stress Distribution in Anisotropic Plate with Welded Rigid Elliptic Nucleus (or Ring) under the Effect of Force or Moment Applied at Some Point of the Plate[1]. We will assume that the above stated anisotropic plate is located under the effect of a force or moment applied at an arbitrary point. The rigid nucleus can be displaced gradually and can rotate by some angle ε. By abstracting from the gradual displacement of the nucleus, the displacements of the contour points of the plate can be represented in the form (III.103). In order to solve this problem we will use the Cartesian coordinate system, as shown in Figure III.17 or III.18.

Effect of Force. By repeating the analysis of the preceding section, we arrive at the following expressions for the stress functions:

$$\Phi(\zeta_1) = A^{(2)} \ln(\zeta_1 - \zeta_{10}) + Q_1^{(1)} \ln \frac{\zeta_1 \bar{\zeta}_{10}}{\zeta_1 \bar{\zeta}_{10} - 1} + Q_1^{(2)} \ln \frac{\zeta_1 \bar{\zeta}_{20}}{\zeta_1 \bar{\zeta}_{20} - 1} - \varepsilon \frac{P_1}{\zeta_1},$$
$$\Psi(\zeta_2) = B^{(2)} \ln(\zeta_2 - \zeta_{20}) - Q_2^{(1)} \ln \frac{\zeta_2 \bar{\zeta}_{10}}{\zeta_2 \bar{\zeta}_{10} - 1} - Q_2^{(2)} \ln \frac{\zeta_2 \bar{\zeta}_{20}}{\zeta_2 \bar{\zeta}_{20} - 1} + \varepsilon \frac{P_2}{\zeta_2}, \quad \text{(III.113)}$$

where P_1 and P_2 have the values (III.106);

$$Q_1^{(1)} = \bar{A}^{(2)} \frac{\bar{p}_1 q_2 - p_2 \bar{q}_1}{p_1 q_2 - p_2 q_1}; \quad Q_1^{(2)} = \bar{B}^{(2)} \frac{\bar{p}_2 q_2 - p_2 \bar{q}_2}{p_1 q_2 - p_2 q_1};$$
$$Q_2^{(1)} = \bar{A}^{(2)} \frac{\bar{p}_1 q_1 - p_1 \bar{q}_1}{p_1 q_2 - p_2 q_1}; \quad Q_2^{(2)} = \bar{B}^{(2)} \frac{\bar{p}_2 q_1 - p_1 \bar{q}_2}{p_1 q_2 - p_2 q_1}; \quad \text{(III.114)}$$

the value of ε, as before, is found from the condition of equality of the moment of the forces acting on the nucleus from the direction of the elastic material to zero:

$$\varepsilon = \frac{1}{\operatorname{Re}\{ i [P_1 (a - is_1 b) - P_2 (a - is_2 b)] \}} \times$$
$$\times \operatorname{Re}\left\{ i \left[\frac{(a+is_1 b) A^{(2)} + (a - is_1 b) Q_1^{(1)} - (a - is_2 b) Q_2^{(1)}}{\zeta_{10}} + \right.\right. \quad \text{(III.115)}$$
$$\left.\left. + \frac{(a+is_2 b) B^{(2)} + (a - is_1 b) Q_1^{(2)} - (a - is_2 b) Q_2^{(2)}}{\zeta_{20}} \right] \right\}.$$

[1]The solution is given by D. V. Grilitskiy [2].

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Effect of Moment. The functions are /256

$$\Phi(\zeta_1) = -\frac{M}{2\pi}\left[\frac{n_1\zeta_{10}^2}{\zeta_1-\zeta_{10}} + \frac{L_1^{(1)}\bar{\zeta}_{10}}{\zeta_1\bar{\zeta}_{10}-1} - \frac{L_1^{(2)}\bar{\zeta}_{20}}{\zeta_1\bar{\zeta}_{20}-1}\right] - \varepsilon\frac{P_1}{\zeta_1},$$
$$\Psi(\zeta_2) = \frac{M}{2\pi}\left[\frac{n_2\zeta_{20}^2}{\zeta_2-\zeta_{20}} + \frac{L_2^{(1)}\bar{\zeta}_{10}}{\zeta_2\bar{\zeta}_{10}-1} - \frac{L_2^{(2)}\bar{\zeta}_{20}}{\zeta_2\bar{\zeta}_{20}-1}\right] + \varepsilon\frac{P_2}{\zeta_2}, \quad \text{(III.116)}$$

where n_1 and n_2 have the values (III.88);

$$L_1^{(1)} = \bar{n}_1\frac{\bar{p}_1q_2-\bar{p}_2q_1}{p_1q_2-p_2q_1};\quad L_2^{(1)} = \bar{n}_1\frac{\bar{p}_1q_1-\bar{p}_1q_1}{p_1q_2-p_2q_1}; \quad \text{(III.117)}$$
$$L_1^{(2)} = \bar{n}_2\frac{\bar{p}_2q_2-\bar{p}_2q_2}{p_1q_2-p_2q_1};\quad L_2^{(2)} = \bar{n}_2\frac{\bar{p}_2q_1-\bar{q}_2p_1}{p_1q_2-p_2q_1};$$

$$\varepsilon = \frac{M}{4\pi}\frac{\operatorname{Re}\{i[(a+is_1b)n_1-(a+is_2b)n_2+(a-is_1b)(L_1^{(1)}-L_1^{(2)})-(a-is_2b)(L_2^{(1)}-L_2^{(2)})]\}}{\operatorname{Re}\{i[P_1(Q-is_1b)-P_2(Q-is_2b)\}} \quad \text{(III.118)}$$

§3. Anisotropic Plate with Welded Round Isotropic Disc[1]

Anisotropic plates with elastic or rigid cores are used as construction elements in technology. The problem of elastic equilibrium of such a heterogeneous plate has been analyzed by several authors[2]. S. G. Lekhnitskiy [1, 3] found the general solution of the problem of stress distribution in an anisotropic plate with a welded or glued elliptic anisotropic core, where the plate is under the effect of arbitrary forces distributed along the edges. Also, several practical partial cases of tension, deflection, and displacement of a plate are analyzed in the same works. Tension of an isotropic plate with a welded isotropic elliptic core is examined by G. Kaiser [1] with the aid of the method of elliptic coordinates. M. P. Sheremet'yev [1, 2] found the solution for an isotropic plate with a round hole, the edge of which is reinforced by a thin elastic isotropic ring, for tension in two directions.

The procedure for solving the problem of elastic equilibrium of an anisotropic plate with a welded round isotropic disc is outlined in this section for the case of concentrated loads applied at an arbitrary point of the plate or core. The proposed procedure is a generalization of the solution of analogous problems for an isotropic plate with a welded elastic core. In solving these problems we used the tables of integrals presented in §4, Chapter I. /257

Force Applied to Plate. With the aid of data found in §1, Chapter III, it is possible to construct the solution of the more general problem of the

[1]The solution is given by G. N. Savin and D. V. Grilitskiy [2].
[2]See, for example, the works of J. Dundurs, M. Hetenyi [1] and M. Hetenyi, J. Dundurs [1].

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interaction of a homogeneous anisotropic plate with a welded round isotropic core. Here we will examine the effect of a force applied at a point of the plate.

The functions for the core (disc) are

$$\varphi_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f\,dt}{t-\zeta} - \bar{a}_1\zeta,$$

$$\psi_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f'\,dt}{t-\zeta} - \frac{1}{\zeta}\cdot\frac{1}{2\pi i}\int_\gamma \frac{f'\,dt}{t-\zeta} + \frac{a_1+a_1}{\zeta}, \qquad \text{(III.119)}$$

where a_1 is a constant that requires definition; t is a point on the unit circle.

The functions $\phi_2(z_1)$, $\psi_2(z_2)$ for the plate are defined on the basis of formulas (III.75) and (III.76), where, in the given case, we assume

$$A^{(1)} = B^{(1)} = C_1 = C_2 = 0:$$

$$\varphi_2(z_1) = A^{(2)}\ln(\zeta_1-\zeta_{10}) + A_2\ln\frac{\zeta_1\bar{\zeta}_{10}}{\zeta_1\bar{\zeta}_{10}-1} + B_2\ln\frac{\zeta_1\bar{\zeta}_{20}}{\zeta_1\bar{\zeta}_{20}-1} + \\ + \frac{l_2}{2\pi i}\int\frac{f\,dt}{t-\zeta_1} - \frac{m_2}{2\pi i}\int_\gamma\frac{\bar{f}\,dt}{t-\zeta_1},$$

$$\psi_2(z_2) = B^{(2)}\ln(\zeta_2-\zeta_{20}) - A_1\ln\frac{\zeta_2\bar{\zeta}_{10}}{\zeta_2\bar{\zeta}_{10}-1} - B_1\ln\frac{\zeta_2\bar{\zeta}_{20}}{\zeta_2\bar{\zeta}_{20}-1} - \\ - \frac{l_1}{2\pi i}\int_\gamma\frac{f\,dt}{t-\zeta_2} + \frac{m_1}{2\pi i}\int_\gamma\frac{f\,dt}{t-\zeta_2}. \qquad \text{(III.120)}$$

Here ζ_k and ζ_{k0} (k = 1, 2) are defined by formulas (III.12), (III.13) and (III.71), assuming that a = b = R.

The condition of discontinuity of the vector of displacement on the line of the weld is found from (I.10) and (I.97). This condition can be represented conveniently in the form of two relations:

$$[\varkappa_1\varphi_1(\sigma) + \varkappa_1\overline{\varphi_1(\sigma)} - \sigma\overline{\varphi_1'(\sigma)} - \bar{\sigma}\varphi_1'(\sigma) - \overline{\psi_1(\sigma)} - \psi_1'(\sigma)]^+ = \\ = 4\mu_1[p_1\varphi_2(\sigma) + \bar{p}_1\overline{\varphi_2(\sigma)} + p_2\psi_2(\sigma) + \bar{p}_2\overline{\psi_2(\sigma)}]^- \quad (\text{on } \gamma); \qquad \text{(III.121)}$$

$$[\varkappa_1\varphi_1(\sigma) - \varkappa_1\overline{\varphi_1(\sigma)} - \sigma\overline{\varphi_1'(\sigma)} + \bar{\sigma}\varphi_1'(\sigma) - \overline{\psi_1(\sigma)} + \psi_1(\sigma)]^+ = \\ = 4\mu i[q_1\varphi_2(\sigma) + \bar{q}_1\overline{\varphi_2(\sigma)} + q_2\psi_2(\sigma) + \bar{q}_2\overline{\psi_2(\sigma)}]^- \quad (\text{on } \gamma).$$

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By satisfying relations (III.121) on unit circle γ, we obtain a system of two singular integral equations for the determination of the desired functions $f(t)$ and $\overline{f(t)}$: /258

$$af(\sigma)+\bar{a}\overline{f(\sigma)}+\frac{b}{\pi i}\int_{\gamma}\frac{f(t)\,dt}{t-\sigma}-\frac{\bar{b}}{\pi i}\int_{\gamma}\frac{\overline{f(t)}\,dt}{t-\sigma}=F_1(\sigma)\quad(\sigma\in\gamma);$$

$$cf(\sigma)-\bar{c}\overline{f(\sigma)}+\frac{d}{\pi i}\int_{\gamma}\frac{f(t)\,dt}{t-\sigma}+\frac{\bar{d}}{\pi i}\int_{\gamma}\frac{\overline{f(t)}\,dt}{t-\sigma}=F_2(\sigma)\quad(\sigma\in\gamma).\qquad\text{(III.122)}$$

The following definitions are introduced in formulas (III.122):

$$p_1A^{(2)}-\bar{p}_1\bar{A}_2+\bar{p}_2\bar{A}_1=A_3,\quad q_1A^{(2)}-\bar{q}_1\bar{A}_2+\bar{q}_2\bar{A}_1=A_4,\qquad\text{(III.123)}$$

$$p_2B^{(2)}-\bar{p}_1\bar{B}_2+\bar{p}_2\bar{B}_1=B_3,\quad q_2B^{(2)}-\bar{q}_1\bar{B}_2+\bar{q}_2\bar{B}_1=B_4;$$

$$p_1\bar{m}_2-\bar{p}_2\bar{m}_1-p_1l_2+p_2l_1=2d_1,\ \bar{q}_1\bar{m}_2-\bar{q}_2\bar{m}_1-q_1l_2+q_2l_1=2d_2,\qquad\text{(III.124)}$$

$$\bar{p}_1\bar{m}_2-\bar{p}_2\bar{m}_1+p_1l_2-p_2l_1=d_3,\quad\bar{q}_1\bar{m}_2-\bar{q}_2\bar{m}_1+q_1l_2-q_2l_1=d_4;$$

$$\frac{\varkappa_1-1}{2}-4\mu_1d_1=a,\quad\frac{\varkappa_1+1}{2}-2\mu_1d_3=b,\qquad\text{(III.125)}$$

$$\frac{\varkappa_1-1}{2}-4\mu_1id_2=c,\quad\frac{\varkappa_1+1}{2}-2\mu_1id_4=d;$$

$$F_1(\sigma)=(\varkappa_1+1)(\bar{a}_1\sigma+a_1\bar{\sigma})+4\mu_1i\Big[A_3\ln(\sigma-\zeta_{10})+$$

$$+B_3\ln(\sigma-\zeta_{20})+\bar{A}_3\ln\frac{1-\sigma\bar{\zeta}_{10}}{\sigma}+\bar{B}_3\ln\frac{1-\sigma\bar{\zeta}_{20}}{\sigma}\Big],\qquad\text{(III.126)}$$

$$F_2(\sigma)=(\varkappa_1+1)(\bar{a}_1\sigma-a_1\bar{\sigma})+4\mu_1i\Big[A_4\ln(\sigma-\zeta_{10})+$$

$$+B_4\ln(\sigma-\zeta_{20})+\bar{A}_4\ln\frac{1-\sigma\bar{\zeta}_{10}}{\sigma}+\bar{B}_4\ln\frac{1-\sigma\bar{\zeta}_{20}}{\sigma}\Big].$$

Equation system (III.122) can be reduced, with the help of some special form of the parameter N, to two independent equations. For this purpose we will proceed in the following manner[1].

We will multiply the second equation of system (III.122) by N and combine it with the first equation:

$$(\alpha+Nc)f(\sigma)+(\bar{a}-N\bar{c})\overline{f(\sigma)}+\frac{b+Nd}{\pi i}\int_{\gamma}\frac{f\,dt}{t-\sigma}+\frac{-\bar{b}+N\bar{d}}{\pi i}\int_{\gamma}\frac{f\,dt}{t-\sigma}=\qquad\text{(III.127)}$$

$$=F_1(\sigma)+NF_2(\sigma)\qquad(\sigma\in\gamma).$$

[1]See D. V. Grilitskiy [6].

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Since the constant N is arbitrary, we will select it in the following manner, in order to satisfy the relation /259

$$\frac{a+Nc}{\bar{a}-N\bar{c}}=\frac{b+Nd}{-\bar{b}+N\bar{d}}=\frac{1}{\lambda}, \tag{III.128}$$

which leads to the following quadratic equation relative to N:

$$(c\bar{d}+\bar{c}d)N^2+(a\bar{d}-c\bar{b}-\bar{a}d+\bar{c}b)N-(a\bar{b}+\bar{a}b)=0. \tag{III.129}$$

By solving equation (III.129), we find two values for N:

$$N_{1,2}=\frac{-(a\bar{d}-c\bar{b}-\bar{a}d+\bar{c}b)\pm\sqrt{(a\bar{d}-c\bar{b}-\bar{a}d+\bar{c}b)^2+4(c\bar{d}+\bar{c}d)(a\bar{b}+\bar{a}b)}}{2(c\bar{d}+\bar{c}d)}. \tag{III.130}$$

By introducing the definitions

$$\begin{aligned} a+N_kc&=Q_k, & \bar{a}-N_k\bar{c}&=\lambda_kQ_k,\\ b+N_kd&=K_k, & -\bar{b}+N_k\bar{d}&=\lambda_kK_k \end{aligned} \qquad (k=1,2), \tag{III.131}$$

equation (III.127) is transformed to two independent equations:

$$Q_k(f+\lambda_k\bar{f})+\frac{K_k}{\pi i}\int_\gamma\frac{(f+\lambda_k\bar{f})\,dt}{t-\sigma}=F_1(\sigma)+N_kF_2(\sigma) \qquad (\sigma\in\gamma)\quad(k=1,2); \tag{III.132}$$

λ_k is found from relation (III.128). Now, to each of equations (III.132) we can apply the formula

$$f(\sigma)=\frac{1}{a^2-b^2}\left[aF(\sigma)-\frac{b}{\pi i}\int_\gamma\frac{F(t)\,dt}{t-\sigma}\right] \qquad (\sigma\in\gamma),$$

on the basis of which

$$f+\lambda_k\bar{f}=\frac{1}{Q_k^2-K_k^2}\left[Q_k(F_1+N_kF_2)-\frac{K_k}{\pi i}\int_\gamma\frac{(F_1+N_kF_2)\,dt}{t-\sigma}\right] \qquad (\sigma\in\gamma). \tag{III.133}$$

By solving equation (III.133) relative to $f(\sigma)$ and $\overline{f(\sigma)}$, we find the desired functions

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$$f(\sigma)=\frac{1}{\lambda_2-\lambda_1}\left\{\frac{\lambda_2}{Q_1^2-K_1^2}\left[Q_1(F_1(\sigma)+N_1F_2(\sigma))-\frac{K_1}{\pi i}\int_\gamma\frac{F_1(t)+N_1F_2(t)}{t-\sigma}\,dt\right]-\right.$$

$$\left.-\frac{\lambda_1}{Q_2^2-K_2^2}\left[Q_2(F_1(\sigma)+N_2F_2(\sigma))-\frac{K_2}{\pi i}\int_\gamma\frac{F_1(t)+N_2F_2(t)}{t-\sigma}\,dt\right]\right\};$$

$$\overline{f(\sigma)}=-\frac{1}{\lambda_2-\lambda_1}\left\{\frac{1}{Q_1^2-K_1^2}\left[Q_1(F_1(\sigma)+N_1F_2(\sigma))-\frac{K_1}{\pi i}\int_\gamma\frac{F_1(t)+N_1F_2(t)}{t-\sigma}\,dt\right]-\right. \quad \text{(III.134)} \quad /260$$

$$\left.-\frac{1}{Q_2^2-K_2^2}\left[Q_2(F_1(\sigma)+N_2F_2(\sigma))-\frac{K_2}{\pi i}\int_\gamma\frac{F_1(t)+N_2F_2(t)}{t-\sigma}\,dt\right]\right\}.$$

On the basis of the known functions $f(\sigma)$ and $\overline{f(\sigma)}$, the stress functions for the disc and plate are determined[1] from formulas (III.119) and (III.120):

$$f(\sigma)=\frac{1}{\lambda_2-\lambda_1}\left[H_1\ln(\sigma-\zeta_{10})+H_2\ln(\sigma-\zeta_{20})+\right.$$

$$\left.+H_3\ln\frac{1-\sigma\bar{\zeta}_{10}}{\sigma}+H_4\ln\frac{1-\sigma\bar{\zeta}_{20}}{\sigma}+H_5\bar{a}_1\sigma+H_6\frac{a_1}{\sigma}\right],$$

$$\overline{f(\sigma)}=-\frac{1}{\lambda_2-\lambda_1}\left[h_1\ln(\sigma-\zeta_{10})+h_2\ln(\sigma-\zeta_{20})+h_3\ln\frac{1-\sigma\bar{\zeta}_{10}}{\sigma}+\right. \quad \text{(III.135)}$$

$$\left.+h_4\ln\frac{1-\sigma\bar{\zeta}_{20}}{\sigma}+h_5\bar{a}_1\sigma+h_6\frac{a_1}{\sigma}\right|;$$

$$\varphi_1(\zeta)=\frac{1}{\lambda_2-\lambda_1}[H_1\ln(\zeta-\zeta_{10})+H_2\ln(\zeta-\zeta_{20})]+\left(\frac{H_5}{\lambda_2-\lambda_1}-1\right)\bar{a}_1\zeta,$$

$$\psi_1(\zeta)=-\frac{1}{\lambda_2-\lambda_1}[h_1\ln(\zeta-\zeta_{10})+h_2\ln(\zeta-\zeta_{20})+h_3\bar{a}_1\zeta]- \quad \text{(III.136)}$$

$$-\frac{1}{(\lambda_2-\lambda_1)\zeta}\left[\frac{H_1}{\zeta-\zeta_{10}}+\frac{H_2}{\zeta-\zeta_{20}}+H_5\bar{a}_1\right]+\frac{a_1+\bar{a}_1}{\zeta};$$

$$\varphi_2(z_1)=A^{(2)}\ln(\zeta_1-\zeta_{10})+\left(A_2+\frac{l_2H_3-m_2h_3}{\lambda_2-\lambda_1}\right)\ln\frac{\zeta_1}{\zeta_1\bar{\zeta}_{10}-1}+$$

$$+\left(B_2+\frac{l_2H_4-m_2h_4}{\lambda_2-\lambda_1}\right)\ln\frac{\zeta_1}{\zeta_1\bar{\zeta}_{20}-1}-\frac{l_2H_6-m_2h_6}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta_1}, \quad \text{(III.137)}$$

$$\psi_2(z_2)=B^{(2)}\ln(\zeta_2-\zeta_{20})-\left(A_1+\frac{l_1H_3-m_1h_3}{\lambda_2-\lambda_1}\right)\ln\frac{\zeta_2}{\zeta_2\bar{\zeta}_{10}-1}-$$

$$-\left(B_1+\frac{l_1H_4-m_1h_4}{\lambda_2-\lambda_1}\right)\ln\frac{\zeta_2}{\zeta_2\bar{\zeta}_{20}-1}+\frac{l_1H_6-m_1h_6}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta_2};$$

[1]Here, and in the rest of this section, we will give the final results, omitting the cumbersome intermediate calculations.

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a_1 is a coefficient for ζ in the expansion of the function $\phi_1(\zeta)$ in the vicinity of the point $\zeta = 0$: /261

$$a_1 = \frac{\dfrac{H_1\zeta_{20} + H_2\zeta_{10}}{(\lambda_2 - \lambda_1)\,\zeta_{10}\zeta_{20}} - \left(1 - \dfrac{H_5}{\lambda_2 - \lambda_1}\right)\dfrac{\overline{H}_1\overline{\zeta}_{20} + \overline{H}_2\overline{\zeta}_{10}}{(\overline{\lambda}_2 - \overline{\lambda}_1)\,\overline{\zeta}_{10}\overline{\zeta}_{20}}}{\left(1 - \dfrac{H_5}{\lambda_2 - \lambda_1}\right)\left(1 - \dfrac{\overline{H}_5}{\overline{\lambda}_2 - \overline{\lambda}_1}\right) - 1} \tag{III.138}$$

where

$$\begin{aligned}
H_1 &= 4\mu_1\left[\frac{\lambda_2(A_3 + N_1 iA_4)}{Q_1 + K_1} - \frac{\lambda_1(A_3 + N_2 iA_4)}{Q_2 + K_2}\right],\\
H_2 &= 4\mu_1\left[\frac{\lambda_2(B_3 + N_1 iB_4)}{Q_1 + K_1} - \frac{\lambda_1(B_3 + N_2 iB_4)}{Q_2 + K_2}\right],\\
H_3 &= 4\mu_1\left[\frac{\lambda_2(\overline{A}_3 + N_1 i\overline{A}_4)}{Q_1 - K_1} - \frac{\lambda_1(\overline{A}_3 + N_2 i\overline{A}_4)}{Q_2 - K_2}\right],\\
H_4 &= 4\mu_1\left[\frac{\lambda_2(\overline{B}_3 + N_1 i\overline{B}_4)}{Q_1 - K_1} - \frac{\lambda_1(\overline{B}_3 + N_2 i\overline{B}_4)}{Q_2 - K_2}\right],\\
H_5 &= (\varkappa_1 + 1)\left[\frac{\lambda_2(1 + N_1)}{Q_1 + K_1} - \frac{\lambda_1(1 + N_2)}{Q_2 + K_2}\right],\\
H_6 &= (\varkappa_1 + 1)\left[\frac{\lambda_2(1 - N_1)}{Q_1 - K_1} - \frac{\lambda_1(1 - N_2)}{Q_2 - K_2}\right];
\end{aligned} \tag{III.139}$$

h_j ($j = 1, 2, \ldots, 6$) are found respectively from H_j, assuming $\lambda_1 = \lambda_2 = 1$.

Moment Applied to Plate. The functions $\phi_1(\zeta)$, $\psi_1(\zeta)$ for the disc are defined by formulas (III.119). The functions $\phi_2(z)$, $\psi_2(z_2)$ for the plate are found from formulas (III.89), assuming that $A^{(1)} = B^{(1)} = C_1 = C_2 = 0$:

$$\begin{aligned}
\varphi_2(z_1) &= -\frac{M}{4\pi}\left(\frac{n_1\zeta_{10}^2}{\zeta_1 - \zeta_{10}} + \frac{\gamma_1\overline{\zeta}_{10}}{\zeta_1\overline{\zeta}_{10} - 1} - \frac{\gamma_2\overline{\zeta}_{20}}{\zeta_1\overline{\zeta}_{20} - 1}\right) + \frac{l^2}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t - \zeta_1} - \frac{m_2}{2\pi i}\int_\gamma \frac{\overline{f(t)}\,dt}{t - \zeta_1},\\
\psi_2(z_2) &= \frac{M}{4\pi}\left(\frac{n_2\zeta_{20}^2}{\zeta_2 - \zeta_{20}} + \frac{\delta_1\overline{\zeta}_{10}}{\zeta_2\overline{\zeta}_{10} - 1} - \frac{\delta_2\overline{\zeta}_{20}}{\zeta_2\overline{\zeta}_{20} - 1}\right) - \frac{l_1}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t - \zeta_2} + \frac{m_1}{2\pi i}\int_\gamma \frac{\overline{f(t)}\,dt}{t - \zeta_2}.
\end{aligned} \tag{III.140}$$

The conditions of discontinuity of displacements on the weld (III.121) lead to a system of two singular integral equations (III.122) with the right hand sides:

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$$F_3(\sigma) = (\varkappa_1 + 1)(\bar{a}_1\sigma + a_1\bar{\sigma}) + \frac{M\mu_1}{\pi}\left[\frac{l_2}{\sigma-\zeta_{20}} - \frac{l_1}{\sigma-\zeta_{10}} + \right.$$
$$\left. - \frac{\bar{l}_2\sigma}{1-\bar{\zeta}_{20}\sigma} - \frac{\bar{l}_1\sigma}{1-\bar{\zeta}_{10}\sigma} - \frac{l_3}{1-\bar{\zeta}_{10}\sigma} + \frac{l_4}{1-\bar{\zeta}_{20}\sigma} - \frac{\bar{l}_3\sigma}{\sigma-\zeta_{10}} + \frac{\bar{l}_4\sigma}{\sigma-\zeta_{20}}\right],$$
$$F_4(\sigma) = (\varkappa_1 + 1)(\bar{a}_1\sigma - a_1\bar{\sigma}) + \frac{M\mu_1 i}{\pi}\left[\frac{f_2}{\sigma-\zeta_{20}} - \frac{f_1}{\sigma-\zeta_{10}} + \frac{\bar{f}_2\sigma}{1-\bar{\zeta}_{20}\sigma} - \right. \quad \text{(III.141)} \quad \underline{/262}$$
$$\left. - \frac{\bar{f}_1\sigma}{1-\bar{\zeta}_{10}\sigma} - \frac{f_3}{1-\bar{\zeta}_{10}\sigma} + \frac{f_4}{1-\bar{\zeta}_{20}\sigma} - \frac{\bar{f}_3\sigma}{\sigma-\zeta_{10}} + \frac{\bar{f}_4\sigma}{\sigma-\zeta_{20}}\right],$$

where

$$l_k = p_k n_k \zeta_{k0}^2, \quad l_{k+2} = (p_2\delta_k - p_1\gamma_k)\bar{\zeta}_{k0} \quad (k = 1, 2); \qquad \text{(III.142)}$$

f_j (j = 1, 2, 3, 4) are found from the expression for l_j by substituting p_k by q_k.

On the basis of formulas (III.134) we find the desired functions:

$$f(\sigma) = \frac{1}{\lambda_2-\lambda_1}\left(\frac{G_1+G_4\sigma}{\sigma-\zeta_{20}} - \frac{G_2+G_3\sigma}{\sigma-\zeta_{10}} + \frac{G_5\sigma+G_8}{1-\bar{\zeta}_{20}\sigma} - \frac{G_6\sigma+G_7}{1-\bar{\zeta}_{10}\sigma} + H_5\bar{a}_1\sigma + H_6 a_1\frac{1}{6}\right)$$
$$\overline{f(\sigma)} = -\frac{1}{\lambda_2-\lambda_1}\left(\frac{g_1+g_4\sigma}{\sigma-\zeta_{20}} - \frac{g_2+g_3\sigma}{\sigma-\zeta_{10}} + \frac{g_5\sigma+g_8}{1-\bar{\zeta}_{20}\sigma} - \right. \qquad \text{(III.143)}$$
$$\left. - \frac{g_6\sigma+g_7}{1-\bar{\zeta}_{10}\sigma} + h_5\bar{a}_1\sigma + h_6 a_1\frac{1}{\sigma}\right).$$

The stress functions for the disc and plate are

$$\varphi_1(\zeta) = \frac{1}{\lambda_2-\lambda_1}\left(\frac{G_1+G_4\zeta}{\zeta-\zeta_{20}} - \frac{G_2+G_3\zeta}{\zeta-\zeta_{10}}\right) + \left(\frac{H_5}{\lambda_2-\lambda_1} - 1\right)\bar{a}_1\zeta,$$
$$\psi_1(\zeta) = \frac{1}{\lambda_2-\lambda_1}\left(\frac{g_1+g_4\zeta}{\zeta-\zeta_{20}} - \frac{g_2+g_3\zeta}{\zeta-\zeta_{10}} + h_5\bar{a}_1\zeta\right) - \qquad \text{(III.144)}$$
$$- \frac{1}{(\lambda_2-\lambda_1)\zeta}\left[\frac{G_2+G_3\zeta_{10}}{(\zeta-\zeta_{10})^2} - \frac{G_1+G_4\zeta_{20}}{(\zeta-\zeta_{20})^2} + H_5\bar{a}_1\right] + \frac{a_1+\bar{a}_1}{\zeta};$$

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$$\varphi_2(z_1) = -\frac{M}{4\pi}\left(\frac{n_1\zeta_{10}^2}{\zeta_1-\zeta_{10}} - \frac{\gamma_1\bar{\zeta}_{10}}{1-\zeta_1\bar{\zeta}_{10}} + \frac{\gamma_2\bar{\zeta}_{20}}{1-\zeta_1\bar{\zeta}_{20}}\right) +$$

$$+\frac{1}{\lambda_2-\lambda_1}\left[(m_2g_5-l_2G_5)\frac{\zeta_1}{1-\zeta_1\bar{\zeta}_{20}} - (m_2g_6-l_2G_6)\frac{\zeta_1}{1-\zeta_1\bar{\zeta}_{10}} - \right.$$

$$\left. - \frac{m_2g_7-l_2G_7}{1-\zeta_1\bar{\zeta}_{10}} + \frac{m_2g_8-l_2G_8}{1-\zeta_1\bar{\zeta}_{20}} + \frac{(m_2h_6-l_2H_6)\,a_1}{\zeta_1}\right],$$

$$\psi_2(z_2) = \frac{M}{4\pi}\left(\frac{n_2\zeta_{20}^2}{\zeta_2-\zeta_{20}} - \frac{\delta_1\bar{\zeta}_{10}}{1-\zeta_2\bar{\zeta}_{10}} + \frac{\delta_2\bar{\zeta}_{20}}{1-\zeta_2\bar{\zeta}_{20}}\right) - \frac{1}{\lambda_2-\lambda_1}\times \qquad \text{(III.145)} \quad \underline{/263}$$

$$\times\left[(m_1g_5-l_1G_5)\frac{\zeta_2}{1-\zeta_2\bar{\zeta}_{20}} - (m_1g_6-l_1G_6)\frac{\zeta_2}{1-\zeta_2\bar{\zeta}_{10}} - \right.$$

$$\left. - \frac{m_1g_7-l_1G_7}{1-\zeta_2\bar{\zeta}_{10}} + \frac{m_1g_8-l_1G_8}{1-\zeta_2\bar{\zeta}_{20}} + \frac{(m_1h_6-l_1H_6)\,a_1}{\zeta_2}\right].$$

The following definitions are made in formulas (III.143)-(III.145):

$$G_1 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(l_2+Nif_2)}{Q_1+K_1} - \frac{\lambda_1(l_2+N_2if_2)}{Q_2+K_2}\right],$$

$$G_2 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(l_1+N_1if_1)}{Q_1+K_1} - \frac{\lambda_1(l_1+N_2if_1)}{Q_2+K_2}\right],$$

$$G_3 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(\bar{l}_3+N_1i\bar{f}_3)}{Q_1+K_1} - \frac{\lambda_1(\bar{l}_3+N_2i\bar{f}_3)}{Q_2+K_2}\right],$$

$$G_4 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(\bar{l}_4+N_1i\bar{f}_4)}{Q_1+K_1} - \frac{\lambda_1(\bar{l}_4+N_2i\bar{f}_4)}{Q_2+K_2}\right], \qquad \text{(III.146)}$$

$$G_5 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(\bar{l}_2+N_1i\bar{f}_2)}{Q_1-K_1} - \frac{\lambda_1(\bar{l}_2+N_2i\bar{f}_2)}{Q_2-K_2}\right],$$

$$G_6 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(\bar{l}_1+N_1\bar{f}_1i)}{Q_1-K_1} - \frac{\lambda_1(\bar{l}_1+N_2i\bar{f}_1)}{Q_2-K_2}\right],$$

$$G_7 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(l_3+N_1if_3)}{Q_1-K_1} - \frac{\lambda_1(l_3+N_2if_3)}{Q_2-K_2}\right],$$

$$G_8 = \frac{M\mu_1}{\pi}\left[\frac{\lambda_2(l_4+N_1if_4)}{Q_1-K_1} - \frac{\lambda_1(l_4+N_2if_4)}{Q_2-K_2}\right];$$

$$a_1 = \left[\frac{1}{\lambda_2-\lambda_1}\left(\frac{G_1+\zeta_{20}\dot{G}_4}{\zeta_{20}^2} - \frac{G_2+\zeta_{10}G_3}{\zeta_{10}^2}\right) - \right.$$

$$\left. - \frac{1}{\bar{\lambda}_2-\bar{\lambda}_1}\left(\frac{\bar{G}_1+\bar{\zeta}_{20}\bar{G}_4}{\bar{\zeta}_{20}^2} - \frac{\bar{G}_2+\bar{\zeta}_{10}\bar{G}_{13}}{\bar{\zeta}_{10}^2}\right)\left(1-\frac{H_5}{\lambda_2-\lambda_1}\right)\right]\times \qquad \text{(III.147)}$$

$$\times\left[\left(1-\frac{H_5}{\lambda_2-\lambda_1}\right)\left(1-\frac{\bar{H}_5}{\bar{\lambda}_2-\bar{\lambda}_1}\right)-1\right]^{-1};$$

g_j (j = 1, 2, ..., 8) are found respectively from G_j for $\lambda_2 = \lambda_1 = 1$.

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Force Applied to Disc. Suppose a concentrated force $\overline{P}(x, y)$ is applied at an arbitrary point of the elastic disc.

The stress functions for the disc are

$$\varphi_1(\zeta) = -\frac{X+iY}{2\pi(1+\varkappa_1)}\ln(\zeta-\zeta_0) + \frac{1}{2\pi i}\int_\gamma \frac{f_1^0(t)\,dt}{t-\zeta} - \bar{a}_1\zeta,$$

$$\psi_1(\zeta) = \frac{\varkappa_1(X-iY)}{2\pi(1+\varkappa_1)}\ln(\zeta-\zeta_0) + \frac{X+iY}{2\pi(1+\varkappa_1)}\frac{\bar{\zeta}_0}{\zeta-\zeta_0} + \quad /264$$

$$+\frac{1}{2\pi i}\int_\gamma \frac{\overline{f_1^0(t)}\,dt}{t-\zeta} - \frac{1}{\zeta}\cdot\frac{1}{2\pi i}\int_\gamma \frac{f_1'^0(t)\,dt}{t-\zeta} + \frac{a_1+\bar{a}_1}{\zeta}.$$

The stress functions for the plate are found on the basis of expressions (III.75) and (III.76), in which we must assume that

$$A^{(2)} = B^{(2)} = A_2 = B_2 \equiv 0:$$

$$\varphi_2(z_1) = C_2\ln\zeta_1 + (A^{(1)} - C_2)\ln(\sigma_1-\zeta_1) + \frac{l_2}{2\pi i}\int_\gamma \frac{f\,dt}{t-\zeta_1} - \frac{m_2}{2\pi i}\int_\gamma \frac{\bar{f}\,dt}{t-\zeta_1}, \qquad \text{(III.148)}$$

$$\psi_2(z_2) = -C_1\ln\zeta_2 + (B^{(1)} + C_1)\ln(\sigma_1-\zeta_{12}) - \frac{l_1}{2\pi i}\int_\gamma \frac{f\,dt}{t-\zeta_2} + \frac{m_1}{2\pi i}\int_\gamma \frac{\bar{f}\,dt}{t-\zeta_2}.$$

The right hand sides of equations (III.122) have the form

$$F_5(\sigma) = (\varkappa_1+1)(\bar{a}_1\sigma + a_1\bar{\sigma}) - \frac{X-iY}{2\pi}\frac{\sigma(\sigma-\zeta_0)}{1-\bar{\zeta}_0\sigma} - \frac{X+iY}{2\pi}\cdot\frac{1-\bar{\zeta}_0\sigma}{\sigma(\sigma-\zeta_0)} +$$

$$+\frac{iY(\varkappa_1-1)+X(\varkappa_1+1)}{2\pi}\ln(1-\bar{\zeta}_0\sigma) - \frac{iY(\varkappa_1-1)-X(\varkappa_1+1)}{2\pi}\ln(\sigma-\zeta_0) +$$

$$+4\mu_1 D_1\ln\sigma - \left[\frac{X(\varkappa_1+1)}{\pi} + 4\mu_1(D_1+\bar{D}_1)\right]\ln(\sigma-\sigma_1), \qquad \text{(III.149)}$$

$$F_6(\sigma) = (\varkappa_1+1)(\bar{a}_1\sigma - a_1\bar{\sigma}) - \frac{X-iY}{2\pi}\frac{\sigma(\sigma-\zeta_0)}{1-\bar{\zeta}_0\sigma} + \frac{X+iY}{2\pi}\cdot\frac{1-\bar{\zeta}_0\sigma}{\sigma(\sigma-\zeta_0)} +$$

$$+\frac{iY(\varkappa_1+1)+X(\varkappa_1-1)}{2\pi}\ln(1-\bar{\zeta}_0\sigma) + \frac{iY(\varkappa_1+1)-X(\varkappa_1-1)}{2\pi}\ln(\sigma-\zeta_0) +$$

$$+4\mu_1 iD_2\ln\vartheta - \left[\frac{iY(\varkappa_1+1)}{\pi} + 4\mu_1 i(D_2+\bar{D}_2)\right]\ln(\sigma-\sigma_1),$$

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where

$$D_1 = p_1C_2 - p_2C_1 - \bar{p}_1\bar{A}^{(1)} - \bar{p}_2\bar{B}^{(1)},$$
$$D_2 = q_1C_2 - q_2C_1 - \bar{q}_1\bar{A}^{(1)} - \bar{q}_2\bar{B}^{(1)}. \qquad \text{(III.150)}$$

We find the formulas for the functions $f(\sigma)$, $\overline{f(\sigma)}$ and $\phi_1(\zeta)$, $\psi_1(\zeta)$: /265

$$f(\sigma) = \frac{1}{\lambda_2 - \lambda_1}\left[R_1\bar{a}_1\sigma + R_2a_1\frac{1}{\sigma} - R_3\frac{\sigma(\sigma-\zeta_0)}{1-\bar{\zeta}_0\sigma} - R_4\frac{1-\bar{\zeta}_0\sigma}{\sigma(\sigma-\zeta_0)} + \right.$$
$$\left. + R_5\ln(1-\bar{\zeta}_0\sigma) - R_6\ln(\sigma-\zeta_0) + R_7\ln\sigma\right],$$

$$\overline{f(\sigma)} = -\frac{1}{\lambda_2 - \lambda_1}\left[r_1\bar{a}_1\sigma + r_2a_1\frac{1}{\sigma} - r_3\frac{\sigma(\sigma-\zeta_0)}{1-\bar{\zeta}_0\sigma} - r_4\frac{1-\bar{\zeta}_0\sigma}{\sigma(\sigma-\zeta_0)} + \right. \qquad \text{(III.151)}$$
$$\left. + r_5\ln(1-\zeta_0\sigma) - r_6\ln(\sigma-\zeta_0) + r_7\ln\sigma\right];$$

$$\varphi_1(\zeta) = -\frac{X+iY}{2\pi(\varkappa_1+1)}\ln(\zeta-\zeta_0) + \left(\frac{R_1}{\lambda_2-\lambda_1} - 1\right)\bar{a}_1\zeta + \frac{1}{\lambda_2-\lambda_1}\left[(R_5 - \right.$$
$$\left. - \frac{\varkappa_1(\lambda_2-\lambda_1)(X+iY)}{2\pi(\varkappa_1+1)}\ln(1-\bar{\zeta}_0\zeta) - \left(R_3 - \frac{(\lambda_2-\lambda_1)(X-iY)}{2\pi(\varkappa_1+1)}\right)\frac{\zeta(\zeta-\zeta_0)}{1-\bar{\zeta}_0\zeta}\right],$$

$$\psi_1(\zeta) = \frac{\varkappa_1(X-iY)}{2\pi(\varkappa_1+1)}\ln(\zeta-\zeta_0) + \frac{X+iY}{2\pi(\varkappa_1+1)}\frac{\bar{\zeta}_0}{\zeta-\zeta_0} + \left(a_1 + \bar{a}_1 - \frac{R_1\bar{a}_1}{\lambda_2-\lambda_1}\right)\times$$
$$\times\frac{1}{\zeta} - \frac{r_1\bar{a}_1}{\lambda_2-\lambda_1}\zeta + \frac{r_3}{\lambda_2-\lambda_1}\frac{\zeta(\zeta-\zeta_0)}{1-\bar{\zeta}_0\zeta} - \left(\frac{\varkappa_1(X+iY)}{2\pi(\varkappa_1+1)} - \frac{R_5}{\lambda_2-\lambda_1}\right)\frac{\bar{\zeta}_0}{\zeta(1-\bar{\zeta}_0\zeta)} - \qquad \text{(III.152)}$$
$$- \left(\frac{X-iY}{2\pi(\varkappa_1+1)} - \frac{R_3}{\lambda_2-\lambda_1}\right)\frac{2\zeta-\zeta^2\bar{\zeta}_0-\zeta_0}{\zeta(1-\bar{\zeta}_0\zeta)^2} + \left(\frac{X-iY}{2\pi(\varkappa_1+1)} - \frac{r_5}{\lambda_2-\lambda_1}\right)\ln(1-\bar{\zeta}_0\zeta);$$

$$\varphi_2(z_1) = \left(C_2 - \frac{l_2R_7+m_2r_7}{\lambda_2-\lambda_1}\right)\ln\zeta_1 + \frac{l_2R_6+m_2r_6}{\lambda_2-\lambda_1}\ln(\zeta_1-\zeta_0) +$$
$$+ \frac{l_2R_4+m_2r_4}{\lambda_2-\lambda_1}\cdot\frac{1-\bar{\zeta}_0\zeta_1}{\zeta_1(\zeta_1-\zeta_0)} - \frac{l_2R_2+m_2r_2}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta},$$

$$\psi_2(z_2) = -\left(C_1 - \frac{l_1R_7+m_1r_7}{\lambda_2-\lambda_1}\right)\ln\zeta_2 - \frac{l_1R_6+m_1r_6}{\lambda_2-\lambda_1}\ln(\zeta_2-\zeta_0) - \qquad \text{(III.153)}$$
$$- \frac{l_1R_4+m_1r_4}{\lambda_2-\lambda_1}\cdot\frac{1-\bar{\zeta}_0\zeta_2}{\zeta_2(\zeta_2-\zeta_0)} + \frac{l_1R_2+m_1r_2}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta_2},$$

$$R_1 = \frac{\lambda_2W_1}{Q_1+K_1} - \frac{\lambda_1U_1}{Q_2+K_2}; \qquad R_2 = \frac{\lambda_2W_2}{Q_1-K_1} - \frac{\lambda_1U_2}{Q_2-K_2};$$

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where

$$R_3 = \frac{\lambda_2 W_3}{Q_1 + K_1} - \frac{\lambda_1 U_3}{Q_2 + K_2}; \quad R_4 = \frac{\lambda_2 W_4}{Q_1 - K_1} - \frac{\lambda_1 U_4}{Q_2 - K_2};$$

$$R_5 = \frac{\lambda_2 W_5}{Q_1 + K_1} - \frac{\lambda_1 U_5}{Q_1 + K_2}; \quad R_6 = \frac{\lambda_2 W_6}{Q_1 - K_1} - \frac{\lambda_1 U_6}{Q_2 - K_2}; \qquad \text{(III.154)}$$

$$R_7 = \frac{\lambda_2 W_7}{Q_1 + K_1} - \frac{\lambda_1 U_7}{Q_2 + K_2};$$

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r_j (j = 1, 2, ..., 7) are found respectively from R_j for $\lambda_2 = \lambda_1 = 1$.

In formulas (III.154) we introduce the following definitions:

$$W_1 = (\varkappa_1 + 1)(1 + N_1); \qquad W_2 = (\varkappa_1 + 1)(1 - N_1);$$

$$W_3 = \frac{1}{2\pi}(1 + N_1)(X - iY); \qquad W_4 = \frac{1}{2\pi}(1 - N_1)(X + iY);$$

$$W_5 = \frac{X}{2\pi}[\varkappa_1(1 + N_1) + (1 - N_1)] + \frac{iY}{2\pi}[\varkappa_1(1 + N_1) - (1 - N_1)]; \qquad \text{(III.155)}$$

$$W_6 = -\frac{X}{2\pi}[\varkappa_1(1 - N_1) + (1 + N_1)] + \frac{iY}{2\pi}[\varkappa_1(1 - N_1) - (1 + N_1)];$$

$$W_7 = 4\mu(D_1 + N_1 i D_2);$$

U_j (j = 1, 2, ..., 7) are found respectively from W_j by substituing N_1 by N_2.

Finally, for determining the coefficient a_1 we have the formula

$$a_1 = \frac{(\lambda_2 - \lambda_1 - R_1)\left\{\bar{\zeta}_0 \bar{R}_3 - \zeta_0 \bar{R}_5 - \frac{\bar{\lambda}_2 - \bar{\lambda}_1}{2\pi(\varkappa_1 + 1)}[\bar{\zeta}_0(X + iY) - \varkappa_1 \zeta_0(X - iY)]\right\}}{(\lambda_2^2 - \lambda_1 - R_1)(\lambda_2 - \bar{\lambda}_1 - \bar{R}_1) - (\bar{\lambda}_2 - \bar{\lambda}_1)(\lambda_2 - \lambda_1)} -$$

$$- \frac{(\bar{\lambda}_2 - \lambda_1)\left\{\zeta_0 R_3 - \bar{\zeta}_0 R_5 - \frac{\lambda_2 - \lambda_1}{2\pi(\varkappa_1 + 1)}[\zeta_0(X - iY) - \varkappa_1 \bar{\zeta}_0(X + iY)]\right\}}{(\lambda_2^2 - \lambda_1 - R_1)(\bar{\lambda}_2 - \bar{\lambda}_1 - \bar{R}_1) - (\lambda_2 - \bar{\lambda}_1)(\lambda_2 - \lambda_1)} \qquad \text{(III.156)}$$

In particular, if the force P is applied at the center of the disc and is directed along the Ox axis, then the first formula of (III.151) acquires the form

$$f(\sigma) = -\frac{P}{2\pi}\left[\ln \sigma + \frac{1}{\lambda_2 - \lambda_1}(R_3' \sigma^2 + R_4' \bar{\sigma}^2)\right], \qquad \text{(III.157)}$$

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$$R_3' = \frac{\lambda_2(1+N_1)}{Q_1+K_1} - \frac{\lambda_1(1+N_2)}{Q_2+K_2},$$
$$R_4' = \frac{\lambda_2(1-N_1)}{Q_1-K_1} - \frac{\lambda_1(1-N_2)}{Q_2-K_2}. \qquad \text{(III.158)}$$

For stresses along the contour of the weld in the polar coordinate system (r, θ), considering the form of the function f(σ) in (III.157), we find the formulas /267

$$\sigma_r = -\frac{P}{2\pi}\left[\frac{2(R_3'+3R_4')}{\lambda_2-\lambda_1} + 1 - \frac{8R_4'}{\lambda_2-\lambda_1}\cos^2\theta\right]\cos\theta,$$
$$\tau_{r\theta} = -\frac{P}{2\pi}\left[\frac{2(R_3'-R_4')}{\lambda_2-\lambda_1} - 1 + \frac{8R_4'}{\lambda_2-\lambda_1}\cos^2\theta\right]\sin\theta. \qquad \text{(III.159)}$$

By way of example, the stresses for a birch veneer plate with elastic constants

$$E_1 = 1.2\cdot 9.81\cdot 10^9, \quad E_2 = 0.6\cdot 9.81\cdot 10^9, \quad \nu_1 = 0.071, \quad \nu_2 = 0.036,$$
$$G_2 = 0.07\cdot 9.81\cdot 10^9, \quad s_1 = 4.11i, \quad s_2 = 0.343i$$

and for a steel core with elastic constants $G_1 = 8.1\cdot 9.81\cdot 10^9$, $\kappa_1 = 2.125$ were calculated by formulas (III.159) for the case where the x axis is directed along the grain, i.e., coincides with the direction corresponding to the greater modulus of elasticity.

The values of normal σ_r and tangential $\tau_{r\theta}$ stresses (in fractions of P) along the contour of the weld at points of the first quadrant of the contour of the weld are presented in Table III.6 for the case where the force P is directed along the grain. The contact stresses for a plywood plate and a steel core differ only slightly from the corresponding values for an isotropic plate with an absolutely rigid core. This is understandable, since the parameter n, equal to the ratio of the modulus of displacement of the material of the core to the modulus of displacement of the plate, will be greater than 100 in the given example. We see from Table III.6 that the anisotropy of the material of the plate also has very little effect on the character of distribution of contact stresses.

Moment Applied to Disc. The stress functions for the core (disc) are

$$\varphi_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f_1(t)\,dt}{t-\zeta} + \frac{iM}{2\pi R}\cdot\frac{\zeta}{1-\bar{\zeta}_0\zeta} - \bar{a}_1\zeta;$$
$$\psi_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{f(t)}\,dt}{t-\zeta} - \frac{1}{\zeta}\cdot\frac{1}{2\pi i}\int_\gamma \frac{f'(t)\,dt}{t-\zeta} + \frac{iM}{2\pi R}\left[\frac{1}{\zeta-\zeta_0} - \frac{1}{\zeta(1-\bar{\zeta}_0\zeta)^2}\right] + \frac{a_1+\bar{a}_1}{\zeta}.$$

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TABLE III.6.

θ°	σ_r	$\tau_{r\theta}$	θ°	σ_r	$\tau_{r\theta}$
5	—0,1657	0,0000	60	—0,08168	0,1328
15	—0,1599	0,04026	75	—0,04213	0,1476
30	—0,1428	0,07748	90	—0,0000	0,1526
45	—0,1161	0,1090			

Tr. Note: Commas indicate decimal points.

The stress functions for the plate are found from formulas (III.89), assuming that $A^{(1)} = B^{(1)} = C_1 = C_2 = 0$:

$$\varphi_2(z_1) = \frac{l_2}{2\pi i}\int_\gamma \frac{f\,dt}{t-\zeta_1} - \frac{m_2}{2\pi i}\int_\gamma \frac{\bar{f}\,dt}{t-\zeta_1},$$
$$\psi_2(z_2) = -\frac{l_1}{2\pi i}\int_\gamma \frac{f\,dt}{t-\zeta_2} + \frac{m_1}{2\pi i}\int_\gamma \frac{\bar{f}\,dt}{t-\zeta_2}. \qquad \text{(III.160)}$$

By satisfying conditions (III.121), we find equations (III.122) with the right hand sides /268

$$F_7(\sigma) = (\varkappa_1+1)(\bar{a}_1\sigma + a_1\bar{\sigma}) - \frac{iM(\varkappa_1+1)}{2\pi R}\left(\frac{\sigma}{1-\bar{\zeta}_0\sigma} - \frac{1}{\sigma-\zeta_0}\right),$$
$$F_8(\sigma) = (\varkappa_1+1)(\bar{a}_1\sigma - a_1\bar{\sigma}) - \frac{iM(\varkappa_1+1)}{2\pi R}\left(\frac{\sigma}{1-\bar{\zeta}_0\sigma} + \frac{1}{\sigma-\zeta_0}\right). \qquad \text{(III.161)}$$

Formulas (III.134) reduce to the following expressions:

$$f(\sigma) = \frac{1}{\lambda_2-\lambda_1}\left(T_3\bar{a}_1\sigma + T_4 a_1\frac{1}{\sigma} + \frac{T_2}{\sigma-\zeta_0} - \frac{T_1\sigma}{1-\bar{\zeta}_0\sigma}\right),$$
$$\overline{f(\sigma)} = -\frac{1}{\lambda_2-\lambda_1}\left(t_3\bar{a}_1\sigma + t_4 a_1\frac{1}{\sigma} + \frac{t_2}{\sigma-\zeta_0} - \frac{t_1\sigma}{1-\bar{\zeta}_0\sigma}\right). \qquad \text{(III.162)}$$

On the basis of (III.160) and (III.162), we find, finally,

$$\varphi_1(\zeta) = \left(\frac{iM}{2\pi R} - \frac{T_1}{\lambda_2-\lambda_1}\right)\frac{\zeta}{1-\bar{\zeta}_0\zeta} + \left(\frac{T_2}{\lambda_2-\lambda_1} - 1\right)\bar{a}_1\zeta,$$
$$\psi_1(\zeta) = \frac{iM}{2\pi R}\cdot\frac{1}{\zeta-\zeta_0} - \left(\frac{iM}{2\pi R} - \frac{T_1}{\lambda_2-\lambda_1}\right)\frac{1}{\zeta(1-\bar{\zeta}_0\zeta)^2} + \frac{t_1}{\lambda_2-\lambda_1}\frac{\zeta}{1-\bar{\zeta}_0\zeta} + \qquad \text{(III.163)}$$
$$+\left[a_1 + \left(1 - \frac{T_2}{\lambda_2-\lambda_1}\right)\bar{a}_1\right]\frac{1}{\zeta} - \frac{t_2}{\lambda_2-\lambda_1}a_1\zeta;$$

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$$\left.\begin{aligned}\varphi_2(z_1) &= -\frac{l_2T_2+m_2t_2}{\lambda_2-\lambda_1}\cdot\frac{1}{\zeta_1-\zeta_0}-\frac{l_2T_4+m_2t_4}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta_1},\\ \psi_2(z_2) &= \frac{l_1T_2+m_1t_2}{\lambda_2-\lambda_1}\cdot\frac{1}{\zeta_2-\zeta_0}+\frac{l_1T_4+m_1t_4}{\lambda_2-\lambda_1}\cdot\frac{a_1}{\zeta_2}.\end{aligned}\right\} \qquad \text{(III.164)}$$

Here we introduce the following definitions:

$$\left.\begin{aligned}T_1 &= \frac{\lambda_2 iM(\varkappa_1+1)(1+N_1)}{2\pi R(Q_1+K_1)}-\frac{\lambda_1 iM(\varkappa_1+1)(1+N_2)}{2\pi R(Q_2+K_2)},\\ T_2 &= \frac{\lambda_2 iM(\varkappa_1+1)(1-N_1)}{2\pi R(Q_1-K_1)}-\frac{\lambda_1 iM(\varkappa_1+1)(1-N_2)}{2\pi R(Q_2-K_2)},\\ T_3 &= \frac{\lambda_2(\varkappa_1+1)(1+N_1)}{Q_1+K_1}-\frac{\lambda_1(\varkappa_1+1)(1+N_2)}{Q_2+K_2},\\ T_4 &= \frac{\lambda_2(\varkappa_1+1)(1-N_1)}{Q_1-K_1}-\frac{\lambda_1(\varkappa_1+1)(1-N_2)}{Q_2-K_2};\end{aligned}\right\} \qquad \text{(III.165)}$$

$$a_1 = -\frac{\left(\frac{iM}{2\pi R}-\frac{T_1}{\lambda_2-\lambda_1}\right)-\left(\frac{iM}{2\pi R}+\frac{\overline{T}_1}{\overline{\lambda}_2-\overline{\lambda}_1}\right)\left(\frac{T_3}{\lambda_2-\lambda_1}-1\right)}{\left(\frac{T_3}{\lambda_2-\lambda_1}-1\right)\left(\frac{\overline{T}_3}{\overline{\lambda}_2-\overline{\lambda}_1}-1\right)-1}; \qquad \text{(III.166)}$$

t_j (j = 1, 2, 3, 4) are found respectively from T_j for $\lambda_2 = \lambda_1 = 1$.

Thus, all problems of the given section are solved in simple closed form. /269 The final formulas for the function $f(\sigma)$ and stress functions in an isotropic core and anisotropic plate are presented for each case.

§4. Mixed Boundary Problem for Orthotropic Plate with Round Hole[1]

Statement and General Solution of Problem. Suppose we have an unbounded homogeneous anisotropic plate with a round hole of radius R. We will assume the plate to be orthotropic, and that the vector components of displacement $u = f_1(t)$, $v = f_2(t)$ are known on a part L_1 of the edge of the round hole, where t are points of the contour of the round hole, and that the componets of external stresses, which, without limiting generality, we will assume to be equal to zero, are given on the remainder L_2 of the edge. A homogeneous stress field is given at a distance from the hole. In particular, we will examine the case of tension of a plate in two mutually perpendicular directions by forces of intensity p and q, parallel to the lines of intersection with the plane of

[1]The solution is given by D. V. Grilitskiy [9]. See also D. V. Grilitskiy and Ya. M. Kizim [1]. This problem for an isotropic plate is analyzed in the works of I. N. Kartsivadze, B. A. Mintsberga [1] and N. I. Muskhelishvili [1].

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elastic symmetry. The given functions $f_1(t)$ and $f_2(t)$ are assumed to be such that their first derivatives $f_1'(t)$ and $f_2'(t)$ satisfy Gel'der's condition on L_1. Furthermore, we will assume as known the resultant vector of all forces applied to L_1. In this problem we are required to determine the law of distribution of forces σ_r and $\tau_{r\theta}$ on segment L_1 of the edge of the hole.

To solve the stated mixed problem, we will use the Cartesian coordinate system with the origin at the center of the hole, and for the direction of the x and y axes, we will use the main directions of elasticity of the material of the plate. We will notice that the requirement of orthotropicity of the material of the plate is not binding, and for simplicity, additional calculations are made.

The elastic state in a flat anisotropic medium is given by two functions $\phi(z_1)$ and $\psi(z_2)$:

$$z_1 = x + s_1 y, \qquad z_2 = x + s_2 y; \tag{III.167}$$

s_1 and s_2 depend on the elastic properties of the medium:

$$s_1 = i\beta_1, \qquad s_2 = i\beta_2. \tag{III.168}$$

The boundary conditions of the problem are written on the basis of formulas (I.93) in the form

$$2\,\mathrm{Re}\,[(\sin\theta - s_1\cos\theta)^2\varphi'(t_1) + (\sin\theta - s_2\cos\theta)^2\psi'(t_2)] = \begin{cases}\sigma_r & \text{on } L_1,\\ 0 & \text{on } L_2;\end{cases}$$

$$2\,\mathrm{Re}\,[(\sin\theta - s_1\cos\theta)(\cos\theta + s_1\sin\theta)\varphi'(t_1) + (\sin\theta - s_2\cos\theta)(\cos\theta + s_2\sin\theta)\psi'(t)] = \begin{cases}\tau_{r\theta} & \text{on } L_1,\\ 0 & \text{on } L_2.\end{cases} \tag{III.169}$$

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The stress functions $\phi(z_1)$ and $\psi(z_2)$, which are included in the boundary conditions (III.169), have the form (I.121), where the constants A and B are defined by formulas (I.127), and the constants B*, B_1^* and C_1^* are found on the basis of formulas (I.126), from which

$$B^* = -\frac{p + \beta_2^2 q}{2(\beta_1^2 - \beta_2^2)}, \quad B_1^* = \frac{p + \beta_1^2 q}{2(\beta_1^2 - \beta_2^2)}, \quad C_1^* = 0. \tag{III.170}$$

We will introduce a new variable ζ by relations (III.4) and (III.5), in which it is assumed that a = b = R:

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$$z_1 = \omega_1(\zeta) = \frac{R}{2}[(1 - is_1)\zeta + (1 + is_1)\zeta^{-1}].$$
$$z_2 = \omega_2(\zeta) = \frac{R}{2}[(1 - is_2)\zeta + (1 + is_2)\zeta^{-1}]. \qquad \text{(III.171)}$$

Then the functions $\Phi(\zeta) = \phi[\omega_1(\zeta)]$ and $\Psi(\zeta) = \psi[\omega_2(\zeta)]$, with consideration of formula (I.124), acquire the form

$$\Phi(\zeta) = A\ln\zeta + a_1\zeta + a_0 + \frac{a_{-1}}{\zeta} + \frac{a_{-2}}{\zeta^2} + \ldots,$$
$$\Psi(\zeta) = B\ln\zeta + b_1\zeta + b_0 + \frac{b_{-1}}{\zeta} + \frac{b_{-2}}{\zeta^2} + \ldots, \qquad \text{(III.172)}$$

where

$$a_1 = \frac{R}{2}(1 - is_1)B^*, \quad b_1 = \frac{R}{2}(1 - is_2)B_1^*. \qquad \text{(III.173)}$$

Considering

$$\frac{d\varphi(z_1)}{dz_1} = \Phi'(\zeta)\frac{2\zeta^2}{R[(1 - is_1)\zeta^2 - (1 + is_1)]},$$
$$\frac{d\psi(z_1)}{dz_2} = \Psi'(\zeta)\frac{2\zeta^2}{R[(1 - is_2)\zeta^2 - (1 + is_2)]}, \qquad \text{(III.174)}$$

we find boundary conditions (III.169) in variable ζ (σ on the contour of γ):

$$2\,\mathrm{Re}\left\{\left[-\frac{i}{2}(\sigma - \sigma^{-1}) - \frac{s_1}{2}(\sigma + \sigma^{-1})\right]^2 \frac{2\sigma^2}{R[(1 - is_1)\sigma^2 - (1 + is_1)]}\Phi'(\sigma) + \right.$$
$$\left. + \left[-\frac{i}{2}(\sigma - \sigma^{-1}) - \frac{s_2}{2}(\sigma + \sigma^{-1})\right]^2 \frac{2\sigma^2}{R[(1 - is_2)\sigma^2 - (1 + is_2)]}\Psi'(\sigma)\right\} = \begin{cases}\sigma_r \text{ on } \gamma_1; \\ 0 \text{ on } \gamma_2;\end{cases}$$

$$2\,\mathrm{Re}\left\{\left[\frac{i(s_1^2 - 1)}{4}(\sigma^2 - \sigma^{-2}) - \frac{s_1}{2}(\sigma^2 + \sigma^{-2})\right]\frac{2\sigma^2}{R[(1 - is_1)\sigma^2 - (1 + is_1)]}\Phi'(\sigma) + \right. \qquad \text{(III.175)} \quad \underline{/271}$$
$$\left. + \left[\frac{i(s_2^2 - 1)}{4}(\sigma^2 - \sigma^{-2}) - \frac{s_2}{2}(\sigma^2 + \sigma^{-2})\right]\frac{2\sigma^2}{R[(1 - is_2)\sigma^2 - (1 + is_2)]}\Psi'(\sigma)\right\} =$$
$$= \begin{cases}\tau_{r\theta} \text{ on } \gamma_1, \\ 0 \text{ on } \gamma_2.\end{cases}$$

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We multiply (III.175) by $1/2\pi i \cdot d\sigma/\sigma - \zeta$, where ζ is located outside of the unit circle γ, and we integrate with respect to γ in the counter clockwise direction. Consequently, we have two equations:

$$-\left[-\frac{i}{2}(\zeta-\zeta^{-1})-\frac{s_1}{2}(\zeta+\zeta^{-1})\right]^2\frac{2\zeta^2}{[(1-is_1)\zeta^2-(1+is_1)]}\Phi'(\zeta)+$$

$$+\frac{s_1^2-1+2is_1}{2(1-is_1)}(A\zeta+a_1\zeta^2-a_{-1})+\frac{1+s_1^2}{2(1-is_1)}a_1-\frac{\bar{s}_1^2-1-2i\bar{s}_1}{2(1+i\bar{s}_1)}\left(\frac{\bar{A}}{\zeta}+\frac{\bar{a}_1}{\zeta^2}\right)-$$

$$-\left[-\frac{i}{2}(\zeta-\zeta^{-1})-\frac{s_2}{2}(\zeta+\zeta^{-1})\right]^2\frac{2\zeta^2}{[(1-is_2)\zeta^2-(1+is_2)]}\Psi'(\zeta)+$$

$$+\frac{s_2^2-1+2is_2}{2(1-is_2)}(B\zeta+b_1\zeta^2-b_{-1})+\frac{1+s_2^2}{2(1-is_2)}b_1-\frac{\bar{s}_2^2-1-2i\bar{s}_2}{2(1+i\bar{s}_2)}\left(\frac{\bar{B}}{\zeta}+\frac{\bar{b}_1}{\zeta^2}\right)=$$

$$=\frac{R}{2\pi i}\int_{\gamma_1}\frac{\sigma_r dt}{t-\zeta};$$

$$-\left[\frac{i(s_1^2-1)}{4}(\zeta^2-\zeta^{-2})-\frac{s_1^2}{2}(\zeta^2+\zeta^{-2})\right]\frac{2\zeta^2}{[(1-is_1)\zeta^2-(1+is_1)]}\Phi'(\zeta)+$$

$$+\frac{i(s_1^2-1)-2s_1}{2(1-is_1)}(A\zeta+a_1\zeta^2-a_{-1})+\frac{s_1-i}{2}a_1+\frac{i(\bar{s}_1^2-1)+2\bar{s}_1}{2(1+i\bar{s}_1)}\left(\frac{\bar{A}}{\zeta}+\frac{\bar{a}_1}{\zeta^2}\right)-$$

$$-\left[\frac{i(s_2^2-1)}{4}(\zeta^2-\zeta^{-2})-\frac{s_2}{2}(\zeta^2+\zeta^{-2})\right]\frac{2\zeta^2}{[(1-is_2)\zeta^2-(1+is_2)]}\Psi'(\zeta)+$$

$$+\frac{i(s_2^2-1)-2s_2}{2(1-is_2)}(B\zeta+b_1\zeta^2-b_{-1})+\frac{s_2-i}{2}b_1+\frac{i(\bar{s}_2^2-1)+2\bar{s}_2}{2(1+i\bar{s}_2)}\left(\frac{\bar{B}}{\zeta}+\frac{\bar{b}_1}{\zeta^2}\right)=$$

$$=\frac{R}{2\pi i}\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t-\zeta}.$$

By solving these equations relative to functions $\Phi'(\zeta)$ and $\Psi'(\zeta)$, we obtain

$$\Phi'(\zeta)=\frac{(1+\beta_1)\zeta^2-(1-\beta_1)}{2\zeta^2\Delta(\zeta)}\left\{\frac{RL_2(\zeta)}{2\pi i}\int_{\gamma_1}\frac{\sigma_r dt}{t-\zeta}-\frac{RM_2(\zeta)}{2\pi i}\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t-\zeta}+\right.$$

$$+[iM_2(\zeta)-L_2(\zeta)][l_1(A\zeta+a_1\zeta^2-a_{-1})+l_2(B\zeta+b_1\zeta^2-b_{-1})]+$$

$$+\left[l_1\left(\frac{\bar{A}}{\zeta}+\frac{\bar{a}_1}{\zeta^2}\right)+l_2\left(\frac{\bar{B}}{\zeta}+\frac{\bar{b}_1}{\zeta^2}\right)\right][iM_2(\zeta)+L_2(\zeta)]- \qquad \text{(III.176)} \quad \underline{/272}$$

$$\left.-\frac{1}{2}[(1-\beta_1)a_1+(1-\beta_2)b_1][L_2(\zeta)+iM_2(\zeta)]\right\};$$

$$\Psi'(\zeta)=-\frac{(1+\beta_2)\zeta^2-(1-\beta_2)}{2\zeta^2\Delta(\zeta)}\left\{\frac{RL_1(\zeta)}{2\pi i}\int_{\gamma}\frac{\sigma_r dt}{t-\zeta}-\frac{RM_1(\zeta)}{2\pi i}\int_{\gamma}\frac{\tau_{r\theta}dt}{t-\zeta}+\right.$$

$$+[iM_1(\zeta)-L_1(\zeta)][l_1(A\zeta+a\zeta^2-a_{-1})+l_2(B\zeta+b_1\zeta^2-b_{-1})]+\left[l_1\left(\frac{\bar{A}}{\zeta}+\frac{\bar{a}_1}{\zeta^2}\right)-\right.$$

$$\left.+l_2\left(\frac{\bar{B}}{\zeta}+\frac{\bar{b}_1}{\zeta^2}\right)\right][iM_1(\zeta)+L_1(\zeta)]-\frac{1}{2}[(1-\beta_1)a_1+(1-\beta_2)b_1][L_1(\zeta)+iM_1(\zeta)] \qquad \text{(III.177)}$$

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Here we introduce the definitions:

$$L_k(\zeta) = -\frac{i}{4}[(1+\beta_k^2)(\zeta^2-\zeta^{-2}) + 2\beta_k(\zeta^2+\zeta^{-2})];$$
$$M_k(\zeta) = -\frac{1}{4}[(\zeta-\zeta^{-1}) + \beta_k(\zeta+\zeta^{-1})]^2;$$
$$\Delta(\zeta) = L_1(\zeta)M_2(\zeta) - M_1(\zeta)L_2(\zeta); \qquad \text{(III.178)}$$
$$l_k = -\frac{1}{2}(1+\beta_k) \qquad (k = 1, 2);$$

γ_1 is the part of the edge of the round hole of unit radius that corresponds to contour L_1.

The formulas for stress functions (III.176) and (III.177) include two derivatives, generally speaking, complex constants a_{-1} and b_{-1}. Since ζ is located outside of γ, expressions (III.176) and (III.177) are not completely equivalent to the original contour conditions (III.175), from which it was obtained. In order to make sure that the contour conditions are satisfied, it is necessary to add to formulas (III.176) and (III.177), equations obtained by multiplying both sides of equations (III.175) by $1/2\pi i \cdot dt/t$ and integrating with respect to γ. As the result we arrive at a relation that connects these two constants:

$$(1+\beta_1)a_{-1} + (1+\beta_2)b_{-1} = \frac{R}{2\pi i}\int_{\gamma_1}\frac{\sigma_r dt}{t} - \frac{R}{2\pi i}\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t} - (1-\beta_1)\bar{a}_1 - (1-\beta_2)\bar{b}_1. \qquad \text{(III.179)}$$

Thus, the derivatives of functions Φ and Ψ are expressed through the desired forces σ_r and $\tau_{r\theta}$ with the aid of integrals of the Cauchy type by formulas (III.176) and (III.177) along with the additional relation (III.179)

In order to determine the conditions from which it would be possible to determine the desired functions, we will make use of the fact that Cartesian components of the vector of displacements are given on part L_1 of the edge of the hole. On the basis of formulas (I.97), /273

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[p_1\varphi(t_1) + p_2\psi(t_2)] &= f_1 \\ 2\,\mathrm{Re}\,[q_1\varphi(t_1) + q_2\psi(t_2)] &= f_2 \end{aligned}\right\} \text{(on } L_1\text{)}, \qquad \text{(III.180)}$$

from which, by converting to variable ζ, we obtain

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[p_1\Phi(\sigma) + p_2\Psi(\sigma)] &= f_1(R\sigma) \\ 2\,\mathrm{Re}\,[q_1\Phi(\sigma) + q_2\Psi(\sigma)] &= f_2(R\sigma) \end{aligned}\right\} \text{(on } \gamma_1\text{)}. \qquad \text{(III.181)}$$

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From conditions (III.181), as the result of differentiation with respect to θ, we find

$$\left.\begin{aligned} 2\,\mathrm{Re}\,\{i\sigma[p_1\Phi'(\sigma)+p_2\Psi'(\sigma)]\} &= Ri\sigma f_1'(R\sigma)\\ 2\,\mathrm{Re}\,\{i\sigma[q_1\Phi'(\sigma)+q_2\Psi'(\sigma)]\} &= Ri\sigma f_2'(R\sigma)\end{aligned}\right\}\ (\text{on } \gamma_1). \qquad \text{(III.182)}$$

We will satisfy conditions (III.182) with the aid of functions (III.176) and (III.177), considering relation (III.179). After certain transformations we obtain a system of two singular integral equations with variable coefficients

$$\begin{aligned} &-4\pi n_2\tau_{r\theta}(\sigma)-[n_1(\sigma+\bar\sigma)^2-n_3(\sigma-\bar\sigma)^2]\int_{\gamma_1}\frac{\sigma_r(t)dt}{t-\sigma}-\\ &-i(n_1-n_3)(\sigma^2-\bar\sigma^2)\int_{\gamma_1}\frac{\tau_{r\theta}(t)\,dt}{t-\sigma}=F_1(\sigma)\qquad(\sigma\in\gamma_1);\\ &-4\pi n_2\sigma_r(\sigma)+i(n_1-n_3)(\sigma^2-\bar\sigma^2)\int_{\gamma_1}\frac{\sigma_r(t)\,dt}{t-\sigma}-\\ &-[n_1(\sigma-\bar\sigma)^2-n_3(\sigma+\bar\sigma)^2]\int_{\gamma_1}\frac{\tau_{r\theta}(t)\,dt}{t-\sigma}=F_2(\sigma)\qquad(\sigma\in\gamma_1), \end{aligned} \qquad \text{(III.183)}$$

where

$$\begin{aligned} F_1(\sigma)&=\frac{(n_1+n_2)Y}{R}(\sigma+\bar\sigma)+\frac{i(n_3+n_3)X}{R}(\sigma-\bar\sigma)-\\ &-[n_1(1+\bar\sigma^2)+n_3(1-\bar\sigma^2)]\int_{\gamma_1}\frac{\sigma_r dt}{t}+i[n_1(1+\bar\sigma^2)+\\ &+n_3(1-\bar\sigma^2)-2n_2]\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t}-\pi i[n_1(n_4p-n_6q)+n_3(n_5p-n_7q)](\sigma^2-\bar\sigma^2)+\\ &+2\pi[(\sigma^2-1)f_2'(R\sigma)+i(\sigma^2+1)f_1'(R\sigma)];\\ F_2(\sigma)&=-\frac{i(n_1+n_2)Y}{R}(\sigma-\bar\sigma)+\frac{(n_2+n_3)X}{R}(\sigma+\bar\sigma)+\\ &+i[n_1(1-\bar\sigma^2)+n_3(1+\bar\sigma^2)]\int_{\gamma_1}\frac{\sigma_r dt}{t}+[n_1(1-\bar\sigma^2)+\\ &+n_3(1+\bar\sigma^2)]\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t}-\pi[n_1(n_4p-n_6q)(\sigma-\bar\sigma)^2+n_3(n_5p-n_7q)(\sigma+\bar\sigma)^2]-\\ &+2\pi[i(\sigma^2+1)f_2'(R\sigma)+(\sigma^2-1)f_1'(R\sigma)]. \end{aligned} \qquad \text{(III.184)}$$

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Here X, Y are components of the resultant vector of the desired forces acting on part L of the edge of the hole; p and q are the forces of tension at infinity of the plate in the direction of the Ox and Oy axes, respectively;

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$$n_1 = a_{11}(\beta_1 + \beta_2), \quad n_2 = a_{12} + \frac{a_{22}}{\beta_1\beta_2} = a_{12} + a_{11}\beta_1\beta_2, \quad n_3 = a_{22}\frac{\beta_1 + \beta_2}{\beta_1\beta_2},$$
$$n_4 = 1 + n_5, \quad n_5 = \frac{1}{\beta_1 + \beta_2}, \quad n_6 = \beta_1\beta_2 n_5, \quad n_7 = 1 + n_6. \tag{III.185}$$

In the case of plane deformation, in formulas (III.185) it is necessary to use, instead of the elastic constants a_{ij}, the adduced elastic constants β_{ij}, defined[1] by relations (I.13).

Equation system (III.183) also represents the relations from which the character of distribution of normal and tangential stresses on segment L_1 of the edge of the round hole in an orthotropic plate must be determined.

We will notice that the requirement of a round hole, rather than, let us say, an elliptical hole, is also not binding, just as the requirement of orthotropicity of the material of the plate. These assumptions are made for simplicity (and brevity) in outlining the method for solving the problem, since as follows from 1, for an anisotropic plate with a round or elliptic hole, the problem is solved simultaneously and identically, which is not the case in an isotropic plate, where the ellipticity of the hole involves considerable difficulties in the solution of the problem.

We will solve equation system (III.183). We multiply the second equation of (III.183) by as yet arbitrary function $N(\sigma)$ and combine it with the first equation:

$$\{-[n_1(\sigma+\bar{\sigma})^2 - n_3(\sigma-\bar{\sigma})^2] + Ni(n_1-n_3)(\sigma^2-\bar{\sigma}^2)\}\int_{\gamma_1}\frac{\sigma_r dt}{t-\sigma} +$$
$$+\{-i(n_1-n_3)(\sigma^2-\bar{\sigma}^2) - N[n_1(\sigma-\bar{\sigma})^2 - n_3(\sigma+\bar{\sigma})^2]\}\int_{\gamma_1}\frac{\tau_{r\theta}dt}{t-\sigma} - \tag{III.186}$$
$$-4\pi n_2 N\sigma_r(\sigma) - 4\pi n_2\tau_{r\theta}(\sigma) = F_1(\sigma) + NF_2(\sigma) \qquad (\sigma \in \gamma_1).$$

We will choose the function $N(\sigma)$ in such a manner that we satisfy the condition

$$\frac{-[n_1(\sigma+\bar{\sigma})^2 - n_3(\sigma-\bar{\sigma})^2] + N(\sigma)\,i(n_1-n_3)(\sigma^2-\bar{\sigma}^2)}{-i(n_1-n_3)(\sigma^2-\bar{\sigma}^2) - N(\sigma)[n_1(\sigma-\bar{\sigma})^2 - n_3(\sigma+\sigma)^2]} = N(\sigma) = \frac{1}{\lambda(\sigma)} \tag{III.187}$$

from which we obtain two values for $N(\sigma)$: /275

[1]Here the symbols a_{ij} are used for elastic constants in the case of the generalized stress state.

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$$N_1(\sigma) = -i\frac{\sigma^2 - z_1^2}{\sigma^2 + z_1^2}, \qquad N_2(\sigma) = -i\frac{\sigma^2 - z_2^2}{\sigma^2 + z_2^2}, \tag{III.188}$$

where

$$z_1 = i\sqrt{\frac{\sqrt{n_1} + \sqrt{n_3}}{\sqrt{n_1} - \sqrt{n_3}}}, \qquad z_2 = i\sqrt{\frac{\sqrt{n_1} - \sqrt{n_3}}{\sqrt{n_1} + \sqrt{n_3}}}. \tag{III.189}$$

We introduce the definitions:

$$\begin{gathered} -[n_1(\sigma + \bar{\sigma})^2 - n_3(\sigma - \bar{\sigma})^2] + N_k(\sigma)\, i(n_1 - n_3)(\sigma^2 - \bar{\sigma}^2) = K_k(\sigma), \\ -4\pi n_2 N_k(\sigma) = Q_k(\sigma) \qquad (k = 1, 2). \end{gathered} \tag{III.190}$$

Then, on the basis of relation (III.187),

$$\begin{gathered} -i(n_1 - n_3)(\sigma^2 - \bar{\sigma}^2) - N_k(\sigma)[n_1(\sigma - \bar{\sigma})^2 - n_3(\sigma + \bar{\sigma})^2] = \lambda_k(\sigma) K_k(\sigma), \\ -4\pi n_2 = \lambda_k(\sigma) Q_k(\sigma), \end{gathered} \tag{III.191}$$

where

$$\lambda_k(\zeta) = i\frac{\zeta^2 + z_k^2}{\zeta^2 - z_k^2} \quad (k = 1, 2). \tag{III.192}$$

Considering definitions (III.190) and (III.191), equation (III.186) can be represented in the form of two independent equations:

$$\begin{gathered} K_k(\sigma)\int_{\gamma_1}[\sigma_r(t) + \lambda_k(\sigma)\tau_{r\theta}(t)]\frac{dt}{t - \sigma} + Q_k(\sigma)[\sigma_r(\sigma) + \lambda_k(\sigma)\tau_{r\theta}(\sigma)] = \\ = F_1(\sigma) + N_k F_2(\sigma) \quad \begin{matrix}(\sigma \in \gamma_1), \\ (k = 1, 2).\end{matrix} \end{gathered} \tag{III.193}$$

We reduce the solution of equations (III.193) to the analysis of two Riemann-Hilbert boundary problems. We write

$$W_k(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1}[\sigma_r(t) + \lambda_k(\zeta)\tau_{r\theta}(t)]\frac{dt}{t - \zeta} \quad (k = 1, 2). \tag{III.194}$$

Using the Sokhotskiy-Plemel' formulas, from (III.194) we find

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$$\left.\begin{aligned} &\sigma_r(\sigma)+\lambda_k(\sigma)\tau_{r\theta}(\sigma)=W_k^+(\sigma)-W_k^-(\sigma)\\ &\int_{\gamma_1}[\sigma_r(t)+\lambda_k(\sigma)\tau_{r\theta}(t)]\frac{dt}{t-\sigma}=\pi i\,[W_k^+(\sigma)+W_k^-(\sigma)] \end{aligned}\right\}_{(k=1,2).}^{(\sigma\in\gamma_1),} \qquad \text{(III.195)}$$

Here $W_k^+(\sigma)$ and $W_k^-(\sigma)$ are the values of functions $W_k(\zeta)$ (k = 1, 2) approaching the point $\zeta = \sigma$, located on contour γ_1, from inside and from outside, respectively circle γ.

As we see, the left hand sides of equations (III.186) can be expressed on /276
the basis of (III.195) through the limit values of the functions $W_k(\zeta)$ (k = = 1, 2):

$$\left.\begin{aligned} &W_1^+(\sigma)+\frac{\sqrt{n_1n_2}-n_2}{\sqrt{n_1n_2}+n_2}W_1^-(\sigma)=\Omega_1(\sigma)\quad(\text{on }\gamma_1),\\ &W_1^+(\sigma)-W_1^-(\sigma)=0\qquad(\text{on }\gamma_2); \end{aligned}\right\} \qquad \text{(III.196)}$$

$$\left.\begin{aligned} &W_2^+(\sigma)+\frac{\sqrt{n_1n_2}+n_2}{\sqrt{n_1n_2}-n_2}W_2^-(\sigma)=\Omega_2(\sigma)\quad(\text{on }\gamma_1),\\ &W_2^+(\sigma)-W_2^-(\sigma)=0\qquad(\text{on }\gamma_2). \end{aligned}\right\} \qquad \text{(III.197)}$$

Here

$$\left.\begin{aligned} \Omega_1(\sigma)&=\frac{F_1(\sigma)+N_1(\sigma)F_2(\sigma)}{i\pi K_1(\sigma)+Q_1(\sigma)},\\ \Omega_2(\sigma)&=\frac{F_1(\sigma)+N_2(\sigma)F_2(\sigma)}{i\pi K_2(\sigma)+Q_2(\sigma)}; \end{aligned}\right\} \qquad \text{(III.198)}$$

γ_2 is the part of circle γ that complements γ_1 to a complete circle.

The solutions of boundary problems (III.196) and (III.197), with simple poles at the given points z_1, $-z_1$ and z_2, $-z_2$ of plane , are represented by the following formulas, respectively:

$$W_1(\zeta)=\frac{X_1(\zeta)}{2\pi i}\int_{\gamma_1}\frac{\Omega_1(t)\,dt}{X_1^+(t)(t-\zeta)}+X_1(\zeta)\left(C_1+\frac{D_1}{\zeta-z_1}+\frac{E_1}{\zeta+z_1}\right); \qquad \text{(III.199)}$$

$$W_2(\zeta)=\frac{X_2(\zeta)}{2\pi i}\int_{\gamma_1}\frac{\Omega_2(t)\,dt}{X_2^+(t)(t-\zeta)}+X_2(\zeta)\left(C_2+\frac{D_2}{\zeta-z_2}+\frac{E_2}{\zeta+z_2}\right). \qquad \text{(III.200)}$$

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Here G_1, D_1, ..., E_2 are constants that are subject to determination;

$$X_1(\zeta) = (\zeta - a)^{-\frac{1}{2}+i\delta}(\zeta - b)^{-\frac{1}{2}-i\delta},$$
$$X_2(\zeta) = (\zeta - a)^{-\frac{1}{2}-i\delta}(\zeta - b)^{-\frac{1}{2}+i\delta}, \qquad \text{(III.201)}$$

where

$$\delta = \frac{1}{2\pi}\ln\frac{\sqrt{n_1 n_3} - n_2}{\sqrt{n_1 n_3} + n_2}; \qquad \text{(III.202)}$$

a and b are points that separate arcs γ_1 and γ_2; by $X_1(\zeta)$ and $X_2(\zeta)$ we mean the arms that are holomorphic on the plane of complex variable ζ, cut along γ_1.

On the basis of the first equation of (III.195), the stresses along contour $\gamma_1(L_1)$ are found by formulas /277

$$\sigma_r(\sigma) = \frac{\lambda_1(\sigma)[W_2^+(\sigma) - W_2^-(\sigma)] - \lambda_2(\sigma)[W_1^+(\sigma) - W_1^-(\sigma)]}{\lambda_1(\sigma) - \lambda_2(\sigma)},$$
$$\tau_{r\theta}(\sigma) = -\frac{[W_2^+(\sigma) - W_1^-(\sigma)] - [W_1^+(\sigma) - W_2^-(\sigma)]}{\lambda_1(\sigma) - \lambda_2(\sigma)}, \qquad \text{(III.203)}$$

in which $W_k^{\pm}(\sigma)$ (k = 1, 2) are the boundary values of the functions defined by expressions (III.199) and (III.200).

Formulas (III.199) and (III.200) for functions $W_1(\zeta)$ and $W_2(\zeta)$ contain seven unknown constants:

$$A_1 = \int_{\gamma_1}\frac{\sigma_r dt}{t}, \quad B_1 = \int_{\gamma_1}\frac{\tau_{r\theta}dt}{t}, C_1, D_1, E_1, C_2, D_2 \text{ and } E_2.$$

Constants C_1 and C_2 are easily found from conditions at infinity. Specifically, on the one hand, on the basis of (III.194),

$$W_k(\zeta) \underset{|\zeta|\to\infty}{\to} -\frac{1}{2\pi i}\left[\int_{\gamma_1}\sigma_r(t)\,dt + i\int_{\gamma_1}\tau_{r\theta}(t)\,dt\right]\frac{1}{\zeta} \quad (k = 1, 2). \qquad \text{(III.204)}$$

On the other hand, from formulas (III.199) and (III.200), we have

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$$W_k(\zeta) \underset{|\zeta|\to\infty}{\to} \frac{C_k}{\zeta} \qquad (k = 1, 2). \tag{III.205}$$

By comparing the right hand sides of the last two equations, we arrive at the relation

$$C_k = -\frac{1}{2\pi i}\left[\int_{\gamma_1} \sigma_r(t)\,dt + i\int_{\gamma_1} \tau_{r\theta}(t)\,dt\right],$$

which is easily transformed to

$$C_k = \frac{X + iY}{2\pi R} \quad (k = 1, 2), \tag{III.206}$$

where X and Y are components of the resultant vector of forces acting on L_1.

From the zero condition we obtain the two following relations:

$$\begin{aligned} A_1 - iB_1 &= X_1(0)\left[\int_{\gamma_1} \frac{\Omega_1(t)\,dt}{tX_1^+(t)} + 2\pi i\left(C_1 + \frac{E_1 - D_1}{z_1}\right)\right], \\ A_1 - iB_1 &= X_2(0)\left[\int_{\gamma_1} \frac{\Omega_2(t)\,dt}{tX_2^+(t)} + 2\pi i\left(C_2 + \frac{E_2 - D_2}{z_2}\right)\right]. \end{aligned} \tag{III.207}$$

We find the last four relations by comparing in formulas (III.194) and (III.199), (III.200), the main part of the poles of functions $W_1(\zeta)$ at the points z_1, $-z_1$, and of the function $W_2(\zeta)$ at the points z_2, $-z_2$: /278

$$\begin{aligned} &\text{I. } 2\pi X_1(z_1) D_1 = z_1\int_{\gamma_1} \frac{\tau_{r\theta}\,dt}{t - z_1}; \quad \text{II. } 2\pi X_1(-z_1) E_1 = -z_1\int_{\gamma_1} \frac{\tau_{r\theta}\,dt}{t + z_1}; \\ &\text{III. } 2\pi X_2(z_2) D_2 = z_2\int_{\gamma_1} \frac{\tau_{r\theta}\,dt}{t - z_2}; \quad \text{IV. } 2\pi X_2(-z_2) E_2 = -z_2\int_{\gamma_1} \frac{\tau_{r\theta}\,dt}{t + z_2}. \end{aligned} \tag{III.208}$$

Here, by $\tau_{r\theta}(t)$ we mean the expression obtained from the second formula of (III.203) and from formulas (III.199), (III.200); z_1 and z_2 are points of plane ζ defined by formulas (III.189). Conditions (III.207), (III.208) also define six constants: A_1, B_1, D_1, E_1, D_2, E_2.

We will examine two examples that are of independent importance.

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Impression of a Rigid Punch into an Elastic Orthotropic Body. Let a rigid punch with the shape of an arc of a hole of the same radius R, and which is immovably affixed to an elastic body, be applied to arc L = ab of a round hole in an orthotropic plate. The punch is impressed into the elastic body by normal force P_0, applied symmetrically and acting in the direction of the Ox axis, which passes through the center of arc ab. There are no stresses at infinity of the plate. We are required to determine the law of distribution of contact stresses σ_r and $\tau_{r\theta}$ between the punch and the elastic body.

In the given case

$$X = P_0; \quad Y = 0; \quad p = q = 0;$$
$$f_1'(R\sigma) = f_2'(R\sigma) = 0; \quad a = e^{-i\theta_0}, \quad b = e^{i\theta_0} \quad (2\theta_0 < \pi); \tag{III.209}$$

$B_1 = \int_{\gamma_1} \frac{\tau_{r\theta}\, dt}{t} = 0$, due to the symmetry of the problem;

$$C_1 = C_2 = \frac{P_0}{2\pi R}. \tag{III.210}$$

Functions (III.198) acquire the values

$$\Omega_1(\sigma) = \frac{\sqrt{n_1}}{\pi i (\sqrt{n_1 n_3} + n_2)(\sqrt{n_1} - \sqrt{n_3})(\sigma^2 - z_1^2)} \left[-i (n_2 + n_3) \frac{P_0}{R} \sigma + \sqrt{n_3}(\sqrt{n_1} + \sqrt{n_3}) A_1 \right].$$

$$\Omega_2(\sigma) = \frac{\sqrt{n_1}}{\pi i (\sqrt{n_1 n_3} - n_2)(\sqrt{n_1} + \sqrt{n_3})(\sigma^2 - z_2^2)} \left[i (n_2 + n_3) \frac{P_0}{R} \sigma + \sqrt{n_3}(\sqrt{n_1} - \sqrt{n_3}) A_1 \right] \tag{III.211}$$

By substituting functions (III.211) into formulas (III.199), (III.200) and after calculating the required integrals of the Cauchy type[1], we obtain

$$W_1(\zeta) = \frac{1}{2\pi i \sqrt{n_3}(\sqrt{n_1} - \sqrt{n_3})} \left\{ \frac{1}{\zeta^2 - z_1^2} \left[-\frac{i (n_2 + n_3) P_0}{R} \zeta + \right. \right.$$
$$\left. - i \sqrt{n_3}(\sqrt{n_1} + \sqrt{n_3}) A_1 \right] + \frac{i}{2} X_1(\zeta) \left[\frac{1}{X_1(z_1)(\zeta - z_1)} \left(\frac{n_2 + n_3}{R} P_0 + \sqrt{n_3}\sqrt{n_1 - n_3} A_1 \right) + \right. \tag{III.212}$$
$$\left. \left. + \frac{1}{X_1(-z_1)(\zeta + z_1)} \left(\frac{n_2 + n_3}{R} P_0 - \sqrt{n_3}\sqrt{n_1 - n_3} A_1 \right) + 2 (n_2 + n_3) \frac{P_0}{R} \right] \right\} +$$
$$+ X_1(\zeta) \left(\frac{P_0}{2\pi R} + \frac{D_1}{\zeta - z_1} + \frac{E_1}{\zeta - z_1} \right);$$

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[1] See table of integrals of Cauchy type, presented in §4, Chapter I.

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$$W_2(\zeta)=\frac{1}{2\pi i\sqrt{n_3}(\sqrt{n_1}+\sqrt{n_3})}\left\{\frac{1}{\zeta^2-z_2^2}\left[\frac{i(n_2+n_3)P_0}{R}\zeta+\right.\right.$$
$$\left.+i\sqrt{n_3}(\sqrt{n_1}-\sqrt{n_3})A_1\right]+\frac{i}{2}X_2(\zeta)\left[\frac{1}{X_2(z_2)(\zeta-z_2)}\left(-\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-\right.$$
$$-\frac{1}{X_2(-z_2)(\zeta+z_2)}\left(\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-2(n_2+n_3)\frac{P_0}{R}\Big]\Big\}+ \qquad \text{(III.213)}$$
$$+X_2(\zeta)\left(\frac{P_0}{2\pi R}+\frac{D_2}{\zeta-z_2}+\frac{E_2}{\zeta+z_2}\right).$$

The components of the stresses on contour γ_1 are defined in accordance with formulas (III.193)

$$\sigma_r=-\frac{\sqrt{n_1n_3}\lambda_2(\sigma)X_1(\sigma)}{\pi i(\sqrt{n_1n_3}-n_2)[\lambda_1(\sigma)-\lambda_2(\sigma)]}\left\{\frac{i}{2\sqrt{n_3}(\sqrt{n_1}-\sqrt{n_3})}\left[\frac{1}{X_1(z_1)(\sigma-z_1)}\right.\right.\times$$
$$\times\left(\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)+\frac{1}{X_1(-z_1)(\sigma+z_1)}\left(\frac{n_2+n_3}{R}P_0-\right.$$
$$\left.\left.-\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)+2(n_2+n_3)\frac{P_0}{R}\right]+\frac{iP_0}{R}+2\pi i\left(\frac{D_1}{\sigma-z_1}+\frac{E_1}{\sigma+z_1}\right)\Big\}+$$
$$+\frac{\sqrt{n_1n_3}\lambda_1(\sigma)X_2(\sigma)}{\pi i(\sqrt{n_1n_3}+n_2)[\lambda_1(\sigma)-\lambda_2(\sigma)]}\left\{\frac{i}{2\sqrt{n_3}(\sqrt{n_1}+\sqrt{n_3})}\left[\frac{1}{X_2(z_2)(\sigma-z_2)}\times\right.\right.$$
$$\times\left(-\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-\frac{1}{X_2(-z_2)(\sigma+z_2)}\left(\frac{n_2+n_3}{R}P_0+\right. \qquad \text{(III.214)}$$
$$\left.\left.+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-2(n_2+n_3)\frac{P_0}{R}\right]+\frac{iP_0}{R}+2\pi i\left(\frac{D_2}{\sigma-z_2}+\frac{E_2}{\sigma+z_2}\right)\Big\};$$
$$\tau_{r\theta}=\frac{\sqrt{n_1n_3}X_1(\sigma)}{\pi i(\sqrt{n_1n_3}-n_2)[\lambda_1(\sigma)-\lambda_2(\sigma)]}\left\{\frac{i}{2\sqrt{n_3}(\sqrt{n_1}-\sqrt{n_3})}\left[\frac{1}{X_1(z_1)(\sigma-z_1)}\times\right.\right.$$
$$\times\left(\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)+\frac{1}{X_1(-z_1)(\sigma+z_1)}\left(\frac{n_1+n_3}{R}P_0-\right.$$
$$\left.\left.-\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)+2(n_2+n_3)\frac{P_0}{R}\right]+\frac{iP_0}{R}+2\pi i\left(\frac{D_1}{\sigma-z_1}+\frac{E_1}{\sigma+z_1}\right)\Big\}-$$

$$-\frac{\sqrt{n_1n_3}X_2(\sigma)}{\pi i(\sqrt{n_1n_3}+n_2)[\lambda_1(\sigma)-\lambda_2(\sigma)]}\left\{\frac{i}{2\sqrt{n_3}(\sqrt{n_1}+\sqrt{n_3})}\left[\frac{1}{X_2(z_2)(\sigma-z_2)}\times\right.\right. \qquad \underline{/280}$$
$$\times\left(-\frac{n_2+n_3}{R}P_0+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-\frac{1}{X_2(-z_2)(\sigma+z_2)}\left(\frac{n_2+n_3}{R}P_0+\right.$$
$$\left.\left.+\sqrt{n_3}\sqrt{n_1-n_3}A_1\right)-2(n_2+n_3)\frac{P_0}{R}\right]+\frac{iP_0}{R}+2\pi i\left(\frac{D_2}{\sigma-z_2}+\frac{E_2}{\sigma+z_2}\right)\Big\}. \qquad \text{(III.215)}$$

The constant A_1 found in the solution of the given problem is determined by any of formulas (III.207). By substituting expression (III.215) for $\tau_{r\theta}$ into formulas (III.208), we find four conditions for the determination of the

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constants D_1, E_1, D_2, E_2. It turns out that these conditions are interrelated by one relation I + II + z_2^2(III + IV) = 0, such that there will only be three independent conditions. The fourth incomplete condition is found by comparing the right hand sides of expressions (III.207). We write the final expressions for the stresses in polar coordinates, i.e., related to the variable θ ($r = 1$, $\sigma = e^{i\theta}$):

$$\sigma_r(\theta) = -\frac{P_0(n_3 - n_1)}{8\pi R\sqrt{n_3}\sqrt{n_1 n_3 - n_2^2}\Delta(\theta)}\left\{\frac{\sqrt{n_1 n_3} + n_2}{\sqrt{n_3} - \sqrt{n_1}} e^{\theta_0\delta}\left[\cos\left(\frac{3}{2}\theta - \theta_1\right) + \right.\right.$$
$$\left.\left. + z_2^2\cos\left(\frac{\theta}{2} + \theta_1\right)\right] + \frac{\sqrt{n_1 n_3} - n_2}{\sqrt{n_3} + \sqrt{n_1}} e^{-\theta_0\delta}\left[\cos\left(\frac{3}{2}\theta + \theta_1\right) + z_1^2\cos\left(\frac{\theta}{2} - \theta_1\right)\right]\right\} \qquad \text{(III.216)}$$
$$(-\theta_0 < \theta < \theta_0);$$

$$\tau_{r\theta}(\theta) = \frac{P_0(n_3 - n_1)}{8\pi R\sqrt{n_3}\sqrt{n_1 n_3 - n_2^2}\Delta(\theta)}\left\{\frac{\sqrt{n_1 n_3} + n_2}{\sqrt{n_3} - \sqrt{n_1}} e^{\theta_0\delta}\left[\sin\left(\frac{3}{2}\theta - \theta_1\right) + \right.\right.$$
$$\left.\left. + z_2^2\sin\left(\frac{\theta}{2} + \theta_1\right)\right] + \frac{\sqrt{n_1 n_3} - n_2}{\sqrt{n_3} + \sqrt{n_1}} e^{-\theta_0\delta}\left[\sin\left(\frac{3}{2}\theta + \theta_1\right) + z_1^2\sin\left(\frac{\theta}{2} - \theta_1\right)\right]\right\} \qquad \text{(III.217)}$$
$$(-\theta_0 < \theta < \theta_0).$$

Here

$$\Delta(\theta) = \sqrt{\sin\frac{1}{2}(\theta_0 + \theta)\sin\frac{1}{2}(\theta_0 - \theta)},$$
$$\theta_1 = \delta\ln\frac{\sin\frac{1}{2}(\theta_0 - \theta)}{\sin\frac{1}{2}(\theta_0 + \theta)}; \qquad \text{(III.218)}$$

z_1 and z_2 are found from formulas (III.189), and δ, from formula (III.202).

In particular, for an isotropic plate, formulas (III.216) and (III.217) are converted to the form

$$\sigma_r(\theta) = -\frac{P_0\sqrt{\varkappa}}{4\pi R\Delta(\theta)}\left[e^{-\frac{\theta_0}{2\pi}\ln\varkappa}\cos\left(\frac{3}{2}\theta - \theta_1\right) + \frac{1}{\varkappa}e^{\frac{\theta_0}{2\pi}\ln\varkappa}\cos\left(\frac{1}{2}\theta - \theta_1\right)\right],$$
$$(-\theta_0 < \theta < \theta_0)$$
$$\tau_{r\theta}(\theta) = \frac{P_0\sqrt{\varkappa}}{4\pi R\Delta(\theta)}\left[e^{-\frac{\theta_0}{2\pi}\ln\varkappa}\sin\left(\frac{3}{2}\theta - \theta_1\right) + \frac{1}{\varkappa}e^{\frac{\theta_0}{2\pi}\ln\varkappa}\sin\left(\frac{1}{2}\theta - \theta_1\right)\right]$$
$$(-\theta_0 < \theta < \theta_0).$$

Here, by δ, which goes into the expression for θ_1, we mean /281

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$$\delta = -\frac{\ln \varkappa}{2\pi}. \qquad \text{(III.221)}$$

The results of calculations by formulas (III.216) and (III.217) for a plate made of plywood II, where the Ox axis is directed along the grain, are presented in Table III.7. Such a plate, as known[1], has the following elastic properties:

$$a_{11} = \frac{10^{-9}}{1{,}2 \cdot 9{,}81}, \quad a_{22} = \frac{10^{-9}}{0{,}6 \cdot 9{,}81}, \quad a_{12} = -\frac{0{,}071}{1{,}2 \cdot 9{,}81 \cdot 10^{9}},$$

$$\beta_1 = 4{,}11; \quad \beta_2 = 0{,}343.$$

The parameters in (III.185) for the given case are

$$n_1 = 3{,}7107\,\frac{10^{-9}}{9{,}81}, \quad n_2 = 1{,}1147\,\frac{10^{-9}}{9{,}81}, \quad n_3 = 5{,}231\,\frac{10^{-9}}{9{,}81}.$$

For comparison, the values of stresses $\sigma_r(\theta)$ of (III.219) and $\tau_{r\theta}(\theta)$ of (III.220) are also presented in Table III.7 for an isotropic plate with $\varkappa = 2$, in fractions P_0/R for the case $\theta_0 = \pi/4$.

TABLE III.7

	Plywood plate		Isotropic plate	
θ°	σ_r	$\tau_{r\theta}$	σ_r	$\tau_{r\theta}$
0	−0,4237	0	−0,4300	0
5	−0,4248	0,0339	−0,4312	0,0324
10	−0,4281	0,0686	−0,4351	0,0655
15	−0,4345	0,1051	−0,4426	0,1002
20	−0,4454	0,1449	−0,4553	0,1376
25	−0,4643	0,1904	−0,4770	0,1800
30	−0,4981	0,2468	−0,5157	0,2314
35	−0,5662	0,3265	−0,5930	0,3015
40	−0,7482	0,4758	−0,7990	0,4242
43	−1,1574	0,7216	−1,1862	0,6002
44	−1,6476	0,9588	−1,8167	0,7392

Tr. Note: Commas indicate decimal points.

TABLE III.8.

θ°	σ_r	$\tau_{r\theta}$	θ°	σ_r	$\tau_{r\theta}$
0	−0,2283	0,00	60	−0,1526	0,2355
10	−0,2264	0,03856	70	−0,1345	0,2926
20	−0,2193	0,07694	80	−0,1289	0,4076
30	−0,2072	0,11498	85	−0,1659	0,5517
40	−0,1897	0,15305	89	−0,4792	1,1537
50	−0,1732	0,1920			

Tr. Note: Commas indicate decimal points.

The values of contact stresses σ_r of (III.219) and $\tau_{r\theta}$ of (III.220) are presented in Table III.8 (in fractions P_0/R) for an isotropic plate with the contact angle $\theta_0 = \pi/2$.

Comparison of the data in Tables III.7 and III.8 for an isotropic plate reveals a qualitative difference in the character of distribution of normal stresses: for contact angle $\theta_0 = \pi/4$, normal stresses increase gradually from

[1]See S. G. Lekhnitskiy [1].

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the center of the arc of contact to the edge, whereas for contact angle $\theta_0 = \pi/4$, normal stresses decrease in value and only near the very edge do they begin to increase rapidly due to the particular solution, which has a singularity on the ends of the arc of contact. There are no differences in the character of distribution of tangential stresses: in each case they increase from the center of the arc of contact to the edge. Such a character of distribution of normal and tangential stresses for the case $\theta_0 = \pi/2$ becomes clear considering that as the point approaches the edge of the arc of contact, the elastic material experiences gradually less pressure from the direction of the rigid punch and gradually more pressure due to the effect of the forces of displacement.

Uniaxial Tension of Orthotropic Plate with Hole, to Part of the Edge of Which is Soldered a Rigid Cover Plate. To segment ab of the edge of a round hole of radius R in an orthotropic plate, let a rigid cover plate with the shape of an arc of a hole of the same radius be soldered. Suppose that the plate with the soldered cover plate is subjected to uniform tension at infinity /282 by forces of intensity p, parallel to the Ox axis. We are required to determine the character of stress distribution between the cover plate and the elastic body.

It is obvious that part of the contour of the hole in contact with the cover plate will be displaced as a whole during deformation of the plate. We will examine only such positions of the cover plate as when it is displaced gradually. This will occur at least when the middle of the cover plate is located at the ends of the vertical or horizontal diameters. For instance, we will examine the case where the x axis passes through the center of the cover plate. In this case

$$\begin{gathered} X = Y = 0; \quad q = 0; \quad f_1'(R\sigma) = f_2'(R\sigma) = 0; \\ B_1 = \int_{\gamma_1} \frac{\tau_{r\theta}\, dt}{t} = 0 \text{ due to symmetry of the problem;} \\ a = e^{-i\theta_0}, \quad b = e^{i\theta_0}\ (2\theta_0 < \pi), \quad C_1 = C_2 = 0. \end{gathered} \tag{III.222}$$

Formulas (III.198) are converted to the form

$$\begin{aligned} \Omega_1(\sigma) = \frac{\sqrt{n_1 n_3}}{\pi i\,(\sqrt{n_1 n_3} + n_2)\,(\sqrt{n_1} - \sqrt{n_3})\,(\sigma^2 - z_1^2)} \{\pi i p\,[\sigma^2 (n_5\sqrt{n_3} + n_4\sqrt{n_1}) + \\ + (n_5\sqrt{n_3} - n_4\sqrt{n_1})] + (\sqrt{n_1} + \sqrt{n_3})\,A_1\}; \end{aligned} \tag{III.223}$$

$$\begin{aligned} \Omega_2(\sigma) = \frac{\sqrt{n_1 n_3}}{\pi i\,(\sqrt{n_1 n_3} - n_2)\,(\sqrt{n_1} + \sqrt{n_3})\,(\sigma^2 - z_2^2)} \{\pi i p\,[\sigma^2 (n_4\sqrt{n_1} - n_5\sqrt{n_3}) - \\ - (n_5\sqrt{n_3} + n_4\sqrt{n_1})] + (\sqrt{n_1} - \sqrt{n_3})\,A_1\}. \end{aligned}$$

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By substituting into formulas (III.199) and (III.200), instead of $\Omega_1(t)$ and $\Omega_2(t)$, their values from (III.223), we obtain the functions

$$W_1(\zeta) = \frac{1}{2\pi i(\sqrt{n_1}-\sqrt{n_3})(\zeta^2-z_1^2)}\{\pi ip[\zeta^2(n_5\sqrt{n_3}+n_4\sqrt{n_1})+(n_5\sqrt{n_3}-n_4\sqrt{n_1})]+$$
$$+(\sqrt{n_1}+\sqrt{n_3})A_1\}+\frac{X_1(\zeta)}{2\pi i}\left\{\frac{1}{X_1(z_1)(\sqrt{n_1}-\sqrt{n_3})^2(\zeta-z_1)}\left[\pi ip(n_1n_4+n_3n_5)-\right.\right.$$
$$\left.-\frac{1}{2}(n_1-n_3)A_1\right]-\frac{1}{X_1(-z_1)(\sqrt{n_1}-\sqrt{n_3})^2(\zeta+z_1)}\left[\pi ip(n_1n_4+n_3n_5)-\right.$$
$$\left.\left.-\frac{1}{2}(n_1-n_3)A_1\right]-\frac{\pi ip}{\sqrt{n_1}-\sqrt{n_3}}(n_5\sqrt{n_3}+n_4\sqrt{n_1})(\zeta+c)\right\}+X_1(\zeta)\left(\frac{D_1}{\zeta-z_1}+\frac{E_1}{\zeta+z_1}\right.$$
(III.224)

$$W_2(\zeta) = \frac{1}{2\pi i(\sqrt{n_1}+\sqrt{n_3})(\zeta^2-z_2^2)}\{\pi ip[\zeta^2(n_4\sqrt{n_1}-n_5\sqrt{n_3})-$$
$$-(n_5\sqrt{n_3}+n_4\sqrt{n_1})]+(\sqrt{n_1}-\sqrt{n_3})A_1\}+\frac{X_2(\zeta)}{2\pi i}\left\{\frac{1}{X_2(z_2)(\sqrt{n_1}+\sqrt{n_3})^2(\zeta-z_2)}\times\right.$$
$$\left.\times\left[\pi ip(n_1n_4+n_3n_5)-\frac{1}{2}(n_1-n_3)A_1\right]-\right.$$
$$-\frac{1}{X_2(-z_2)(\sqrt{n_1}+\sqrt{n_3})^2(\zeta+z_2)}\left[\pi ip(n_1n_4+n_3n_5)-\frac{1}{2}(n_1-n_3)A_1\right]-$$
$$\left.-\frac{\pi ip}{\sqrt{n_1}+\sqrt{n_3}}(n_4\sqrt{n_1}-n_5\sqrt{n_3})(\zeta+d)\right\}+X_2(\zeta)\left(\frac{D_2}{\zeta-z_2}+\frac{E_2}{\zeta+z_2}\right).$$
(III.225) /283

Here

$$c = -(\cos\theta_0 - 2\delta\sin\theta_0),$$
$$d = -(\cos\theta_0 + 2\delta\sin\theta_0).$$
(III.226)

The forces of tension along the contour of the weld are defined by formulas (III.203) by substituting there the values found for W_1 from (III.224) and W_2 from (III.225):

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$$\sigma_r = -\frac{\sqrt{n_1 n_3}\lambda_2(\sigma) X_1(\sigma)}{\pi i(\sqrt{n_1 n_3} - n_2)[\lambda_1(\sigma) - \lambda_2(\sigma)]}\left\{\frac{1}{X_1(z_1)(\sqrt{n_1} - \sqrt{n_3})^2(\sigma - z_1)}\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \frac{1}{X_1(-z_1)(\sqrt{n_1} - \sqrt{n_3})^2(\sigma + z_1)} \times \right.$$
$$\times\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \frac{\pi ip}{\sqrt{n_1} - \sqrt{n_3}}(n_5\sqrt{n_3} + n_4\sqrt{n_1})(\sigma + c) +$$
$$\left. + 2\pi i\left(\frac{D_1}{\sigma - z_1} + \frac{E_1}{\sigma + z_2}\right)\right\} + \frac{\sqrt{n_1 n_3}\lambda_1(\sigma) X_2(\sigma)}{\pi i(\sqrt{n_1 n_3} + n_2)[\lambda_1(\sigma) - \lambda_2(\sigma)]} \times$$
$$\times\left\{\frac{1}{X_2(z_2)(\sqrt{n_1} + \sqrt{n_3})^2(\sigma - z_2)}\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \right.$$
$$- \frac{1}{X_2(-z_2)(\sqrt{n_1} + \sqrt{n_3})^2(\sigma + z_2)}\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] -$$
$$\left. - \frac{\pi ip}{\sqrt{n_1} + \sqrt{n_3}}(n_4\sqrt{n_1} - n_5\sqrt{n_3})(\sigma + d) + 2\pi i\left(\frac{D_2}{\sigma - z_2} + \frac{E_2}{\sigma + z_2}\right)\right\};$$
$$\tau_{r\theta} = \frac{\sqrt{n_1 n_3} X_1(\sigma)}{\pi i(\sqrt{n_1 n_3} - n_2)[\lambda_1(\sigma) - \lambda_2(\sigma)]}\left\{\frac{1}{X_1(z_1)(\sqrt{n_1} - \sqrt{n_3})^2(\sigma - z_1)} \times \right. \qquad \text{(III.227)}$$
$$\times\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \frac{1}{X_1(-z_1)(\sqrt{n_1} - \sqrt{n_3})^2(\sigma + z_1)} \times$$
$$\times\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \frac{\pi ip}{\sqrt{n_1} - \sqrt{n_3}}(n_4\sqrt{n_1} + n_5\sqrt{n_3})(\sigma + c) +$$
$$\left. + 2\pi i\left(\frac{D_1}{\sigma - z_1} + \frac{E_1}{\sigma + z_1}\right)\right\} - \frac{\sqrt{n_1 n_3} X_2(\sigma)}{\pi i(\sqrt{n_1 n_3} + n_2)[\lambda_1(\sigma) - \lambda_2(\sigma)]} \times$$
$$\times\left\{\frac{1}{X_2(z_2)(\sqrt{n_1} + \sqrt{n_3})^2(\sigma - z_2)} \times \left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] - \right.$$
$$- \frac{1}{X_2(-z_2)(\sqrt{n_1} + \sqrt{n_3})^2(\sigma + z_2)}\left[\pi ip(n_1 n_4 + n_3 n_5) - \frac{1}{2}(n_1 - n_3)A_1\right] -$$
$$\left. - \frac{\pi ip}{\sqrt{n_1} + \sqrt{n_3}}(n_4\sqrt{n_1} - n_5\sqrt{n_3})(\sigma + d) + 2\pi i\left(\frac{D_2}{\sigma - z_2} + \frac{E_2}{\sigma + z_2}\right)\right\}. \qquad \text{(III.228)}$$

The constants in the solution of the given problem are determined in the manner analogous to that described in the preceding case, and therefore we will introduce only the final formulas for the stresses, written in the variable θ: /284

$$\sigma_r(\theta) = \frac{p(n_3 - n_1)}{8\sqrt{n_1 n_3 - n_2^2}\,\Delta(\theta)}\left\{\frac{n_5\sqrt{n_3} + \sqrt{n_1}n_4}{\sqrt{n_3} - \sqrt{n_1}} e^{\theta_0\delta}\left[z_2^2\cos\left(\frac{\theta}{2} - \theta_1\right) + \right.\right.$$
$$+ cz_2^2\cos\left(\frac{\theta}{2} + \theta_1\right) + c\cos\left(\frac{3}{2}\theta - \theta_1\right) + \cos\left(\frac{5}{2}\theta - \theta_1\right)\Big] + \qquad \text{(III.229)}$$
$$+ \frac{n_4\sqrt{n_1} - n_5\sqrt{n_3}}{\sqrt{n_1} + \sqrt{n_3}} e^{-\theta_0\delta}\left[z_1^2\cos\left(\frac{\theta}{2} + \theta_1\right) + dz_1^2\cos\left(\frac{\theta}{2} - \theta_1\right) + \right.$$
$$\left.\left. + d\cos\left(\frac{3}{2}\theta + \theta_1\right) + \cos\left(\frac{5}{2}\theta + \theta_1\right)\right]\right\} \qquad (-\theta_0 < \theta < \theta_0);$$

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$$\tau_{r\theta}(\theta)=\frac{p(n_3-n_1)}{8\sqrt{n_1n_3-n_2^2\Delta(\theta)}}\left\{\frac{n_5\sqrt{n_3}+n_4\sqrt{n_1}}{\sqrt{n_3}-\sqrt{n_1}}e^{\theta_0\delta}\left[z_2^2\sin\left(\frac{\theta}{2}-\theta_1\right)-\right.\right.$$
$$-cz_2^2\sin\left(\frac{\theta}{2}+\theta_1\right)-c\sin\left(\frac{3}{2}\theta-\theta_1\right)-\sin\left(\frac{5}{2}\theta-\theta_1\right)\Big]-$$
$$-\frac{n_4\sqrt{n_1}-n_5\sqrt{n_3}}{\sqrt{n_1}+\sqrt{n_3}}e^{-\theta_0\delta}\left[-z_1^2\sin\left(\frac{\theta}{2}+\theta_1\right)+dz_1^2\sin\left(\frac{\theta}{2}-\theta_1\right)+\right. \quad \text{(III.230)}$$
$$\left.\left.+d\sin\left(\frac{3}{2}\theta+\theta_1\right)+\sin\left(\frac{5}{2}\theta+\theta_1\right)\right]\right\} \qquad (-\theta_0<\theta<\theta_0).$$

For an isotropic plate, formulas (III.229) and (III.230) acquire the form

$$\sigma_r(\theta)=\frac{p}{2\sqrt{(1+\nu)(3-\nu)\Delta(\theta)}}\left\{2e^{-\frac{\theta_0}{2\pi}\ln\varkappa}\left[\cos\left(\frac{5}{2}\theta-\theta_1\right)-\right.\right.$$
$$-\left(\cos\theta_0+\frac{\ln\varkappa}{\pi}\sin\theta_0\right)\cos\left(\frac{3}{2}\theta-\theta_1\right)+e^{\frac{\theta_0}{2\pi}\ln\varkappa}\times$$
$$\left.\left.\times\left[\cos\left(\frac{\theta}{2}+\theta_1\right)-\left(\cos\theta_0-\frac{\ln\varkappa}{\pi}\sin\theta_0\right)\cos\left(\frac{\theta}{2}-\theta_1\right)\right]\right\} \quad \text{(III.231)}$$
$$(-\theta_0<\theta<\theta_0);$$
$$\tau_{r\theta}(\theta)=-\frac{p}{2\sqrt{(1+\nu)(3-\nu)\Delta(\theta)}}\left\{2e^{-\frac{\theta_0}{2\pi}\ln\varkappa}\left[\sin\left(\frac{5}{2}\theta-\theta_1\right)-\right.\right.$$
$$-\left(\cos\theta_0+\frac{\ln\varkappa}{\pi}\sin\theta_0\right)\sin\left(\frac{3}{2}\theta-\theta_1\right)-e^{\frac{\theta_0}{2\pi}\ln\varkappa}\times$$
$$\left.\left.\times\left[\sin\left(\frac{\theta}{2}+\theta_1\right)+\left(\cos\theta_0-\frac{\ln\varkappa}{\pi}\sin\theta_0\right)\sin\left(\frac{\theta}{2}-\theta_1\right)\right]\right\} \quad \text{(III.232)}$$
$$(-\theta_0<\theta<\theta_0).$$

Here δ is defined by formula (III.221).

5. Pressure of a Rigid Disc on Edge of Round Hole in Orthotropic Plate /285

Statement of Problem and Derivation of Integral Fredholm's Equation. We will analyze the problem of compression of two bodies[1], one of which represents an unbounded homogeneous orthotropic plate with a round hole of radius R_1, and the other, a rigid round disc of radius R_2. Let R_1 differ little from R_2, such that the difference $(R_1 - R_2) = \varepsilon$ represents a magnitude of the order of the

[1]This problem was analyzed by D. V. Grilitskiy [3-5]. Contact stresses for the case of isotropic compressed bodies are analyzed in the works of I. Ya. Shtayerman [1, 2], M. Z. Narodetskiy [1], M. P. Sheremet'yev [2-4], V. V. Panasyuk [1, 2], A. I. Kalandiya [1, 2] and B. L. Ramalis [1, 2].

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elastic displacements. We will assume that the force P that presses one body toward the other acts along the diameter of the disc and is directed along one of the lines of intersection of the planes of elastic symmetry of the material from which the plate is made. Moreover, we will assume that there are no forces of friction acting between the bodies. We are required to determine the size of the area of contact and distribution of pressure upon it.

To solve this problem we will use a rectangular Cartesian coordinate system xOy with the origin at the center of the hole, and the coordinate axes directed along the principal directions of elasticity such that the Ox axis coincides with the direction of compressive force P.

The equations of contour of the rigid disc prior to deformation and after deformation are, respectively,

$$[x-(R_1-R_2)]^2+y^2=R_2^2, \qquad [x-(R_1-R_2)-d]^2+y^2=R_2^2. \tag{III.233}$$

where d is displacement of the center of the disc due to deformation.

The contour of the hole in the elastic plane prior to deformation is determined by the equation

$$x^2+y^2=R_1^2; \tag{III.234}$$

after deformation the equation of contour can be represented in the form

$$\xi=x+u, \qquad \eta=y+v, \tag{III.235}$$

where u and v are elastic displacements of the points of contour of the hole of the plate due to deformation.

The coordinates ξ and η should satisfy the second equation of (III.233) on the arc of contact with the plate:

$$[\xi-(R_1-R_2)-d]^2+\eta^2=R_2^2. \tag{III.236}$$

By substituting into (III.236), instead of ξ and η, their values from (III.235), and disregarding the values of the second order of smallness in comparison with u and v, we find the condition which must be satisfied for the points of the area of contact:

$$v_r=u\cos\theta+v\sin\theta=d\cos\theta-\varepsilon(1-\cos\theta). \tag{III.237}$$

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The radial displacement v_r is expressed with the aid of two functions of /286
complex variable, through the first formula of (I.94), on the basis of which the boundary condition of problem (III.237) acquires the form

$$v_r = 2\mathrm{Re}\left[(p_1\cos\theta + q_1\sin\theta)\,\varphi(t_1) + (p_2\cos\theta + q_2\sin\theta)\,\psi(t_2)\right] = \\ = d\cos\theta - \varepsilon(1-\cos\theta). \tag{III.238}$$

Earlier, by formula (III.78), we established the form of the stress functions for the case where a concentrated force $\overline{P}(x, y)$ is applied to the point σ_0 of the contour of the hole:

$$\varphi(z_1) = (A_2 + B_2)\ln\zeta_1 + (A^{(2)} - A_2 - B_2)\ln(\zeta_1 - \sigma_0),$$
$$\psi(z_2) = -(A_1 + B_1)\ln\zeta_2 + (B^{(2)} + A_1 + B_1)\ln(\zeta_2 - \sigma_0), \tag{III.239}$$

where A_k and B_k (k = 1, 2) are expressed on the basis of formulas (III.74); $A^{(2)}$ and $B^{(2)}$ are determined according to formulas (I.127), and ζ_1 and ζ_2, by formulas (III.12) and (III.13) for $a = b = R_1$.

When force P is applied normally to the contour of the hole at angle α to the Ox axis, the determination of $A^{(2)}$ and $B^{(2)}$ requires the use of expression (I.128). In this case formulas (III.239) acquire the form

$$\varphi(z_1) = A^{(2)}\ln\zeta_1 + \frac{P(\cos\alpha + i\beta_2\sin\alpha)}{2\pi(\beta_1 - \beta_2)}\ln\frac{\zeta_1 - \sigma_0}{\zeta_1},$$
$$\psi(z_2) = B^{(2)}\ln\zeta_2 - \frac{P(\cos\alpha + i\beta_1\sin\alpha)}{2\pi(\beta_1 - \beta_2)}\ln\frac{\zeta_2 - \sigma_0}{\zeta_2}. \tag{III.240}$$

If, into the left hand side of the boundary condition (III.238), we substitute formulas (III.240) and separate the real part, we find an expression for radial displacement of the point of the contour of the round hole defined by the coordinate θ, due to the effect of the normal concentrated force applied to the edge of the hole at the point defined by the coordinate α:

$$v_r = \frac{P}{\pi}\left[\frac{n_2}{2}(|\theta-\alpha| - \pi)\sin|\theta-\alpha| - (n_1\cos\theta\cos\alpha + \right. \\ \left. + n_3\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}\right]. \tag{III.241}$$

Here, the symbols used in formula (III.185) are taken for the constants n_k (k = 1, 2, 3).

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In the following, we will make no distinction between the radii of the hole R_1 and disc R_2 so that we can use $R_1 = R_2 = R$, preserving, however, the difference $\varepsilon = R_1 - R_2$ in boundary condition (III.238).

If $p(\theta)$ is normal pressure on the arc of contact, then upon element $Rd\alpha$ of the arc will act a force $p(\alpha)Rd\alpha$. We will use this force as a concentrated force applied on element $Rd\alpha$ of the arc. This force, at the point defined by coordinate θ, will cause radial displacement. /287

$$dv_r = \frac{p(\alpha)R}{\pi}\left[\frac{n_3}{2}(|\theta-\alpha|-\pi)\sin|\theta-\alpha| - (n_1\cos\theta\cos\alpha + n_3\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}\right]d\alpha. \tag{III.242}$$

If to the area of contact there corresponds a change in angle θ within the range $-\theta_0$ to $+\theta_0$, then the total radial displacement of the point is

$$v_r = \frac{R}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)\left[\frac{n_3}{2}(|\theta-\alpha|-\pi)\sin|\theta-\alpha| - (n_1\cos\theta\cos\alpha + n_3\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}\right]d\alpha. \tag{III.243}$$

By substituting (III.243) into boundary condition (III.238), we find an integral Fredholm's equation of the first kind:

$$-\frac{R}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)(n_1\cos\theta\cos\alpha + n_3\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}\,d\alpha + \frac{Rn_3}{2\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)(|\theta-\alpha|-\pi)\sin|\theta-\alpha|\,d\alpha = d\cos\theta - \varepsilon(1-\cos\theta) \tag{III.244}$$

$$(-\theta_0 < \theta < \theta_0),$$

which, with the condition

$$P = R\int_{-\theta_0}^{\theta_0} p(\alpha)\cos\alpha\,d\alpha \tag{III.245}$$

determines pressure $p(\theta)$ and the area of contact.

For an isotropic plate ($n_1 = n_3$) equation (III.244) acquires the form

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$$-\frac{Rn_1}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)\cos(\theta-\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}\,d\alpha + \tag{III.246}$$

$$+\frac{Rn_2}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)(|\theta-\alpha|-\pi)\sin|\theta-\alpha|\,d\alpha = d\cos\theta-\varepsilon(1-\cos\theta)$$

$$(-\theta_0<\theta<\theta_0),$$

where

$$n_1=\frac{2}{E};\qquad n_2=\frac{1-\nu}{E}; \tag{III.247}$$

E is Young's modulus; ν is Poisson's ratio.

Integral equation (III.246), for $\varepsilon = 0$, is equivalent to an integral-differential equation of the Prandtl type, and one can be transformed into the other[1]. /288

In view of the great complexity of integral equation (III.244), its precise solution cannot be assured at any given moment. Therefore, we give below the approximate solution of the above equation, reducing it to the solution of a linear system of algebraic equations that is convenient for use by modern computer technology. In doing this we will use, as did I. Ya. Shtayerman [2], the method of finite differences, which boils down to the fact that the range of change of the desired function (area of contact) is broken down into n equal parts, and in each part obtained, the desired function is assumed to be constant. In solving the solution in this manner, we will replace the continuous function $p(\theta)$ by a piece-wise continuous (piece-wise constant) function that changes in jumps during transition from one section to the next.

Approximate Solution of Integral Equation (III.244). From equation (III.244) we will exclude the known constant d. For this purpose we will assume that $\theta = 0$. Consequently, we have

$$d=-\frac{Rn_1}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)\cos\alpha\ln 2\sin\frac{|\alpha|}{2}\,d\alpha+\frac{Rn_2}{2\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)(|\alpha|-\pi)\sin|\alpha|\,d\alpha. \tag{III.248}$$

After substitution of (III.248) into (III.244), we arrive at an integral equation that contains no d:

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[1]See D. V. Grilitskiy [8].

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$$\frac{R}{\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)\left[(n_1\cos\theta\cos\alpha+n_2\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}-\right.$$

$$\left.-n_1\cos\theta\cos\alpha\ln 2\sin\frac{|\alpha|}{2}\right]d\alpha-\frac{Rn_2}{2\pi}\int_{-\theta_0}^{\theta_0} p(\alpha)[(|\theta-\alpha|-\pi)\sin|\theta-\alpha|- \qquad \text{(III.249)}$$

$$-(|\alpha|-\pi)\cos\theta\sin|\alpha|]\,d\alpha=\varepsilon(1-\cos\theta)$$

$$(-\theta_0<\theta<\theta_0),$$

which, together with condition (III.245), solves the stated problem.

From the condition of evenness of the function $p(\theta)$ in the area of contact, we have the relation

$$\frac{R}{\pi}\int_{-\theta_0}^{0} p(\alpha)\left[(n_1\cos\theta\cos\alpha+n_2\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}-\right.$$

$$\left.-n_1\cos\theta\cos\alpha\ln 2\sin\frac{|\alpha|}{2}\right]d\alpha-\frac{Rn_2}{2\pi}\int_{-\theta_0}^{0} p(\alpha)[(|\theta-\alpha|-\pi)\sin|\theta-\alpha|-(|\alpha|-$$

$$-\pi)\cos\theta\sin|\alpha|]\,d\alpha=\frac{R}{\pi}\int_{0}^{\theta_0} p(\alpha)[(n_1\cos\theta\cos\alpha-n_2\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta+\alpha|}{2}-$$

$$-n_1\cos\theta\cos\alpha\ln 2\sin\frac{\alpha}{2}\Big]d\alpha- \qquad \text{/289}$$

$$-\frac{Rn_2}{2\pi}\int_{0}^{\theta_0} p(\alpha)[(|\theta+\alpha|-\pi)\sin|\theta+\alpha|-(\alpha-\pi)\cos\theta\sin\alpha]\,d\alpha,$$

with consideration of which integral equation (III.249) acquires the form

$$\frac{R}{\pi}\int_{0}^{\theta_0} p(\alpha)\left[(n_1\cos\theta\cos\alpha+n_2\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta-\alpha|}{2}+(n_1\cos\theta\cos\alpha-\right.$$

$$\left.-n_2\sin\theta\sin\alpha)\ln 2\sin\frac{|\theta+\alpha|}{2}-2n_1\cos\theta\cos\alpha\ln 2\sin\frac{\alpha}{2}\right]d\alpha-$$

$$-\frac{Rn_2}{2\pi}\int_{0}^{\theta_0} p(\alpha)[(|\theta-\alpha|-\pi)\sin|\theta-\alpha|+(|\theta+\alpha|-\pi)\sin|\theta+\alpha|- \qquad \text{(III.250)}$$

$$-2(\alpha-\pi)\cos\theta\sin\alpha]\,d\alpha=\varepsilon(1-\cos\theta)$$

$$(-\theta_0<\theta<\theta_0).$$

We will divide the interval $(0, \theta_0)$ into n equal parts and assume that in each part obtained the function $p(\theta)$ is constant:

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$$p(\theta) = p_k \quad \text{and} \quad (k-1)\vartheta < \theta < k\vartheta \qquad (k = 1, 2, \ldots, n), \qquad \vartheta = \frac{\theta_0}{n}. \tag{III.251}$$

By substituting $p(\theta)$ from (III.251) into (III.250), assuming $\theta = l\vartheta$ ($l = 1, 2, \ldots, n$), we obtain the following equation system

$$\frac{R}{\pi}\sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} \left[(n_1 \cos l\vartheta \cos\alpha + n_2 \sin l\vartheta \sin\alpha) \ln 2\sin\frac{|l\vartheta-\alpha|}{2} + \right.$$

$$\left. + (n_1 \cos l\vartheta \cos\alpha - n_2 \sin l\vartheta \sin\alpha) \ln 2\sin\frac{|l\vartheta+\alpha|}{2} - 2n_1 \cos l\vartheta \cos\alpha \ln 2\sin\frac{\alpha}{2}\right] d\alpha -$$

$$-\frac{Rn_2}{2\pi}\sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} [(|l\vartheta-\alpha|-\pi)\sin|l\vartheta-\alpha| + (l\vartheta+\alpha-\pi)\sin(l\vartheta+\alpha) - \tag{III.252}$$

$$-2(\alpha-\pi)\cos l\vartheta \sin\alpha]\, d\alpha = \varepsilon(1-\cos l\vartheta) \qquad (l = 1, 2, \ldots n).$$

Omitting the intermediate calculations, we will give the values of the integrals in equation (III.252):

$$\int_{(k-1)\vartheta}^{k\vartheta} (n_1 \cos l\vartheta \cos\alpha + n_2 \sin l\vartheta \sin\alpha) \ln 2\sin\frac{|l\vartheta-\alpha|}{2} d\alpha =$$

$$= [n_2 \sin l\vartheta \cos(k-1)\vartheta - n_1 \cos l\vartheta \sin(k-1)\vartheta] \ln 2\sin\frac{|l-k+1|\vartheta}{2} - \qquad /290$$

$$-(n_2 \sin l\vartheta \cos k\vartheta - n_1 \cos l\vartheta \sin k\vartheta) \ln 2\sin\frac{|l-k|\vartheta}{2} +$$

$$+(n_1-n_2)\sin l\vartheta \cos l\vartheta \left[\ln\sin\frac{|l-k+1|\vartheta}{2} - \ln\sin\frac{|l-k|\vartheta}{2} - \right.$$

$$\left. -\sin\frac{\vartheta}{2}\sin\left(l-k+\frac{1}{2}\right)\vartheta\right] + \frac{1}{2}(n_1\cos^2 l\vartheta + n_2\sin^2 l\vartheta) \times$$

$$\times [\sin(l-k)\vartheta - \sin(l-k+1)\vartheta - \vartheta]; \tag{III.253}$$

$$\int_{(k-1)\vartheta}^{k\vartheta} (n_1 \cos l\vartheta \cos\alpha - n_2 \sin l\vartheta \sin\alpha) \ln 2\sin\frac{(l\vartheta+\alpha)}{2} d\alpha =$$

$$= (n_2 \sin l\vartheta \cos k\vartheta + n_1 \cos l\vartheta \sin k\vartheta) \ln 2\sin\frac{(l+k)\vartheta}{2} -$$

$$-[n_2 \sin l\vartheta \cos(k-1)\vartheta + n_1 \cos l\vartheta \sin(k-1)\vartheta] \ln 2\sin\frac{(l+k-1)\vartheta}{2} +$$

$$+(n_1-n_2)\sin l\vartheta \cos l\vartheta \left[\ln\sin\frac{(l+k)\vartheta}{2} - \ln\sin\frac{(l+k-1)\vartheta}{2} - \right.$$

$$\left. -\sin\frac{\vartheta}{2}\sin\left(l+k-\frac{1}{2}\right)\vartheta\right] - \frac{1}{2}(n_1\cos^2 l\vartheta + n_2\sin^2 l\vartheta) \times \tag{III.254}$$

$$\times [\sin(l+k)\vartheta - \sin(l+k-1)\vartheta + \vartheta];$$

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$$\int_{(k-1)\vartheta}^{k\vartheta} \cos\alpha \ln 2\sin\frac{\alpha}{2}\,d\alpha = \sin k\vartheta \ln 2\sin\frac{k\vartheta}{2} - \sin(k-1)\vartheta \ln 2\sin\frac{(k-1)\vartheta}{2} - \\ - \frac{1}{2}[\sin k\vartheta - \sin(k-1)\vartheta] - \frac{\vartheta}{2}; \tag{III.255}$$

$$\int_{(k-1)\vartheta}^{k\vartheta} [(|l\vartheta-\alpha|-\pi)\sin|l\vartheta-\alpha| + (l\vartheta+\alpha-\pi)\sin(l\vartheta+\alpha) - \\ - 2(\alpha-\pi)\cos l\vartheta\sin\alpha]\,d\alpha = \tau(l-k)[(|l-k|\vartheta-\pi)\cos(l-k)\vartheta - \\ - (|l-k+1|\vartheta-\pi)\cos(l-k+1)\vartheta] + [(l+k-1)\vartheta-\pi]\cos(l+k-1)\vartheta - \\ - [(l+k)\vartheta-\pi]\cos(l+k)\vartheta + \sin(l-k+1)\vartheta - \sin(l-k)\vartheta + \sin(l+k)\vartheta - \\ - \sin(l+k-1)\vartheta - 2\cos l\vartheta\{[(k-1)\vartheta-\pi]\cos(k-1)\vartheta - (k\vartheta-\pi)\cos k\vartheta + \\ + \sin k\vartheta - \sin(k-1)\vartheta\}, \tag{III.256}$$

where

$$\tau(l-k) = 1 \quad \text{for} \quad (l-k) \geqslant 0, \\ \tau(l-k) = -1 \quad \text{for} \quad (l-k) \leqslant -1. \tag{III.257}$$

By substituting (III.253)-(III.256) into (III.252), introducing the definitions

$$\Delta_{k,l} = \{n_1\sin(k+1)\vartheta - (n_1-n_3)\sin l\vartheta[\cos(l-k-1)\vartheta - \cos l\vartheta]\} \times \\ \times \ln 2\sin\frac{|k+1|\vartheta}{2} - \{n_1\sin k\vartheta - (n_1-n_3)\sin l\vartheta[\cos(l-k)\vartheta - \\ - \cos l\vartheta]\}\ln 2\sin\frac{|k|\vartheta}{2} - \frac{\vartheta}{2}(n_1\cos^2 l\vartheta + n_3\sin^2 l\vartheta) + n_2\tau(k)\times \\ \times\left[\left(\frac{|k+1|\vartheta}{2} - \frac{\pi}{2}\right)\cos(k+1)\vartheta - \left(\frac{|k|\vartheta}{2} - \frac{\pi}{2}\right)\cos k\vartheta\right]; \tag{III.258}$$

$$\Delta_{k,l} = (n_1-n_3)\sin 2l\vartheta\left\{[\cos(l-k+1)\vartheta - \cos l\vartheta]\ln 2\sin\frac{(k-1)\vartheta}{2} - \right. \\ \left. - [\cos(l-k)\vartheta - \cos l\vartheta]\ln 2\sin\frac{k\vartheta}{2} + \frac{\vartheta}{2}\sin l\vartheta\right\} \qquad k \geqslant 1, \quad l \geqslant 1$$

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recalling that

$$\tau(l+k-1) = \tau(k-1) = 1 \quad \text{for} \quad k \geqslant 1; \quad l \geqslant 0,$$

as follows from (III.257), we find the basic system of equations:

$$\sum_{k=1}^{n} p_k[\Delta_{l-k,l} + \Delta_{l+k-1,l} - 2\cos l\vartheta\Delta_{k-1,l} + \Delta_{k,l}] = \frac{\pi\varepsilon}{R}(1-\cos l\vartheta) \\ (l = 1, 2, \ldots, n). \tag{III.259}$$

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We will simplify this system. When $k \geqslant 0$, $l \geqslant 0$, we have from (III.258)

$$\Delta_{k,l} = \{n_1 \sin (k+1)\,\vartheta - (n_1 - n_3) \sin l\vartheta\,[\cos[l-k-1)\,\vartheta - \cos l\vartheta]\} \times$$
$$\times \ln 2 \sin \frac{(k+1)\,\vartheta}{2} - \{n_1 \sin k\vartheta - (n_1 - n_3) \sin l\vartheta[\cos(l-k)\,\vartheta - \cos l\vartheta]\} \times$$
$$\times \ln 2 \sin \frac{k\vartheta}{2} - \frac{\vartheta}{2}(n_1 \cos^2 l\vartheta + n_3 \sin^2 l\vartheta) + n_2\left[\left(\frac{(k+1)\,\vartheta}{2} - \frac{\pi}{2}\right)\cos(k+1)\,\vartheta - \right. \tag{III.260}$$
$$\left. - \left(\frac{k\vartheta}{2} - \frac{\pi}{2}\right)\cos k\vartheta\right].$$

From (III.258) we also have

$$\Delta_{-k,l} = \Delta_{k-1,l} = 2(n_1 - n_3) \sin 2l\vartheta \left[\sin^2 \frac{(k-1)\,\vartheta}{2} \ln 2 \sin \frac{(k-1)\,\vartheta}{2} - \right.$$
$$\left. - \sin^2 \frac{k\vartheta}{2} \ln 2 \sin \frac{k\vartheta}{2}\right] \text{ for } k \geqslant 1; \quad l \geqslant 0. \tag{III.261}$$

If we introduce the definition

$$F(k,l) = n_1\left(\sin k\vartheta \ln 2 \sin \frac{k\vartheta}{2} - \frac{k\vartheta}{2}\right) + n_2\left(\frac{k\vartheta}{2} - \frac{\pi}{2}\right)\cos k\vartheta +$$
$$+ (n_1 - n_3) \sin l\vartheta \left\{k \frac{\vartheta}{2} \sin l\vartheta - [\cos(l-k)\,\vartheta - \cos l\vartheta] \ln 2 \sin \frac{k\vartheta}{2}\right\} \tag{III.262}$$
$$\text{for } k \geqslant 0; \quad l \geqslant 0,$$

then (III.260) can be represented in the form

$$\Delta_{k,l} = F(k+1, l) - F(k, l) \quad \text{for} \quad k \geqslant 0; \quad l \geqslant 0.$$

The function (III.262) is represented in the form /292

$$F(k, l) = -\frac{1}{2} n_1 f(k, l) \quad \text{for} \quad k \geqslant 0; \quad l \geqslant 0,$$

where

$$f(k,l) = \varphi_*(k) - \psi_*(k, l) \quad \text{for} \quad k \geqslant 0; \quad l \geqslant 0;$$
$$\varphi_*(k) = C_1(\pi - k\vartheta)\cos k\vartheta + k\vartheta - 2 \sin k\vartheta \ln 2 \sin \frac{k\vartheta}{2}$$
$$\text{for } k \geqslant 0;$$

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If we assume that

$$\Delta_{k,l} = -\frac{1}{2} n_1 \delta_{k,l},$$

we will have

$$\delta_{k,l} = f(k+1, l) - f_1(k, l) \quad \text{for} \quad k \geqslant 0; \quad l \geqslant 0;$$

$$\delta_{-k,l} = \delta_{k-1,l} + 4C_2 \sin 2l\vartheta \left[\sin^2 \frac{k\vartheta}{2} \ln 2\sin \frac{k\vartheta}{2} - \sin^2 \frac{(k-1)\vartheta}{2} \ln 2\sin \frac{(k-1)\vartheta}{2} \right]$$

for $k \geqslant 1$; $l \geqslant 0$.

Finally, if we introduce the definition

$$\Omega_k = \varphi_0(k+1) - \varphi_0(k) \quad \text{for} \quad k \geqslant 0,$$

then equation system (III.259) can be represented in final form:

$$\sum_{k=1}^{n} p_k (2\cos l\vartheta \Omega_{k-1} - \delta_{l-k,l} - \delta_{l+k-1,l}) = \frac{2\pi\varepsilon}{n_1 R} (1 - \cos l\vartheta) \qquad (l = 1, 2, \ldots, n). \qquad \text{(III.263)}$$

For an isotropic plate, $n = n_3$, and consequently, $C_2 = 0$. The relationship between compressive force P and the contact angle θ_0 is given by formula (III.245), which, assuming $p(\theta) = p_k$, when $(k-1)\vartheta < \theta < k\vartheta$ $(k = 1, 2, \ldots, n)$, can be reduced to the form

$$2R \sum_{k=1}^{n} p_k [\sin k\vartheta - \sin (k-1)\vartheta] = P. \qquad \text{(III.264)}$$

Equations (III.263) and formula (III.264) can be used for determining pressure in the are of contact (the unknown $p_1, p_2, \ldots, p_n$) and the relationship between angle θ_0 and force P.

The solutions of equation system (III.263) as obtained with the Ural-1 computer are presented below for n = 10 for five angles θ_0: 30, 40, 50, 60 and 70°. The calculations were made for an orthotropic plate[1] made of plywood I, and also for an orthotropic plate for $\nu = 0.3$ and $E = 2.1 \cdot 9.81 \cdot 10^{10}$.

Tables III.9-III.11 show the effect of anisotropy of an elastic material on the size and character of distribution of contact stresses between a disc

[1] See S. G. Lekhnitskiy [1].

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and a plate, as well as the relationship between the contact angle and acting force. The pressure is given in fractions of $9.81 \cdot 10^9$ ε/R n/m^2; the force in fractions of $9.81 \cdot 10^9$ ε n/m^2. Tables III.9 and III.10 also include data concerning distribution of pressure in plywood I, where the x axis is directed, respectively, with and against the grain for $C_1 = 0.7118$, $C_2 = -2.4496$, $n_1 = 3.0001\ 10^{-9}/9.81$ (Table III.9) and $C_1 = 0.2037$, $C_2 = 0.7152$, $n_1 = 10.3714 \times 10^{-9}/9.81$ (Table III.10). Table III.11 includes the results of calculations for an isotropic plate when $C_1 = 0.3500$, $C_2 = 0$, $n_1 = 0.09524\ 10^{-9}/9.81$. /293

TABLE III.9

k	θ°				
	30	40	50	60	70
1	0,2056	0,3389	0,5596	1,0359	2,3862
2	0,2004	0,3261	0,5296	0,9604	2,1594
3	0,1905	0,3026	0,4768	0,8356	1,8080
4	0,1766	0,2717	0,4123	0,6943	1,4422
5	0,1597	0,2368	0,3453	0,5598	1,1222
6	0,1406	0,2006	0,2817	0,4426	0,8633
7	0,1199	0,1650	0,2240	0,3438	0,6585
8	0,09805	0,1305	0,1711	0,2604	0,4937
9	0,07443	0,09625	0,1292	0,1865	0,3527
10	0,04662	0,05888	0,06698	0,1123	0,2133
P	0,1434	0,2825	0,5194	1,0338	2,4909

Tr. Note: Commas indicate decimal points.

TABLE III.10

k	θ°				
	30	40	50	60	70
1	0,05661	0,08574	0,1311	0,2159	0,4341
2	0,05636	0,08567	0,1315	0,2176	0,4398
3	0,05583	0,08549	0,1323	0,2211	0,4513
4	0,05497	0,08508	0,1334	0,2260	0,4685
5	0,05366	0,08428	0,1344	0,2322	0,4914
6	0,05175	0,08278	0,1349	0,2388	0,5186
7	0,04895	0,08006	0,1340	0,2442	0,5468
8	0,04479	0,07521	0,1299	0,2448	0,5671
9	0,03834	0,06636	0,1188	0,2324	0,5570
10	0,02721	0,04876	0,09084	0,1850	0,4554
P	0,04921	0,1012	0,1966	0,3919	0,8899

Tr. Note: Commas indicate decimal points.

The relationship between the applied force P and angle θ_0 for known pressure p_k is given by formula (III.264).

By analyzing the data in Tables III.9 and III.11, we see that the contact stresses, after reaching maximum values at the center of the area of contact, gradually diminish toward the edges. At the edges of the area of contact, due to discontinuity, they acquire zero values. For only one of the given cases (Table III.10) is the character of distribution of the contact stresses somewhat different: for contact angles 100°, 120° and 140°, the points at which the pressure is maximum are displaced closer to the edge. For the given anisotropic materials, the compressive force that corresponds to a given angle of rotation is only one-tenth as great as that for an isotropic plate. It can be reasoned that the contact stresses are limited to about the same quantitative ratio. Consequently, the anisotropy of the body that encompasses the disc has a considerable effect on both the magnitude and the character of distribution of pressure in the contact area. /294

The approximate value of the desired function is given in Tables III.9-III.11 in the form

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TABLE III.11

k	θ°				
	30	40	50	60	70
1	6,2236	9,7747	15,7380	27,9147	64,3923
2	6,1666	9,6835	15,5849	27,6205	63,6321
3	6,0507	9,4989	15,2756	27,0268	62,1056
4	5,8718	9,2152	14,8000	26,1215	59,7958
5	5,6232	8,8213	14,1441	24,8825	56,6719
6	5,2939	8,3000	13,2819	23,2735	52,6739
7	4,8651	7,6238	12,1714	21,2289	47,6897
8	4,3044	6,7425	10,7373	18,6283	41,4920
9	3,5456	5,5524	8,8187	15,2134	33,5666
10	2,4116	3,7766	5,9842	10,2635	22,4179
P	5,0854	10,3375	19,9569	40,2305	101,2639

Tr. Note: Commas indicate decimal points.

of a piece-wise constant function that changes in jumps at the points of separation of the range of integration. The data in these tables make it possible to construct step-by-step pressure graphs for the five respective angles θ_0. By smoothing off the jumps we can obtain a smooth curve that expresses the approximate solution of integral equation (III.244). By constructing the graph for σ_r in this manner, we can represent the character of pressure distribution for the case where the x axis is directed along and against the grain. For comparison (by the data in Table III.11) it is interesting to construct the corresponding curves that characterize the pressure distribution for an isotropic plate. By using formula (III.264) it is possible to construct the curves that express the relationship between angle θ_0 and the ratio P/ε. These curves make it possible to determine angle θ_0, i.e., the sizes of the area of contact, on the basis of the given difference in radii ε and amount of force P applied.

§6. Solution of First Basic Problem for Area with Nearly Elliptical Hole[1]

We will examine an infinite anisotropic plate with a hole, the contour of which is nearly elliptical. The equation of this contour is taken in the form

$$x = R\left[\cos\vartheta + \varepsilon\sum_{k=2}^{N}(\alpha_k\cos k\vartheta - \beta_k\sin k\vartheta)\right],$$
$$y = R\left[-c\sin\vartheta + \varepsilon\sum_{k=2}^{N}(\alpha_k\sin k\vartheta + \beta_k\cos k\vartheta)\right]. \qquad \text{(III.265)}$$

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Here x and y are rectangular Cartesian coordinates; ε is a small parameter; α_k, β_k are constants; ϑ is an angle which, in passing around the contour, changes from zero to 2π; $c = b/a$ ($a = R$), where a and b are the semiaxes of the ellipse; N is any whole number.

[1]The approximate solution of this problem was first found by S. G. Lekhnitskiy [2]. Another approximation method for the solution of the analogous problem was proposed by A. S. Kosmodamianskiy [8].

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As in §1, we will assume that forces X_n and Y_n, the resultant vector of which is equal to zero, are applied to the examined contour of the hole. The problem of the stress state of such a plate is reduced to the determination of functions $\phi(z_1)$ and $\psi(z_2)$ from boundary conditions (III.6).

We will map conformally the interior of the unit circle on an infinite area with a hole of the type (III.265):

$$z = x + iy = \omega(\zeta) = R\left[\frac{1-c}{2}\zeta + \frac{1+c}{2}\zeta^{-1} + e\chi(\zeta)\right]. \qquad \text{(III.266)}$$

where

$$\chi(\zeta) = \sum_{k=2}^{N}(\alpha_k + i\beta_k)\zeta^k. \qquad \text{(III.267)}$$

On the contour of the unit circle, $\zeta = \sigma = e^{i\vartheta}$.

By tracing S. G. Lekhnitskiy's method [2], we represent the functions $\phi(z_1)$ and $\psi(z_2)$ in the form

$$\begin{aligned}
&\varphi(z_1) = f_{10}(\zeta_1) + e\{f_{11}(\zeta_1) + [\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]f'_{10}(\zeta_1)\} + \ldots + \\
&+ e^k\left\{f_{1k}(\zeta_1) + [\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]f'_{1,k-1}(\zeta_1) + \ldots + \frac{1}{k!}[\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]^k f^{(k)}_{10}(\zeta_1)\right\} + \ldots, \\
&\psi(z_2) = f_{20}(\zeta_2) + e\{f_{21}(\zeta_2) + [\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]f'_{20}(\zeta_2)\} + \ldots + \\
&+ e^k\left\{f_{2k}(\zeta_2) + [\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]f'_{2,k-1}(\zeta_2) + \ldots + \frac{1}{k!}[\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]^k f^{(k)}_{20}(\zeta_2)\right\} + \ldots,
\end{aligned} \qquad \text{(III.268)}$$

Here

$$z_p = z + \lambda_p\bar{z}, \quad \lambda_p = \frac{1+is_p}{1-is_p}; \qquad \text{(III.269)}$$

$f_{pk}(\zeta_p)$ (k = 0, 1, 2, ...; p = 1, 2) are analytical functions of arguments;

$$\zeta_p = \frac{1-c}{2}\zeta + \frac{1+c}{2}\zeta^{-1} + \lambda_p\left(\frac{1-c}{2}\bar{\zeta} + \frac{1+c}{2}\bar{\zeta}^{-1}\right). \qquad \text{(III.270)}$$

Functions $f_{pk}(\zeta_p)$ are defined in regions S_p, which represent infinite planes with elliptical holes.

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We will assume that the forces applied to the contour of the hole can be expanded into series by degress of small parameter ε: /296

$$\begin{aligned} -\int_0^s Y_n ds + C_1 &= \sum_{k=0} p_{1k}\varepsilon^k, \\ \int_0^s X_n ds + C_2 &= \sum_{k=0} p_{2k}\varepsilon^k. \end{aligned} \tag{III.271}$$

Here p_{1k} and p_{2k} are known functions defined for each partial case.

We will substitute expressions (III.268) and (III.271) into boundary conditions (III.6) and equate the terms for identical degrees of ε. Then, for the determination of functions $f_{pk}(\zeta_p)$, we obtain the boundary conditions

$$\begin{aligned} 2\,\mathrm{Re}\,[f_{10}(\zeta_1) + f_{20}(\zeta_2)] &= p_{10}, \\ 2\,\mathrm{Re}\,[s_1 f_{10}(\zeta_1) + f_{20}(\zeta_2)\, s_2] &= p_{20}; \end{aligned} \tag{III.272}$$

. . . . . .

$$\begin{aligned} 2\mathrm{Re}\,[f_{1k}(\zeta_1) + f_{2k}(\zeta_2)] = p_{1k} - 2\,\mathrm{Re}\,\Big\{&[\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]\, f'_{1,k-1}(\zeta_1) + \ldots + \\ &+ \frac{1}{k!}[\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]^k f_{10}^{(k)}(\zeta_1) + [\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]\, f'_{2,k-1}(\zeta_2) + \ldots + \\ &+ \frac{1}{k!}[\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]^k f_{20}^{(k)}(\zeta_2)\Big\}, \end{aligned} \tag{III.273}$$

$$\begin{aligned} 2\mathrm{Re}\,[s_1 f_{1k}(\zeta_1) + s_2 f_{2k}(\zeta_2)] = p_{2k} - 2\,\mathrm{Re}\,\Big\{&s_1[\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]\, f'_{1,k-1}(\zeta_1) + \ldots + \\ &+ \frac{s_1}{k!}[\chi(\zeta) + \lambda_1\overline{\chi(\zeta)}]^k f_{10}^{(k)}(\zeta_1) + s_2[\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]\, f'_{2,k-1}(\zeta_2) + \ldots + \\ &+ \frac{s_2}{k!}[\chi(\zeta) + \lambda_2\overline{\chi(\zeta)}]^k f_{20}^{(k)}(\zeta_2)\Big\}. \end{aligned}$$

Since the functions $f_{pk}(\zeta_p)$ are defined, as mentioned above, in planes with elliptical holes, they can be found by the same method[1] as in §1.

We will map conformally the interior of unit circle γ onto the exterior of the above stated elliptical holes:

---

[1] To determine the functions $f_{pk}(\zeta_p)$, L. G. Lekhnitskiy [2] used the method of series. For this purpose, A. S. Kosmodamianskiy [1] used integrals of the Cauchy type. Here, for the determination of the above stated functions, we use Schwartz' formula (III.8).

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$$\zeta_p = \omega_p(\zeta) = \left(\frac{1+c}{2} + \frac{1-c}{2}\lambda_p\right)\frac{1}{\zeta} + \left(\frac{1+c}{2}\lambda_p + \frac{1-c}{2}\right)\zeta. \tag{III.274}$$

We introduce the definitions

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$$f_{pk}(\zeta_p) = f_{pk}[\omega_p(\zeta)] = \psi_{pk}(\zeta). \tag{III.275}$$

Now the boundary conditions (III.272) and (III.273) acquire the form

$$\begin{aligned} &2\mathrm{Re}[\psi_{10}(\sigma) + \psi_{20}(\sigma)] = f_{10}(\vartheta), \\ &2\mathrm{Re}[s_1\psi_{10}(\sigma) + s_2\psi_{20}(\sigma)] = f_{20}(\vartheta); \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{III.276}$$

$$\begin{aligned} &2\mathrm{Re}[\psi_{1k}(\sigma) + \psi_{2k}(\sigma)] = f_{1k}(\vartheta), \\ &2\mathrm{Re}[s_1\psi_{1k}(\sigma) + s_2\varphi_{2k}(\sigma)] = f_{2k}(\vartheta), \\ &f_{10}(\vartheta) = p_{10}, \quad f_{20}(\vartheta) = p_{20}, \end{aligned} \tag{III.277}$$

where

$$\begin{aligned} f_{1k}(\vartheta) = p_{1k} - 2\mathrm{Re}\Big\{&(\chi + \lambda_1\chi)\frac{d\psi_{1,k-1}}{d\sigma_1} + \ldots + \frac{1}{k!}(\chi + \lambda_1\bar{\chi})^k\frac{d^k\psi_{10}}{d\sigma_1^k} + \\ &+ (\chi + \lambda_2\bar{\chi})\frac{d\psi_{2,k-1}}{d\sigma_2} + \ldots + \frac{1}{k!}(\chi + \lambda_2\bar{\chi})^k\frac{d^k\psi_{20}}{d\sigma_2^k}\Big\}, \\ f_{2k}(\vartheta) = p_{2k} - 2\mathrm{Re}\Big\{&s_1(\chi + \lambda_1\bar{\chi})\frac{d\psi_{1,k-1}}{d\sigma_1} + \ldots + \frac{s_1}{k!}(\chi + \lambda_1\bar{\chi})^k\frac{d^k\psi_{10}}{d\sigma_1^k} + \\ &+ s_2(\chi + \lambda_2\bar{\chi})\frac{d\psi_{2,k-1}}{d\sigma_2} + \ldots + \frac{s_2}{k!}(\chi + \lambda_2\bar{\chi})^k\frac{d^k\psi_{20}}{d\sigma_2^k}\Big\}. \end{aligned} \tag{III.278}$$

Here

$$\sigma_p = \left(\frac{1+c}{2} + \frac{1-c}{2}\lambda_p\right)\frac{1}{\sigma} + \left(\frac{1-c}{2} + \frac{1+c}{2}\lambda_p\right)\sigma, \tag{III.279}$$

and therefore, in differentiating the functions ψ_{pk} with respect to σ_p, it is necessary to use the formula

$$\frac{d\psi_{pk}}{d\sigma_p} = \frac{d\psi_{pk}}{d\sigma}\cdot\frac{d\sigma}{d\sigma_p} = -\frac{\omega_p\sigma^2}{1-\varkappa_p\sigma^2}\cdot\frac{d\psi_{pk}}{d\sigma}, \tag{III.280}$$

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where

$$\omega_p = \frac{2}{1+c+\lambda_p(1-c)}, \quad \varkappa_p = \frac{1-c+\lambda_p(1+c)}{1+c+\lambda_p(1-c)} \tag{III.281}$$

To determine the functions $\psi_{pk}(\zeta_p)$ that are holomorphic within unit circle γ and which satisfy on γ the boundary conditions (III.276) and (III.277), we will use, as earlier, Schwartz's formula. Then, on the basis of formulas (III.9) and (III.10), we will have

$$\begin{aligned}\psi_{1k}(\zeta) &= \frac{i}{4\pi(s_1-s_2)}\int_\gamma [s_2 f_{1k}(\vartheta) - f_{2k}(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_{1k},\\ \psi_{2k}(\zeta) &= \frac{-i}{4\pi(s_1-s_2)}\int_\gamma [s_1 f_{1k}(\vartheta) - f_{2k}(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_{2k}.\end{aligned} \tag{III.282}$$

We will return now to the variables ζ_1 and ζ_2. From the expression (III.274) we have /298

$$\zeta = \frac{\zeta_p - i\sqrt{\zeta_p^2 - [(1+\lambda_p^2)(1-c^2)+2\lambda_p(1+c^2)]}}{1-c+\lambda_p(1+c)}. \tag{III.283}$$

We will substitute the expression for ζ, for p = 1, into the function $\psi_{1k}(\zeta)$ and for p = 2, into the function $\psi_{2k}(\zeta)$. Then we find, respectively, the expressions for $f_{1k}(\zeta_1)$ and $f_{2k}(\zeta_2)$, the knowledge of which, on the basis of formulas (III.268), enable us to find the functions $\phi(z_1)$ and $\psi(z_2)$. After finding them, the stresses in the plate can be found by formulas (I.90).

In analyzing the problem of stress concentration near a hole in a plate, the determination of normal stress σ_ϑ, which acts on the surfaces normal to the contour of the hole, is quite important[1]:

$$\sigma_\vartheta = 2\,\mathrm{Re}\left[(1+\lambda_1)\left(\frac{dy}{ds}-s_1\frac{dx}{ds}\right)^2\varphi'(z_1) + (1+\lambda_2)\left(\frac{dy}{ds}-s_2\frac{dx}{ds}\right)^2\psi'(z_2)\right]. \tag{III.284}$$

§7. Tension of Anisotropic Plate with Common Curvilinear Hole

We will analyze an infinite homogeneous anisotropic plate weakened by a hole, the contour of which is given by equations

[1]See S. G. Lekhnitskiy [2].

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$$x = R\left(\cos\vartheta + \varepsilon\sum_{k=2}^{5} a_k \cos k\vartheta\right),$$
$$y = R\left(-c\sin\vartheta + \varepsilon\sum_{k=2}^{5} a_k \sin k\vartheta\right). \qquad \text{(III.285)}$$

The function that conformally maps the interior of the unit circle on an infinite area with the examined hole (III.285) consists of six terms and has the form

$$z = \omega(\zeta) = R\left(\frac{1-c}{2}\zeta + \frac{1+c}{2}\zeta^{-1} + \varepsilon\sum_{k=2}^{5} a_k\zeta^k\right). \qquad \text{(III.286)}$$

The choice of the mapping functions (III.286) permits us to obtain quite easily the formulas for determining the stress state in a plate with various holes.

Let the contour of a hole be free of external forces, and let, at infinity, uniformly distributed forces of tension p, parallel to the Ox axis, and forces q, parallel to the Oy axis, act on the plate (Figure III.19). /299

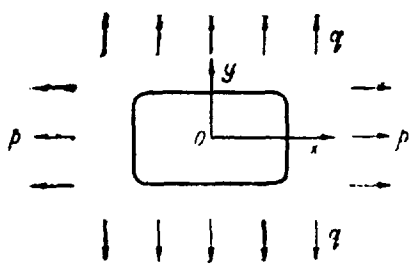


Figure III.19.

The solution of the problem of the stress state of a plate is found by combining two stress fields:

1) the stress field that develops in a thick plate

$$\sigma_x^0 = p, \quad \sigma_y^0 = q, \quad \tau_{xy}^0 = 0; \qquad \text{(III.287)}$$

in this case the projections of the forces acting on the contour of the assumed hole have the form

$$X_n^0 = -p\frac{dy}{ds}, \quad Y_n^0 = q\frac{dx}{ds}; \qquad \text{(III.288)}$$

2) the stress field occurring in a plate with a hole, to the contour of which are applied the forces

$$X_n = -X_n^0 = p\frac{dy}{ds},$$
$$Y_n = -Y_n^0 = -q\frac{dx}{ds}, \qquad \text{(III.289)}$$

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there being no forces at infinity.

We will determine the stress state of a plate when forces X_n, Y_n act on the contour.

The boundary values (III.271) are represented in the form

$$-\int_0^s Y_n ds + C_1 = p_{10} + \varepsilon p_{11},$$

$$\int_0^s X_n ds + C_2 = p_{20} + \varepsilon p_{21}, \tag{III.290}$$

where

$$p_{10} = -\frac{qR}{2}\left(\sigma + \frac{1}{\sigma}\right), \quad p_{11} = -\frac{qR}{2}\sum_{k=2}^{5} a_k\left(\sigma^k + \frac{1}{\sigma^k}\right),$$

$$p_{20} = -\frac{pRl}{2}\,c\left(\sigma - \frac{1}{\sigma}\right), \quad p_{21} = \frac{pRl}{2}\sum_{k=2}^{5} a_k\left(\sigma^k - \frac{1}{\sigma^k}\right).$$

In solving this problem we will confine ourselves to the second approximation. We will preserve in expressions (III.268) the terms that contain ε in the second power, discarding those that contain ε in powers $\geqslant 3$. We will assume that the complex parameters for an orthotropic plate are purely imaginary ($s_1 = i\beta_1$, $s_2 = i\beta_2$). We will confine ourselves to the determination of the functions $\phi(z_1)$ and $\psi(z_2)$ for the points of the contour of the given hole.

By using the theory outlined above, after several transformations we obtain /300

$$\varphi(\sigma) = \frac{R}{2}\left[N_0\sigma + \varepsilon\sum_{k=1}^{5}(N_k\sigma^k + N_{-k}\sigma^{-k}) + \varepsilon^2\sum_{k=1}^{7}(A_k\sigma^k + A_{-k}\sigma^{-k})\right]. \tag{III.291}$$

Here we introduce the definitions:

$$N_0 = \frac{1}{\beta_1 - \beta_2}\,m_1,$$

$$N_{-1} = -\frac{1}{\beta_1 - \beta_2}\,\delta_1(a_3 + \varkappa_1 a_5)\,m_1, \quad N_{-2} = -\frac{1}{\beta_1 - \beta_2}\,\delta_1 a_4 m_1,$$

$$N_{-3} = -\frac{1}{\beta_1 - \beta_2}\,\delta_1 a_5 m_1, \qquad N_{-4} = N_{-5} = 0,$$

$$N_1 = \frac{1}{(\beta_1 - \beta_2)^3}\left[2\beta_2\delta_2(a_3 + \varkappa_2 a_5)\,m_2 - (\beta_1 + \beta_2)\,\delta_1(a_3 + \varkappa_1 a_5)\,m_1\right], \tag{III.292}$$

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$$N_2 = \frac{1}{(\beta_1-\beta_2)^2}[2\beta_2\delta_2 a_4 m_2 - (\beta_1+\beta_2)\delta_1 a_4 m_1 + (\beta_1-\beta_2)a_2 n_1],$$

$$N_3 = \frac{1}{(\beta_1-\beta_2)^2}[2\beta_2\delta_2 a_5 m_2 - (\beta_1+\beta_2)\delta_1 a_5 m_1 + (\beta_1-\beta_2)a_3 n_1],$$

$$N_4 = \frac{1}{\beta_1-\beta_2}a_4 n_1, \quad N_5 = \frac{1}{\beta_1-\beta_2}a_5 n_1;$$

$$A_{-1} = \delta_1[-3a_5 N_3 - 2a_4 N_2 - (a_3 + \varkappa_1 a_5)N_1 +$$

$$+ \frac{\delta_1}{\beta_1-\beta_2}(a_2^2 + 4\varkappa_1 a_2 a_4 + 2\varkappa_1 a_3^2 + 7\varkappa_1^2 a_3 a_5 + 4\varkappa_1^2 a_4^2 + 6\varkappa_1^3 a_5^2)m_1],$$

$$A_{-2} = \delta_1[-2a_5 N_2 - a_4 N_1 + \frac{\delta_1}{\beta_1-\beta_2}(2a_2 a_3 + 4\varkappa_1 a_2 a_5 + 5\varkappa_1 a_3 a_4 + 9\varkappa_1^2 a_4 a_5)m_1],$$

$$A_{-3} = \delta[-a_5 N + \frac{\delta_1}{\beta_1-\beta_2}(2a_2 a_4 + a_3^2 + 5\varkappa_1 a_3 a_5 + 3\varkappa_1 a_4^2 + 5\varkappa_1^2 a_5^2)m_1], \qquad \text{(III.293)}$$

$$A_{-4} = \frac{\delta_1^2}{\beta_1-\beta_2}(2a_2 a_5 + 2a_3 a_4 + 6\varkappa_1 a_4 a_5)m_1,$$

$$A_{-5} = \frac{\delta_1^2}{\beta_1-\beta_2}(2a_3 a_5 + a_4^2 + 3\varkappa_1 a_5^2)m_1,$$

$$A_{-6} = \frac{\delta_1^2}{\beta_1-\beta_2}2a_4 a_5 m_1,$$

$$A_{-7} = \frac{\delta_1^2}{\beta_1-\beta_2}a_5^2 m_1,$$

$$A_k = \frac{1}{\beta_1-\beta_2}[(\beta_1+\beta_2)A_{-k} + 2\beta_2 B_{-k}] \quad (k = 1, \ldots, 7).$$

Thus /301

$$m_1 = -pc + q\beta_2, \qquad n_1 = p + q\beta_2,$$
$$m_2 = -pc + q\beta_1, \qquad n_2 = p + q\beta_1,$$
$$\varkappa_p = \frac{1-c\beta_p}{1+c\beta_p}, \qquad \delta_p = \omega_p\lambda_p = \frac{1-\beta_p}{1+c\beta_p} \qquad (p = 1.2). \qquad \text{(III.294)}$$

The coefficients B_{-k} and B_k are found from A_{-k} and A_k if, in the latter, we substitute β_1, δ_1, κ_1, m_1, n_1, respectively, by β_2, δ_2, κ_2, m_2, n_2, and conversely.

The function $\psi(\sigma)$ is found from (III.291), if, in the expressions of coefficients (III.292) and (III.293) we perform circular permutation of the subscript i in β_i, δ_i, κ_i, m_i, n_i (i = 1, 2).

The expression for the normal stress acting on the areas normal to the contour of the hole acquire the form

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$$\sigma_\vartheta = p\left\{\frac{B^2}{C^2} + \frac{A^2}{C^2L}\left[A^4\beta_1^2 + A^2B^2(2\beta_1^2 + 1 - \beta_1^2\beta_2^2) + B^4(2 - \beta_2^2)\right] + \right.$$
$$\left. + \frac{(\beta_1 + \beta_2)C^2}{L}(B\varphi_1 - \beta_1 A\psi_1)\right\} +$$
$$+ q\left\{\frac{A^2}{C^2} + \frac{B^2}{C^2L}\left[A^4\beta_1^2(2\beta_2^2 - 1) + A^2B^2(2\beta_2^2 - 1 + \beta_1^2\beta_2^2) + B^4\beta_2^2\right] + \right. \quad \text{(III.295)}$$
$$\left. + \frac{(\beta_1 + \beta_2)\beta_2 C^2}{L}(B\varphi_2 - \beta_1 A\psi_2)\right\}.$$

Here we introduce the following definitions:

$$A = -c\cos\vartheta + \varepsilon\sum_{k=2}^{5} ka_k\cos k\vartheta, \quad \text{(III.296)}$$
$$B = \sin\vartheta + \varepsilon\sum_{k=2}^{5} ka_k\sin k\vartheta;$$

$$C^2 = A^2 + B^2, \qquad L = (A^2\beta_1^2 + B^2)(A^2\beta_2^2 + B^2); \quad \text{(III.297)}$$

$$\varphi_1 = -B + (1 + c)\sin\vartheta - \frac{2\beta_1 c}{\beta_1 - \beta_2}\left[\varepsilon\sum_{k=1}^{3} k\left(\delta_1 M_k - \delta_2\frac{\beta_2}{\beta_1}p_k\right)\sin k\vartheta - \right.$$
$$\left. - \varepsilon^2\sum_{k=1}^{7} k\left(\delta_1^2 A_{1k} - \delta_2^2\frac{\beta_2}{\beta_1}B_{1k}\right)\sin k\vartheta\right],$$

$$\psi_1 = A + \frac{2\beta_2 C}{\beta_1 - \beta_2}\left[\varepsilon\sum_{k=1}^{3} k\left(\delta_1 M_k - \delta_2 p_k\right)\cos k\vartheta - \right. \quad \text{(III.298)}$$
$$\left. - \varepsilon^2\sum_{k=1}^{7} k(\delta_1^2 A_{1k} - \delta_2^2 B_{1k})\cos k\vartheta\right];$$

$$\varphi_2 = -B + \frac{2\beta_1}{\beta_1 - \beta_2}\left[\varepsilon\sum_{k=1}^{3} k(\delta_1 M_k - \delta_2 p_k)\sin k\vartheta - \varepsilon^2\sum_{k=1}^{7} k(\delta_1^2 A_{2k} - \delta_2^2 B_{2k})\sin k\vartheta\right.$$

$$\psi_2 = A + (1 + c)\cos\vartheta - \frac{2\beta_2}{\beta_1 - \beta_2}\left[\varepsilon\sum_{k=1}^{3} k\left(\delta_1 M_k - \delta_2\frac{\beta_1}{\beta_2}p_k\right)\cos k\vartheta - \right. \quad \text{(III.299)} \quad \underline{/302}$$
$$\left. - \varepsilon^2\sum_{k=1}^{7} k\left(\delta_1^2 A_{2k} - \delta_2^2\frac{\beta_1}{\beta_2}B_{2k}\right)\cos k\vartheta\right].$$

Hence

$$M_1 = a_3 + \varkappa_1 a_5,$$
$$M_2 = a_4,$$
$$M_3 = a_5,$$

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$$A_{11} = a_2^2 + 4a_2a_4\varkappa_1 + 2a_3^2\varkappa_1 + 7a_3a_5\varkappa_1^2 + 4a_4^2\varkappa_1^2 + 6a_5^2\varkappa_1^3 +$$
$$+ \frac{3a_3a_5 + 2a_2a_4}{c\delta_1} + [2a_4^2 + 3a_5^2 + (a_3 + \varkappa_1 a_5)^2 \frac{\beta_1 + \beta_2}{\beta_1 - \beta_2} -$$
$$- [2a_4^2 + 3a_5^2 + (a_3 + \varkappa_1 a_5)(a_3 + \varkappa_2 a_5)] \frac{2\beta_2}{\beta_1 - \beta_2} \cdot \frac{\delta_2}{\delta_1},$$
$$A_{12} = 2a_2a_3 + 4a_2a_5\varkappa_1 + 5a_3a_4\varkappa_1 + 9a_4a_5\varkappa_1^2 + \frac{2a_2a_5}{c\delta_1} +$$
$$+ [2a_4a_5 + a_4(a_3 + \varkappa_1 a_5)] \frac{\beta_1 + \beta_2}{\beta_1 - \beta_2} - [2a_4a_5 + a_4(a_3 + \varkappa_2 a_5)] \frac{2\beta_2}{\beta_1 - \beta_2} \cdot \frac{\delta_2}{\delta_1},$$
$$A_{13} = a_3^2 + 2a_2a_4 + 5a_3a_5\varkappa_1 + 3a_4^2\varkappa_1 + 5a_5^2\varkappa_1^2 + a_5(a_3 + \varkappa_1 a_5) \frac{\beta_1 + \beta_2}{\beta_1 - \beta_2} -$$
$$- a_5(a_3 + \varkappa_2 a_5) \frac{2\beta_2}{\beta_1 - \beta_2} \cdot \frac{\delta_2}{\delta_1}, \qquad \text{(III.300)}$$
$$A_{14} = 2a_2a_5 + 2a_3a_4 + 6a_4a_5\varkappa_1,$$
$$A_{15} = 2a_3a_5 + a_4^2 + 3a_5^2\varkappa_1,$$
$$A_{16} = 2a_4a_5,$$
$$A_{17} = a_5^2;$$
$$A_{21} = A_{11} - 2[2a_4^2 + 3a_5^2 + (a_3 + \varkappa_1 a_5)(a_3 + \varkappa_2 a_5)] \frac{\delta_2}{\delta_1} - \frac{3a_3a_5 + 2a_4a_2}{c\delta_1}(1 + c),$$
$$A_{22} = A_{12} - 2[2a_4a_5 + a_4(a_3 + \varkappa_2 a_5)] \frac{\delta_2}{\delta_1} - \frac{2a_2a_5}{c\delta_1}(1 + c), \qquad \text{(III.301)}$$
$$A_{23} = A_{13} - 2a_5(a_3 + \varkappa_2 a_5) \frac{\delta_2}{\delta_1},$$
$$A_{2k} = A_{1k} \; (k = 4, \ldots, 7).$$

The coefficients p_k, B_{1k}, B_{2k} are found from the expressions for M_k, A_{1k}, A_{2k} by substitution of β_1, β_2, δ_1, δ_2, κ_1, κ_2, respectively, by β_2, β_1, δ_2, δ_1, κ_2, κ_1. /303

In the examples given below, the stresses that occur on the contour of the hole are determined for the case where the plate is made of aviation plywood[1]:

1) elastic constants

$$a_{11} = 0{,}83333 \cdot \frac{10^{-9}}{9{,}81}, \qquad a_{12} = -0{,}05917 \cdot \frac{10^{-9}}{9{,}81},$$
$$a_{22} = 1{,}66667 \cdot \frac{10^{-9}}{9{,}81}, \qquad a_{66} = 14{,}2857 \cdot \frac{10^{-9}}{9{,}81}$$

[1] See S. G. Lekhnitskiy [1], p. 54.

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and complex parameters

$$s_1 = 4{,}11i, \qquad s_2 = 0{,}343i,$$

if the x axis is directed with the grain ($E_x = E_{max}$);

2) elastic constants

$$a_{11} = 1{,}66667 \cdot \frac{10^{-9}}{9{,}81}, \qquad a_{12} = -0{,}05917 \cdot \frac{10^{-9}}{9{,}81},$$

$$a_{22} = 0{,}83333 \cdot \frac{10^{-9}}{9{,}81}, \qquad a_{66} = 14{,}2857 \cdot \frac{10^{-9}}{9{,}81}$$

and complex parameters

$$s_1 = 0{,}243i, \qquad s_2 = 2{,}91i,$$

if the x axis is directed against the grain ($E_x = E_{min}$).

The results of calculations for plywood and isotropic plates are illustrated by the graphs presented below. The heavy lines on these figures correspond to the case where $E_x = E_{max}$, the dot-dash lines correspond to the case $E_x = E_{min}$, and the dotted line pertains to an isotropic plate.

The formula for stress σ_ϑ in the case of an isotropic plate can be found from approximation formula (III.295) by way of limit transition, assuming that $\beta_1 = \beta_2 = 1$. However, this problem also has a precise solution[1].

Hole with the Shape of a Right Triangle[2]. The contour is given by the equations

$$\begin{aligned} x &= R(\cos\vartheta + \varepsilon\cos 2\vartheta), \\ y &= R(-\sin\vartheta + \varepsilon\sin 2\vartheta). \end{aligned} \qquad \text{(III.302)}$$

Equations (III.302) are found from equations (III.285), assuming in the latter that $c = 1$, $a_2 = 1$, $a_3 = a_4 = a_5 = 0$. When $\varepsilon = 0.25$ we obtain a hole which differs only slightly from an equilateral triangle with rounded corners. Hence the curvature at the center of the sides of the triangle is equal to zero[3].

We will consider two partial cases.

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[1]See Ye. F. Burmistrov [1].
[2]See S. G. Lekhnitskiy [1], p. 220.
[3]See M. I. Nayman [1].

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Tension of Plate by Uniformly Distributed Force p Acting Along the Ox Axis (Figure III.20). Assuming that q = 0 in formula (III.295), we obtain /304

$$\sigma_\vartheta = p\left\{\frac{B^2}{C^2} + \frac{A^2}{C^2L}\left[A^4\beta_1^2 + A^2B^2(1 + 2\beta_1^2 - \beta_1^2\beta_2^2) + B^4(2 - \beta_2^2)\right] + \frac{C^2(\beta_1 + \beta_2)}{L}(B\varphi_1 - \beta_1 A\psi_1)\right\}, \tag{III.303}$$

where

$$\varphi_1 = -B + 2\sin\vartheta + \frac{2\beta_1}{\beta_1 - \beta_2}\varepsilon^2\left(\delta_1^2 - \delta_2^2\frac{\beta_2}{\beta_1}\right)\sin\vartheta,$$
$$\psi_1 = A - \frac{2\beta_2}{\beta_1 - \beta_2}\varepsilon^2(\delta_1^2 - \delta_2^2)\cos\vartheta. \tag{III.304}$$

The values C^2, L are defined by formulas (III.297), in which it is necessary to assume that

$$A = -\cos\vartheta + 2\varepsilon\cos 2\vartheta,$$
$$B = \sin\vartheta + 2\varepsilon\sin 2\vartheta. \tag{III.305}$$

Expressions (III.304) and (III.305) are found from (III.296), (III.297) and (III.300), assuming in the latter that c = 1, a_2 = 1 and $a_3 = a_4 = a_5 = 0$.

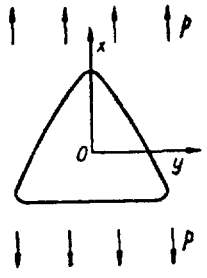


Figure III.20.

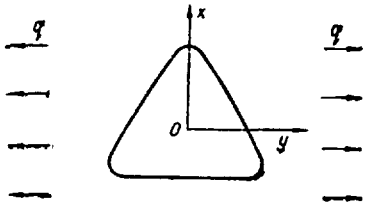


Figure III.21.

Tension of Plate by Uniformly Distributed Forces q Acting Along the Oy Axis (Figure III.21). The formula for the determination of stress σ_ϑ is found from (III.295), assuming in the latter that p = 0:

$$\sigma_\vartheta = q\left\{\frac{A^2}{C^2} + \frac{B^2}{C^2L}\left[A^4\beta_1^2(2\beta_2^2 - 1) + A^2B^2(2\beta_2^2 + \beta_1^2\beta_2^2 - 1) + B^4\beta_2^2\right] + \frac{(\beta_1 + \beta_2)\beta_2C^2}{L}(B\varphi_2 - \beta_1 A\psi_2)\right\}, \tag{III.306}$$

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where

$$\varphi_2 = -B - \frac{2\beta_1}{\beta_1 - \beta_2}\varepsilon^2(\delta_1^2 - \delta_2^2)\sin\vartheta,$$
$$\psi_2 = A + 2\cos\vartheta + \frac{2\beta_2}{\beta_1 - \beta_2}\varepsilon^2\left(\delta_1^2 - \delta_2^2\frac{\beta_1}{\beta_2}\right)\cos\vartheta. \qquad \text{(III.307)}$$

The coefficients C^2, L, A and B are defined, as before, by formulas (III.297) and (III.305). /305

Figure III.22 illustrates the graphs of distribution of stresses σ_ϑ along the contour of the triangular hole.

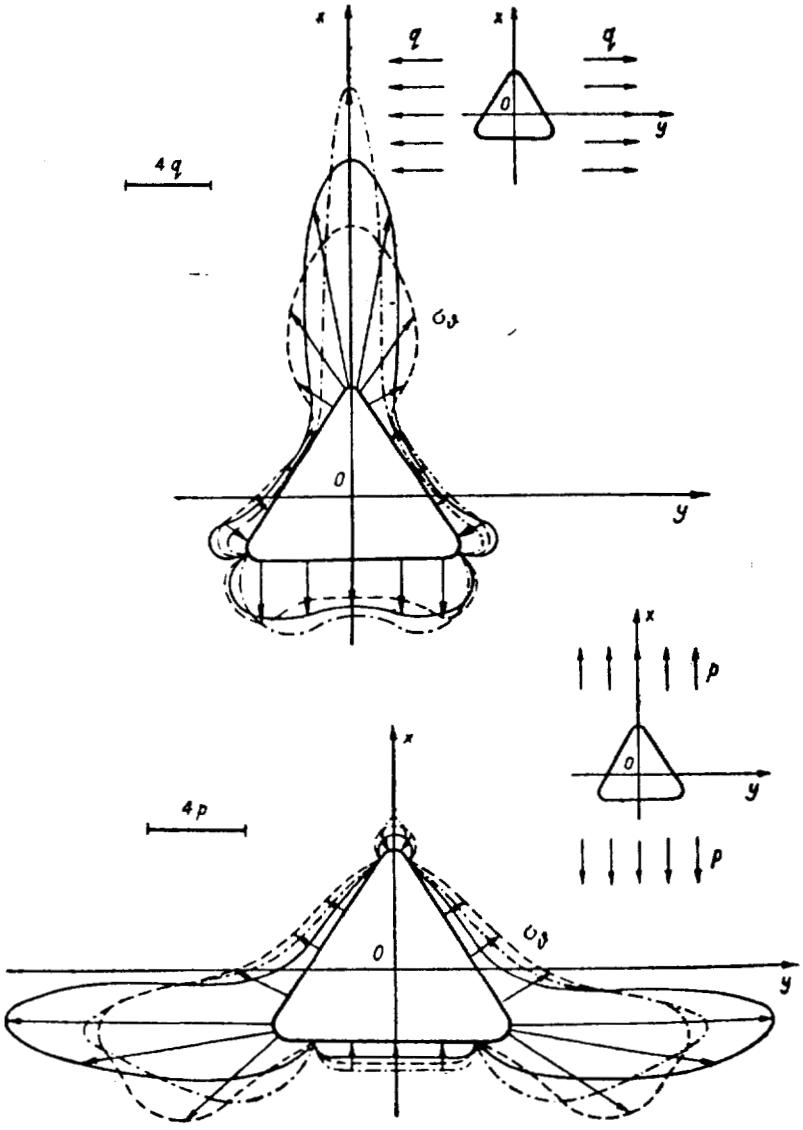


Figure III.22.

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Hole with the Shape of an Isosceles Triangle[1]. We have an orthotropic plate weakened by a hole, the contour of which is defined by the equation

$$x = R[\cos\vartheta + \varepsilon(a_2\cos 2\vartheta + a_3\cos 3\vartheta)],$$
$$y = R[-c\sin\vartheta + \varepsilon(a_2\sin 2\vartheta + a_3\sin 3\vartheta)]. \qquad \text{(III.308)}$$

With the appropriate selection of constants c, a_2, a_3 and the parameter ε, we obtain a hole that differs but little from an isosceles triangle with rounded corners. The formula for stress σ_ϑ is found from expressions (III.295)-(III.300) assuming in the latter that $a_4 = a_5 = 0$.

We will consider two partial cases.

Stress of Plate in Direction Perpendicular to Base of Isosceles Triangle (Figure III.23). Stress σ_ϑ on the contour of the hole is found from formula (III.303). Hence

$$A = -c\cos\vartheta + \varepsilon(2a_2\cos 2\vartheta + 3a_3\cos 3\vartheta),$$
$$B = \sin\vartheta + \varepsilon(2a_2\sin 2\vartheta + 3a_3\sin 3\vartheta); \qquad \text{(III.309)}$$

C^2 and L are defined by formula (III.297);

$$\varphi_1 = -B + (1+c)\sin\vartheta - \frac{2\beta_1 c}{\beta_1-\beta_2}\left\{\varepsilon\left(\delta_1 - \frac{\beta_2}{\beta_1}\delta_2\right)a_3\sin\vartheta - \right.$$
$$\left. -\varepsilon^2\left[\left(\delta_1^2 - \frac{\beta_2}{\beta_1}\delta_2^2\right)(3a_3^2\sin 3\vartheta + 4a_2a_3\sin 2\vartheta) + \left(\delta_1^2 A_{11} - \delta_2^2\frac{\beta_2}{\beta_1}B_{11}\right)\sin\vartheta\right]\right\}, \qquad \text{(III.310)}$$
$$\psi_1 = A + \frac{2\beta_2 c}{\beta_1-\beta_2}\{\varepsilon(\delta_1-\delta_2)a_3\cos\vartheta - \varepsilon^2[(\delta_1^2-\delta_2^2)(3a_3^2\cos 3\vartheta + $$
$$+ 4a_2a_3\cos 2\vartheta) + (\delta_1^2A_{11} - \delta_2^2B_{11})\cos\vartheta]\}.$$

Here

$$A_{11} = a_2^2 + 2a_3^2\varkappa_1 + a_3^2\left(\frac{\beta_1+\beta_2}{\beta_1-\beta_2} - \frac{2\beta_2}{\beta_1-\beta_2}\cdot\frac{\delta_2}{\delta_1}\right). \qquad \text{(III.311)}$$

[1]The problem of the stress state of a plate with a hole close in shape to an isosceles triangle is examined in the work of A. S. Kosmodamianskiy, V. V. Meglinskiy and V. A. Shvetsov [1].

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The expression for B_{11} is found from (III.311) by substituting κ_1, β_1, δ_1 by κ_2, β_2, δ_2 and conversely.

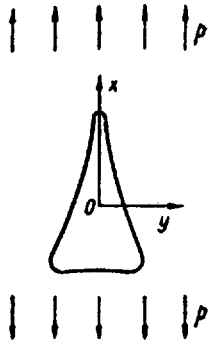


Figure III.23.

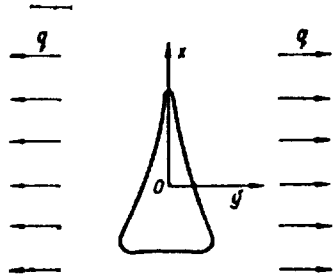


Figure III.24.

Stress of Plate in Direction Parallel to Base of Isosceles Triangle (Figure III.24). The stress σ_ϑ on the contour of the hole is defined by formula (III.306), where /307

$$\varphi_2 = -B + \frac{2\beta_1}{\beta_2 - \beta_1}\{\varepsilon(\delta_1 - \delta_2)a_3 \sin\vartheta - \varepsilon^2[(\delta_1^2 - \delta_2^2)(3a_3^2 \sin 3\vartheta + 4a_2a_3 \sin 2\vartheta) + (\delta_1^2 A_{21} - \delta_2^2 B_{21})\sin\vartheta]\},$$

$$\psi_2 = A + (1+c)\cos\vartheta - \frac{2\beta_2}{\beta_1 - \beta_2}\left\{\varepsilon\left(\delta_1 - \delta_2\frac{\beta_1}{\beta_2}\right)a_3\cos\vartheta - \right. \tag{III.312}$$

$$\left. - \varepsilon^2\left[\left(\delta_1^2 - \delta_2^2\frac{\beta_1}{\beta_2}\right)(3a_3^2\cos 3\vartheta + 4a_2a_3\cos 2\vartheta) + \left(\delta_1^2 A_{21} - \delta_2^2\frac{\beta_1}{\beta_2}B_{21}\right)\cos\vartheta\right]\right\}.$$

Hence

$$A_{21} = a_2^2 + 2a_3^2\kappa_1 + a_3^2\left(\frac{\beta_1+\beta_2}{\beta_1-\beta_2} - \frac{2\beta_1}{\beta_1-\beta_2}\cdot\frac{\delta_2}{\delta_1}\right). \tag{III.313}$$

The expression for B_{21} is found from (III.313) by substituting κ_1, β_1, δ_1, by κ_2, β_2, δ_2 and conversely.

We will examine the following holes: a) a hole that differs slightly from an isosceles right triangle, for which c = 2; ε = 0.44; a_2 = 1; a_3 = 0.168; b) a hole that differs slightly from an isosceles triangle with a vertex angle $\pi/6$, for which c = 0.4; ε = 0.1905; a_2 = 1; a_3 = -0.2143. To the first case correspond the graphs in Figure III.25, and to the second, the graphs in Figure III.26.

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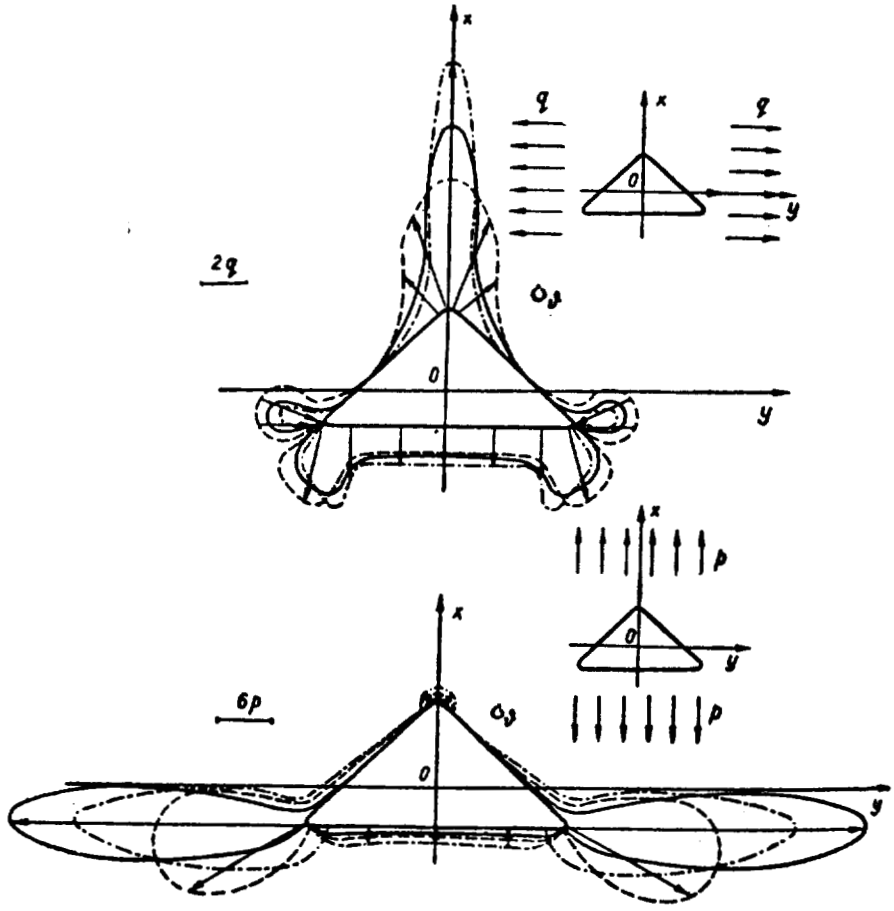


Figure III.25.

Square Hole[1]. Let the equation of contour of the hole have the form

$$\begin{aligned} x &= R(\cos\vartheta + \varepsilon\cos 3\vartheta), \\ y &= R(-\sin\vartheta + \varepsilon\sin 3\vartheta), \end{aligned} \qquad \text{(III.314)}$$

where $\varepsilon = \pm 1/9$. This hole differs but little from a square with rounded corners. For negative ε, the sides of the square are parallel to the coordinate axes, and for positive ε, the corners of the square lie on the coordinate axes (Figure III.27). /308

[1]See S. G. Lekhnitskiy [1], p. 235.

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We will assume that at infinity the plate is subjected to tension by uniformly distributed forces p, parallel to the Ox axis (see Figure III.27).

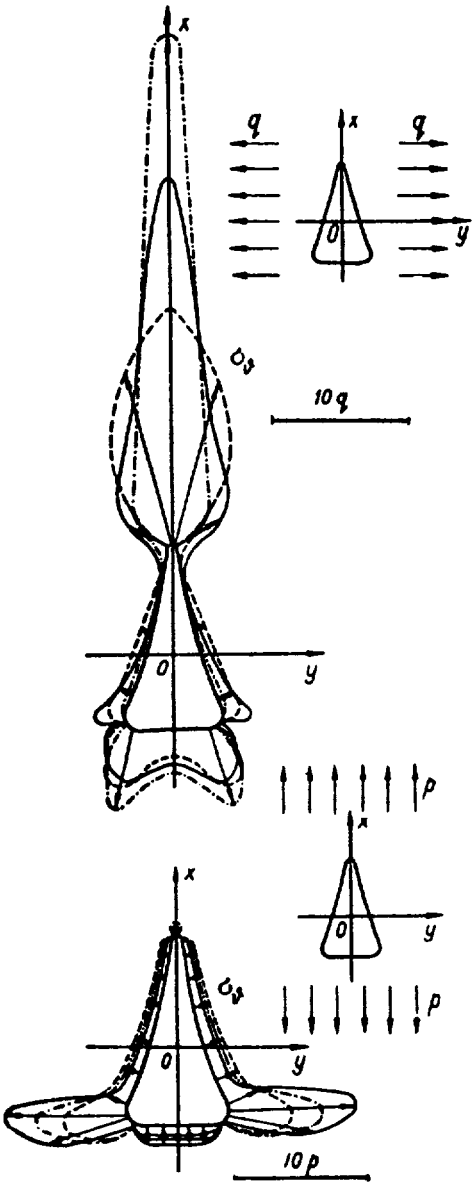


Figure III.26.

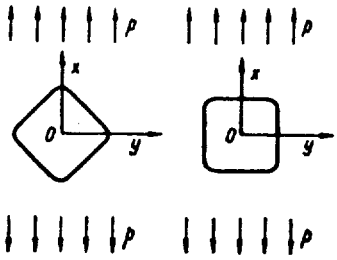


Figure III.27.

Stresses σ_ϑ along the contour of the hole are defined by formula (III.303), whence

$$A = -\cos\vartheta + 3\varepsilon\cos 3\vartheta, \quad B = \sin\vartheta + 3\varepsilon\sin 3\vartheta,$$

$$\varphi_1 = -B + 2\sin\vartheta - \frac{2\beta_1}{\beta_1-\beta_2}\left\{\varepsilon\left(\delta_1 - \frac{\beta_2}{\beta_1}\delta_2\right)\sin\vartheta - \right.$$

$$\left. - \varepsilon^2\left[\left(\delta_1^2 - \frac{\beta_2}{\beta_1}\delta_2^2\right)3\sin 3\vartheta + \left(\delta_1^2 A_{11} - \frac{\beta_2}{\beta_1}\delta_2^2 B_{11}\right)\sin\vartheta\right]\right\}, \quad \text{(III.315)} \quad \underline{/309}$$

$$\psi_1 = A + \frac{2\beta_2}{\beta_1-\beta_2}\{\varepsilon(\delta_1-\delta_2)\times$$

$$\times\cos\vartheta - \varepsilon^2[(\delta_1^2-\delta_2^2)3\cos 3\vartheta +$$

$$+(\delta_1^2 A_{11} - \delta_2^2 B_{11})\cos\vartheta]\},$$

where

$$A_{11} = 2\varkappa_1 + \frac{\beta_1+\beta_2}{\beta_1-\beta_2} - \frac{2\beta_2}{\beta_1-\beta_2}\cdot\frac{\delta_2}{\delta_1}, \quad \text{(III.316)}$$

and the expression for B_{11} is found from (III.316) by substituting $\varkappa_1$, β_1, δ_1 by $\varkappa_2$, β_2, δ_2, and conversely. The stresses (in fractions of p), calculated by formulas (III.303), (III.315) and (III.316), are presented in Figure III.28.

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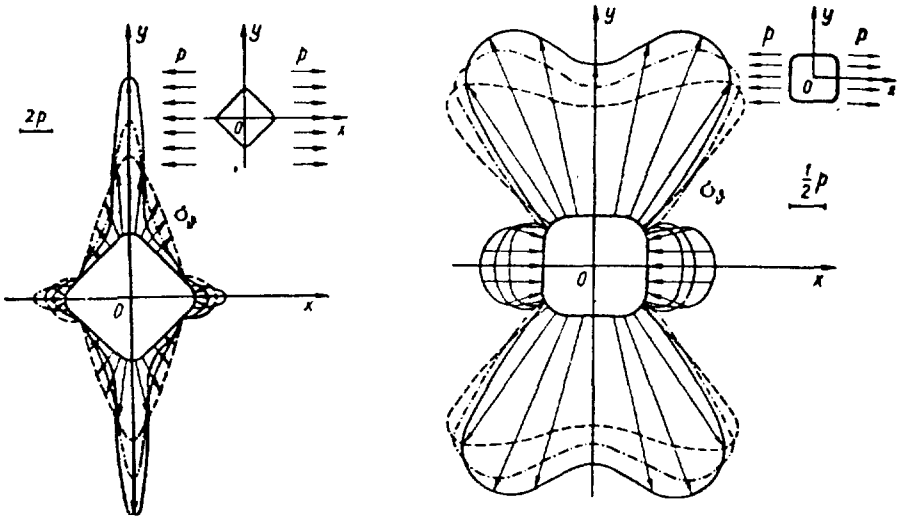


Figure III.28.

Rectangular Hole with Ratio of Sides a/b = 2. Such a contour (Figure III.29) is found by assuming in formulas (III.285) that

$$c = 0{,}52788; \quad a_2 = a_4 = 0; \quad a_3 = -1;$$
$$a_5 = -0{,}11979; \quad \varepsilon = 0{,}088006. \qquad \text{(III.317)}$$

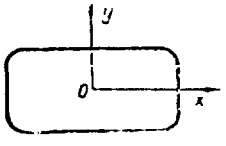


Figure III.29.

By formulas (III.295)-(III.300), we calculate[1] the stresses σ_ϑ near the contour of the hole, when the plate is made of aviation plywood, the elastic constants of which are shown above, and also for an isotropic plate. Here we examine the cases where the forces of tension act in the direction of the large or small sides of the rectangle. The values σ_ϑ (in fractions of p and q) are shown in Figure III.30, where the solid lines correspond to the case $E_x = E_{max}$, the dot-dash lines, to the case $E_x = E_{min}$, and the broken lines, to an isotropic plate. /310

Trapezoidal Hole. A hole with the shape of an equilateral trapezoid (Figure III.31) is found[2] by assuming in formulas (III.285) that

[1]See A. S. Kosmodamianskiy [7].
[2]See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [3].

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$$c = 1{,}0438; \qquad a_2 = 1;$$
$$a_3 = -1{,}47619; \qquad a_4 = -0{,}25867;$$
$$a_5 = -0{,}019698; \qquad \varepsilon = 0{,}10427. \qquad \text{(III.318)}$$

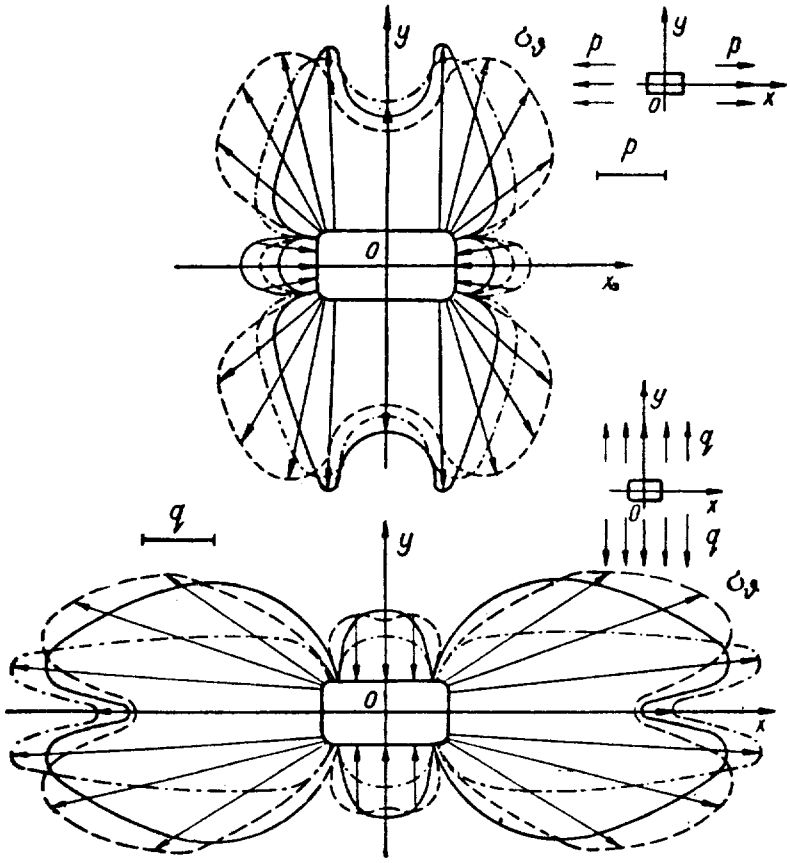


Figure III.30.

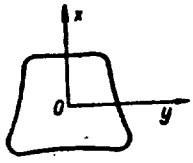


Figure III.31.

The function $\phi(\sigma)$ is given by formulas (III.291)-(III.294); the function $\psi(\sigma)$ is found from the expression for $\phi(\sigma)$, if for the values β_i, δ_i, κ_i, m_i, n_i we carry out circular permutation of the subscript i (i = 1, 2).

Stress σ_ϑ is defined by formulas (III.295)-(III.301). The values σ_ϑ/p and σ_ϑ/q are given in Figure III.32, respectively, for the case where the forces p are applied /312

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both in the direction of the Ox axis (perpendicular to the base of the trapezoid) and in the direction of the Oy axis (parallel to the base of the trapezoid).

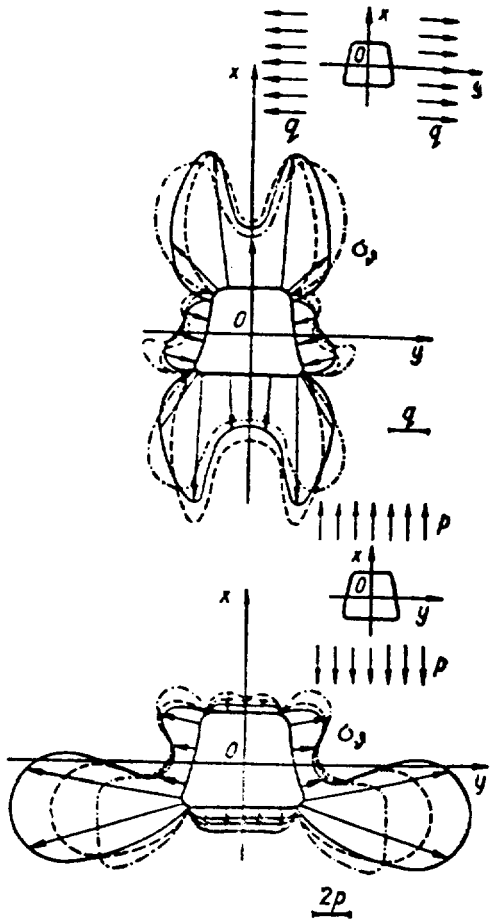


Figure III.32.

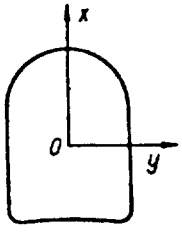


Figure III.33.

Hole in the Shape of an Arch[1]. A plate is weakened by a hole, the shape of is given by equations (III.285), where

$$c = 0{,}752035; \quad a_2 = 1;$$
$$a_3 = -1{,}4908; \quad a_4 = 0{,}56998; \qquad \text{(III.319)}$$
$$a_5 = 0{,}02855; \quad \varepsilon = 0{,}059525.$$

For these parameters[2] we find an arch-shaped hole (Figure III.33).

The normal stresses σ_ϑ are calculated for plywood and isotropic plates. Figure III.34 gives the values σ_ϑ/p and σ_ϑ/q, respectively, for the case where forces p are applied in the direction of the Ox axis, and forces q, in the direction of the Oy axis. The concentration of stresses is great if the plate is under tension with the grain; when the plate is under tension against the grain, the stress distribution of σ_ϑ is more uniform along the contour of the hole. Moreover, the concentration of stresses in an anisotropic plate is much greater than in an isotropic plate. /313

§8. Pure Deflection of Anisotropic Plate with Curvilinear Hole

We will assume that forces that produce deflecting moment M act on a rectangular anisotropic plate with a hole the contour of which is given by equations (III.285). The dimensions of the hole are small in comparison with the dimensions of the plate, and it is located so far from the edge that in the solution of the problem we may assume that the plate is infinite.

[1]See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [2].
[2]See I. S. Khara [1].

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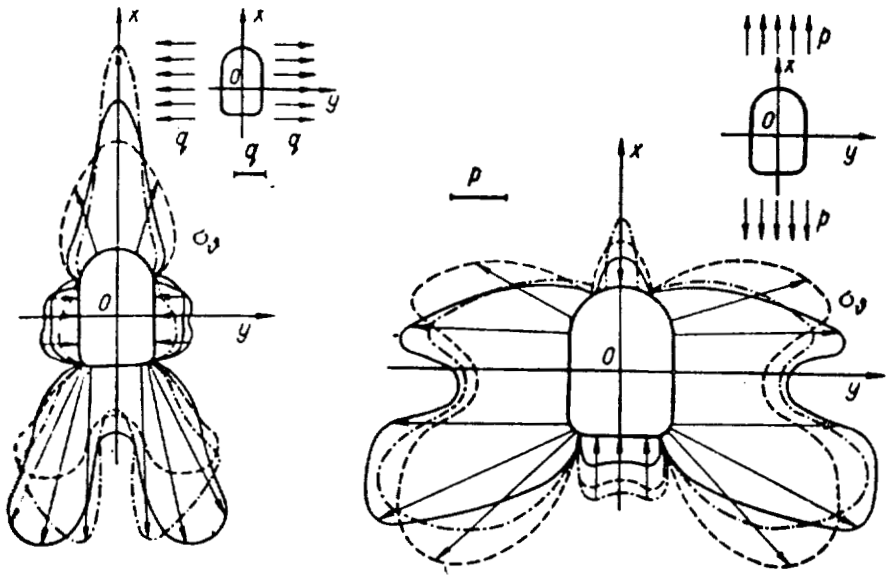


Figure III.34.

We will consider two cases. In the first case the deflecting moments are applied as indicated in Figure III.35. Then

$$p_{20} = \frac{MR^2}{8J} c^2 \left(\sigma - \frac{1}{\sigma}\right)^2,$$
$$p_{21} = -\frac{MR^2}{4J} c \left(\sigma - \frac{1}{\sigma}\right) \sum_{k=2}^{5} a_k \left(\sigma^k - \frac{1}{\sigma^k}\right), \qquad \text{(III.320)}$$
$$p_{22} = \frac{MR^2}{8J} \left[\sum_{k=2}^{5} a_k \left(\sigma^k - \frac{1}{\sigma^k}\right)\right]^2.$$

The other coefficients p_{2k} and all p_{1k} are equal to zero. Here J is the moment /314 of inertia of the cross section of the plate, normal to the Ox axis.

The functions $\phi(z_1)$ and $\psi(z_2)$ for the points of the contour of the hole have the form

$$\varphi(\sigma) = \frac{MR^2}{8J} c^2 i \left\{ -\frac{1}{\beta_1 - \beta_2} \sigma^2 + \varepsilon \left[\sum_{k=1}^{2} (N_{1k}\sigma^k + N_{-1k}\sigma^{-k}) - \frac{1}{c(\beta_1 - \beta_2)} \sum_{k=1}^{6} m_{1k}\sigma^k \right] + \right.$$
$$\left. + \varepsilon^2 \left[\sum_{k=1}^{6} (A_{1k}\sigma^k + A_{-1k}\sigma^{-k}) - \frac{1}{c^2(\beta_1 - \beta_2)} \sum_{k=1}^{10} n_{1k}\sigma^k \right] \right\},$$

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$$\psi(\sigma)=\frac{MR^2}{8J}c^2i\left\{\frac{1}{\beta_1-\beta_2}\sigma^2+\varepsilon\left[\sum_{k=1}^{2}(M_{1k}\sigma^k+M_{-1k}\sigma^{-k})+\right.\right. \tag{III.321}$$

$$\left.\left.+\frac{1}{c(\beta_1-\beta_2)}\sum_{k=1}^{6}m_{1k}\sigma^k\right]+\varepsilon^2\left[\sum_{k=1}^{6}(B_{1k}\sigma^k+B_{-1k}\sigma^{-k})+\frac{1}{c^2(\beta_1-\beta_2)}\sum_{k=1}^{10}n_{1k}\sigma^k\right]\right\}.$$

Here we introduce the following definitions:

$$N_{-11}=\frac{2\delta_1}{\beta_1-\beta_2}a_4,\quad N_{-12}=\frac{2\delta_1}{\beta_1-\beta_2}a_5,\quad N_{1k}=-\frac{\beta_1+\beta_2}{\beta_1-\beta_2}N_{-1k}-\frac{2\beta_2}{\beta_1-\beta_2}M_{-1k}$$

$$(k=1,2),$$

$$A_{-11}=-\frac{\delta_1^2}{\beta_1-\beta_2}(4a_2a_3+8a_2a_5\varkappa_1+10a_3a_4\varkappa_1+18a_4a_5\varkappa_1^2)-$$

$$-\frac{\delta_1^2}{c(\beta_1-\beta_2)}6a_5(a_2-a_4)-N_{11}\delta_1(a_3+a_5\varkappa_1)-2N_{12}\delta_1a_4,$$

$$A_{-12}=-\frac{\delta_1^2}{\beta_1-\beta_2}(4a_2a_4+3a_3^2+12a_3a_5\varkappa_1+6a_4^2\varkappa_1+11a_5^2\varkappa_1^2)-$$

$$-N_{11}\delta_1a_4-2N_{12}\delta_1a_5, \tag{III.322}$$

$$A_{-13}=-\frac{\delta_1^2}{\beta_1-\beta_2}(4a_2a_5+6a_3a_4+14a_4a_5\varkappa_1)-N_{11}\delta_1a_5,$$

$$A_{-14}=-\frac{\delta_1^2}{\beta_1-\beta_2}(6a_3a_5+3a_4^2+8a_5^2\varkappa_1),$$

$$A_{-15}=-\frac{\delta_1^2}{\beta_1-\beta_2}6a_4a_5,\quad A_{-16}=-\frac{\delta_1^2}{\beta_1-\beta_2}3a_5^2,$$

$$A_{1k}=-\frac{\beta_1+\beta_2}{\beta_1-\beta_2}A_{-1k}-\frac{2\beta_2}{\beta_1-\beta_2}B_{-1k}\quad(k=1,\ldots,6);$$

$$m_{11}=2a_2,\quad m_{12}=2a_3,\quad m_{13}=-2(a_2-a_4),$$

$$m_{14}=-2(a_3-a_5),\quad m_{15}=-2a_4,\quad m_{16}=-2a_5,$$

$$n_{11}=-2(a_2a_3+a_3a_4+a_4a_5),\quad n_{12}=-2(a_2a_4+a_3a_5),$$

$$n_{13}=-2a_2a_5,\quad n_{14}=a_2^2,\quad n_{15}=2a_2a_3,\quad n_{16}=a_3^2+2a_2a_4, \tag{III.323}$$

$$n_{17}=2(a_2a_5+a_3a_4),\quad n_{18}=a_4^2+3a_3a_5,\quad n_{19}=2a_4a_5,\quad n_{1,10}=a_5^2.$$

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The coefficients M_{1k}, M_{-1k}, B_{1k}, B_{-1k} are found from N_{1k}, N_{-1k}, A_{1k}, A_{-1k}, if, in the latter, we substitute β_1, δ_1, κ_1, respectively, by β_2, δ_2, κ_2 and conversely.

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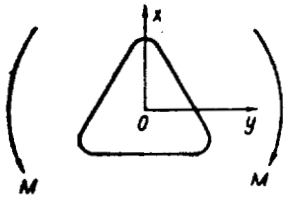


Figure III.35.

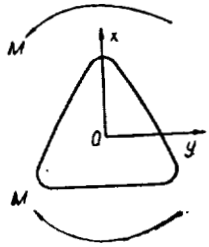


Figure III.36.

The stresses along the contour of the hole are

$$\sigma_\vartheta = \frac{MR}{J}\left(-c\sin\vartheta + \varepsilon\sum_{k=2}^{5} a_k \sin k\vartheta\right)\frac{B^2}{C_2} +$$

$$+\frac{MR}{2J}\cdot\frac{c_2}{LC^2}\{A[-A^4\beta_1\beta_2 + A^2B^2(1-2\beta_1\beta_2-\beta_1^2\beta_2^2) + B^4(2-\beta_1\beta_2-\beta_1^2-\beta_2^2)]\psi_1^{(2)} + \quad \text{(III.324)}$$

$$+C^4(\beta_1+\beta_2)(B\varphi_1 - A\beta_1\beta_2\psi_1^{(1)})\},$$

where

$$\varphi_1 = \cos 2\vartheta + \varepsilon\left[\sum_{k=1}^{2}(\beta_1 N_{-1k} + \beta_2 M_{-1k})\,k\cos k\vartheta + \frac{1}{2c}\sum_{k=1}^{6} km_{1k}\cos k\vartheta\right] +$$

$$+\varepsilon^2\left[\sum_{k=1}^{6}(\beta_1 A_{-1k} + \beta_2 B_{-1k})\,k\cos k\vartheta + \frac{1}{2c^2}\sum_{k=1}^{10} kn_{1k}\cos k\vartheta\right], \quad \text{(III.325)}$$

$$\psi_1^{(1)} = \varepsilon\sum_{k=1}^{2}(N_{-1k} + M_{-1k})\,k\sin k\vartheta + \varepsilon^2\sum_{k=1}^{6}(A_{-1k} + B_{-1k})\,k\sin k\vartheta,$$

$$\psi_1^{(2)} = \sin 2\vartheta + \frac{\varepsilon}{2c}\sum_{k=1}^{6} km_{1k}\sin k\vartheta + \frac{\varepsilon^2}{2c^2}\sum_{k=1}^{10} kn_{1k}\sin k\vartheta:$$

A, B, C^2, L are defined by formulas (III.296) and (III.297). In the second case, the deflecting moments are applied as indicated in Figure III.36. Then /316

$$p_{10} = -\frac{MR_2}{8J}\left(\sigma + \frac{1}{\sigma}\right)^2,$$

$$p_{11} = -\frac{MR^2}{4J}\left(\sigma + \frac{1}{\sigma}\right)\sum_{k=2}^{5} a_k\left(\sigma^k + \frac{1}{\sigma^k}\right), \quad \text{(III.326)}$$

$$p_{12} = -\frac{MR^2}{8J}\left[\sum_{k=2}^{5} a_k\left(\sigma^k + \frac{1}{\sigma^k}\right)\right]^2, \qquad p_{1k} = 0 \quad (k = 3, 4, \ldots),$$

$$p_{2k} = 0 \quad (k = 0, 1, 2, \ldots).$$

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Here J is the moment of inertia of the cross section of the plate normal to the Oy axis.

Further, we obtain

$$\varphi(\sigma)=\frac{MR^2}{8J}\left\{\frac{\beta_2}{\beta_1-\beta_2}\sigma^2+\varepsilon\left[\sum_{k=1}^{2}(N_{2k}\sigma^k+N_{-2k}\sigma^{-k})+\frac{\beta_2}{\beta_1-\beta_2}\sum_{k=1}^{6}m_{2k}\sigma^k\right]+\right.$$
$$\left.+\varepsilon^2\left[\sum_{k=1}^{6}(A_{2k}\sigma^k+A_{-2k}\sigma^{-k})+\frac{\beta_2}{\beta_1-\beta_2}\sum_{k=1}^{10}n_{2k}\sigma^k\right]\right\},$$
$$\psi(\sigma)=\frac{MR^2}{8J}\left\{-\frac{\beta_1}{\beta_1-\beta_2}\sigma^2+\varepsilon\left[\sum_{k=1}^{2}(M_{2k}\sigma^k+M_{-2k}\sigma^{-k})-\frac{\beta_1}{\beta_1-\beta_2}\sum_{k=1}^{6}m_{2k}\sigma^k\right]+\right. \qquad \text{(III.327)}$$
$$\left.+\varepsilon^2\left[\sum_{k=1}^{6}(B_{2k}\sigma^k+B_{-2k}\sigma^{-k})-\frac{\beta_1}{\beta_1-\beta_2}\sum_{k=1}^{10}n_{2k}\sigma^k\right]\right\};$$

hence

$$N_{-21}=-\frac{2\beta_2\delta_1}{\beta_1-\beta_2}a_4,\quad N_{-22}=-\frac{2\beta_2\delta_1}{\beta_1-\beta_2}a_5,\quad N_{2k}=\frac{\beta_1+\beta_2}{\beta_1-\beta_2}N_{-2k}+\frac{2\beta_2}{\beta_1-\beta_2}M_{-2k}$$
$$(k=1,2),$$
$$A_{-21}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}(4a_2a_3+8a_2a_5\varkappa_1+10a_3a_4\varkappa_1+18a_4a_5\varkappa_1^2)-$$
$$-\frac{\delta_1}{\beta_1-\beta_2}6a_5(a_2+a_4)-N_{21}\delta_1(a_3+a_5\varkappa_1)-2N_{22}\delta_1a_4, \qquad \text{(III.328)}$$
$$A_{-22}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}(4a_2a_4+3a_3^2+12a_3a_5\varkappa_1+6a_4^2\varkappa_1+11a_5^2\varkappa_1^2)-$$
$$-N_{21}\delta_1a_4-2N_{22}\delta_1a_5,$$
$$A_{-23}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}(4a_2a_5+6a_3a_4+14a_3a_5\varkappa_1)-N_{21}\delta_1a_4,$$
$$A_{-24}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}(3a_4^2+6a_3a_5+8a_5^2\varkappa_1),\quad A_{-25}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}6a_4a_5,$$
$$A_{-26}=\frac{\beta_2\delta_1^2}{\beta_1-\beta_2}3a_5^2,\quad A_{2k}=\frac{\beta_1+\beta_2}{\beta_1-\beta_2}A_{-2k}+\frac{2\beta_2}{\beta_1-\beta_2}B_{-2k}\quad(k=1,\ldots,6);$$

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$$m_{21}=2a_2,\quad m_{22}=2a_3,\quad m_{23}=2(a_2+a_4),\quad m_{24}=2(a_3+a_4),$$
$$m_{25}=2a_4,\quad m_{26}=2a_5, \qquad \text{(III.329)}$$
$$n_{21}=2(a_2a_3+a_3a_4+a_4a_5),\quad n_{22}=2(a_2a_4+a_3a_5),$$
$$n_{23}=2a_2a_5,\quad n_{24}=a_2^2,\quad n_{25}=2a_2a_3,$$
$$n_{26}=a_3^2+2a_2a_4,\quad n_{27}=2(a_2a_5+a_3a_4).$$
$$n_{28}=a_4^2+2a_3a_5,\quad n_{29}=2a_4a_5,\quad n_{2,10}=a_5^2.$$

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The coefficients M_{-2k}, M_{2k}, B_{-2k} and B_{2k} are found from N_{-2k}, N_{2k}, A_{-2k} and A_{2k} after circular substitution of the subscript i in the values β_i, δ_i, and κ_i (i = 1, 2).

The stresses along the contour of the hole are

$$\sigma_\vartheta = \frac{MR}{J}\left(\cos\vartheta + \varepsilon\sum_{k=2}^{5} a_k \cos k\vartheta\right)\frac{A_2}{C^2} + \frac{MR}{2J}\cdot\frac{1}{C^2L}\{B[A^4(2\beta_1^2\beta_2^2 - \\ -\beta_1^2 - \beta_1\beta_2 - \beta_2^2) + A^2B^2(\beta_1^2\beta_2^2 - 2\beta_1\beta_2 - 1) - B^4\beta_1\beta_2]\varphi_2^{(2)} + \\ + C^4(\beta_1 + \beta_2)(B\varphi_2^{(1)} - A\beta_1\beta_2\psi_2)\}, \tag{III.330}$$

where

$$\varphi_2^{(1)} = -\varepsilon\sum_{k=1}^{2}(\beta_1 N_{-2k} + \beta_2 M_{-2k})\,k\sin k\vartheta - \varepsilon^2\sum_{k=1}^{6}(\beta_1 A_{-2k} + \beta_2 B_{-2k})\,k\sin k\vartheta,$$

$$\varphi_2^{(2)} = \sin\vartheta + \frac{\varepsilon}{2}\sum_{k=1}^{6} m_{2k}k\sin k\vartheta + \frac{\varepsilon^2}{2}\sum_{k=1}^{10} n_{2k}k\sin k\vartheta,$$

$$\psi_2 = \cos 2\vartheta + \varepsilon\left[\sum_{k=1}^{2}(N_{-2k} + M_{-2k})\,k\cos k\vartheta + \frac{1}{2}\sum_{k=1}^{6} m_{2k}k\cos k\vartheta\right] + \\ + \varepsilon^2\left[\sum_{k=1}^{6}(A_{-2k} + B_{-2k})\,k\cos k\vartheta + \frac{1}{2}\sum_{k=1}^{10} n_{2k}k\cos k\vartheta\right]. \tag{III.331}$$

Hole with the Shape of a Right Triangle[1]. An orthotropic plate is weakened by a hole, the contour of which is given by equations (III.302).

In the case where the forces are applied on the sides of the plate parallel to the Oy axis (see Figure III.35), the stresses are

$$\sigma_\vartheta = \frac{MR}{J}(-\sin\vartheta + \varepsilon\sin 2\vartheta)\frac{B^2}{C^2} + \frac{MR}{2J}\cdot\frac{1}{C^2L}\{A[-A^4\beta_1\beta_2 + \\ + A^2B^2(1 - 2\beta_1\beta_2 - \beta_1^2\beta_2^2) + B^4(2 - \beta_1\beta_2 - \beta_1^2 - \beta_2^2)[\sin 2\vartheta + \varepsilon(\sin\vartheta - 3\sin 3\vartheta) + \\ + 2\varepsilon^2\sin 4\vartheta] + BC^4(\beta_1 + \beta_2)[\cos 2\vartheta + \varepsilon(\cos\vartheta - 3\cos 3\vartheta) + 2\varepsilon^2\cos 4\vartheta]\}. \tag{III.332}$$

[1] See S. G. Lekhnitskiy [1], p. 224.

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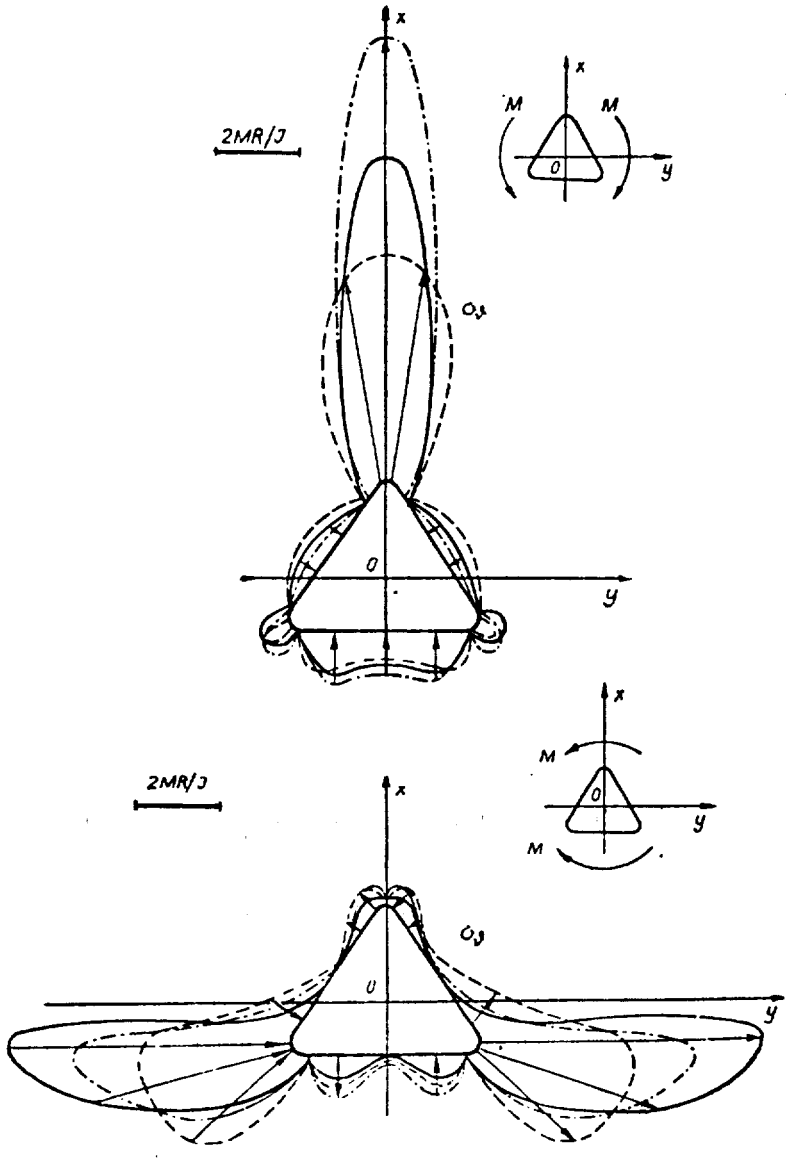


Figure III.37.

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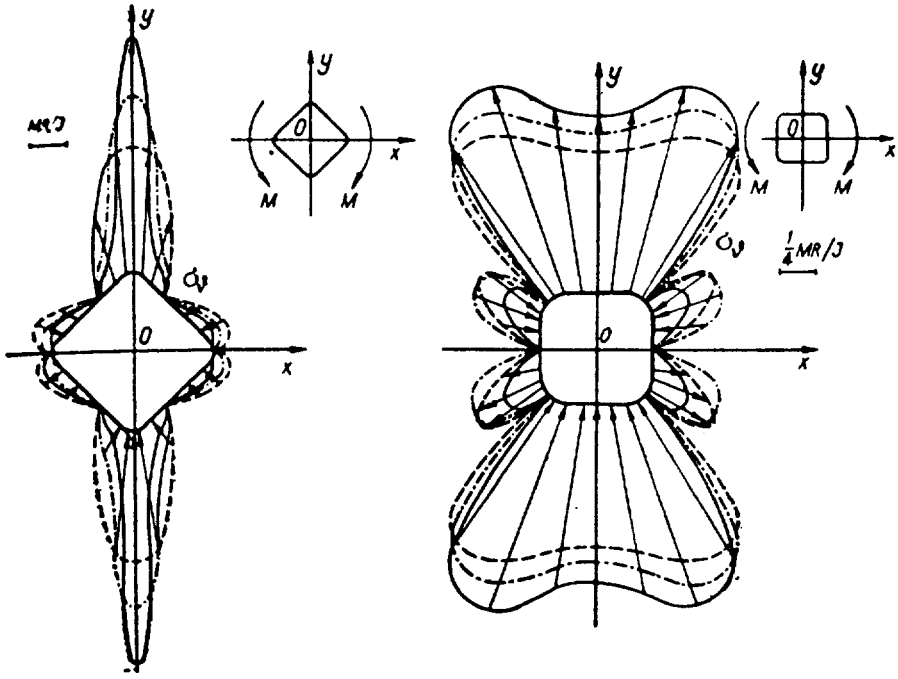


Figure III.38.

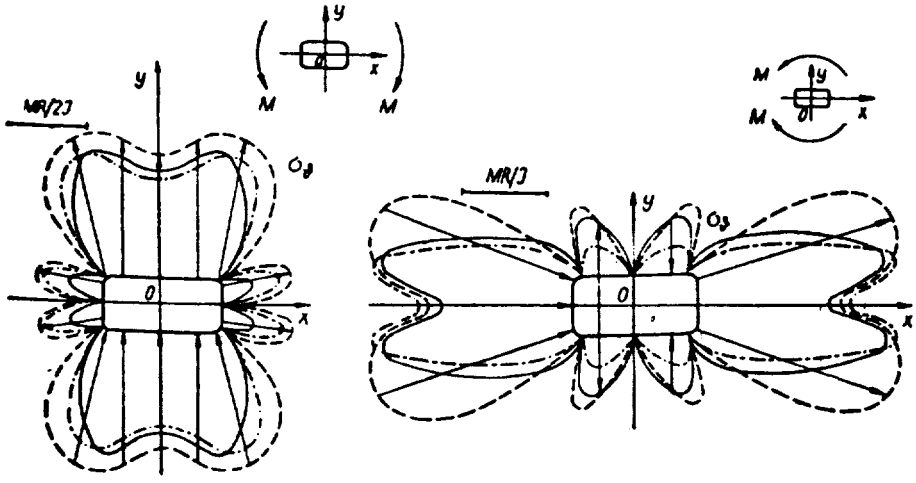


Figure III.39.

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If the forces are applied on the sides of the plate that are parallel to the Ox axis (see Figure III.36), then the stress is /320

$$\sigma_\vartheta = \frac{MR}{J}(\cos\vartheta + \varepsilon\cos 2\vartheta)\frac{A^2}{C^2} + \frac{MR}{2J}\cdot\frac{1}{C^2L}\{B[A^4(2\beta_1^2\beta_2^2 - \beta_1^2 - \beta_1\beta_2 - \beta_2^2) + A^2B^2(\beta_1^2\beta_2^2 - 2\beta_1\beta_2 - 1) - B^4(\beta_1\beta_2)][\sin 2\vartheta + \varepsilon(\sin\vartheta + 3\sin 3\vartheta) + 2\varepsilon^2\sin 4\vartheta] + AC^4\beta_1\beta_2(\beta_1 + \beta_2)[\cos 2\vartheta + \varepsilon(\cos\vartheta + 3\cos 3\vartheta) + 2\varepsilon^2\cos 4\vartheta]\}. \quad \text{(III.332a)}$$

The stresses σ_ϑ calculated by formula (III.332a) are presented in Figure III.37 and pertain to a plate made of aviation plywood, the elastic constants of which were given earlier.

Square Hole[1]. The contour of the hole is given by equations (III.314).

In the first case,

$$\sigma_\vartheta = \frac{MR}{J}(-\sin\vartheta + \varepsilon\sin 3\vartheta)\frac{B^2}{C^2} + \frac{MR}{2J}\cdot\frac{1}{C^2L}\left\{A[-A^4\beta_1\beta_2 + A^2B^2(1 - \beta_1^2\beta_2^2 - 2\beta_1\beta_2) + B^4(2 - \beta_1\beta_2 - \beta_1^2 - \beta_2^2)][\sin 2\vartheta + 2\varepsilon(\sin 2\vartheta - 2\sin 4\vartheta) + 3\varepsilon^2\sin 6\vartheta] + 6\varepsilon^2AC^4(\beta_1 + \beta_2)\beta_1\beta_2\frac{\delta_1^2 - \delta_2^2}{\beta_1 - \beta_2}\sin 2\vartheta + BC^4(\beta_1 + \beta_2)\left[\cos 2\vartheta + 2\varepsilon(\cos 2\vartheta - 2\cos 4\vartheta) + 3\varepsilon^2\left(2\frac{\beta_2\delta_2^2 - \beta_1\delta_1^2}{\beta_1 - \beta_2}\cos 2\vartheta + \cos 6\vartheta\right)\right]\right\}. \quad \text{(III.333)}$$

In the second case,

$$\sigma_\vartheta = \frac{MR}{J}(\cos\vartheta + \varepsilon\cos 3\vartheta)\frac{A^2}{C^2} + \frac{MR}{2J}\cdot\frac{1}{C^2L}\left\{B[A^4(2\beta_1^2\beta_2^2 - \beta_1^2 - \beta_1\beta_2 - \beta_2^2) + A^2B^2(\beta_1^2\beta_2^2 - 2\beta_1\beta_2 - 1) - B^4\beta_1\beta_2][\sin 2\vartheta + 2\varepsilon(\sin 2\vartheta + 2\sin 4\vartheta) + 3\varepsilon^2\sin 6\vartheta] - 6\varepsilon^2BC^4(\beta_1 + \beta_2)\beta_1\beta_2\frac{\delta_1^2 - \delta_2^2}{\beta_1 - \beta_2}\sin 2\vartheta + AC^4(\beta_1 + \beta_2)\beta_1\beta_2\left[\cos 2\vartheta + 2\varepsilon(\cos 2\vartheta + 2\cos 4\vartheta) + 3\varepsilon^2\left(2\frac{\beta_2\delta_1^2 - \beta_1\delta_2^2}{\beta_1 - \beta_2}\cos 2\vartheta + \cos 6\vartheta\right)\right]\right\}. \quad \text{(III.334)}$$

[1]See S. G. Lekhnitskiy [1], p. 238.

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The results of calculations for the above-described plywood and isotropic plates are presented in Figure III.38.

Rectangular Hole with Ratio of Sides a/b = 2. The contour is given by equations (III.285), in which the constants acquire the values of (III.317). In the case where the forces are applied on the sides of the plate that are parallel to the Oy axis, stresses σ_ϑ on the contour of the hole are determined by formulas (III.324) and (III.325). If the forces are applied on the sides parallel to the Ox axis, then the stresses are determined by formulas (III.330) and (III.331). The results of calculations[1] are presented in Figure III.39. The principles outlined in the preceding section remain in force in the case of deflection of a plate. /321

§9. Solution of Second Basic Problem for Region with Nearly Elliptical Hole

As in the case of the first basic problem, we will examine[2] an infinite anisotropic plate with a hole whose contour is given by equations (III.265).

Let the displacements of points of the contour of the hole be known. We will also assume that the resultant vector of external forces causing the given displacements of the points of the contour are equal to zero, and that there are no forces at infinity.

The functions $\phi(z_1)$ and $\psi(z_2)$ through which the stresses are expressed are found from boundary conditions (III.92), the right hand sides of which can be expanded by degrees of the small parameter ε:

$$g_1(s) = \sum_{k=0} t_{1k}\varepsilon^k,$$

$$g_2(s) = \sum_{k=0} t_{2k}\varepsilon^k, \qquad \text{(III.335)}$$

where t_{1k}, t_{2k} are known functions.

Then, using reasoning analogous to that used in §6, we determine the functions $\psi_{pk}(\zeta_p)$ (p = 1, 2) that are holomorphic within unit circle γ and which satisfy the boundary conditions on the contour of γ:

$$\begin{aligned} &2\mathrm{Re}\,[p_1\psi_{10}(\sigma) + p_2\psi_{20}(\sigma)] = g_{10}(\vartheta),\\ &2\mathrm{Re}\,[q_1\psi_{10}(\sigma) + q_2\psi_{20}(\sigma)] = g_{20}(\vartheta),\\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad \text{(III.336)}$$

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[1] See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [4].
[2] The solution was found by A. S. Kosmodamianskiy [2].

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$$2\mathrm{Re}\,[p_1\psi_{1k}(\sigma)+p^2\psi^{2q}(\sigma)]=g_{ik}(\vartheta),$$
$$2\mathrm{Re}\,[q_1\psi_{1k}(\sigma)+q_2\psi_{2k}(\sigma)]=g_{2k}(\vartheta), \qquad \text{(III.337)}$$

where /322

$$g_{10}(\vartheta)=t_{10},\qquad g_{20}(\vartheta)=t_{20},$$
$$g_{1k}(\vartheta)=t_{1k}-2\mathrm{Re}\left\{p_1(\chi+\lambda_1\bar{\chi})\frac{d\psi_{1,k-1}}{d\sigma_1}+\ldots+\frac{p_1}{k!}(\chi+\lambda_1\bar{\chi})^k\frac{d^k\psi_{10}}{d\sigma_1^k}+\right.$$
$$\left.+p_2(\chi+\lambda_2\bar{\chi})\frac{d\psi_{2,k-1}}{d\sigma_2}+\ldots+\frac{p_2}{k!}(\chi+\lambda_2\bar{\chi})^k\frac{d^k\psi_{20}}{d\sigma_2^k}\right\}, \qquad \text{(III.338)}$$
$$g_{2k}(\vartheta)=t_{2k}-2\mathrm{Re}\left\{q_1(\chi+\lambda_1\bar{\chi})\frac{d\psi_{1,k-1}}{d\sigma_1}+\ldots+\frac{q_1}{k!}(\chi+\lambda_1\bar{\chi})^k\frac{d^k\psi_{10}}{d\sigma_1^k}+\right.$$
$$\left.+q_2(\chi+\lambda_2\bar{\chi})\frac{d\psi_{2,k-1}}{d\sigma_2}+\ldots+\frac{q_2}{k!}(\chi+\lambda_2\bar{\chi})^k\frac{d^k\psi_{20}}{d\sigma_2^k}\right\}$$
$$(\sigma=e^{i\vartheta}),\quad (\zeta=\varrho e^{i\vartheta}).$$

From boundary conditions (III.336)-(III.338), as before, we find after simple transformations,

$$\psi_{1k}(\zeta)=\frac{i}{4\pi(p_1q_2-p_2q_1)}\int_\gamma[p_2g_{2k}(\vartheta)-q_2g_{1k}(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+\lambda_{1k},$$
$$\psi_{2k}(\zeta)=\frac{-i}{4\pi(p_1q_2-p_2q_1)}\int_\gamma[p_1g_{2k}(\vartheta)-q_1g_{1k}(\vartheta)]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma}+\lambda_{2k}. \qquad \text{(III.339)}$$

On the basis of formula (III.283) in expressions (III.339) we convert from the variable ζ to ζ_1 and ζ_2, respectively. Using further formulas (III.268), we find finally functions $\phi(z_1)$ and $\psi(z_2)$.

The stress state in the plate near the contour of the hole is defined by formulas[1]

$$\sigma_\varrho=2\mathrm{Re}\left[(1+\lambda_1)\left(\frac{dx}{ds}+s_1\frac{dy}{ds}\right)^2\varphi'(z_1)+(1+\lambda_2)\left(\frac{dx}{ds}+s_2\frac{dy}{ds}\right)^2\psi'(z_2)\right].$$
$$\sigma_\vartheta=2\mathrm{Re}\left[(1+\lambda_1)\left(\frac{dy}{ds}-s_1\frac{dx}{ds}\right)^2\varphi'(z_1)+(1+\lambda_2)\left(\frac{dy}{ds}-s_2\frac{dx}{ds}\right)^2\psi'(z_2)\right].$$

[1]See S. G. Lekhnitskiy [2].

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$$\tau_{\rho\theta} = 2\mathrm{Re}\left[(1+\lambda_1)\left(\frac{dx}{ds}+s_1\frac{dy}{ds}\right)\left(s_1\frac{dx}{ds}-\frac{dy}{ds}\right)\varphi'(z_1)+\right.$$
$$\left.+(1+\lambda_2)\left(\frac{dx}{ds}+s_2\frac{dy}{ds}\right)\left(s_2\frac{dx}{ds}-\frac{dy}{ds}\right)\psi'(z_2)\right]. \qquad \text{(III.340)}$$

§10. Tension of Anisotropic Plate with Curvilinear Hole Reinforced by Rigid Ring /323

We will examine a homogeneous anisotropic plate with a hole, the contour of which is given by equations (III.285). In the hole of the plate is soldered a rigid ring. The plate at infinity is under tension by uniformly distributed forces p parallel to the Ox axis and by forces q parallel to the Oy axis (Figure III.40).

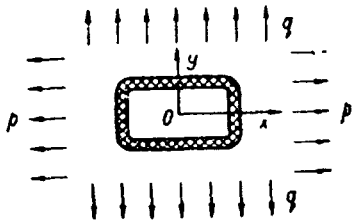


Figure III.40.

The solution of the problem, as in §7, will be found by superposing (combining two stress fields)[1]. The stress state of a thick plate is characterized by stress field (III.287); the projections of displacements are

$$u^0 = (a_{11}p + a_{12}q)x,$$
$$v^0 = (a_{12}p + a_{22}q)y. \qquad \text{(III.341)}$$

We will determine the stress state of the plate for the case when displacements $-u^0$, $-v^0$ on the contour of the hole are given, but when there are no forces at infinity. This problem reduces to the determination of functions $\phi(z_1)$ and $\psi(z_1)$ from boundary conditions (III.92).

We will expand the right hand sides of the boundary conditions by degrees of small parameter ε:

$$g_1(s) = t_{10} + \varepsilon t_{11},$$
$$g_2(s) = t_{20} + \varepsilon t_{21}, \qquad \text{(III.342)}$$

where
$$t_{10} = -\frac{R}{2}(a_{11}p + a_{12}q)\left(\sigma + \frac{1}{\sigma}\right), \qquad t_{20} = -\frac{Ri}{2}c(a_{12}p + a_{22}q)\left(\sigma - \frac{1}{\sigma}\right),$$
$$t_{11} = -\frac{R}{2}(a_{11}p + a_{12}q)\sum_{k=2}^{5} a_k\left(\sigma^k + \frac{1}{\sigma^k}\right),$$
$$t_{21} = \frac{Ri}{2}(a_{12}p + a_{22}p)\sum_{k=2}^{5} a_k\left(\sigma^k - \frac{1}{\sigma^k}\right), \qquad \text{(III.343)}$$
$$t_{1k} = t_{2k} = 0 \qquad (k = 2, 3, \ldots).$$

[1] See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [1].

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Then, on the basis of (III.339)

$$\psi_{10} = -\frac{R}{2}\frac{1}{p_1q_2 - p_2q_1}\frac{1}{4\pi i}\int_\gamma \left[q_2(a_{11}p + a_{12}q)\left(\sigma + \frac{1}{\sigma}\right) - \right.$$

$$\left. - p_2 ci\,(a_{12}p + a_{22}q)\left(\sigma - \frac{1}{\sigma}\right)\right]\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_{10},$$

$$\psi_{11} = -\frac{1}{4\pi i}\int_\gamma \left\{\frac{R}{2}\cdot\frac{1}{p_1q_2 - p_2q_1}\left[q_2(a_{11}p + a_{12}q)\sum_{k=2}^{5} a_k\left(\sigma^k + \frac{1}{\sigma^k}\right) + \right.\right.$$

$$+ ip_2(a_{12}p + a_{22}q)\sum_{k=2}^{5} a_k\left(\sigma^k - \frac{1}{\sigma^k}\right)\Big] + (\varphi + \lambda_1\bar{\varphi})\frac{d\psi_{10}}{d\sigma_1} + \frac{\bar{p}_1q_2 - p_2\bar{q}_1}{p_1q_2 - p_2q_1}(\bar{\varphi} + \bar{\lambda}_1\varphi)\frac{d\bar{\psi}_{10}}{d\bar{\sigma}_1} + \quad \text{(III.344)}$$

$$\left. + \frac{\bar{p}_2q_2 - p_2\bar{q}_2}{p_1q_2 - p_2q_1}(\bar{\varphi} + \bar{\lambda}_2\varphi)\frac{d\bar{\psi}_{20}}{d\bar{\sigma}_2}\right\}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_{11},$$

$$\psi_{12} = -\frac{1}{4\pi i}\int_\gamma \left\{\left[(\varphi + \lambda_1\bar{\varphi})\frac{d\psi_{11}}{d\sigma_1} + \frac{1}{2!}(\varphi + \lambda_1\bar{\varphi})^2\frac{d\psi_{10}}{d\sigma_1^2}\right] + \frac{\bar{p}_1q_2 - p_2\bar{q}_1}{p_1q_2 - p_2q_1}\left[(\bar{\varphi} + \bar{\lambda}_1\varphi)\frac{d\bar{\psi}_{11}}{d\bar{\sigma}_1} + \right.\right.$$

$$+ \frac{1}{2!}(\bar{\varphi} + \bar{\lambda}_1\varphi)^2\frac{d^2\bar{\psi}_{10}}{d\bar{\sigma}_1^2}\Big] + \frac{\bar{p}_2q_2 - p_2\bar{q}_2}{p_1q_2 - p_2q_1}\left[(\bar{\varphi} + \bar{\lambda}_2\varphi)\frac{d\bar{\psi}_{21}}{d\bar{\sigma}_2} + \right.$$

$$\left.\left. + \frac{1}{2!}(\bar{\varphi} + \bar{\lambda}_2\varphi)^2\frac{d^2\bar{\psi}_{20}}{d\bar{\sigma}_2^2}\right]\right\}\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + \lambda_{12}.$$

The expressions for ψ_{20}, ψ_{21} and ψ_{22} are found from (III.344) by substituting p_1, q_1, λ_1, σ_1, ψ_{1k} by p_r, q_2, λ_2, σ_2, ψ_{2k}, respectively, and conversely.

Proceeding as in §6, we find

$$\varphi(\sigma) = \frac{R}{2}\left\{N_0\sigma + \varepsilon\sum_{k=1}^{5}(N_k\sigma^k + N_{-k}\sigma^{-k}) + \varepsilon^2\sum_{k=1}^{7}(A_k\sigma^k + A_{-k}\sigma^{-k})\right\}, \quad \text{(III.345)}$$

where

$$N_0 = -k_1m_1, \quad N_{-1} = k_1\delta_1(a_3 + \varkappa_1a_5)m_1,$$

$$N_{-2} = k_1\delta_1a_4m_1, \quad N_{-3} = k_1\delta_1a_5m_1, \quad N_{-4} = N_{-5} = 0,$$

$$N_1 = k_1^2[2p_2q_2^*\delta_2(a_3 + \varkappa_2a_5)m_2 - (p_1q_2^* + p_2q_1^*)\delta_1(a_3 + \varkappa_1a_5)m_1],$$

$$N_2 = k_1^2\left[2p_2q_2^*\delta_2a_4m_2 - (p_1q_2^* + p_2q_1^*)\delta_1a_4m_1 - \frac{a_2}{k_1}n_1\right], \quad \text{(III.346)}$$

$$N_3 = k_1^2\left[2p_2q_2^*\delta_2a_5m_2 - (p_1q_2^* + p_2q_1^*)\delta_1a_5m_1 - \frac{a_2}{k_1}n_1\right],$$

$$N_4 = -k_1a_4n_1, \quad N_5 = -k_1a_5n_1;$$

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$$A_{-1} = -\delta_1 [3a_5N_3 + 2a_4N_2 + (a_3 + \varkappa_1 a_5) N_1 + k_1\delta_1 m_1 (a_2^2 + 4\varkappa_1 a_2 a_4 + 2\varkappa a_3^2 + 7\varkappa_1^2 a_3 a_5 + 4\varkappa_1^2 a_4^2 + 6\varkappa_1^3 a_5^2)],$$

$$A_{-2} = -\delta_1 [2a_5N_2 + a_4N_1 + k_1\delta_1 m_1 (2a_2a_3 + 4\varkappa_1 a_2 a_5 + 5\varkappa_1 a_3 a_4 + 9\varkappa_1 a_4 a_5)].$$

$$A_{-3} = -\delta_1 [a_5N_1 + k_1\delta_1 m_1 (2a_2a_4 + a_3^2 + 5\varkappa_1 a_3 a_5 + 3\varkappa_1 a_4^2 + 5\varkappa_1^2 a_5^2)]. \qquad \text{(III.347)}$$

$$A_{-4} = -k_1\delta_1^2 m_1 (2a_2a_5 + 2a_3a_4 + 6\varkappa_1 a_4 a_5).$$

$$A_{-5} = -k_1\delta_1^2 m_1 (2a_3a_5 + a_4^2 + 3\varkappa_1 a_5^2),$$

$$A_{-6} = -2k_1\delta_1^2 m_1 a_4 a_5, \qquad A_{-7} = -k_1\delta_1^2 m_1 a_5^2,$$

$$A_k = -k_1 [(p_1q_2^* + p_2q_1^*) A_{-k} + 2p_2q_2^* B_{-k}] \qquad (k = 1, 2, 3, 4, 5, 6, 7).$$

Hence

$$m_1 = p(a_{11}q_2^* - a_{12}cp_2) + q(a_{12}q_2^* - a_{22}cp_2),$$

$$m_2 = p(a_{11}q_1^* - a_{12}cp_1) + q(a_{12}q_1^* - a_{22}cp_1),$$

$$n_1 = p(a_{11}q_2^* + a_{12}p_2) + q(a_{12}q_2^* + a_{22}p_2),$$

$$n_2 = p(a_{11}q_1^* + a_{12}p_1) + q(a_{12}q_1^* + a_{22}p_1), \qquad \text{(III.348)}$$

$$q_1^* = -iq_1, \qquad q_2^* = -iq_2, \qquad k_1 = (p_1q_2^* - p_2q_1^*)^{-1},$$

$$\delta_\varrho = \frac{1 - \beta_\varrho}{1 + c\beta_\varrho}, \qquad \varkappa_\varrho = \frac{1 - c\beta_\varrho}{1 + c\beta_\varrho} \qquad (\varrho = 1, 2).$$

The function $\psi(\sigma)$ is found from (III.345) by substituting in the latter N_k and A_k, respectively, by M_k and B_k ($k = 0, \pm 1, \ldots, \pm 7$), where M_k and B_k are found from N_k and A_k by substituting p_1, q_1^*, δ_1, κ_1 by p_2, q_2^*, δ_2, κ_2 and conversely.

We introduce the following definitions:

$$\frac{d\varphi}{d\vartheta} = \frac{R}{2}(\varphi_1 + i\psi_1), \qquad \frac{d\psi}{d\vartheta} = \frac{R}{2}(\varphi_2 + i\psi_2),$$

$$A = -c\cos\vartheta + \varepsilon\sum_{k=2}^{5} ka_k \cos k\vartheta, \qquad B = \sin\vartheta + \varepsilon\sum_{k=2}^{5} ka_k \sin k\vartheta, \qquad \text{(III.349)}$$

$$C^2 = A^2 + B^2, \qquad \alpha = \frac{B}{A}.$$

Then, recalling (III.349), we find the stresses

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$$\sigma_\varrho = \frac{1}{C^2}\{(pA^2 + qB^2) - A(\beta_1\psi_1 + \beta_2\psi_2) - B(\varphi_1 + \varphi_2)\},$$
$$\sigma_\vartheta = \frac{1}{C^2}\{(qA^2 + pB^2) + A_{11}\varphi_1 + A_{12}\varphi_2 + A_{13}\psi_1 + A_{14}\psi_2\}, \qquad \text{(III.350)}$$
$$\tau_{\varrho\vartheta} = \frac{1}{C^2}\{AB(p - q) + A(\varphi_1 + \varphi_2) - B(\beta_1\psi_1 + \beta_2\psi_2)\},$$

where

$$A_{11} = B\frac{\alpha^2\beta_1^2 + 2\beta_1^2 - 1}{\alpha^2 + \beta_1^2}, \qquad A_{12} = B\frac{\alpha^2\beta_2^2 + 2\beta_2^2 - 1}{\alpha^2 + \beta_2^2},$$
$$A_{13} = A\beta_1\frac{2\alpha^2 - \alpha^2\beta_1^2 + 1}{\alpha^2 + \beta_1^2}, \qquad A_{14} = A\beta_2\frac{2\alpha^2 - \alpha^2\beta_2^2 + 1}{\alpha^2 + \beta_2^2}; \qquad \text{(III.351)}$$

$$\varphi_1 = \sum_{k=1}^{7} b_k \sin k\vartheta, \qquad \psi_1 = \sum_{k=1}^{7} c_k \cos k\vartheta. \qquad \text{(III.352)}$$

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Here

$$b_1 = -[N_0 + \varepsilon(N_1 + N_{-1}) + \varepsilon^2(A_1 + A_{-1})],$$
$$b_m = -m[\varepsilon(N_m + N_{-m}) + \varepsilon^2(A_m + A_{-m})] \qquad (m = 2, 3, 4, 5),$$
$$b_n = -n\varepsilon^2(A_n - A_{-n}) \qquad (n = 6, 7),$$
$$c_1 = [-N_0 + \varepsilon(N_1 - N_{-1}) + \varepsilon^2(A_1 - A_{-1})], \qquad \text{(III.353)}$$
$$c_m = m[\varepsilon(N_m - N_{-m}) + \varepsilon^2(A_m - A_{-m})] \qquad (m = 2, 3, 4, 5),$$
$$c_n = n\varepsilon^2(A_n - A_{-n}) \qquad (n = 6, 7).$$

The expressions for ϕ_2, ψ_2 are found from (III.352) and (III.353) by substituting in the latter N_k, A_k by M_k, B_k, respectively.

By way of example, consider the stress state along the contour of the seam of a rigid ring and plate made of aviation plywood ($a_{11} = 0.83333 \cdot 10^{-9}/9.81$, $a_{12} = -0.05917 \cdot 10^{-9}/9.81$, $a_{22} = 1.66667 \cdot 10^{-9}/9.81$, $a_{66} = 14.2857 \cdot 10^{-9}/9.81$); for comparison, the corresponding data for an isotropic plate are also given[1]. For holes in the form of a right triangle, square and rectangle, Young's modulus and Poisson's ratio for an isotropic plate were assumed to be equal to the arithmetic mean of these values for plywood. In other cases $\nu = 0.3$.

[1]The formulas for an isotropic plate are given by Ye. F. Burmistrov [1].

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Right Triangular Hole. An orthotropic plate is weakened by a hole, the contour of which is given by equations (III.302). The function $\phi(\zeta)$ on the contour of the hole has the form[1]

$$\varphi(\sigma) = -\frac{Rk_1}{2}\left\{m_1\sigma + \varepsilon n_1\sigma^2 + \varepsilon^2 k_1 [2p_2q_2^*\lambda_2^2 m_2 - (p_1q_2^* + p_2q_1^*)\lambda_1^2 m_1]\sigma + \right.$$
$$\left. + \varepsilon^2\lambda_1^2 m_1 \frac{1}{\sigma}\right\}. \qquad \text{(III.354)}$$

The expression for the function $\psi(\zeta)$ is found from (III.354) by substituting m_1, n_1, p_1, q_1^*, λ_1, k_1 by m_2, n_2, p_2, q_2^*, λ_2, $-k_1$, respectively, and conversely.

The values of stresses σ_ρ/p and σ_ρ/q along the seam of the ring and plate are presented in Figure III.41[2] for cases where the forces are applied in the direction of the Ox axis (q = 0) and along the Oy axis (p = 0), respectively.

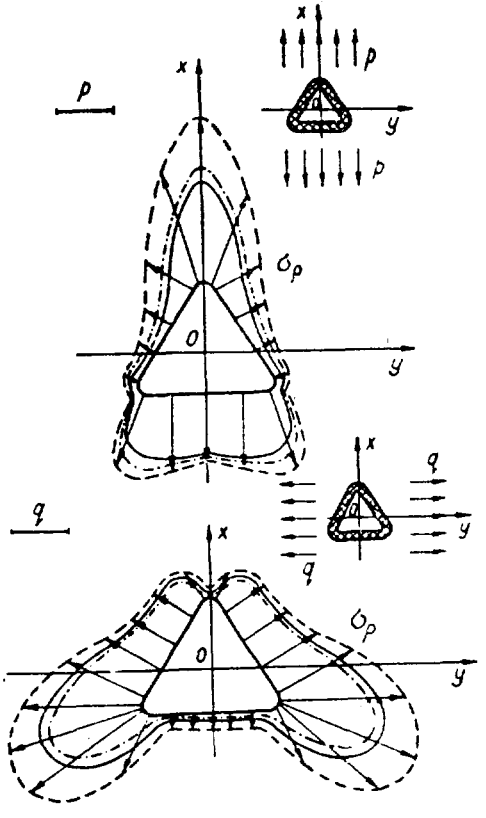


Figure III.41.

Hole with the Shape of an Isosceles Triangle[3]. The contour of the hole is given by equations (III.308). The expression for the function $\phi(\zeta)$ is found from (III.345)-(III.347), where $a_4 = a_5 = 0$:

$$\varphi(\sigma) = \frac{R}{2}\left\{N_0\sigma + \varepsilon\sum_{k=1}^{3}(N_k\sigma^k + N_{-k}\sigma^{-k}) + \varepsilon^2\sum_{k=1}^{3}(A_k\sigma^k + A_{-k}\sigma^{-k})\right\}, \qquad \text{(III.355)}$$

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where

$$N_0 = -k_1 m_1, \quad N_{-1} = k_1\delta_1 m_1 a_3, \quad N_{-2} = N_{-3} = 0,$$
$$N_1 = k_1^2[2p_2q_2^*\delta_2 m_2 - (p_1q_2^* + p_2q_1^*)\delta_1 m_1]a_3,$$
$$N_2 = -k_1 n_1 a_2, \quad N_3 = -k_1 n_1 a_3,$$

$$A_{-1} = -\delta_1 N_1 a_3 - k_1\delta_1^2 m_1(a_2^2 + 2\varkappa_1 a_3^2), \qquad \text{(III.356)}$$
$$A_{-2} = -k_1\delta_1^2 m_1 2a_2 a_3,$$
$$A_{-3} = -k_1\delta_1^2 m_1 a_3^2,$$
$$A_k = -k_1[2p_2q_2^* B_{-k} + (p_1q_2^* + p_2q_1^*)A_{-k}]$$
$$(k = 1, 2, 3).$$

[1] See A. S. Kosmodamianskiy [8].
[2] The symbols are the same as those used for the graphs presented in §7.
[3] See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [1].

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We will examine the following holes: a) hole with the shape of an isosceles right triangle ($c = 2$, $\varepsilon = 0.44$, $a_2 = 1$, $a_3 = 0.168$); b) hole with the shape of an isosceles triangle with the vertex angle $\pi/6$ ($c = 0.4$, $\varepsilon = 0.1905$, $a_2 = 1$, $a_3 = -0.2143$).

The stresses σ_ρ along the seam of the ring and plate, when the forces of tension are parallel to one of the axes, are defined. The results of these calculations for the first and second cases are presented in Figures III.42 and III.43, respectively.

Square Hole[1]. By taking the equations of the contour of the hole in the form (III.314) and assuming in (III.355) and (III.356) that $c = 1$, $a_2 = 0$, $a_3 = 1$, we obtain the function $\phi(\zeta)$ on the contour of the hole:

$$\varphi(\sigma) = \frac{R}{2}\left\{N_0\sigma + \varepsilon \sum_{k=1,3} (N_k\sigma^k + N_{-k}\sigma^{-k}) + \varepsilon^2 \sum_{k=1,3} (A_k\sigma^k + A_{-k}\sigma^{-k}).\right. \qquad \text{(III.357)}$$

Here

$$N_0 = -k_1 m_1, \quad N_{-1} = k_1 m_1 \lambda_1, \quad N_{-3} = 0,$$

$$N_1 = k_1^2 [2p_2 q_2^* \lambda_2 m_2 - (p_1 q_2^* + p_2 q_1^*)\lambda_1 m_1], \quad N_3 = -k_1 n_1,$$

$$A_{-1} = -\lambda_1 N_1 - 2k_1 \lambda_1^3 m_1, \quad A_{-3} = -k_1 \lambda_1^2 m_1, \qquad \text{(III.358)}$$

$$A_k = -k_1 [2p_2 q_2^* B_{-k} + (p_1 q_2^* + p_2 q_1^*) A_{-k}] \quad (k = 1, 3).$$

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The expression for the function $\psi(\zeta)$ on the contour of the hole, as before, is found from formula (III.357) by substituting in it λ_1, p_1, q_1^*, k_1, m_1, n_1 by λ_2, p_2, q_2^*, $-k_1$, m_2, n_2, respectively, and conversely.

Stresses σ_ρ, σ_ϑ, $\tau_{\rho\vartheta}$ are defined by formulas (III.350).

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In Figure III.44 are presented the values of stresses σ_ρ/p along the seam for the cases where the forces of tension p are applied in the direction of the side of the square, and in the direction of the diagonal of the square.

Rectangular Hole with Side Ratio a/b = 2. The contour of the hole is determined by equations (III.285), in which the constants acquire the values (III.317). The function $\phi(\zeta)$ is found from (III.345) assuming that $a_2 = a_4 = 0$ in expressions (III.346) and (III.347).

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[1]See A. S. Kosmodamianskiy [8].

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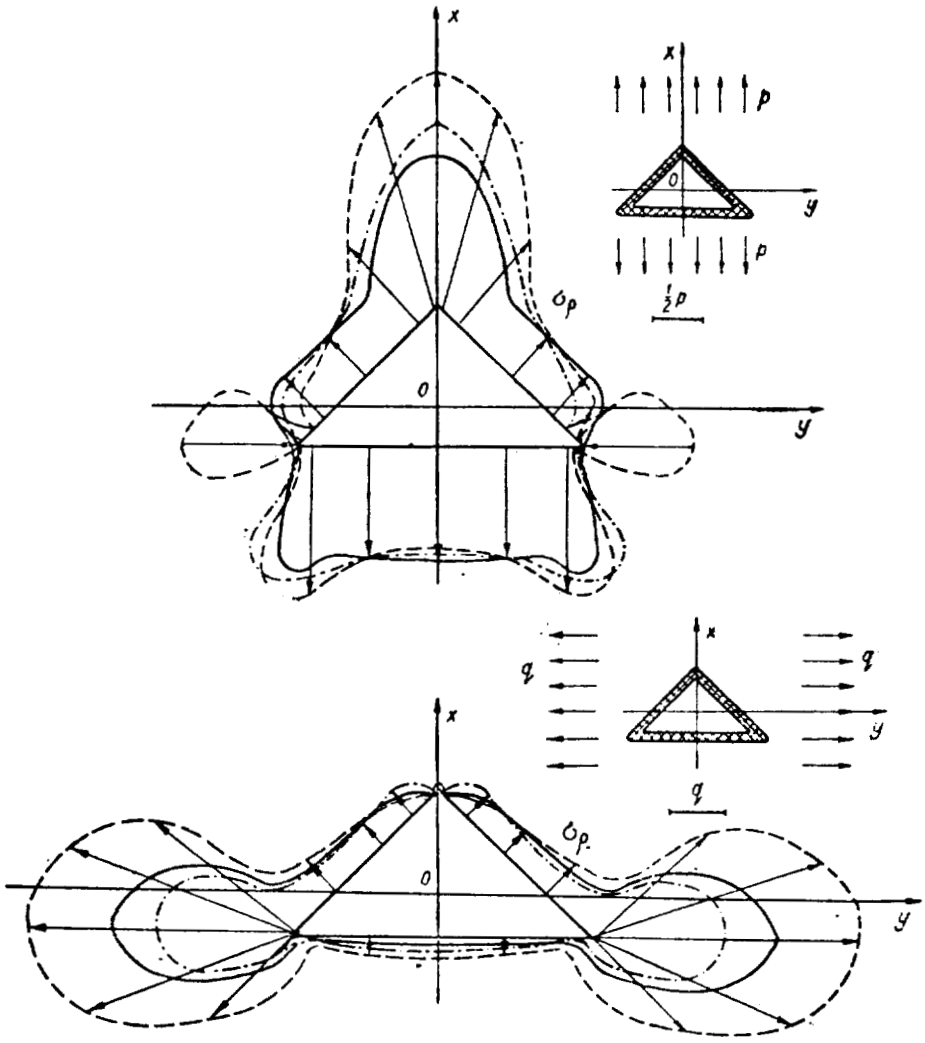


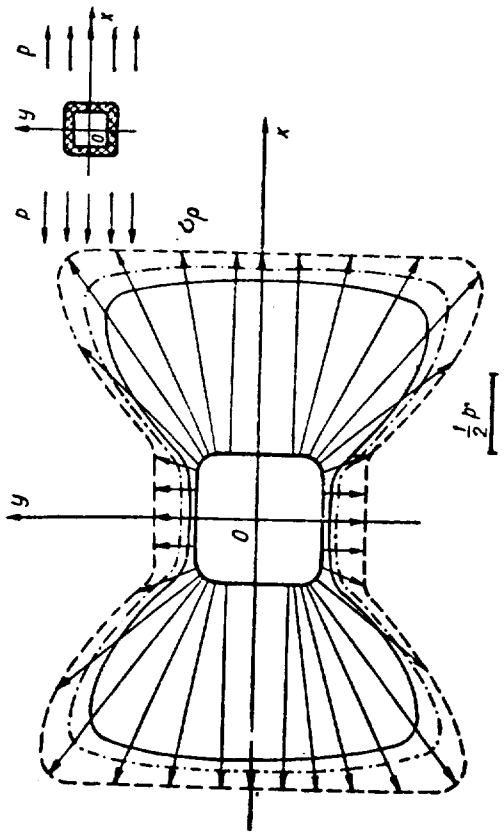
Figure III.42.

The results of calculations[1] are presented in Figure III.45, where the distribution of stresses σ_ρ along the contour of the seam of the rigid ring with the anisotropic plate is shown.

Trapezoidal Hole[2]. In equations (III.285), as before, c = 1.04380, a_2 = 1, a_3 = -1.47619, a_4 = 0.25867, a_5 = -0.019698, ε = 0.10428.

[1]See A. S. Kosmodamianskiy [8].
[2]See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [3].

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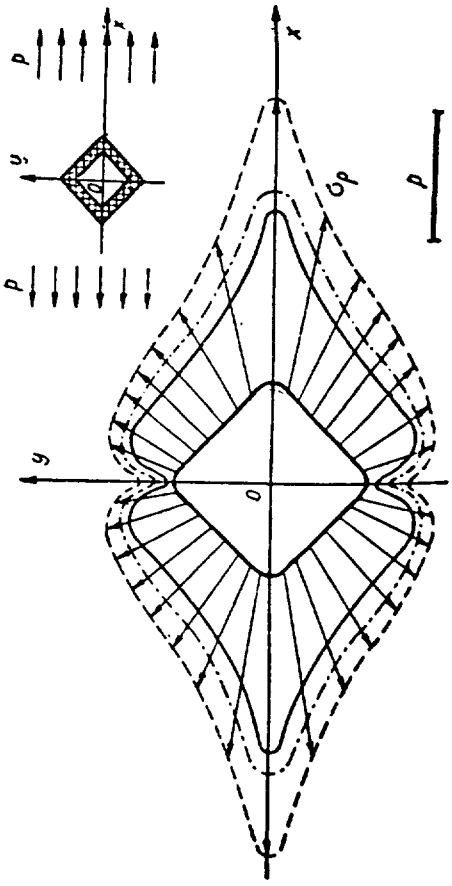
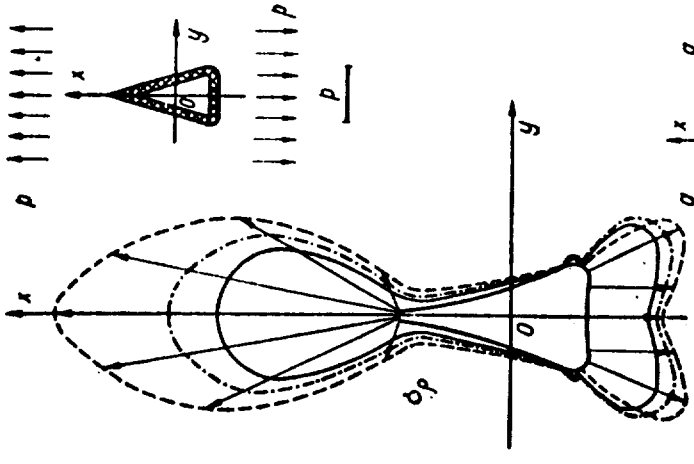


Figure III.44.



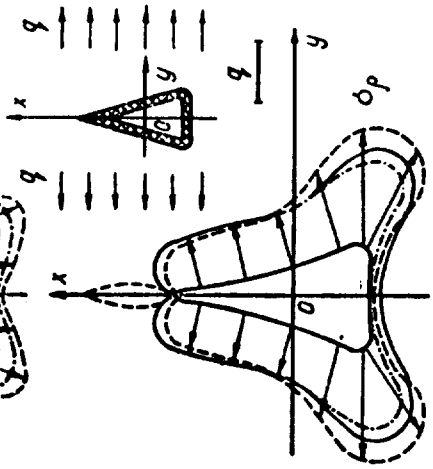


Figure III.43.

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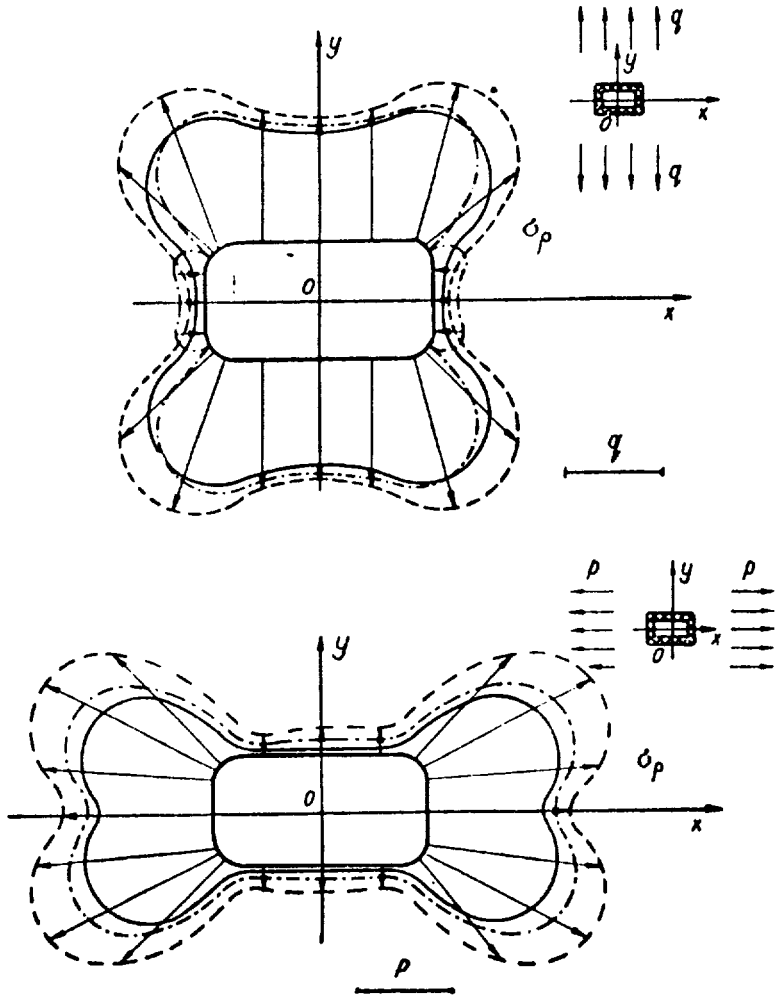


Figure III.45.

The graphs characterizing the distribution of stresses σ_ρ (in fractions of p and q) along the contour of the seam are shown in Figure III.46.

Arch-Shaped Hole[1]. In this case c = 0.752035, a_2 = 1, a_3 = -1.4908, a_4 = = 0.56998, a_5 = 0.02855, ε = 0.059525.

Stresses σ_ρ/p and σ_ρ/q along the contour of the seam of the plate with the ring are given in Figure III.47.

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[1]See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [2].

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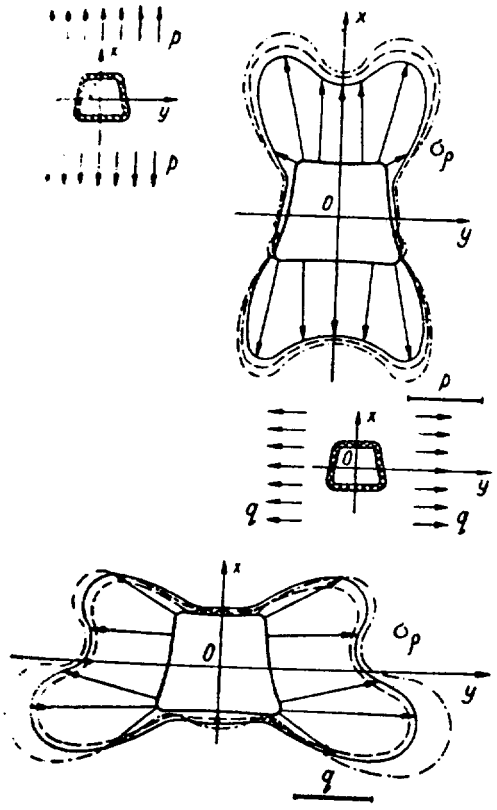


Figure III.46.

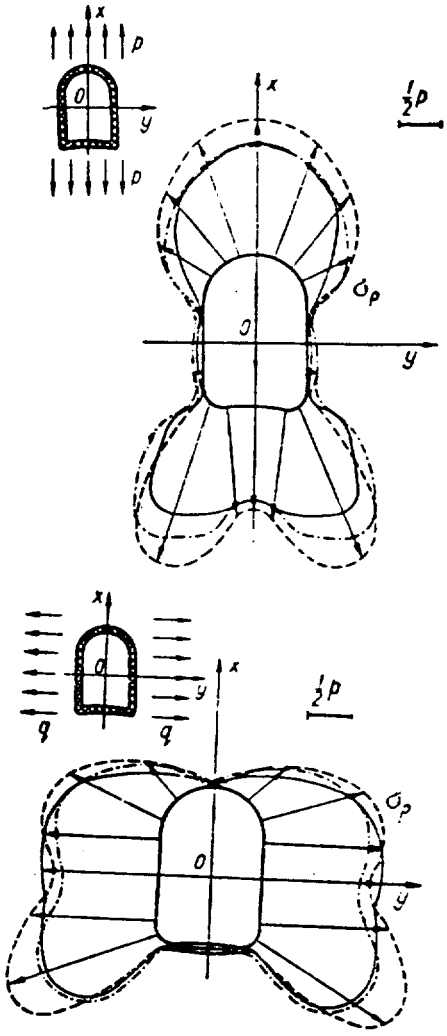


Figure III.47.

Thus, the cited analyses show that a rigid ring in a hole sharply decreases the concentration of stresses near its contour. The stress concentration in an anisotropic plate is less than in an isotropic plate, whereas in the case of an arch-shaped hole, the opposite is true.

§11. Pure Deflection of Anisotropic Plate with Curvilinear Hole Reinforced by a Rigid Ring[1] /332

We will assume that an anisotropic plate with a curvilinear hole, the contour of which is given by equations (III.285), is in the state of pure

[1]See A. S. Kosmodamianskiy, V. V. Meglinskiy, V. A. Shvetsov [4].

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deflection. Deflecting moments M_1 and M_2 act as illustrated in Figure III.48. A rigid ring is affixed in the hole. We will analyze the stress state in the plate near the contour of the given hole, and we will assume (theoretically), as before, that the plate is unbounded.

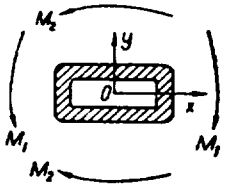


Figure III.48.

We will examine two cases.

Moments Act on the Sides of the Plate Parallel to the Oy Axis ($M_1 \neq 0$, $M_2 = 0$). The stress field in a heavy plate is

$$\sigma_x^0 = \frac{M_1}{J} y, \qquad \sigma_y^0 = \tau_{xy}^0 = 0. \tag{III.359}$$

and the projections of displacements are

$$u^0 = \frac{M_1}{J} a_{11} xy,$$
$$v^0 = \frac{M_1}{2J} (a_{12} y^2 - a_{11} x^2). \tag{III.360}$$

The solution of the problem of the stress state of the plate for the case where displacements $-u^0$, $-v^0$ on the contour of the hole are given, there being no forces at infinity, reduces, as we know, to the determination of the functions $\phi(z_1)$ and $\psi(z_2)$ from boundary conditions (III.92). By expanding the right hand sides of these equations into series by degrees of small parameter ε, we find the functions t_{1x}, t_{2k}:

$$t_{10} = -\frac{M_1 R^2}{4J} i a_{11} c \left(\sigma^2 - \frac{1}{\sigma^2}\right),$$

$$t_{20} = \frac{M_1 R^2}{8J} \left[(a_{11} + a_{12} c^2)\left(\sigma^2 + \frac{1}{\sigma^2}\right) + 2(a_{11} - a_{12} c^2)\right],$$

$$t_{11} = -\frac{M_1 R^2}{4J} i a_{11} \left\{(c-1) a_5 \left(\sigma^6 - \frac{1}{\sigma^6}\right) + (c-1) a_4 \left(\sigma^5 - \frac{1}{\sigma^5}\right) + \left[(c-1) a_3 - \right.\right.$$
$$\left. - (c+1) a_5\right]\left(\sigma^4 - \frac{1}{\sigma^4}\right) + [(c-1) a_2 - (c+1) a_4]\left(\sigma^3 - \frac{1}{\sigma^3}\right) -$$
$$\left. - (c+1) a_3 \left(\sigma^2 - \frac{1}{\sigma^2}\right) - (c+1) a_2 \left(\sigma - \frac{1}{\sigma}\right)\right\},$$

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$$t_{21}=\frac{M_1R^2}{4J}\left\{(a_{11}-a_{12}c)\,a_5\left(\sigma^6+\frac{1}{\sigma^6}\right)+(a_{11}-a_{12}c)\,a_1\left(\sigma^5+\frac{1}{\sigma^5}\right)+[(a_{11}-a_{12}c)a_3+\right.$$
$$+(a_{11}+a_{12}c)\,a_5]\left(\sigma^4+\frac{1}{\sigma^4}\right)+[(a_{11}-a_{12}c)\,a_2+(a_{11}+a_{12}c)\,a_4]\left(\sigma^3+\frac{1}{\sigma^3}\right)+$$
$$\left.+(a_{11}+a_{12}c)\,a_3\left(\sigma^2+\frac{1}{\sigma^2}\right)+(a_{11}+a_{12}c)\,a_2\left(\sigma+\frac{1}{\sigma}\right)\right\},$$
$$t_{12}=\frac{M_1R^2}{4J}ia_{11}\left[a_5^2\left(\sigma^{10}-\frac{1}{\sigma^{10}}\right)+2a_4a_5\left(\sigma^9-\frac{1}{\sigma^9}\right)+(a_4^2+2a_3a_5)\left(\sigma^8-\frac{1}{\sigma^8}\right)+\right.$$
$$+2(a_2a_5+a_3a_4)\left(\sigma^7-\frac{1}{\sigma^7}\right)+(a_3^2+2a_2a_4)\left(\sigma^6-\frac{1}{\sigma^6}\right)+2a_2a_3\left(\sigma^5-\frac{1}{\sigma^5}\right)+$$
$$\left.+a_2^2\left(\sigma^4-\frac{1}{\sigma^4}\right)\right], \qquad \text{(III.361)}$$
$$t_{22}=\frac{M_1R^2}{8J}\left\{(a_{11}+a_{12})\left[a_5^2\left(\sigma^{10}+\frac{1}{\sigma^{10}}\right)+2a_4a_5\left(\sigma^9+\frac{1}{\sigma^9}\right)+(a_4^2+2a_3a_5)\left(\sigma^8+\frac{1}{\sigma^8}\right)+\right.\right.$$
$$+2(a_2a_5+a_3a_4)\left(\sigma^7+\frac{1}{\sigma^7}\right)+(a_3^2+2a_2a_4)\left(\sigma^6+\frac{1}{\sigma^6}\right)+2a_2a_3\left(\sigma^5+\frac{1}{\sigma^5}\right)+$$
$$\left.+a_2^2\left(\sigma^4+\frac{1}{\sigma^4}\right)\right]+2(a_{11}-a_{12})\left[a_2a_5\left(\sigma^3+\frac{1}{\sigma^3}\right)+(a_2a_4+a_3a_5)\left(\sigma^2+\frac{1}{\sigma^2}\right)+\right.$$
$$\left.\left.+(a_2a_3+a_3a_4+a_4a_5)\left(\sigma+\frac{1}{\sigma}\right)+(a_2^2+a_3^2+a_4^2+a_5^2)\right]\right\},$$
$$t_{1k}=t_{2k}=0 \qquad (k\geqslant 3).$$

By substituting functions (III.337) into (III.338), and the latter into (III.339) and then integrating, we obtain the expressions for the functions $\psi_{\rho k}(\zeta)$.

Further, by the method given in §6, we obtain

$$\varphi=\frac{M_1R^2}{8J}i\left\{N_0\sigma^2-\varepsilon\sum_{k=1}^{6}(N_k\sigma^k-N_{-k}\sigma^{-k})-\varepsilon^2\sum_{k=1}^{10}(A_k\sigma^k-A_{-k}\sigma^{-k})\right\}, \qquad \text{(III.362)}$$

where

$$N_0=-k_1m_1, \qquad N_{-1}=2k_1m_1\delta_1a_4, \qquad N_{-2}=2k_1m_1\delta_1a_5,$$
$$N_{-3}=N_{-4}=N_{-5}=N_{-6}=0,$$
$$N_1=2k_1^2\left\{[2p_2q_2^*m_2\delta_2-(p_1q_2^*+p_2q_1^*)\,m_1\delta_1]\,a_4-\frac{n_1}{k_1}a_2\right\},$$
$$N_2=2k_1^2\left\{[2p_2q_2^*m_2\delta_2-(p_1q_2^*+p_2q_1^*)\,m_1\delta_1]\,a_5-\frac{n_1}{k_1}a_3\right\}, \qquad \text{(III.363)}$$
$$N_3=-2k_1(d_1a_2+n_1a_4), \qquad N_4=-2k_1(d_1a_3+n_1a_5),$$
$$N_5=-2k_1d_1a_4, \qquad N_6=-2k_1d_1a_5;$$

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$$A_{-1} = -\delta_1 [3N_3a_5 + 2N_2a_4 + N_1(a_3 + \varkappa_1 a_5) + 2k_1m_1\delta_1 (2a_2a_3 + 4\varkappa_1 a_2a_5 + 5\varkappa_1 a_3a_4 + 9\varkappa_1^2 a_4a_5)],$$

$$A_{-2} = -\delta_1 [2N_2a_5 + N_1a_4 + k_1m_1\delta_1 (3a_3^2 + 4a_2a_4 + 12\varkappa_1 a_3a_5 + 6\varkappa_1 a_4^2 + 11\varkappa_1^2 a_5^2)],$$

$$A_{-3} = -\delta_1 [N_1a_5 + 2k_1m_1\delta_1 (2a_2a_5 + 3a_3a_4 + 7\varkappa_1 a_4a_5)], \qquad \text{(III.364)}$$

$$A_{-4} = -k_1 m_1 \delta_1^2 (3a_4^2 + 6a_3a_5 + 8\varkappa_1 a_5^2),$$

$$A_{-5} = 6k_1m_1\delta_1^2 a_4a_5, \quad A_{-6} = -3k_1m_1\delta_1^2 a_5^2, \quad A_{-7} = A_{-8} = A_{-9} = A_{-10} = 0,$$

$$A_k = -k_1 [(p_1q_2^* + p_2q_1^*) A_{-k} + 2p_2q_2^* B_{-k} + P_k] \quad (k = 1, 2, \ldots, 10);$$

$$P_1 = 2h_1(a_2a_3 + a_3a_4 + a_4a_5), \quad P_2 = 2h_1(a_2a_4 + a_3a_5), \quad P_3 = 2h_1a_2a_5,$$

$$P_4 = r_1a_2^2, \quad P_5 = 2r_1a_2a_3, \quad P_6 = r_1(a_3^2 + 2a_2a_4), \qquad \text{(III.365)}$$

$$P_7 = 2r_1(a_2a_5 + a_3a_4), \quad P_8 = r_1(a_4^2 + 2a_3a_5), \quad P_9 = 2r_1a_4a_5, \quad P_{10} = r_1a_5^2.$$

Hence

$$m_1 = 2q_2^*a_{11}c - p_2(a_{11} + a_{12}c^2), \quad n_1 = q_2^*a_{11}(c + 1) + p_2(a_{11} + a_{12}c),$$

$$d_1 = -q_2^*a_{11}(c - 1) + p_2(a_{11} - a_{12}c), \quad r_1 = 2q_2^*a_{11} + p_2(a_{11} + a_{12}),$$

$$h_1 = p_2(a_{11} - a_{12}),$$

$$q_p^* = -iq_p \quad (p = 1, 2), \quad k_1 = (p_1q_2^* - p_2q_1^*)^{-1}, \qquad \text{(III.366)}$$

$$\varkappa_p = \frac{1 - c\beta_p}{1 + c\beta_p}, \quad \delta_p = \frac{1 - \beta_p}{1 + c\beta_p}.$$

The function ψ is found from expressions (III.362)-(III.365) by substituting in them p_1, q_1^*, κ_1, δ_1, k_1, respectively, by p_2, q_2^*, κ_2, δ_2, $-k_1$ and conversely.

We will denote

$$\frac{d\varphi}{d\vartheta} = \frac{M_kR^2}{2J}(\varphi_1 + i\psi_1), \quad \frac{d\psi}{d\vartheta} = \frac{M_kR^2}{2J}(\varphi_2 + i\psi_2) \quad (k = 1, 2). \qquad \text{(III.367)}$$

Then the stress components are

$$\sigma_\vartheta = \frac{M_1R}{J} \cdot \frac{1}{C^2}\left\{A^2\frac{y}{R} - [B(\varphi_1 + \varphi_2) + A(\beta_1\psi_1 + \beta_2\psi_2)]\right\},$$

$$\tau_{\vartheta\phi} = -\frac{M_1R}{J} \cdot \frac{1}{C^2}\left\{AB\frac{y}{R} + A(\varphi_1 + \varphi_2) - B(\beta_1\psi_1 + \beta_2\psi_2)\right\}, \qquad \text{(III.368)}$$

$$\sigma_\phi = \frac{M_1R}{J} \cdot \frac{1}{C^2}\left\{B^2\frac{y}{R} + A_{11}\varphi_1 + A_{12}\varphi_2 + A_{13}\psi_1 + A_{14}\psi_2\right\},$$

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where the coefficients A, B, C^2, A_{11}, A_{12}, A_{13} and A_{14} are defined by formulas (III.328) and (III.330).

Moments Act on Sides of Plate Parallel to Ox Axis ($M_1 = 0$, $M_2 \neq 0$). The stress field in a dense plate is /335

$$\sigma_y^0 = \frac{M_2}{J} x, \qquad \sigma_x^0 = \tau_{xy}^0 = 0, \tag{III.369}$$

and the projections of displacements are

$$u^0 = \frac{M_2}{2J}(a_{12}x^2 - a_{22}y^2),$$
$$v^0 = \frac{M_2}{J} a_{22} xy. \tag{III.370}$$

The functions t_{1k} and t_{2k} in this case are

$$t_{10} = -\frac{M_2R^2}{8J}\left[(a_{12} + a_{22}c^2)\left(\sigma^2 + \frac{1}{\sigma^2}\right) + 2(a_{12} - a_{22}c^2)\right].$$

$$t_{20} = -\frac{M_2R^2}{4J} a_{22} ic \left(\sigma^2 - \frac{1}{\sigma^2}\right),$$

$$t_{11} = -\frac{M_2R^2}{4J}\left\{(a_{12} - a_{22}c) a_5\left(\sigma^6 + \frac{1}{\sigma^6}\right) + (a_{12} - a_{22}c) a_4\left(\sigma^5 + \frac{1}{\sigma^5}\right) + \right.$$
$$+ [(a_{12} - a_{22}c) a_3 + (a_{12} + a_{22}c) a_5]\left(\sigma^4 + \frac{1}{\sigma^4}\right) + [(a_{12} - a_{22}c) a_2 +$$
$$\left. + (a_{12} + a_{22}c) a_4]\left(\sigma^3 + \frac{1}{\sigma^3}\right) + (a_{12} + a_{22}c) a_3\left(\sigma^2 + \frac{1}{\sigma^2}\right) + (a_{12} + a_{22}c) a_2\left(\sigma + \frac{1}{\sigma}\right)\right\},$$

$$t_{21} = -\frac{M_2R^2}{4J} i a_{22}\left\{(c-1) a_5\left(\sigma^6 - \frac{1}{\sigma^6}\right) + (c-1) a_4\left(\sigma^5 - \frac{1}{\sigma^5}\right) + \right.$$
$$+ [(c-1) a_3 - (c+1) a_5]\left(\sigma^4 - \frac{1}{\sigma^4}\right) + [(c-1) a_2 - (c+1) a_4]\left(\sigma^3 - \frac{1}{\sigma^3}\right) -$$
$$\left. - (c+1) a_3\left(\sigma^2 - \frac{1}{\sigma^2}\right) - (c+1) a_2\left(\sigma - \frac{1}{\sigma}\right)\right\},$$

$$t_{12} = -\frac{M_2R^2}{8J}\left\{(a_{12} + a_{22})\left[a_5^2\left(\sigma^{10} + \frac{1}{\sigma^{10}}\right) + 2a_4a_5\left(\sigma^9 + \frac{1}{\sigma^9}\right) + \right.\right. \tag{III.371}$$
$$+ (a_4^2 + 2a_3a_5)\left(\sigma^8 + \frac{1}{\sigma^8}\right) + 2(a_2a_5 + a_3a_4)\left(\sigma^7 + \frac{1}{\sigma^7}\right) + (a_3^2 + 2a_2a_4)\left(\sigma^6 + \frac{1}{\sigma^6}\right) +$$
$$+ 2a_2a_3\left(\sigma^5 + \frac{1}{\sigma^5}\right) + a_2^2\left(\sigma^4 + \frac{1}{\sigma^4}\right)\Big] + 2(a_{12} - a_{22})\left[a_2a_5\left(\sigma^3 + \frac{1}{\sigma^3}\right) + \right.$$
$$+ (a_2a_4 + a_3a_5)\left(\sigma^2 + \frac{1}{\sigma^2}\right) + (a_2a_3 + a_3a_4 + a_4a_5)\left(\sigma + \frac{1}{\sigma}\right) +$$
$$\left.\left. + a_2^2 + a_3^2 + a_4^2 + a_5^2\right]\right\},$$

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$$t_{22} = \frac{M_2R^2}{8J} ia_{22}\left(a_5^2\left(\sigma^{10} - \frac{1}{\sigma^{10}}\right) + 2a_4a_5\left(\sigma^9 - \frac{1}{\sigma^9}\right) + (a_4^2 + 2a_3a_5)\left(\sigma^8 - \frac{1}{\sigma^8}\right) + \right.$$

$$+ 2(a_2a_5 + a_3a_4)\left(\sigma^7 - \frac{1}{\sigma^7}\right) + (a_3^2 + 2a_2a_4) \times$$

$$\times \left(\sigma^6 - \frac{1}{\sigma^6}\right) + 2a_2a_3\left(\sigma^5 - \frac{1}{\sigma^5}\right) + a_2^2 \times$$

$$\left. \times \left(\sigma^4 - \frac{1}{\sigma^4}\right)\right],$$

$$t_{1k} = t_{2k} = 0 \qquad (k \geqslant 3).$$

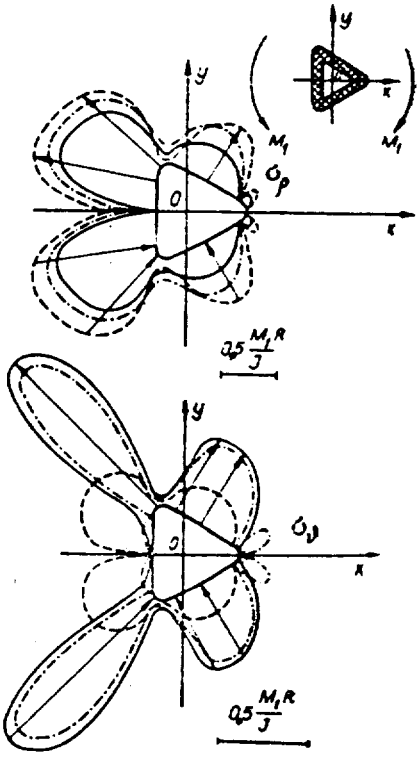


Figure III.49.

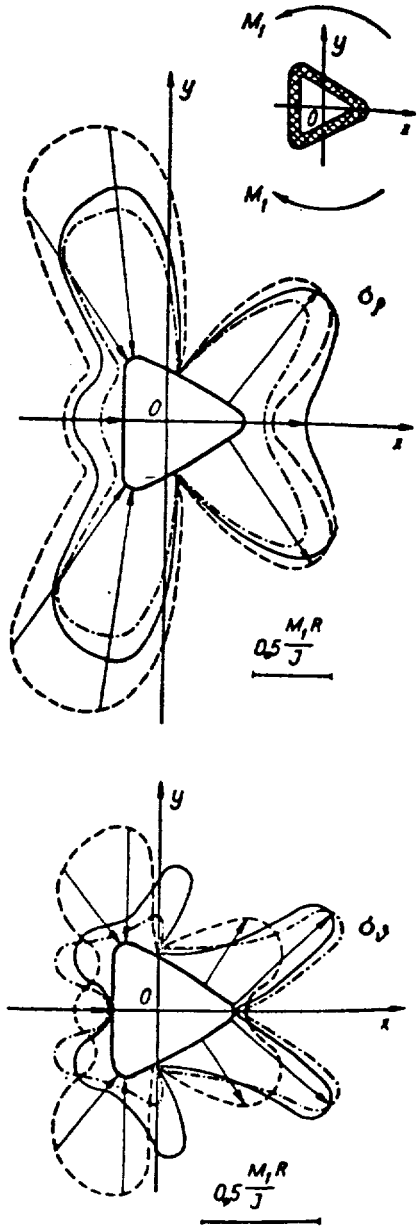


Figure III.50.

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Proceeding further as in the preceding case, we obtain /337

$$\psi = \frac{M_2 R^2}{8J}\left\{N_0\sigma^2 + \varepsilon\sum_{k=1}^{6}(N_k\sigma^k + N_{-k}\sigma^{-k}) + \varepsilon^2\sum_{k=1}^{10}(A_k\sigma^k + A_{-k}\sigma^{-k})\right\}. \qquad \text{(III.372)}$$

The coefficients N_0, $N_{\pm k}$, $A_{\pm k}$ are defined by formulas (III.363) and (III.364). Hence the values m_1, n_1, d_1, r_1, h_1 that are essential for calculating the coefficient P_k must be taken in the following form:

$$\begin{aligned} m_1 &= q_2^*(a_{12} + a_{22}c) - 2p_2a_{22}c, \qquad n_1 = q_2^*(a_{12} + a_{22}c) + p_2a_{22}(c+1), \\ d_1 &= q_2^*(a_{12} - a_{22}c) - p_2a_{22}(c-1), \qquad r_1 = q_2^*(a_{12} + a_{22}) + 2p_2a_{22}, \\ h_1 &= q_2^*(a_{12} - a_{22}). \end{aligned} \qquad \text{(III.373)}$$

The function ψ can be found from expression (III.372) just as in the preceding case.

Recalling (III.367), we find such expressions for the stresses:

$$\begin{aligned} \sigma_\rho &= \frac{M_2R}{J}\cdot\frac{1}{C^2}\left\{B^2\frac{x}{R} - [B(\varphi_1 + \varphi_2) + A(\beta_1\psi_1 + \beta_2\psi_2)]\right\}, \\ \tau_{\rho\theta} &= \frac{M_2R}{J}\cdot\frac{1}{C^2}\left\{AB\frac{x}{R} + A(\varphi_1 + \varphi_2) - B(\beta_1\psi_1 + \beta_2\psi_2)\right\}, \\ \sigma_\vartheta &= \frac{M_2R}{J}\cdot\frac{1}{C^2}\left(A^2\frac{x}{R} + A_{11}\varphi_1 + A_{12}\varphi_2 + A_{13}\psi_1 + A_{14}\psi_2\right). \end{aligned} \qquad \text{(III.374)}$$

Figures III.49-III.54 show stresses σ_ρ and σ_ϑ ($\tau_{\rho\theta}$ were considerably less than σ_ρ, σ_ϑ) along the contour of the seam of the plate and ring for the very same plywood and isotropic ($\nu = 1/3$) plates as in §7.

§12. Stress State of Anisotropic Plate Near Curvilinear Hole with Elastic Core[1]

We will analyze an infinite anisotropic plate with a hole that differs but little from a round or elliptic hole, into which is sealed an elastic core made of some other anisotropic material. The stresses that occur in such a plate under the effect of external forces will be found, as earlier, in the form of the sum of stresses in a homogeneous thick plate and additional stresses caused by the core.

Stresses σ_x^0, σ_y^0, τ_{xy}^0 in the thick plate are defined through stress function $U^0(x, y)$ by formulas (I.81).

---

[1]The problem was solved by A. S. Kosmodamianskiy [6].

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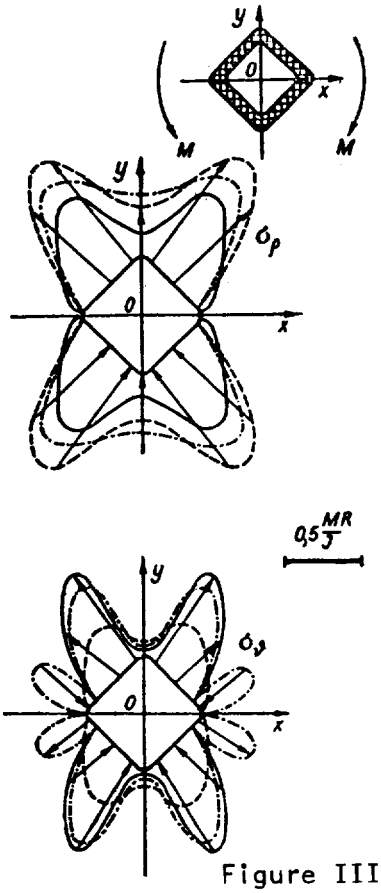


Figure III.51.

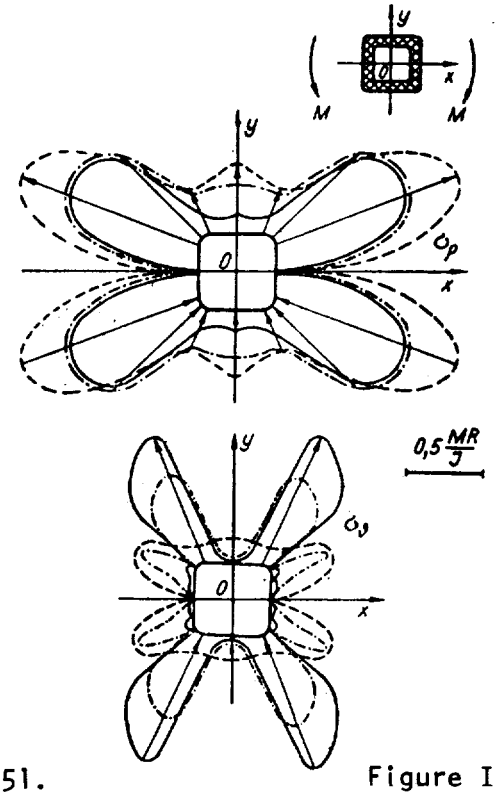


Figure III.52.

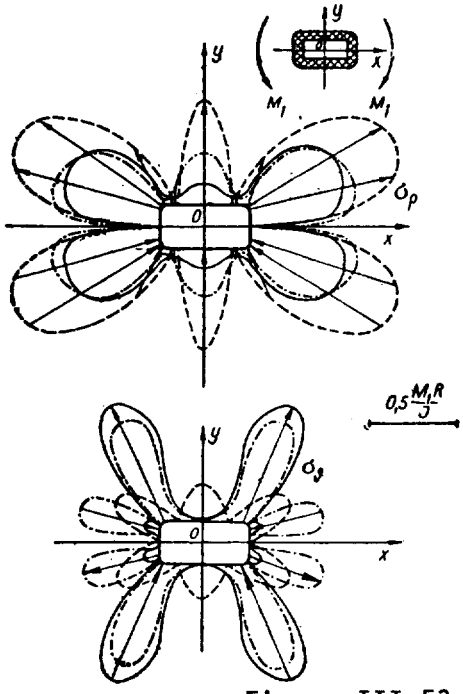


Figure III.53.

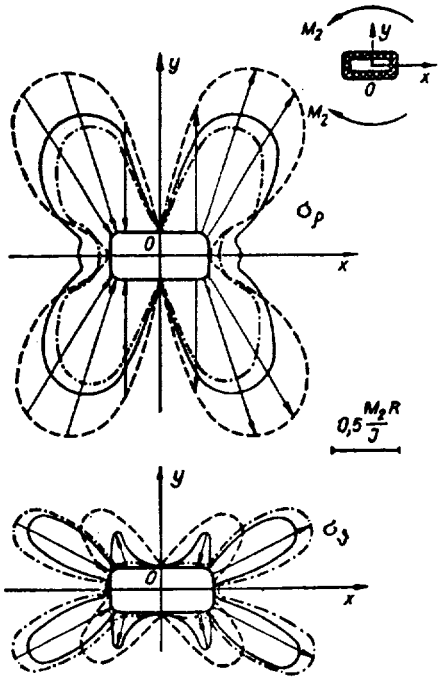


Figure III.54.

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From the conjugate conditions of the plate and core, we find the functions /339
$\phi(z_1)$ and $\psi(z_2)$, through which the additional stresses occurring within the plate are determined. Since by definition the core is sealed in the hole of the plate, then displacements of the plate and core at the points of contact are identical. Moreover, to insure that elastic equilibrium is not disrupted, the conditions in the contiguous points of the core and plate should be equal and opposite in direction. Therefore the boundary conditions are

$$X_n = -X'_n, \quad Y_n = -Y'_n,$$
$$u = u', \quad v = v'. \tag{III.375}$$

These conditions can be represented[1] in the form

$$2\mathrm{Re}\,[\varphi(z_1) + \psi(z_2)] = \frac{\partial(U' - U^0)}{\partial x},$$
$$2\mathrm{Re}\,[s_1\varphi(z_1) + s_2\psi(z_2)] = \frac{\partial(U' - U^0)}{\partial y}; \tag{III.376}$$

$$2\mathrm{Re}\,[p_1\varphi(z_1) + p_2\psi(z_2)] = u' - u^0 - \gamma_0 y + \alpha_0,$$
$$2\mathrm{Re}\,[q_1\varphi(z_1) + q_2\psi(z_2)] = v' - v^0 + \gamma_0 x + \beta_0. \tag{III.377}$$

The terms $-\gamma_0 y + \alpha_0$ and $\gamma_0 x + \beta_0$ in (III.377) characterize rigid displacements that have no effect on the stress distribution in the plate. We will disregard them in the following.

We will take the function U'(x, y) in the form of the sum of polynomials with undefined coefficients from the second to the k-th powers (the magnitude of k depends on the form of contour of the hole and form of the function $U^0(x, y)$. Considering the above coefficients as known, we find from conditions (III.376) the functions $\phi(z_1)$ and $\psi(z_2)$, in the same manner as in §6. The coefficients in the polynomials are found from conditions (III.377), considering that the function U'(x, y) must satisfy[2] the equation

$$a'_{22}\frac{\partial^4 U'}{\partial x^4} - 2a'_{26}\frac{\partial^4 U'}{\partial x^3 \partial y} + (2a'_{12} + a'_{16})\frac{\partial^4 U'}{\partial x^2 \partial y^2} - 2a'_{16}\frac{\partial^4 U'}{\partial x \partial y^3} + a'_{11}\frac{\partial^4 U'}{\partial y^4} = 0. \tag{III.378}$$

---

[1] See S. G. Lekhnitskiy [3].
[2] Here, and in the following, the shaded areas will denote values pertaining to the core.

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We will assume that the elastic core is sealed in a hole of a finite orthotropic[1] plate, the contour of which is defined by the equations

$$x = R(\cos\vartheta + \varepsilon\cos 3\vartheta), \qquad y = R(-\sin\vartheta + \varepsilon\sin 3\vartheta). \tag{III.379}$$

When $\varepsilon = \pm 1/9$, this contour, as we pointed out earlier, is similar to a square with rounded corners. For simplicity we will assume that the core is also made of an orthotropic material. At infinity let the plate be under tension by forces p parallel to the Ox axis (Figure III.55). /340

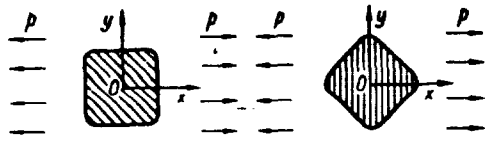


Figure III.55.

Stress distribution in a dense plate in the given case is characterized by the stress function

$$U^0 = \frac{p}{2}y^2, \tag{III.380}$$

which corresponds to elastic displacements

$$u^0 = a_{11}px, \qquad v^0 = a_{12}py. \tag{III.381}$$

Limiting calculations to the second approximation, we assume

$$\varphi = \varphi_{10} + \varepsilon\varphi_{11} + \varepsilon^2\varphi_{12}, \qquad \psi = \psi_{20} + \varepsilon\psi_{21} + \varepsilon^2\psi_{22},$$
$$u' = u'_{10} + \varepsilon u'_{11} + \varepsilon^2 u'_{12}, \qquad v' = v'_{10} + \varepsilon v'_{11} + \varepsilon^2 v'_{12}, \tag{III.382}$$

and we take the expression for the function U'(x, y) in the form

$$U'(x, y) = U'_{10} + \varepsilon U'_{11} + \varepsilon^2 U'_{12} = \frac{1}{2}(B_0x^2 + A_0y^2) + \varepsilon\left[\frac{1}{2}(B_1x^2 + A_1y^2) + (\varepsilon_1x^4 + k_1x^2y^2 + R_1y^4)\right] + \varepsilon^2\left[\frac{1}{2}(B_2x^2 + A_2y^2) + (\varepsilon_2x^4 + k_2x^2y^2 + R_2y^4) + (F_2x^6 + C_2x^4y^2 + D_2x^2y^4 + P_2y^6)\right]. \tag{III.383}$$

For projections of the elastic displacements that correspond to the function (III.383), we find the expressions

[1]For this plate, with the appropriate selection of the coordinate axes, the complex parameters s_1 and s_2 will be purely imaginary, i.e. $s_i = i\beta_i$ (i = 1, 2).

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$$u'_{10} = (a'_{11}A_0 + a'_{12}B_0)\,x, \qquad v'_{10} = (a'_{12}A_0 + a'_{22}B_0)\,y,$$

$$u'_{11} = (a'_{11}A_1 + a'_{12}B_1)\,x + \left(\frac{2}{3}a'_{11}k_1 + 4a'_{12}\varepsilon_1\right)x^3 + 2(6a'_{11}R_1 + a'_{12}k_1)\,xy^2,$$

$$v'_{11} = (a'_{12}A_1 + a'_{22}B_1)\,y + 2(a'_{12}k_1 + 6a'_{22}\varepsilon_1)\,x^2y + \frac{2}{3}(6a'_{12}R_1 + a'_{22}k_1)\,y^3,$$

$$u'_{12} = (a'_{11}A_2 + a'_{12}B_2)\,x + \left(\frac{2}{3}a'_{11}k_2 + 4a'_{12}\varepsilon_2\right)x^3 + 2(6a'_{11}R_2 + a'_{12}k_2)\,xy^2 + \tag{III.384}$$

$$+ \frac{2}{5}(a'_{11}C_2 + 15a'_{12}F_2)\,x^5 + 4(a'_{11}D_2 + a'_{12}C_2)\,x^3y^2 + 2(15a'_{11}P_2 + a'_{12}D_2)\,xy^4,$$

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$$v'_{12} = (a'_{12}A_2 + a'_{22}B_2)\,y + 2(a'_{12}k_2 + 6a'_{22}\varepsilon_2)\,x^2y + \frac{2}{3}(6a'_{12}R_2 + a'_{22}k_2)\,y^3 +$$

$$+ 2(a'_{12}C_2 + 15a'_{22}F_2)\,x^4y + 4(a'_{12}D_2 + a'_{22}C_2)\,x^2y^3 + \frac{2}{5}(15a'_{12}P_2 + a'_{22}D_2)\,y^5.$$

Then, from condition (III.376), by the method outlined in §6, limiting ourselves to the first approximation[1], we find

$$\varphi(\sigma) = \frac{R}{2}k_0\sigma - \frac{Re}{2(\beta_1 - \beta_2)}\left\{k_0\lambda_1(\beta_1 - \beta_2)\frac{1}{\sigma} + [k_0\lambda_1(\beta_1 + \beta_2) + 2M_0\lambda_2\beta_2 + B_1\beta_2 - \right.$$

$$- A_1 + 2k_1(\beta_2 - 1) + 12(\varepsilon_1\beta_2 - R_1)]\,\sigma + [k_0(\beta_1 - \beta_2) + 2\beta_2 B_0 - 2k_1(1 + \beta_2) + \tag{III.385}$$

$$\left. + 4(\varepsilon_1\beta_2 + R_1)]\,\sigma^3\right\}.$$

Here the constant $a^2/4$ is introduced into the coefficients k_1, ε_1, R_1, and

$$k_0 = \frac{A_0 - p - \beta_2 B_0}{\beta_1 - \beta_2}, \qquad M_0 = -\frac{A_0 - p - \beta_1 B_0}{\beta_1 - \beta_2}.$$

The expression for the function $\psi(\sigma)$ is found from formula (III.385) by substituting in it k_0, M_0, β_1, β_2, λ_1, λ_2, respectively, by M_0, k_0, β_2, β_1, λ_2, λ_1. The constants A_0, B_0, A_1, B_1, k_1, ε_1, R_1 are found from conditions (III (III.377), and also from the condition that the function U'(x, y) satisfies equation (III.378). In this manner we obtain the following systems of algebraic equations:

[1]The expression for the function $\phi(z_1)$ in the second approximation will not be written out due to the great complexity of the system for the determination of the coefficients A_2, B_2, ..., D_2, R_2.

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$$A_0[p_1 - p_2 - a'_{11}(\beta_1 - \beta_2)] - B_0[p_1\beta_2 - p_2\beta_1 + a'_{12}(\beta_1 - \beta_2)] = p[p_1 - p_2 - a_{11}(\beta_1 - \beta_2)],$$

$$A_0[q_1^* - q_2^* - a'_{12}(\beta_1 - \beta_2)] - B_0[q_1^*\beta_2 - q_2^*\beta_1 + a'_{22}(\beta_1 - \beta_2)] = p[q_1^* - q_2^* - a_{12}(\beta_1 - \beta_2)]; \quad \text{(III.386)}$$

$$2k_1\left[p_1 - p_2 + (p_1\beta_2 - p_2\beta_1) - (\beta_1 - \beta_2)\left(\frac{1}{3}a'_{11} - a'_{12}\right)\right] - 4R_1[p_1 - p_2 - 3a'_{11}(\beta_1 - \beta_2)] - 4\varepsilon_1[p_1\beta_2 - p_2\beta_1 + a'_{12}(\beta_1 - \beta_2)] = (A_0 - p)(p_1 - p_2) + B_0(p_1\beta_2 - p_2\beta_1) + (\beta_1 - \beta_2)(a'_{11}A_0 + a'_{12}B_0 - a_{11}p),$$

$$2k_1\left[q_1^* - q_2^* + (q_1^*\beta_2 - q_2^*\beta_1) - (\beta_1 - \beta_2)\left(a'_{12} - \frac{1}{3}a'_{22}\right)\right] - 4R_1[q_1^* - q_2^* - a'_{12}(\beta_1 - \beta_2)] - 4\varepsilon_1[q_1^*\beta_2 - q_2^*\beta_1 + 3a'_{22}(\beta_1 - \beta_2)] = (A_0 - p)(q_1^* - q_2^*) + B_0(q_1^*\beta_2 - q_2^*\beta_1) - (\beta_1 - \beta_2)(a'_{12}A_0 + a'_{22}B_0 - a_{12}p),$$

$$k_1(2a'_{12} + a'_{66}) + 6R_1a'_{11} + 6\varepsilon_1a'_{22} = 0; \quad \text{(III.387)}$$

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$$A_1[p_1 - p_2 - a'_{11}(\beta_1 - \beta_2)] - B_1[p_1\beta_2 - p_2\beta_1 + a'_{12}(\beta_1 - \beta_2)] = 2(p_1 - p_2)(k_0\lambda_1\beta_1 + M_0\lambda_2\beta_2 - k_1 - 6R_1) + 2(p_1\beta_2 - p_2\beta_1)(k_1 + 6\varepsilon_1) + 2(\beta_1 - \beta_2)[k_1(a'_{11} + a'_{12}) + 6(\varepsilon_1a'_{12} + R_1a'_{11})],$$

$$A_1[q_1^* - q_2^* - a'_{12}(\beta_1 - \beta_2)] - B_1[q_1^*\beta_2 - q_2^*\beta_1 + a'_{22}(\beta_1 - \beta_2)] = -2(q_1^* - q_2^*)(k_1 + 6R_1) + 2(q_1^*\beta_2 - q_2^*\beta_1)(k_0\lambda_1 + M_0\lambda_2 + k_1 + 6\varepsilon_1) + 2(\beta_1 - \beta_2)[k_1(a'_{12} + a'_{22}) + 6(R_1a'_{12} + \varepsilon_1a'_{22})]. \quad \text{(III.388)}$$

Here

$$p_i = a_{12} - a_{11}\beta_i^2, \quad q_i^* = a_{12}\beta_i - \frac{a_{22}}{\beta_i} \qquad (i = 1,2).$$

Stresses σ_ϑ, σ_ρ and $\tau_{\rho\vartheta}$ on the contour of the seam are found from formulas (III.350), where definitions (III.349) are used. The formulas for finding the stresses in a plate with an absolutely rigid ($a'_{ij} = 0$) or absolute flexible core (no core, $a'_{ij} \to \infty$), can be obtained in the form of partial cases.

Figures III.56 and III.57 show the values of stresses σ_ρ, σ_ϑ (in fractions of p on the contour of the seam of the core with the plate for the case where the elastic constants of the orthotropic plate are /343

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$$a_{11} = 0{,}83333\,\frac{10^{-9}}{9{,}81}, \qquad a_{12} = -\,0{,}05917\,\frac{10^{-9}}{9{,}81}$$

$$a_{22} = 1{,}66667\,\frac{10^{-9}}{9{,}81}, \qquad a_{66} = 14{,}2857\,\frac{10^{-9}}{9{,}81}, \tag{III.389}$$

and the complex parameters are

$$s_1 = 4{,}11i, \quad s_2 = 0{,}343i.$$

The elastic constants of the core are $a'_{ij} = 0.5a_{ij}$.

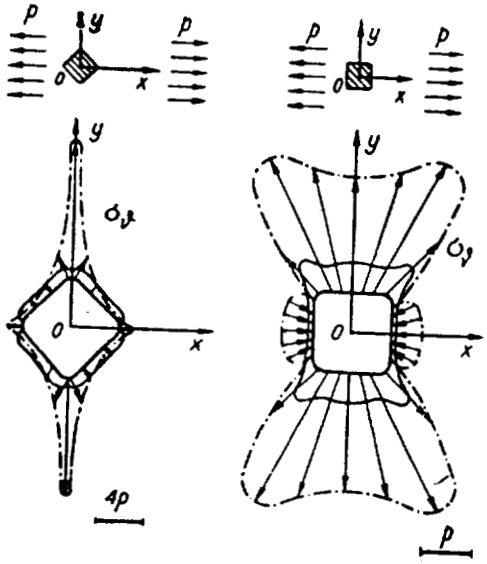


Figure III.56.

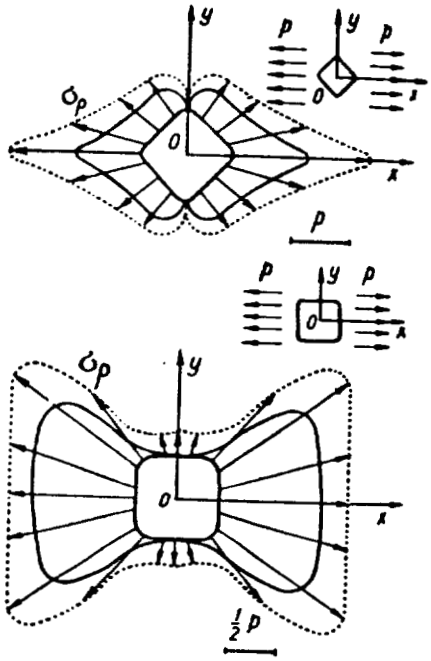


Figure III.57.

These figures show, for comparison, the values of stresses for an absolutely rigid and an absolutely flexible core. The stresses σ_ϑ and $\tau_{\rho\vartheta}$ for a plate with an elastic core were much smaller than σ_ρ, and therefore the numerical values of $\tau_{\rho\vartheta}$ are not included. The solid lines in Figures III.56 and III.57 correspond to an elastic core, the dot-dash line in Figure III.56 corresponds to an absolutely flexible core, i.e., to a free hole, and the dotted line in Figure III.57 corresponds to a rigid core.

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§13. Stresses in Elastic Anisotropic Plate Weakened by Several Elliptic Holes

Analysis of the stress state of elastic anisotropic multiply-connected media involves specific difficulties attributed to the necessity of analyzing two additional surfaces that are related to the original surface by the affine relations discussed earlier. Other holes are found in these regions instead of the given holes; here distances between the holes change, depending on the type of anisotropy of the medium. If the distance between the holes in the wayside areas are increased, the effective construction of the solution of the problem concerning the stress state of such a medium can be found quite easily. In the opposite case, considerable difficulties arise in the solution of the problem.

D. I. Sherman [1] reduced the problem of the stress state of an anisotropic multiply-connected medium to the solution of an integral Fredholm's equation. L. N. Nagibin [1, 2], using D. I. Sherman's method, solved the problem of the stress state of a heavy anisotropic half plane with two identical and two unequal round holes. The analogous problem for a beam with two round holes was solved by Hayashi Takuo [1]. Using the known method of R. C. J. Howzand [1], Kubo Toshihiko [1] analyzed the periodic problem for an anisotropic medium with an infinite row of round holes. A. S. Kosmodamianskiy solved the problem of the stress state of an anisotropic medium for the case where the latter is /344
weakened by a finite number of elliptic holes [1-3], as well as by one or two rows of identical elliptic holes [4]. In the latter work, the method for analyzing a medium weakened by strong anisotropy is described. The case where elliptic holes are reinforced by absolutely rigid rings is examined by A. S. Kosmodamianskiy and V. V. Meglinskiy [1].

Tension of Anisotropic Medium with Infinite Row of Elliptical Holes[1]. Let an elastic anisotropic medium be weakened by an infinite row of identical elliptical holes, into which are soldered or glued absolutely rigid rings (Figure III.58). The distances between the holes are identical and equal to l. The semiaxes of the elliptical holes are denoted through a and b. At infinity is given a homogeneous stress state $\sigma_x^\infty = p$ along the center line of the holes and $\sigma_y^\infty = q$ transverse to the center line, i.e.,

$$\sigma_x^0 = p,\ \sigma_y^0 = q,\ \tau_{xy}^0 = 0. \qquad \text{(III.390)}$$

The complex potentials characterizing the stress state of a heavy medium, as follows from formulas (I.90), are

[1]The solution of problems derived in this and the following sections were found by A. S. Kosmodamianskiy and V. V. Meglinskiy [1]. In the case of an isotropic medium, this problem was examined for round free holes by R. C. J. Howland [1]; for an orthotropic medium, by Kubo Toshihiko [1].

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$$\varphi^0(z_1) = -\frac{p + q\beta_2^2}{2(\beta_1^2 - \beta_2^2)} z_1,$$

$$\psi^0(z_2) = \frac{p + q\beta_1^2}{2(\beta_1^2 - \beta_2^2)} z_2. \qquad \text{(III.391)}$$

To determine these functions it is assumed that the complex parameters s_k are purely imaginary ($s_1 = i\beta_1$, $s_2 = i\beta_2$).

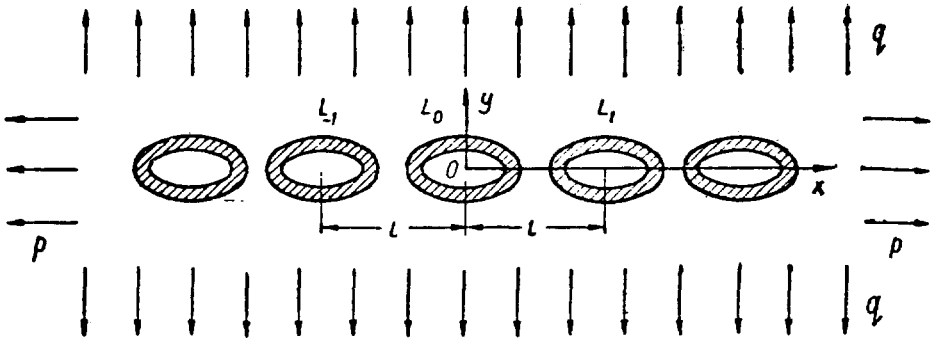


Figure III.58.

In order to determine the stress field occuring in the medium after the formation within it of elliptical holes, it is necessary to determine, as follows from Chapter I, the functions $\phi(z_1)$ and $\psi(z_2)$ from the following boundary conditions on the contours of the holes: /345

$$2\,\mathrm{Re}\,[p_1\varphi(z_1) + p_2\psi(z_2)] = -2\,\mathrm{Re}\,[p_1\varphi^0(z_1) + p_2\psi^0(z_2)],$$
$$2\,\mathrm{Re}\,[q_1\varphi(z_1) + q_2\psi(z_2)] = -2\,\mathrm{Re}\,[q_1\varphi^0(z_1) + q_2\psi^0(z_2)]. \qquad \text{(III.392)}$$

In connection with the fact that the given problem is periodic, the functions $\phi(z_1)$ and $\psi(z_2)$ can be represented in the form

$$\varphi(z_1) = \sum_{k=1}^{\infty} \frac{\varphi_k}{[\zeta(z_1)]^k} + \sum_{n=-\infty}^{\infty}{}^{*} \sum_{k=1}^{\infty} \frac{\psi_k}{[\zeta(z_1 + nl)]^k},$$
$$\psi(z_2) = \sum_{k=1}^{\infty} \frac{\psi_k}{[\zeta(z_2)]^k} + \sum_{n=-\infty}^{\infty}{}^{*} \sum_{k=1}^{\infty} \frac{\psi_k}{[\zeta(z_2 + nl)]^k}. \qquad \text{(III.393)}$$

The * in the second sums of (III.393) denotes the lack of the term corresponding to the value n = 0.

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We will expand the second sums of (III.393) into series by degrees of small parameter $\varepsilon = \tau^{-1}$, confining ourselves in the expansions to the terms containing ε in powers not exceeding six. Then, from the boundary condition on the contour of the hole, in the center of which is located the origin of the coordinate system (this hole will be called the basic hole; it can be any hole desired), to determine the unknown coefficients ϕ_k and ψ_k, we find by the usual method of series the following algebraic system:

$$\begin{gathered}
p_1[(1+a_{11})\varphi_1 + a_{13}\varphi_3 + a_{15}\varphi_5] + p_2[(1+b_{11})\psi_1 + b_{13}\psi_3 + b_{15}\psi_5] = \\
= -(p_1A_0 + p_2B_0), \\
p_1[c_{13}a_{31}\varphi_1 + (1+c_{13}a_{33})\varphi_3] + p_2[c_{23}b_{31}\psi_1 + (1+c_{23}b_{33})\psi_3] = 0, \\
p_1[c_{15}a_{51}\varphi_1 + \varphi_5] + p_2[c_{25}b_{51}\psi_1 + \psi_5] = 0, \\
q_1^*[(1-c_{11}^*a_{11})\varphi_1 - c_{11}^*a_{13}\varphi_3 - c_{11}^*a_{15}\varphi_5] + q_2^*[(1-c_{21}^*b_{11})\psi_1 + \\
+ c_{21}^*b_{13}\psi_3 - c_{21}^*b_{15}\psi_5] = c[q_1^*\beta_1A_0 + q_2^*\beta_2B_0], \\
q_1^*[-c_{13}^*a_{31}\varphi_1 + (1-c_{13}^*a_{33})\varphi_3] + q_2^*[-c_{23}^*b_{31}\psi_1 + (1-c_{23}^*b_{33})\psi_3] = 0, \\
q_1^*(-c_{15}^*a_{51}\varphi_1 + \varphi_5) + q_2^*(-c_{25}^*b_{51}\psi_1 + \psi_5) = 0.
\end{gathered} \tag{III.394}$$

Here

$$A_0 = -\frac{p+q\beta_2^2}{2(\beta_1^2-\beta_2^2)}; \quad B_0 = \frac{p+q\beta_1^2}{2(\beta_1^2-\beta_2^2)}; \quad q_1^* = -iq_1; \quad q_2^* = -iq_2;$$

$$c = \frac{b}{a}; \quad c_{1k} = m_0^k + m_1^k; \quad c_{2k} = n_0^k + n_1^k; \quad c_{1k}^* = m_0^k - m_1^k;$$

$$c_{2k}^* = n_0^k - n_1^k;$$

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$$\begin{gathered}
a_{11} = -2m_0\varepsilon^2(\alpha_2 + 6\alpha_4 m_0 m_1\varepsilon^2 + 50\alpha_6 m_0^2 m_1^2\varepsilon^4); \\
a_{13} = -6m_0^3\varepsilon^4(\alpha_4 + 15\alpha_6 m_0 m_1\varepsilon^2); \quad a_{15} = -10\alpha_6 m_0^5\varepsilon^6; \\
a_{31} = -2m_0\varepsilon^4(\alpha_4 + 15\alpha_6 m_0 m_1\varepsilon^2); \quad a_{33} = -20\alpha_6 m_0^3\varepsilon^6; \\
a_{51} = -2\alpha_6 m_0\varepsilon^6; \quad \alpha_p = \sum_{n=1}^{\infty}\frac{1}{n^p} \quad (p = 2, 4, 6);
\end{gathered} \tag{III.395}$$

$$m_0 = 0.5(1+c\beta_1); \quad m_1 = 0.5(1-c\beta_1); \quad n_0 = 0.5(1+c\beta_2); \quad n_1 = 0.5(1-c\beta_2).$$

The coefficients b_{ij} are found from the expressions for a_{ij} by substituting m_0 and m_1 in them by n_0 and n_1, respectively.

After determining the coefficients ϕ_k and ψ_k from system (III.394), we determine on the basis of (III.393) the functions $\phi(z_1)$ and $\psi(z_2)$ through which the stresses occurring in the medium are expressed by formulas

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$$\begin{aligned}\sigma_x &= \sigma_x^0 - 2\,\mathrm{Re}\,[\beta_1^2\varphi'(z_1) + \beta_2^2\psi'(z_2)],\\ \sigma_y &= \sigma_y^0 + 2\,\mathrm{Re}\,[\varphi'(z_1) + \psi'(z_2)],\\ \tau_{xy} &= \tau_{xy}^0 - 2\,\mathrm{Re}\,\{i\,[\beta_1\varphi'(z_1) + \beta_2\psi'(z_2)]\}.\end{aligned} \tag{III.396}$$

The stresses acting on the areas normal and tangential to the contour of the basic hole[1] are defined by formulas

$$\begin{aligned}\sigma_\varrho &= \frac{1}{C^2}\{(pA^2 + qB^2) - 2[A(\beta_1\psi_1^* + \beta_2\psi_2^*) + B(\varphi_1^* + \varphi_2^*)]\},\\ \tau_{\varrho\vartheta} &= \frac{1}{C^2}\{AB(q-p) - 2[A(\varphi_1^* + \varphi_2^*) - B(\beta_1\psi_1^* + \beta_2\psi_2^*)]\},\\ \sigma_\vartheta &= \frac{1}{C^2}\{qA^2 + pB^2 - 2(A_{11}\varphi_1^* + A_{12}\varphi_2^* + A_{13}\psi_1^* + A_{14}\psi_2^*)\},\end{aligned} \tag{III.397}$$

where

$$\begin{aligned}&A = c\cos\vartheta, \quad B = \sin\vartheta,\\ &A_{11} = B\,\frac{\alpha^2\beta_1^2 + 2\beta_1^2 - 1}{\alpha^2 + \beta_1^2}, \quad A_{12} = B\,\frac{\alpha^2\beta_2^2 + 2\beta_2^2 - 1}{\alpha^2 + \beta_2^2},\\ &A_{13} = A\beta_1\,\frac{2\alpha^2 - \alpha^2\beta_1^2 + 1}{\alpha^2 + \beta_1^2}, \quad A_{14} = A\beta_2\,\frac{2\alpha^2 - \alpha^2\beta_2^2 + 1}{\alpha^2 + \beta_2^2},\\ &\alpha = \frac{B}{A}, \quad C^2 = A^2 + B^2;\end{aligned} \tag{III.398}$$

$$\begin{aligned}&\varphi_1^* = \sum_{k=1}^{\infty} k\,(\varphi_k + c_{1k}A_{kt})\sin k\vartheta, \quad \varphi_2^* = \sum_{k=1}^{\infty} k\,(\psi_k + c_{2k}B_{kt})\sin k\vartheta,\\ &\psi_1^* = \sum_{k=1}^{\infty} k\,(\varphi_k - c_{1k}^*A_{kt})\cos k\vartheta, \quad \psi_2^* = \sum_{k=1}^{\infty} k\,(\psi_k - c_{2k}^*B_{kt})\cos k\vartheta,\\ &A_{kt} = \sum_{t=1}^{\infty}\alpha_{kt}\varphi_t, \quad B_{kt} = \sum_{t=1}^{\infty} b_{kt}\psi_t.\end{aligned} \tag{III.399}$$

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In the case where the holes of the plates are not reinforced by rigid rings and are free of external forces, the formulas obtained remain in force. It is necessary only to assume formally that $p_1 = p_2 = 1$, $q_1 = s_1$, $q_2 = s_2$ in them.

[1] We have the identical picture near the other holes.

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The results of calculations for stresses occurring near a free hole are presented in Table III.12 for the case where c = 0.5, ε = 1/3 (the distance between the holes is equal to one-half the large axis of one of the ellipses). The plate is made of SVAM [Anisotropic glass-fiber material], for which

$$p_1 = -1{,}07004 \cdot 10^{-5}, \quad p_2 = -0{,}119187 \cdot 10^{-5},$$
$$q_1^* = -0{,}225486 \cdot 10^{-5}, \quad q_2^* = -0{,}568983 \cdot 10^{-5}, \qquad \text{(III.400)}$$
$$\beta_1 = 1{,}89, \quad \beta_2 = 0{,}531.$$

In the case where the holes are reinforced by rigid rings, the stresses σ_ρ greatly exceed the stresses σ_ϑ and $\tau_{\rho\vartheta}$. Therefore, Table III.12 includes only the values of σ_ρ for plates with reinforced holes. If, on the other hand, the holes are not reinforced but are free of external forces, the stresses $\sigma_\rho = \tau_{\rho\vartheta} = 0$, and Table III.12 includes the values of σ_ϑ. Moreover, for comparison, this table also includes the values σ_ρ^* and σ_ϑ^*, which pertain to a medium with one elliptical hole.

TABLE III.12

	Plate with reinforced holes				Plate with free holes			
ϕ°	σ_ρ/p	σ_ρ/q	σ_ρ^*/p	σ_ρ^*/q	σ_ϑ/p	σ_ϑ/q	σ_ϑ^*/p	σ_ϑ^*/q
0	3,30	—0,20	1,93	0,02	—0,35	7,21	—1,00	5,83
15	2,50	0,12	1,53	0,31	—0,07	4.28	—0,22	3,27
30	1,32	0,60	0,91	0,72	0,31	2,10	0,57	1,33
45	0,62	0,90	0,50	0,98	0,76	0,89	1,21	0,29
60	0,29	1,06	0,28	1,13	1,21	0,04	1,74	—0,42
75	0,15	1,13	0,17	1,20	1,53	—0,48	2,09	—0,86
90	0,11	1,16	0,14	1,22	1,65	—0,65	2,21	—1,00

Tr. Note: Commas indicate decimal points.

Figures III.59 and III.60 show the graphs characterizing the stress distribution of σ_ρ around reinforced holes. The shaded areas correspond to the case where the plate is weakened by one reinforced hole. Figures III.61 and III.62 represent the analogous graphs for the case where the holes in the plate are not reinforced and are free of external forces. /348

Tension of Anisotropic Medium with Two Identical Elliptical Holes. Let there be an elastic anisotropic plate that is weakened by two elliptical holes, reinforced by rigid rings (Figure III.63). The distance between the centers of the holes is $2l$. The basic stress state of the plate and the sizes of the holes remain the same as in the preceding case.

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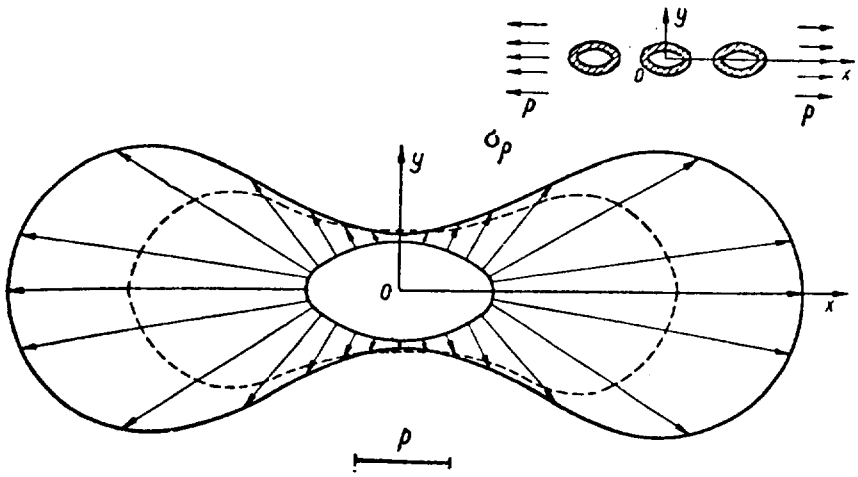


Figure III.59.

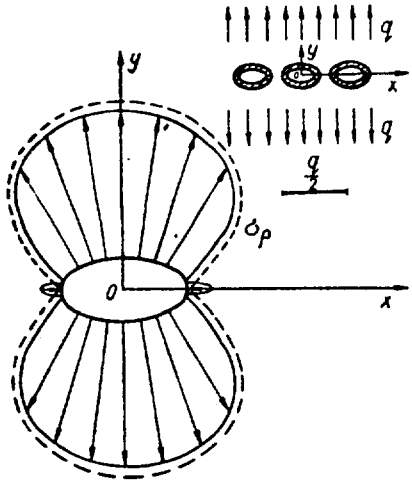


Figure III.60.

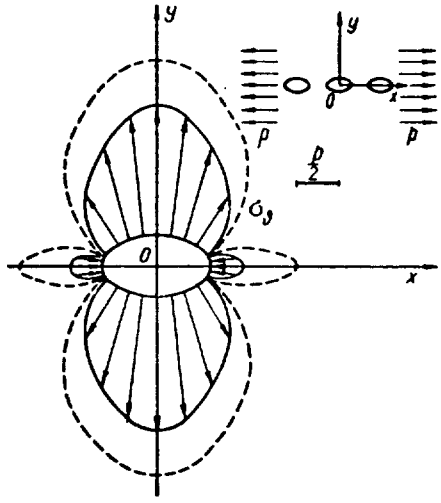


Figure III.61.

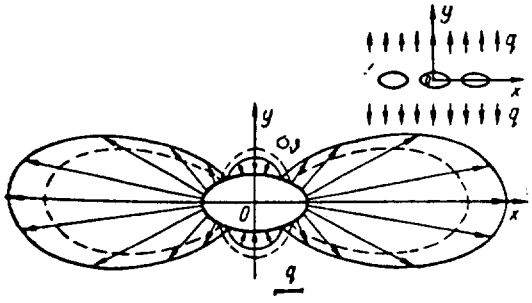


Figure III.62.

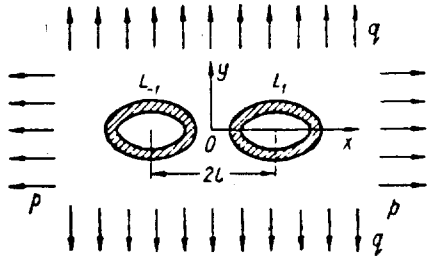


Figure III.63.

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The functions $\phi(z_1)$ and $\psi(z_2)$ can be represented in the following form

$$\varphi(z_1) = \sum_{k=1}^{\infty} \frac{\varphi_k}{[\zeta(z_1^*)]^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\varphi_k}{[\zeta(z_1^* + 2l)]^k},$$
$$\psi(z_2) = \sum_{k=1}^{\infty} \frac{\psi_k}{[\zeta(z_2^*)]^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\psi_k}{[\zeta(z_2^* + 2l)]^k}, \qquad \text{(III.401)}$$

where $z_1^* = z_1 - l$, $z_2^* = z_2 - l$.

The coefficients ϕ_k and ψ_k, due to geometric and force symmetry, can be determined from the boundary conditions (III.392) on the contour of the right hole. The boundary conditions on the contour of the left hole will be satisfied automatically.

Proceeding just as in the preceding case, we expand the second sums of expression (III.401) into series with respect to the small parameter $\varepsilon = 1/2\ l$ and terminate these expansions, preserving the terms containing ε in powers not exceeding four. Then, from boundary conditions (III.392), we obtain for the determination of coefficients ϕ_k and ψ_k, the following algebraic system:

$$\begin{gathered}
p_1[(1+a_{11})\varphi_1 + a_{12}\varphi_2 + a_{13}\varphi_3] + p_2[(1+b_{11})\psi_1 + b_{12}\psi_2 + b_{13}\psi_3] = \\
= -(p_1A_0 + p_2B_0), \\
p_1[c_{12}a_{21}\varphi_1 + (1 + c_{12}a_{22})\varphi_2] + p_2[c_{22}b_{21}\psi_1 + (1 + c_{22}b_{22})\psi_2] = 0, \\
p_1(c_{13}a_{31}\varphi_1 + \varphi_3) + p_2(c_{23}b_{31}\psi_1 + \psi_3) = 0, \\
q_1^*[(1 - c_{11}^*a_{11})\varphi_1 - c_{11}^*a_{12}\varphi_2 - c_{11}^*a_{13}\varphi_3] + q_2^*[(1 - c_{21}^*e_{11})\psi_1 - c_{21}^*b_{12}\psi_2 - \\
- c_{21}^*b_{13}\psi_3] = c(q_1^*\beta_1A_0 + q_2^*\beta_2B_0), \\
q_1^*[-c_{12}^*a_{21}\varphi_1 + (1 - c_{12}^*a_{22})\varphi_2] + q_2^*[-c_{22}^*b_{21}\psi_1 + (1 - c_{22}^*b_{22})\psi_2] = 0, \\
q_1^*(-c_{13}^*a_{31}\varphi_1 + \varphi_3) + q_2^*(-c_{23}^*b_{31}\psi_1 + \psi_3) = 0,
\end{gathered} \qquad \text{(III.402)}$$

where

$$a_{11} = -m_0\varepsilon^2(1 + 6m_0m_1\varepsilon^2), \quad a_{21} = m_0\varepsilon^3, \quad a_{31} = -m_0\varepsilon^4,$$
$$a_{12} = 2m_0^2\varepsilon^3, \quad a_{22} = -3m_0^2\varepsilon^4, \quad a_{13} = -3m_0^3\varepsilon^4. \qquad \text{(III.403)}$$

The coefficients b_{ij} are found from expressions for a_{ij} after substituting /350 in the latter m_0 and m_1 by n_0 and n_1, respectively. The other symbols remain the same as in the preceding case.

After find the coefficients ϕ_k and ψ_k, the stresses can be found from formulas (III.397) and (III.399).

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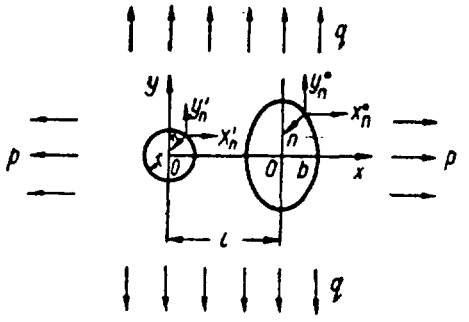


Figure III.64.

From Table III.13 we see that the character of distribution of the stresses occurring near the right hand hole is the same as in the case where the plate is weakened by an infinite row of these holes.

Stresses in Anisotropic Plate with Two Nonidentical Holes[1]. Let an elastic anisotropic plate be weakened by one round and one elliptical hole. The distance between the centers of the holes is l. The radius of the round hole is r = 1, and the semiaxes of the ellipse are a and b. To the contour of the elliptical hole are applied forces X^0_n and Y^0_n, and to the contour of the round hole, forces X'_n and Y'_n. Moreover, at infinity there are forces of tension p = const and q = const (Figure III.64).

TABLE III.13

ϑ°	σ_ρ/p	σ_ρ/q	σ_ϑ/p	σ_ϑ/q	ϑ°	σ_ρ/p	σ_ρ/q	σ_ϑ/p	σ_ϑ/q
0	2,09	0,02	—0,89	6,10	105	0,13	1,16	1,91	—0,75
15	1,67	0,28	—0,24	3,50	120	0,21	1,09	1,55	—0,26
30	1,01	0,68	0,40	1,56	135	0,45	0,94	1,04	0,51
45	0,58	0,94	0,96	0,51	150	0,98	0,67	0,49	1,59
60	0,34	1,08	1,49	—0,25	165	1,84	0,24	—0,09	3,55
75	0,21	1,15	1,88	—0,75	180	2,41	—0,04	—0,60	6,15
90	0,14	1,20	2,04	—0,92					

Tr. Note: Commas indicate decimal points.

To determine stresses σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ we will use Schwartz' method of series approximations.

We will assume that the stress field occurring in the plate with one elliptical hole is characterized by the functions of complex variables $\phi_0(z_1)$ and $\psi_0(z_2)$. When the round hole is made in the medium, a second stress field occurs that is characterized by functions $\phi_1(z_1)$ and $\psi_1(z_2)$, which are found from the following conditions on the contour of the round hole

[1]The solution of this problem was found by A. S. Kosmodamianskiy [3].

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$$2\,\mathrm{Re}\,[\varphi_1(z_1)+\psi_1(z_2)]=\int_0^s Y_n'ds-2\,\mathrm{Re}\,[\varphi_0(z_1)+\psi_0(z_2)],$$
$$2\,\mathrm{Re}\,[i\beta_1\varphi_1(z_1)+i\beta_2\psi_1(z_2)]=-\int_0^s X_n'ds-2\,\mathrm{Re}\,[i\beta_1\varphi_0(z_1)+i\beta_2\psi_0(z_2)]. \quad \text{(III.404)}$$

If the large axis of the elliptical hole is much greater than the radius of the round hole, then we simply determine the functions ϕ_0, ϕ_1, ψ_0 and ψ_1 to find the approximate solution of the problem, if the distance between the holes exceeds two diameters of the round hole[1]. /351

In the case where the holes in the medium are free of external forces,

$$\varphi_0=-\frac{p+q\beta_2^2}{2(\beta_1^2-\beta_2^2)}z_1-\frac{pb-qa\beta_2}{2(\beta_1-\beta_2)}\zeta_1^{-1},$$
$$\psi_0=\frac{p+q\beta_1^2}{2(\beta_1^2-\beta_2^2)}z_2+\frac{pb-qa\beta_1}{2(\beta_1-\beta_2)}\zeta_2^{-1}. \quad \text{(III.405)}$$

The functions ζ_k (k = 1, 2) are related to z_k by the relation

$$z_k-l=m_{0k}\zeta_k+m_{1k}\zeta_k^{-1}, \quad \text{(III.406)}$$

where

$$m_{01}=\frac{a+\beta_1 b}{2},\quad m_{02}=\frac{a+\beta_2 b}{2},\quad m_{11}=\frac{a-\beta_1 b}{2},\quad m_{12}=\frac{a-\beta_2 b}{2}. \quad \text{(III.407)}$$

We will expand functions ζ_k^{-1} into Taylor's series by powers of z_k and limit ourselves in these expansions, for instance, to four terms:

$$\zeta_k^{-1}=p_k+\sum_{n=1}^{3}A_{nk}z_k^n, \quad \text{(III.408)}$$

where

$$p_k=\frac{1}{2m_{1k}}(-l+\sqrt{l^2-4m_{0k}m_{1k}}),$$
$$A_{1k}=p_k(l+2m_{1k}p_k)^{-1},$$
$$A_{2k}=A_{1k}^2p_k^{-1}(1-m_{1k}A_{1k}), \quad \text{(III.409)}$$
$$A_{3k}=A_{1k}A_{2k}p_k^{-1}(1-2m_{1k}A_{1k}).$$

[1] See A. S. Kosmodamianskiy [3].

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From boundary conditions (III.404) we find, by the method outlined earlier,

$$\varphi_1(z_1) = -\frac{1}{2}(p+q\beta_2)(\beta_1^2-\beta_2^2)^{-1} - \alpha_1(m_{01}\zeta_1^2 - m_{11})^{-1} + \\ + \zeta_1^{*-2}(N_{11}\zeta_1^{*2} + N_{21}\zeta_1^* + N_{31})(m_{01}'\zeta_1^{*2} - m_{11}')^{-1},$$

$$\psi_1(z_2) = -\frac{1}{2}(p+q\beta_1)(\beta_2^2-\beta_1^2)^{-1} - \alpha_2(m_{02}\zeta_2^2 - m_{12})^{-1} + \\ + \zeta_2^{*2}(N_{12}\zeta_2^{*2} + N_{22}\zeta_2^* + N_{32})(m_{02}'\zeta_2^{*2} - m_{12}')^{-1}. \qquad \text{(III.410)}$$

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$$N_{1k} = \frac{p-\beta_{k-1}^* q}{2(\beta_k^*-\beta_{k-1}^*)} + \alpha_k[A_{1k}(m_{1k}' + n_{1k}m_{0k}') + \\ + 3A_{3k}m_{0k}'m_{1k}'(m_{1k}' - n_{1k}m_{0k}')] - \alpha_{k-1}n_{2k}m_{0,k-1}'(A_{1,k-1} + 3m_{0,k-1}'m_{1,k-1}'A_{3,k-1}),$$

$$N_{2k} = 2\alpha_k A_{2k}(m_{1k}'^2 - n_{1k}m_{0k}'^2) - 2A_{2,k-1}\alpha_{k-1}m_{0,k-1}'^2 n_{2k},$$

$$N_{3k} = 3\alpha_k A_{3k}(m_{1k}'^3 - n_{1k}m_{0k}'^3) - 3A_{3,k-1}\alpha_{k-1}m_{0,k-1}'^3 n_{2k},$$

$$\beta_0^* = \beta_2^* = \beta_2,\ \beta_1^* = \beta_1,\ \alpha_k = -\frac{pb - qa\beta_{k-1}^*}{2(\beta_k^*-\beta_{k-1}^*)}, \qquad \text{(III.411)}$$

$$m_{0k}' = \frac{1+\beta_k^*}{2},\ m_{1k}' = \frac{1-\beta_k^*}{2},\ n_{1k} = \frac{\beta_k^*+\beta_{k-1}^*}{\beta_k^*-\beta_{k-1}^*},$$

$$n_{2k} = \frac{2\beta_{k-1}^*}{\beta_k^*-\beta_{k-1}^*},\quad A_{i0} = A_{i2}\quad (i=1,2,3),$$

and the function ζ_k^* is related to z_k by the relation

$$z_k = m_{0k}'\zeta_k^* + m_{1k}'\zeta_k^{*-1} \qquad \text{(III.412)}$$

After finding functions $\phi(z_1) = \phi_0(z_1) + \phi_1(z_1)$ and $\psi(z_2) = \psi_0(z_2) + \psi_1(z_2)$, we can find the stresses by formulas (III.397). The results of calculations of stresses occurring in a plate made of SVAM with constants (III.400), for various dimensions of the holes, are presented in Table III.14 (a/b = 20, b/r = 1, l/r = 26) and III.15 (b/a = 20, a/r = 1, l/r = 7). Figure III.65 (for p ≠ 0; q = 0) and Figure III.66 (for p = 0; q ≠ 0) show the graphs corresponding to the data in Tables III.14 and III.15, which characterize the stress state in a plate between holes.

Concluding Comments. If the number of identical holes in an anisotropic plate is increased, the concentration of stresses when the plate is under

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tension transverse to the center line of the holes increases, but decreases when the plate is under tension in the direction of the center line. /353

TABLE III.14

Basic stress state		z							
		−1	±i	1	2	3	4	5	6
$p\neq0$ $q=0$	σ_x/p	0	3,18	0	0,34	0,56	0,63	0,59	−0,09
	σ_y/p	−0,94	0	−0,89	0,02	0,03	0,02	0,02	−0,99
$p=0$ $q\neq0$	σ_x/q	0	0,41	0	0,88	1,14	1,53	2,41	0,01
	σ_y/q	4,73	0	4,70	1,16	1,17	1,49	2,36	48,30

Tr. Note: Commas indicate decimal points.

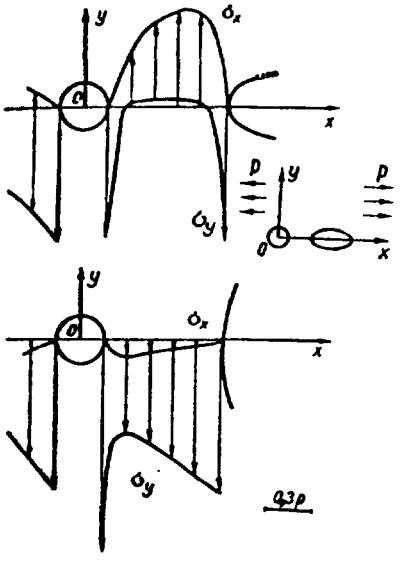


Figure III.65.

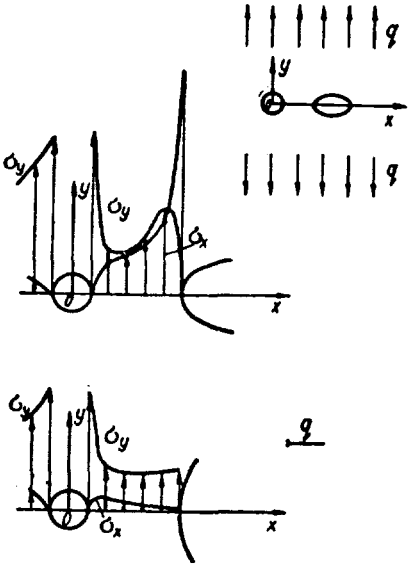


Figure III.66.

If, on the other hand, an anisotropic plate is weakened by two nonidentical holes, then the small hole has only a slight effect on the stress state near the large one. As concerns the effect of the large hole on the stress state of the medium near the small one, it is very slight when the plate is under tension in the direction of the center line of the holes ($b/a \gg r$) and when the medium is under tension transverse to the center line of the holes ($b/a \ll r$). The same principles also hold true for an isotropic medium with holes.

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TABLE III.15

Basic stress state		z							
		−1	±l	1	2	3	4	5	6
$p\neq0$	σ_x/p	0	0,44	0	−0,10	−0,06	−0,04	−0,03	−0,02
$q=0$	σ_y/p	−1,13	0	−1,40	−0,61	−0,66	−0,76	−0,87	−1,00
$p=0$	σ_x/q	0	−1,02	0	0,28	0,16	0,10	0,07	0,05
$q\neq0$	σ_y/q	3,57	0	3,61	1,30	1,16	1,14	1,13	1,14

Tr. Note: Commas indicate decimal points.

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CHAPTER IV. EFFECT OF PHYSICAL NONLINEARITY OF MATERIAL ON STRESS DISTRIBUTION NEAR HOLES

Abstract. Chapter IV deals with the basic equations of the plane problem of the theory of elasticity for physically nonlinear materials slightly deviating from Hooke's law. Complex representation of the principal problems are obtained and the nature of complex potentials for the multiply-connected region is investigated. Two methods are suggesed for solving nonlinear problems: 1) the method of conformal mappings combined with the iteration method; 2) the method of "boundary form perturbation." Examples are discussed on the stress distribution near the circular, elliptical, and square with rounded corners, holes under different types of loads.

The construction of classical elasticity theory is based on three /358
assumptions: deformation components are expressed through derivatives of displacements by linear homogeneous Cauchy's formulas; in the formulation of the conditions of equilibrium of a small element of a body, geometrical changes that occur with the element during transition of the body from the initial state into the deformed state are ignored; the ratios between the component stresses and deformations are linear. Hence the basic equations of elasticity theory for the statement of problems in displacements or in stresses will be linear. The disregarding of any of the above assumptions will result in nonlinearity of the basic equations and additional mathematical complications in the solution of the problems.

Many nonferrous metals and plastics experience deviations from the linear law of relationship between stresses and deformations, even though deformation remains elastic and reversible, while the magnitudes of displacements and deformations are rather small. By departing from linearity of the relationships between the stresses and deformations, but retaining the assumption of smallness of deformations and displacements, it is possible to obtain a variant of the nonlinear statement of problems of elasticity theory, which is known as physically nonlinear elasticity theory. Because of nonlinearity of the law of elasticity, the basic equations of the problem, as a rule, will be nonlinear, permitting only an approximate solution.

Recently, plastic materials with high strength properties and great deformation capability in the range of elastic deformations have found ever increasing application in various branches of technology. For many of them the relationships between the principal stresses and elongations can be assumed to be linear in a great range of variation. With regard to such physically linear materials, it is very interesting to examine problems of stress concentration near holes with consideration of great deformations, i.e. to solve the

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problem of stress concentration near holes in a geometrically nonlinear, but physically linear statement[1].

Also of particular importance are the solutions of problems obtained with the simultaneous consideration of both physical and geometrical nonlinearity. Examples of solutions of problems in such a statement are presented in Chapter IX. However, tremendous computational difficulties compel the use and analysis of the solution with differentiation between physical and geometrical factors of nonlinearity. /359

Below we will examine problems of stress concentration around holes in one of these two partial statements, specifically, we will find the solutions of plane (both for plane deformation and for plane stress state) problems for the physically nonlinear problem only. Two approximate methods are proposed for the solution of these problems: series approximations (§1, 2) and perturbation of boundary shape (§3).

§1. Basic Equations of Plane Problem of Physically Nonlinear Elasticity Theory

Relationship Between Stresses and Deformations. For many physically nonlinear elastic materials, the relationship between stresses and deformations can be represented in the form[2]

$$\varepsilon_{ij} = \frac{k(s_0)}{3K}\sigma_0\delta_{ij} + \frac{g(t_0^2)}{2G}(\sigma_{ij} - \sigma_0\delta_{ij}) \qquad (i, j = 1, 2, 3), \tag{IV.1}$$

where K and G are the moduli of three-dimensional deformation and displacement, respectively, in the case of vanishing small deformations; $k(s_0)$ and $g(t_0^2)$ are the functions of average stress s_0 and intensity of tangential stresses t_0^2, respectively, which can be represented by the power series

$$k(s_0) = 1 + k_1 s_0 + k_2 s_0^2 + \ldots,$$
$$g(t_0^2) = 1 + g_2 t_0^2 + g_4 t_0^4 + \ldots. \tag{IV.2}$$

The presentation of the law of shape change, i.e., of the function $g(t_0^2)$, in the form (IV.2) will facilitate in the following the solution of partial problems, since it does not involve the irrationalities related to the calculation of intensity. The use of this method in the general nonlinear statement of problems can involve erroneous conclusions and over estimation of geometrical nonlinearity in comparison with physical nonlinearity.

---

[1]The approximation method for the solution of such problems, for the case of plane deformation, developed in the works of I. N. Slezinger [1, 2], I. N. Slezinger and S. Ya. Barskaya [1-4], in our opinion, requires additional analysis of the starting positions.

[2]See G. Kauderer [1], p. 37.

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Experimental data show that the relationship between the average stress and the average deformation, for most materials, is close to linear, and therefore we will assume in the following that

$$k(s_0) = 1. \tag{IV.3}$$

In this section we will solve the corresponding problems of stress concentration near curvilinear holes for materials of which the uniaxial stress-strain diagram can be represented schematically as shown in Figure IV.1, where curves 1 and 2, for small deformations, deviate slightly from Hooke's straight (stress-strain) line. Curve 1, which lies below Hooke's line, characterizes the so-called "soft" nonlinearity, while curve 2, which is located above Hooke's line, characterizes "hard" nonlinearity (see §3, Chapter IX). We will assume that these curves can be approximated with sufficient accuracy by retaining in the expansion of (IV.2) of the intensity function of tangential stresses, only two terms: /360

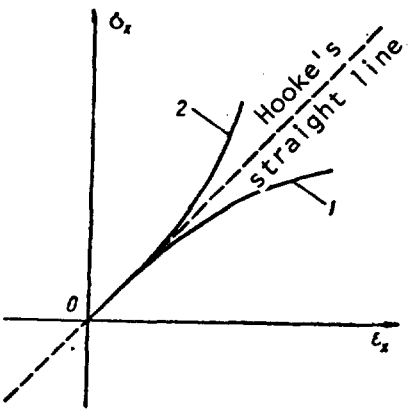


Figure IV.1.

$$g(t^2)_0 = 1 + g_2 t_0^2. \tag{IV.4}$$

In this case we will have a physically nonlinear theory with three elastic constants: K, G, g_2.

The order of magnitudes of these constants can be judged on the basis of experimental data[1] of tension of samples of pure copper:

$$\begin{aligned} K &= 1{,}37 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2; \\ G &= 0{,}46 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2; \\ g_2 &= 0{,}18 \cdot 10^6. \end{aligned} \tag{IV.5}$$

Basic Equations of Plane Problem. In contrast to linear (classical) theory, in physically nonlinear elasticity theory different equation systems correspond to the plane stress state and plane deformation. However, in the case of small deviation from Hooke's law, it is possible, with the same degree of accuracy, to derive equations for both the plane stress state and plane deformation in the very same form. These equations will differ only in relation to the constants which they contain. Actually, in the case of plane stress state, the stress-deformation relations, as indicated by (IV.1), (IV.3) and (IV.4), will be of the form

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{3K-2G}{18KG}\sigma_{kk}\delta_{ij} - \frac{g_2}{2G}t_0^2\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right), \tag{IV.6}$$

[1]See G. Kauderer [1], p. 78.

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where

$$t_0^2 = \frac{2}{9G^2}[(\sigma_{11} + \sigma_{22})^2 + 3(\sigma_{12}^2 - \sigma_{11}\sigma_{22})],$$

$$\sigma_{11} = \sigma_x, \quad \sigma_{22} = \sigma_y, \quad \sigma_{12} = \tau_{xy}, \qquad \text{(IV.7)}$$

$$\varepsilon_{11} = \varepsilon_x, \quad \varepsilon_{22} = \varepsilon_y, \quad \varepsilon_{12} = \varepsilon_{xy}.$$

In the case of plane deformation, by ignoring powers of t_0 higher than two, /361 we find from relations (IV.1), (IV.3) and (IV.4)

$$\sigma_{33} = \frac{1}{2}\left[\frac{3K-2G}{3K-G} + \frac{9KG}{(3K+G)^2} g_2 t_0^2\right](\sigma_{11} + \sigma_{22}). \qquad \text{(IV.8)}$$

Consequently, the stress-deformation relations for both cases can be written in the form

$$\varepsilon_{ij} = \frac{1}{2G}[\sigma_{ij} - a\sigma_{kk}\delta_{ij} + g_2 t_0^2(\sigma_{ij} - b\sigma_{kk}\delta_{ij})], \qquad \text{(IV.9)}$$

where, in the case of plane deformation,

$$t_0^2 = \frac{2}{9G^2}[(1+c)(\sigma_{11} + \sigma_{22})^2 + 3(\sigma_{12}^2 - \sigma_{11}\sigma_{22})],$$

$$a = \frac{3K-2G}{2(3K+G)}, \quad b = \frac{9K^2 + 6KG - 2G^2}{2(3K+G)^2}, \qquad \text{(IV.10)}$$

$$c = \frac{8G^2 - 6KG - 9K^2}{4(3K+G)^2}.$$

If in (IV.9) and (IV.10) we assume

$$a = \frac{3K-2G}{9K}, \quad b = \frac{1}{3}, \quad c = 0, \qquad \text{(IV.11)}$$

we find relations (IV.6) and (IV.7) which correspond to the plane stress state.

The plane static problem of physically nonlinear elasticity theory, in the absence of mass forces, reduces, as in the case of linear theory, to the solution of equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad \text{(IV.12)}$$

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and equations of deformation compatibility

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\varepsilon x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{IV.13}$$

under the corresponding law of elasticity.

In the variant of nonlinear theory of interest to us, Hooke's law is replaced by the nonlinear law of elasticity (IV.9). Thus, the problem reduces to the joint solution of equations (IV.9), (IV.12) and (IV.13). This equation system, generally speaking, is nonlinear. It is convenient to introduce the following stress functions:

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}; \tag{IV.14}$$

then the problem reduces to the solution of a nonlinear equation of the fourth power in partial derivatives under the corresponding boundary conditions and in the given stress state "at infinity," i.e., at sufficiently distant points of the elastic plane. However, nonlinear equation system (IV.9), (IV.12) and (IV.13) can be reduced to the solution of a series of boundary problems of the theory of complex variable functions[1]. G. N. Savin's solution and L. P. Khoroshun's solution [1], which differs somewhat[2] from G. N. Savin's solution [2, 3], are presented in §1 and 2 of this chapter. /362

Complex Potentials. We proceed from the variables x, y to new independent variables

$$z = x + iy, \qquad \bar{z} = x - iy.$$

The form of z and $\bar{z}$ in the new variables[3] is given by formulas (IV.14):

$$\sigma_x + \sigma_y = 4 \frac{\partial^2 U}{\partial z \partial \bar{z}}, \tag{IV.15}$$

$$\sigma_x - \sigma_y + 2i\tau_{xy} = -4 \frac{\partial^2 U}{\partial \bar{z}^2}.$$

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[1] See G. N. Savin [2, 3].

[2] We cite also the work of L. P. Khoroshun [1] in which the complex representation of the plane problem of nonlinear creep is given.

[3] In deriving the complex representations of the basic equations of the problem, we will follow the method of Stevenson (See I. N. Sneddon, D. S. Berri [1]), in which the equation of deformation compatibility (IV.13) will be satisfied automatically.

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From relations (9) and (15) we obtain expressions for deformations ε_x, ε_y and ε_{xy} through stress function U:

$$\varepsilon_x + \varepsilon_y = \frac{2}{G}\left[(1-2a)\frac{\partial^2 U}{\partial z \partial \bar{z}} + (1-2b)\, g_2 t_0^2 \frac{\partial^2 U}{\partial z \partial \bar{z}}\right],$$
$$\varepsilon_x - \varepsilon_y + 2i\varepsilon_{xy} = \frac{2}{G}\left[-\frac{\partial^2 U}{\partial \bar{z}^2} - g_2 t_0^2 \frac{\partial^2 U}{\partial \bar{z}^2}\right]. \qquad \text{(IV.16)}$$

We integrate the second equation in (IV.16), recalling the identity

$$2\frac{\partial}{\partial \bar{z}}(u+iv) = \varepsilon_x - \varepsilon_y + 2i\varepsilon_{xy}, \qquad \text{(IV.17)}$$

where u and v are components of the displacement vector:

$$2G(u+iv) = -2\frac{\partial U}{\partial \bar{z}} + f(z) - 2g_2 \int t_0^2 \frac{\partial^2 U}{\partial \bar{z}^2}\, d\bar{z}, \qquad \text{(IV.18)}$$

where f(z) is a function only of variable z.

We will differentiate (IV.18) with respect to z:

$$2G\frac{\partial(u+iv)}{\partial z} = -2\frac{\partial^2 U}{\partial z \partial \bar{z}} + f'(z) - 2g_2 \frac{\partial}{\partial z}\int t_0^2 \frac{\partial^2 U}{\partial \bar{z}^2}\, d\bar{z}. \qquad \text{(IV.19)}$$

On the other hand, by using the identity

$$2\frac{\partial(u+iv)}{\partial z} = \varepsilon_x + \varepsilon_y + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(IV.20)}$$

and the first expression of (IV.16), we find /363

$$2G\frac{\partial(u+iv)}{\partial z} = 2(1-2a)\frac{\partial^2 U}{\partial z \partial \bar{z}} + Gi\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + 2(1+2b)\, g_2 t_0^2 \frac{\partial^2 U}{\partial z \partial \bar{z}}. \qquad \text{(IV.21)}$$

By excluding from (IV.15) and (IV.21) the function $\partial(u + iv)/\partial z$, we find

$$4\frac{\partial^2 U}{\partial z \partial \bar{z}} = \sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)} + A(z, \bar{z})], \qquad \text{(IV.22)}$$

where, for brevity, we denote

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$$\varphi'(z) = \frac{1}{8(1-a)} f'(z),$$

$$A(z, \bar{z}) = -\frac{g_2}{1-a} \operatorname{Re}\left[(1-2b)\, t_0^2 \frac{\partial^2 U}{\partial z \partial \bar{z}} + \frac{\partial}{\partial z} \int t_0^2 \frac{\partial^2 U}{\partial \bar{z}^2}\, d\bar{z}\right]. \tag{IV.23}$$

After integrating (IV.22), we obtain

$$2\frac{\partial U}{\partial \bar{z}} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \int A(z, \bar{z})\, dz. \tag{IV.24}$$

From (IV.15), (IV.16), (IV.18) and (IV.24) it follows that

$$\begin{aligned} \sigma_x - \sigma_y + 2i\tau_{xy} &= -2\left[z\overline{\varphi''(z)} + \overline{\psi'(z)} + \int \frac{\partial A}{\partial z}\, dz\right], \\ \sigma_y - \sigma_x - 2i\tau_{xy} &= 2\left[\bar{z}\varphi''(z) + \psi'(z) + \int \frac{\partial A}{\partial \bar{z}}\, d\bar{z}\right], \\ 2G(u + iv) &= \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + 2B(z, \bar{z}), \end{aligned} \tag{IV.25}$$

where

$$\varkappa = 3 - 4a, \qquad B(z, \bar{z}) = -g_2 \int t_0^2 \frac{\partial^2 U}{\partial \bar{z}^2}\, d\bar{z} - \frac{1}{2}\int A dz. \tag{IV.26}$$

The expressions for components X, Y on the coordinate axes of the resultant force and for the resultant pair M acting from some side on arc $s_0 s$, taken within the region, will have the form

$$X + iY = -i\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \int A dz\right]_{s_0}^{s}; \tag{IV.27}$$

$$M = \operatorname{Re}\left[\int \psi(z)\, dz - z\psi(z) - z\bar{z}\varphi'(z) - \bar{z}\int A dz + \frac{1}{2}\iint A dz d\bar{z}\right]_{s_0}^{s} \tag{IV.28}$$

In the case of the first problem, where conditions $X_n(s)$ and $Y_n(s)$ on contour L are given, the boundary condition is

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \int A dz = i\int_0^s (X_n + iY_n)\, ds + \text{const} = f_1(s) + if_2(s) + \text{const}. \tag{IV.29}$$

In the case of the second problem, where on contour L are given the vector components of displacement $g^1(s)$ and $g^2(s)$, the boundary condition is /364

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$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + 2B(t, \bar{t}) = 2G[g^{(1)}(s) + ig^{(2)}(s)]. \tag{IV.30}$$

In this section we will solve the problem of stress concentration near curvilinear holes by the method of series approximations, using, as the first approximation, the solution of the corresponding problem of linear (classical) plane elasticity theory. By this method, the stresses and displacements in the n-th (n = 1, 2) approximation[1] are defined by formulas

$$\begin{aligned}
&\overset{(n)}{\sigma}_x + \overset{(n)}{\sigma}_y = 2[\varphi^{(n)'}(z) + \overline{\varphi^{(n)'}(z)} + A^{(n-1)}(z, \bar{z})],\\
&\overset{(n)}{\sigma}_y - \overset{(n)}{\sigma}_x + 2i\overset{(n)}{\tau}_{xy} = 2\left[\bar{z}\varphi^{(n)''}(z) + \psi^{(n)'}(z) + \int \frac{\partial A^{(n-1)}}{\partial z} d\bar{z}\right],\\
&2G(\overset{(n)}{u} + i\overset{(n)}{v}) = \varkappa\varphi^{(n)}(z) - z\overline{\varphi^{(n)'}(z)} - \overline{\psi^{(n)}(z)} + 2B^{(n-1)}(z, \bar{z}),
\end{aligned} \tag{IV.31}$$

where the complex potentials $\phi^{(n)}(z)$ and $\psi^{(n)}(z)$ in the case of the first problem are determined from boundary condition (IV.29):

$$\varphi^{(n)}(t) + t\overline{\varphi^{(n)'}(t)} + \overline{\psi^{(n)}(t)} = f_1(s) + if_2(s) - \int A^{(n-1)}dz + \text{const}, \tag{IV.32}$$

and in the case of the second problem, from boundary condition (IV.30):

$$\varkappa\varphi^{(n)}(t) - t\overline{\varphi^{(n)'}(t)} - \overline{\psi^{(n)}(t)} = 2G(g^{(1)} + ig^{(2)}) - 2B^{(n-1)}(t, \bar{t}). \tag{IV.33}$$

General Formulas for Multiply-connected Region. Case of Infinite Region. We will examine a region bounded by simple closed contours L_1, L_2, ..., L_m, L_{m+1}, of which the latter encompasses all the preceding, and we will analyze the character of multiformity of complex potentials $\phi^{(n)}(z)$, $\psi^{(n)}(z)$.

Since the stresses are identical, in passing around closed contour L_j, ambiguity of the function $A(z, \bar{z})$ from (IV.23) can occur only as the result of the integration of the expression

$$\int t_0^2 \frac{\partial^2 U}{\partial z^2} d\bar{z} = -\sum_{k=1}^{m} q_{1k}(z) \ln(\overline{z - z_k}) + R(z, \bar{z}). \tag{IV.34}$$

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[1] Strictly speaking, in the given statement of the problem, it is necessary to determine only the first and second approximations, since the original assumptions are limited by such a degree of accuracy.

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Consequently, the function $A^{(n)}(z, \bar{z})$ can be represented in the form

$$A^{(n)} = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} [q_{1k}^{(n)'}(z) \ln(\overline{z - z_k}) + \overline{q_{1k}^{(n)'}(z)} \ln(z - z_k)] + R_1^{(n)}(z, \bar{z}). \quad \text{(IV.35)}$$

Here z_k is a point within contour L_k; $q_{1k}^{(n)}(z)$ is a holomorphic function, and $R(z, \bar{z})$ and $R_1(z, \bar{z})$ are unique functions in the region of interest. /365

As a consequence of (IV.35), we have

$$\int A^{(n)} dz = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} [\overline{q_{1k}^{(n)'}(z)} z \ln(z - z_k) + q_{1k}^{(n)}(z) \ln(\overline{z - z_k}) + \overline{q_{2k}^{(n)}(z)} \ln(z - z_k)] + R_2^{(n)}(z, \bar{z}), \quad \text{(IV.36)}$$

$$\int \frac{\partial A^{(n)}}{\partial z} d\bar{z} = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} [q_{1k}^{(n)''}(z) \bar{z} \ln(\overline{z - z_k}) + q_{2k}^{(n)}(z) \ln(\overline{z - z_k})] + R_3^{(n)}(z, \bar{z}),$$

where $q_{2k}^{(n)}(z)$ is holomorphic, and $R_2^{(n)}(z, \bar{z})$ and $R_3^{(n)}(z, \bar{z})$ are unique functions in the examined region.

From the condition of uniqueness of stresses (IV.31) in passing along each contour L_k, recalling relations (IV.35) and (IV.36), we obtain

$$\varphi^{(n)'}(z) = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} q_{1k}^{(n-1)'}(z) \ln(z - z_k) + \Phi_*^{(n)}(z),$$
$$\psi^{(n)'}(z) = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} q_{2k}^{(n-1)'}(z) \ln(z - z_k) + \Psi_*^{(n)}(z), \quad \text{(IV.37)}$$

where $\Phi_*^{(n)}(z)$ and $\Psi_*^{(n)}(z)$ are holomorphic functions in the given region.

Consequently, the complex potentials are

$$\Phi^{(n)}(z) = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} q_{1k}^{(n-1)}(z) \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k^{(n)} \ln(z - z_k) + \varphi_*^{(n)}(z),$$
$$\Psi^{(n)}(z) = \frac{g_2}{2(1-a)} \sum_{k=1}^{m} q_{2k}^{(n-1)}(z) \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k'^{(n)} \ln(z - z_k) + \psi_*^{(n)}(z), \quad \text{(IV.38)}$$

where $\phi_*^{(n)}(z)$, $\psi_*^{(n)}(z)$ are functions that are holomorphic in the region of interest.

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Complex constants $\gamma_k^{(n)}$ and $\gamma_k^{'(n)}$ are determined from the condition of uniqueness of displacements (IV.31) and from the expression for the resultant vector (IV.27) in passing around each contour.

$$\gamma_k^{(n)} = -\frac{\overline{X}_k + iY_k}{2\pi(1+\varkappa)}, \qquad \gamma_k^{'(n)} = \frac{\varkappa(X_k + iY_k)}{2\pi(1+\varkappa)}. \tag{IV.39}$$

From (IV.39) we see that these complex constants are independent of the order of approximation.

Consider the case of an infinite region, where contour L_{m+1} extends to infinity. From the origin of the coordinate system we construct circle L_R of radius R, which contains all contours L_1, L_2, ..., L_m. /366

From the condition of boundedness of stresses σ_x, σ_y and τ_{xy} through the entire region, we know that the functions $A^{(n)}$ and $\partial/\partial z \int A^{(n)} d\bar{z}$ are also bounded. Hence functions $q_{1k}^{(n)}(z)$ and $q_{2k}^{(n)}(z)$ will be holomorphic outside of L_R, including the infinitely distant point, i.e., for rather large $|z|$ we have the following expansions:

$$q_{1k}^{(n)} = \sum_{r=0}^{\infty} \frac{a_{kr}^{(n)}}{z^r}, \qquad q_{2k}^{(n)} = \sum_{r=0}^{\infty} \frac{b_{kr}^{(n)}}{z^r}. \tag{IV.40}$$

Functions (IV.38) in this case are

$$\begin{aligned} \varphi_*^{(n)}(z) &= \Gamma_n z + \sum_{r=0}^{\infty} \frac{c_r^{(n)}}{z^r}, \\ \psi_*^{(n)}(z) &= \Gamma_n' z + \sum_{r=0}^{\infty} \frac{c_r^{'(n)}}{z^r}. \end{aligned} \tag{IV.41}$$

Thus, as follows from (IV.40) and (IV.41), complex potentials (IV.38), for rather large $|z|$, have the form

$$\begin{aligned} \varphi^{(n)}(z) &= \sum_{k=1}^{\infty} (a_{k_0}^{(n-1)} + \gamma_k) \ln z + \Gamma_n z + \varphi_0^{(n)}(z), \\ \psi^{(n)}(z) &= \sum_{k=1}^{\infty} (b_{k_0}^{(n-0)} + \gamma_k') \ln z + \Gamma_n' z + \psi_0^{(n)}(z). \end{aligned} \tag{IV.42}$$

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Here $\phi_0^{(n)}(z)$ and $\psi_0^{(n)}(z)$ are functions that are holomorphic outside of L_R, including the infinitely distant point, whereupon, as in the case of the linear problem, without changing the stress state, we may assume:

$$\varphi_0^{(n)}(\infty) = \psi_0^{(n)}(\infty) = 0, \qquad \operatorname{Im}\Gamma_n = 0. \tag{IV.43}$$

The constants Γ_n and Γ_n' are found from the conditions of the stress state at infinity.

We introduce the following definitions

$$\begin{aligned} A^{(n)}(z, \bar{z}) &= Q_n, \\ \int \frac{\partial A^{(n)}(z, \bar{z})}{\partial z}\, d\bar{z} &= Q_n' \end{aligned} \quad \text{for } |z| \to \infty; \tag{IV.44}$$

then, from (IV.31), (IV.42) and (IV.44), we find /367

$$\begin{aligned} \operatorname{Re}\Gamma_n &= \frac{1}{4}(\sigma_x^{(\infty)} + \sigma_y^{(\infty)}) - \frac{1}{2}Q_{n-1}, \\ \Gamma_n' &= \frac{1}{2}(\sigma_y^{(\infty)} - \sigma_x^{(\infty)} + 2i\tau_{xy}^{(\infty)}) - Q_{n-1}. \end{aligned} \tag{IV.45}$$

§2. Method of Series Approximations

General Formulas. Statement of Problem[1]. We will assume that the basic stress state in an infinite elastic plane is characterized by components σ_x^*, σ_y^* and τ_{xy}^*, which, according to (IV.22) and (IV.25), can be represented in the form

$$\begin{aligned} \sigma_x^* + \sigma_y^* &= 2[\varphi_{1*}'(z) + \overline{\varphi_{1*}'(z)} + A_{1*}(z, \bar{z})], \\ \sigma_y^* - \sigma_x^* + 2i\tau_{xy}^* &= 2\left[\bar{z}\varphi_{1*}'(z) + \psi_{1*}'(z) + \int \frac{\partial A_{1*}}{\partial z}\, d\bar{z}\right]. \end{aligned} \tag{IV.46}$$

If, in this plane, we make a hole of some shape, then around the hole there will be a new distribution of stresses, which in the n-th approximation, according to (IV.31), can be determined by formulas

[1] See G. N. Savin and L. P. Khoroshun [1].

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$$\overset{(n)}{\sigma}_x + \overset{(n)}{\sigma}_y = 2\left[\varphi_1^{(n)'}(z) + \overline{\varphi_1^{(n)'}(z)} + A_1^{(n-1)}(z, \bar{z})\right],$$

$$\overset{(n)}{\sigma}_y - \overset{(n)}{\sigma}_x + 2i\overset{(n)}{\tau}_{xy} = 2\left[\bar{z}\varphi_1^{(n)''}(z) + \psi_1^{(n)'}(z) + \int \frac{\partial A_1^{(n-1)}}{\partial z}\, d\bar{z}\right], \qquad \text{(IV.47)}$$

and at infinity the components $\sigma_x^{(n)}$, $\sigma_y^{(n)}$ and $\tau_{xy}^{(n)}$ acquire assigned values σ_x^*, σ_y^* and τ_{xy}^*.

Recalling (IV.44), we will represent the potentials as follows:

$$\varphi_1^{(n)}(z) = \varphi_{10}^{(n)}(z) + \varphi_{1*}(z) + \frac{1}{2}(Q_* - Q_{n-1})z,$$
$$\psi_1^{(n)}(z) = \psi_{10}^{(n)}(z) + \psi_{1*}(z) + (Q_*' - Q_{n-1}')z, \qquad \text{(IV.48)}$$

where $\phi_{10}^{(n)}(z)$ and $\psi_{10}^{(n)}(z)$ are holomorphic functions that are converted to zero at infinity.

In this manner the stated problem reduces to the determination of functions /368 $\phi_{10}^{(n)}(z)$ and $\psi_{10}^{(n)}(z)$ that satisfy, on the contour of the hole, the condition

$$\varphi_{10}^{(n)}(t) + t\overline{\varphi_{10}^{(n)'}(t)} + \overline{\psi_{10}^{(n)}(t)} = f_{11}^{(n-1)}(s) + if_{12}^{(n-1)}(s), \qquad \text{(IV.49)}$$

where the function that represents the adduced contour load acquires the form

$$f_{11}^{(n-1)}(s) + if_{12}^{(n-1)}(s) = f_{11}(s) + if_{12}(s) - [\varphi_{1*}(t) + t\overline{\varphi_{1*}'(t)} +$$
$$+ \overline{\psi_{1*}(t)} + t(Q_* - Q_{n-1}) + \bar{t}(Q_*' - Q_{n-1}') + \int A_1^{(n-1)}dz. \qquad \text{(IV.50)}$$

Let the function

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + a_1\zeta + \ldots + a_n\zeta^n\right) \qquad \text{(IV.51)}$$

map conformally the examined region, i.e., the surface outside of the hole, onto the interior of the unit circle. Then, converting to the variable $\zeta = \rho e^{i\vartheta}$, we obtain the expressions for stresses and displacements:

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$$\overset{(n)}{\sigma}_{\varrho} + \overset{(n)}{\sigma}_{\theta} = 2\left[\Phi^{(n)}(\zeta) + \overline{\Phi^{(n)}(\zeta)} + A^{(n-1)}(\zeta, \bar{\zeta})\right],$$

$$\overset{(n)}{\sigma}_{\theta} - \overset{(n)}{\sigma}_{\varrho} + 2i\overset{(n)}{\tau}_{\varrho\theta} = \frac{2\zeta^2}{\varrho^2\omega'(\zeta)}\left[\overline{\omega(\zeta)}\,\Phi^{(n)\prime}(\zeta) + \omega'(\zeta)\,\Psi^{(n)}(\zeta) + \int \frac{\partial A^{(n)}}{\partial \zeta}\,\overline{\omega'(\zeta)}\,d\bar{\zeta}\right], \tag{IV.52}$$

$$2G\left(u_{\varrho}^{(n)} + iv_{\theta}^{(n)}\right) = \frac{\bar{\zeta}\,\overline{\omega'(\zeta)}}{\varrho\,|\,\omega'(\zeta)\,|}\left[\varkappa\varphi^{(n)}(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\,\overline{\varphi^{(n)\prime}(\zeta)} - \overline{\psi^{(n)}(\zeta)} + 2B^{(n-1)}(\zeta, \bar{\zeta})\right]$$

Complex potentials (IV.48) and the conditions on the contour of the hole (IV.49) and (IV.50) in the transformed region, i.e., in variable ζ, acquire, accordingly, the form

$$\varphi^{(n)}(\zeta) = \varphi_0^{(n)}(\zeta) + \varphi_*(\zeta) + \frac{1}{2}(Q_* - Q_{n-1})\,\omega(\zeta); \tag{IV.53}$$

$$\psi^{(n)}(\zeta) = \psi_0^{(n)}(\zeta) + \psi_*(\zeta) + (Q_*' - Q_{n-1}')\,\omega(\zeta);$$

$$\varphi_0^{(n)}(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_0^{(n)\prime}(\sigma)} + \overline{\psi_0^{(n)}(\sigma)} = f_1^{(n-1)}(\sigma) + if_2^{(n-1)}(\sigma); \tag{IV.54}$$

$$f_1^{(n-1)}(\sigma) + if_2^{(n-1)}(\sigma) = f_1(\sigma) + if_2(\sigma) - \left[\varphi_*(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_*'(\sigma)} + \psi_*(\sigma)\right] +$$

$$+ \omega(\sigma)(Q_* - Q_{n-1}) + \overline{\omega(\sigma)}\,(Q_*' - Q_{n-1}') + \int A^{(n-1)}(\zeta, \bar{\zeta})\,\omega'(\zeta)\,d\zeta\Big]. \tag{IV.55}$$

Here we make the following definitions: /369

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}; \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)};$$

$$\varphi_1[\omega(\zeta)] = \varphi(\zeta); \qquad \psi(\zeta) = \psi_1[\omega(\zeta)]; \tag{IV.56}$$

$$A_1[\omega(\zeta)] = A(\zeta); \qquad f_{11}[\omega(s)] = f_1(\sigma);$$

$$f_{12}[\omega(s)] = f_2(\sigma); \qquad \sigma = e^{i\theta}.$$

As in the linear problem, boundary condition (IV.54) can be substituted by two equivalent functional equations:

$$\varphi_0^{(n)}(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{f_1^{(n-1)} + if_2^{(n-1)}}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \frac{\overline{\varphi_0^{(n)\prime}(\sigma)}}{\sigma - \zeta}\,d\sigma - \bar{\beta}_0^{(n)},$$

$$\psi_0^{(n)}(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{f_1^{(n-1)} - if_2^{(n-1)}}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \cdot \frac{\varphi_0^{(n)\prime}(\sigma)}{\sigma - \zeta}\,d\sigma. \tag{IV.57}$$

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Thus, the problem of determining the stress concentration around a curvilinear hole in the n-th approximation is reduced to determining the functions $\phi_0^{(n)}(\zeta)$ and $\psi_0^{(n)}(\zeta)$ from functional equations (IV.57).

Stresses $\sigma_x^{(n)}$, $\sigma_y^{(n)}$ and $\tau_{xy}^{(n)}$ in the n-th approximation, according to the functions $\phi_0^{(n)}(\zeta)$ and $\psi_0^{(n)}(\zeta)$ that we have found, are defined by formulas (IV.52) and (IV.53).

If the contour of the hole is free of external forces, then, for stresses along the contour of the hole (since $\sigma_\rho^{(n)} = 0$), from the first formula of (IV.52) we obtain

$$\sigma_\phi^{(n)} = 2\left[\frac{\varphi_0^{(n)'}(\sigma)+\varphi_*'(\sigma)}{\omega'(\sigma)}+\frac{\overline{\varphi_0^{(n)'}(\sigma)}+\overline{\varphi_*'(\sigma)}}{\overline{\omega'(\sigma)}}+(Q_*-Q_{n-1})+A^{(n-1)}(\sigma,\bar{\sigma})\right]. \qquad \text{(IV.58)}$$

Hole Reinforced by Elastic Ring. We will assume that a wide elastic ring of a different material, with nonlinear elasticity, i.e. the elastic properties of this material also deviate only slightly from Hooke's law, is sealed into a curvilinear hole.

The solution of the problem of stress concentration near the hole reinforced by the elastic ring reduces to the determination of complex potentials $\phi_{10}^{(n)}(z)$ and $\psi_{10}^{(n)}(z)$ in the plane outside of the hole and ring, and also the determination of the potentials $\phi_{11}^{(n)}(z)$ and $\psi_{11}^{(n)}(z)$ within the ring[1].

The boundary conditions for $\phi_{11}^{(n)}(z)$ and $\psi_{11}^{(n)}(z)$ on the contour of hole L will have the form (IV.29) or (IV.30), depending on whether the stresses or displacements are given on the contour.

If the ring is soldered to the plate, then on contour L_1 of the seam of the ring and plate, the following conjugation conditions must be satisfied: /370

$$\begin{aligned}\varphi_{10}^{(n)}(z)+\varphi_{1*}(z)+z\left[\overline{\varphi_{10}^{(n)'}(z)}+\overline{\varphi_{1*}'(z)}\right]+\overline{\psi_{10}^{(n)}(z)}+\overline{\psi_{1*}(z)}+\\ +z(Q_*-Q_{n-1})+\bar{z}(Q_*'-Q_{n-1}')+\int A_1^{(n-1)}dz=\varphi_{11}^{(n)}(z)+\\ +z\,\overline{\varphi_{11}^{(n)'}(z)}+\overline{\psi_{11}^{(n)}(z)}+\int A_{11}^{(n-1)}dz;\end{aligned} \qquad \text{(IV.59)}$$

[1]The values pertaining to the ring are denoted by an additional subscript 1.

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$$\varkappa[\varphi_{10}^{(n)}(z)+\varphi_{1*}(z)]-z[\overline{\varphi_{10}^{(n)'}(z)}+\overline{\varphi_{1*}'(z)}]-\overline{\psi_{10}^{(n)}(z)}-\overline{\psi_{1*}(z)}+$$

$$+\frac{\varkappa-1}{2}z(Q_*-Q_{n-1})-\bar{z}(Q_*'-Q_{n-1}')+2B_1^{(n-1)}(z,\bar{z})=\varkappa_1\varphi_{11}^{(n)}(z)-$$

$$-z\overline{\varphi_{11}^{(n)}(z)}-\overline{\psi_{11}^{(n)}(z)}+2B_{11}^{(n-1)}(z,\bar{z}),$$

which express equality of displacements of the points of the seam of the ring and plate and equality of the forces applied to the plate and ring along the contour of the seam.

Three-Dimensional Tension of Elastic Plate with Round Hole with Simultaneous[1] Uniform Pressure on the Contour of the Hole. Let an elastic plate with a round hole be subjected at infinity to multifold tension:

$$\sigma_x^\infty=\sigma_y^\infty=p,\qquad \tau_{xy}^{(\infty)}=0,$$

and let uniform normal pressure

$$\sigma_\varrho=-p_0.$$

be given on the contour of the hole.

We are required to determine the stress state in the elastic plane near the hole. The functions $\phi_{1*}(z)$ and $\psi_{1*}(z)$ in this case are

$$\varphi_{1*}=\left[\frac{p}{2}+\frac{g_2(1-2b)(1+4c)}{18(1-a)G^2}p^3\right]z,$$

$$\psi_{1*}=0,\qquad A_{1*}=-\frac{g_2(1-2b)(1+4c)}{9(1-a)G^2}p^3, \tag{IV.60}$$

while the function that conformally maps the region outside a hole of radius R on the unit circle is

$$\omega(\zeta)=R\frac{1}{\zeta}. \tag{IV.61}$$

In the first approximation, i.e. for the linear problem,

$$\varphi^{(1)}(\zeta)=\frac{Rp}{2}\cdot\frac{1}{\zeta},\qquad \psi^{(1)}(\zeta)=-R(p_0+p)\zeta. \tag{IV.62}$$

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[1] In nonlinear problems, as we know, the principle of superposing does not apply.

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Using functions (IV.62), we find the values A_1', Q_1, Q_1' and $f_1^{(1)} + if_2^{(1)}$ /371 for the determination of the second approximation:

$$A^{(1)} = -\frac{g_2}{9(1-a)G^2}[(1-2b)(1+4c)p^3 + 3(1-2b)p(p_0+ \\ + p)^2\zeta^2\bar{\zeta}^2 + 2(p_0+p)^3\zeta^3\bar{\zeta}^3];$$

$$Q_1 = -\frac{g_2(1-2b)(1+4c)p^3}{9(1-a)G^2}; \qquad Q_1' = 0; \tag{IV.63}$$

$$f_1^{(1)}(\sigma) + if_2^{(1)}(\sigma) = -\frac{(p_0+p)R}{\sigma} - \frac{g_2R}{9(1-a)G^2}[3(1-2b)p(p_0+ \\ + p)^2 + (p_0+p)^3]\frac{1}{\sigma}.$$

On the basis of relations (IV.54), (IV.57) and (IV.63), we find the functions in the second approximation:

$$\varphi^{(2)}(\zeta) = \frac{pR}{2\zeta} + \frac{g_2(1-2b)(1+4c)p^3R}{18(1-a)G^2\zeta},$$

$$\psi^2(\zeta) = -(p_0+p)R\zeta - \frac{g_2R}{9(1-a)G^2}[3(1-2b)p(p_0+p)^2 + (p_0+p)^3]. \tag{IV.64}$$

Stresses in the second approximation are

$$\overset{(2)}{\sigma}_\varrho = p - (p_0+p)\varrho^2 - \frac{g_2}{9(1-a)G^2}[3(1-2b)p(p_0+p)^2(\varrho^2-\varrho^4) + (p_0+p)^3(\varrho^2-\varrho^6)],$$

$$\overset{(2)}{\sigma}_\vartheta = p + (p_0+p)\varrho^2 - \frac{g_2}{9(1-a)G^2}[3(1-2b)p(p_0+p)^2(3\varrho^4-\varrho^2) + \\ + (p_0+p)^3(5\varrho^6-\varrho^2)]. \tag{IV.65}$$

On the contour of the hole the stress is[1]

$$\overset{(2)}{\sigma}_\vartheta = 2p + p_0 - \frac{2g_2}{9(1-a)G^2}[3(1-2b)p(p_0+p)^2 + 2(p_0+p)^3]. \tag{IV.66}$$

After substituting in (IV.66) the values of constants a and b from (IV.10) and (IV.11), we obtain the formulas for stresses $\overset{(2)}{\sigma}_\vartheta$ both for plane deformation:

$$\overset{(2)}{\sigma}_\vartheta = 2p + p_0 - \frac{8(1+\nu)^2g_2}{9(1-\nu)E^2}[(1-2\nu)^2p(p_0+p)^2 + 2(p_0+p)^3], \tag{IV.67}$$

[1]We will recall that the superscript above the stress component indicates the order of approximation.

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and also for the plane stress state:

$$\overset{(2)}{\sigma_\vartheta} = 2p + p_0 - \frac{8(1+\nu)^3 g_2}{9E^2}[p(p_0+p)^2 + 2(p_0+p)^3]. \qquad \text{(IV.68)}$$

Here elastic constants K and G are expressed through Young's modulus E and Poisson's ratio ν. /372

We see from (IV.67) and (IV.68) that in the case of a free contour, i.e. when $p_0 = 0$, stresses $\overset{(2)}{\sigma_\vartheta}$ for plane deformation are greater than in the case of the plane stress state for all $\nu > 0$. They will coincide for $\nu = 0$. The maximum difference between them will occur when $\nu = 0.5$ and is equal to $q_2 p^3/E^2$.

Uniaxial Tension of Elastic Plate with Free Round Hole. Let an elastic plate with a round hole of radius R, free of external forces, be subjected at infinity to uniaxial tension:

$$\sigma_x^{(\infty)} = p, \qquad \sigma_y^{(\infty)} = 0, \qquad \tau_{xy}^{(\infty)} = 0.$$

In the manner analogous to the above, we find in the second approximation stresses $\overset{(2)}{\sigma_\vartheta}$ on the contour of the hole at the point $\vartheta = \pi/2$, both for plane deformation:

$$\overset{(2)}{\sigma_\vartheta} = 3p - \frac{(1+\nu)^2 g_2 p^3}{945(1-\nu)E^2}[8683 - 16807\nu(1+\nu) + 5880\nu^2(1-\nu)^2], \qquad \text{(IV.69)}$$

and for the plane stress state:

$$\overset{(2)}{\sigma_\vartheta} = 3p - \frac{8683(1+\nu)^3 g_2 p^3}{945E^2}. \qquad \text{(IV.70)}$$

Pure Displacement. Let the basic stress state of an elastic plate with a round hole of radius R be characterized by stress components

$$\sigma_x^{(\infty)} = p, \qquad \sigma_y^{(\infty)} = -p, \qquad \tau_{xy}^{(\infty)} = 0.$$

Stresses $\overset{(2)}{\sigma_\vartheta}$ on the contour of the hole at the point $\vartheta = \pi/2$, in the second approximation for plane deformation and for the plane stress state, are, respectively

$$\overset{(2)}{\sigma_\vartheta} = 4p - \frac{8(1+\nu)^2 g_2 p^3}{945(1-\nu)E^2}[2475 - 5264\nu(1-\nu) + 2240\nu^2(1-\nu^2)]; \qquad \text{(IV.71)}$$

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$$\overset{(2)}{\sigma_\vartheta} = 4p - \frac{440(1+\nu)^2 g_2 p^3}{21E^2}. \tag{IV.72}$$

Multifold Tension of Elastic Plate with Elliptical Hole. Let an elastic plate with a free elliptical hole be subjected at infinity to multifold tension:

$$\sigma_x^{(\infty)} = \sigma_y^{(\infty)} = p, \qquad \tau_{xy}^{(\infty)} = 0.$$

The mapping function is

$$\omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right). \tag{IV.73}$$

The basic stress state is characterized by the functions /373

$$\varphi_{1*} = \left[\frac{p}{2} + \frac{g_2(1-2b)(1+4c)}{18(1-a)G^2}\right]z; \qquad \psi_{1*} = 0;$$
$$A_{1k} = -\frac{g_2(1-2b)(1+4c)p^3}{9(1-a)G^2}. \tag{IV.74}$$

The complex potentials in the first approximation, corresponding to the classical linear problem, are, as we know[1],

$$\varphi^{(1)}(\zeta) = \frac{pR}{2}\cdot\frac{1-m\zeta^2}{\zeta}, \qquad \psi^{(1)}(\zeta) = -\frac{pR(1+m^2)\zeta}{1-m\zeta}. \tag{IV.75}$$

Parameter m, which is the eccentricity of the elliptical hole, is assumed to be so small that we can disregard its powers above two.

Retaining the stated accuracy, we find from (IV.58) the stresses on the contour of the hole in the second approximation, both for plane deformation:

$$\overset{(2)}{\sigma_\vartheta} = 2p(1+2m\cos 2\vartheta + 2m^2\cos 4\vartheta) - \frac{8(1+\nu)^2 g_2 p^3}{9(1-\nu)E^2}\left\{2 + (1-2\nu)^2 + \right.$$
$$+ m\left[\frac{16}{3}(1-2\nu)^2 + 8\nu(1-\nu) + \frac{74}{5}\right]\cos 2\vartheta +$$
$$+ m^2\left[\frac{4}{3}(1-2\nu)(5+6\nu^2-6\nu) + \frac{40}{3}(1-\nu+\nu^2) + \right. \tag{IV.76}$$
$$\left.\left. + \left(\frac{101}{15}(1-2\nu)^2 + 30\nu(\nu-1) + \frac{983}{35}\right)\cos 4\vartheta\right]\right\},$$

[1]See §2, Chapter II.

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and for the plane stress state:

$$\overset{(2)}{\sigma_\theta} = 2p(1 + 2m\cos 2\theta + 2m^2\cos 4\theta) - \frac{8(1+\nu)^2 g_0 p^2}{9E^2}\left[3 + \frac{302}{15}m\cos 2\theta + \right.$$
$$\left. + m^2\left(20 + \frac{3656}{105}\cos 4\theta\right)\right]. \qquad \text{(IV.77)}$$

By assuming in (IV.76) and (IV.77) that m = 0, we obtain, respectively, formulas (IV.67) and (IV.68) for $p_0 = 0$, as we expected.

§3. Method of Perturbation of Shape of Boundary

Basic Equations of Problem in Polar Coordinates. The approximation method developed in this section for the solution of the problem[1] requires the representation of the corresponding basic equations of §1 in the polar coordinate system.

We will write these equations of the plane problem in a "dimensionless" polar coordinate system $r = r^*/R$, θ. Hence, for definition, we will analyze the case of the plane stress state[2], i.e. we will represent the law of elasticity in the form[3] (IV.6) and (IV.7) in coordinates r and θ: /374

$$\varepsilon_\theta = \frac{1}{3K}k(s_0)\sigma_0 + \frac{1}{2G}g(t_0^2)(\sigma_\theta - \sigma_0),$$
$$\varepsilon_r = \frac{1}{3K}k(s_0)\sigma_0 + \frac{1}{2G}g(t_0^2)(\sigma_r - \sigma_0), \qquad \text{(IV.78)}$$
$$\varepsilon_{r\theta} = \frac{1}{G}g(t_0^2)\tau_{r\theta},$$

where ε_r, ε_θ and $\varepsilon_{r\theta}$, and also σ_r, σ_θ and $\tau_{r\theta}$ are components of deformations and stresses, respectively, in the polar coordinate system (r, θ).

The dimensionless values are expressed through the invariants:

$$s_0 = \frac{\sigma_0}{3K} = \frac{1}{9K}(\sigma_r + \sigma_\theta),$$
$$t_0^2 = \frac{\tau_0^2}{G^2} = \frac{2}{9G^2}(\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta + 3\tau_{r\theta}^2). \qquad \text{(IV.79)}$$

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[1] See A. N. Guz', G. N. Savin, I. A. Tsurpal [1]. The method of perturbation of shape of boundary is outlined in greater detail in application to problems of stress concentration around holes in Chapter VI (for the case of the plane problem with an asymmetrical stress tensor), and in Chapter X (for the case of thin elastic shells weakened by holes).

[2] The law of elasticity for plane deformation must be taken intthe form (IV.9) and (IV.10).

[3] See G. Kauderer [1], p. 117.

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As in §1, we will assume that small deviations from the linear relationship between stresses and deformations in the elasticity relations (IV.79) can be expressed with a sufficient degree of accuracy by the function $g(t_0^2) = 1 + g_2 t_0^2$. Therefore, in the following we will assume in the elasticity relations (IV.79) that

$$k(s_0) \equiv 1, \qquad g(t_0^2) = 1 + g_2 t_0^2, \tag{IV.80}$$

where g_2 is a dimensionless constant.

After such a selection of nonlinear elasticity law (IV.78), and under conditions (IV.80), the problem of the stress state of a thin plate is reduced to the determination of stress function $U(r, \theta)$ from a nonlinear equation[1] of the fourth power:

$$\begin{aligned}\Delta\Delta U + \frac{2\lambda}{R^4}\Big[&\frac{1}{r^4}U_{,\theta}T_{,\theta} - \frac{1}{r^3}(U_{,r\theta}T_{,\theta} + U_{,\theta}T_{,r\theta}) - \\ &- \frac{1}{r^2}\Big(\frac{1}{2}U_{,rr}T_{,\theta\theta} - U_{,r\theta}T_{,r\theta} + \frac{1}{2}U_{,\theta\theta}T_{,rr}\Big) - \frac{1}{2r}(U_{,r}T_{,rr} + \\ &+ U_{,rr}T_{,r}) - \frac{1}{3}\Delta(T\Delta U)\Big] = 0\end{aligned} \tag{IV.81}$$

under the corresponding boundary conditions on the contour of the hole and at infinity. If the function $U(r, \theta)$ is known, then the stress components are /375

$$\begin{aligned}\sigma_r &= \frac{1}{R^2}\Big[\frac{1}{r}\cdot\frac{\partial U}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 U}{\partial\theta^2}\Big] \equiv \frac{1}{R^2}\Big[\frac{1}{r}U_{,r} + \frac{1}{r^2}U_{,\theta\theta}\Big], \\ \sigma_\theta &= \frac{1}{R^2}U_{,rr}, \\ \tau_{r\theta} &= \frac{1}{R^2}\Big[\frac{1}{r^2}U_{,\theta} - \frac{1}{r}U_{,r\theta}\Big].\end{aligned} \tag{IV.82}$$

In equation (IV.81) and relation (IV.82), r is a dimensionless coordinate pertaining to R, which characterizes the absolute dimensions of the hole; Δ is Laplace's operator, which, in dimensionless polar coordinates, has the form

$$\Delta \equiv \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2}. \tag{IV.83}$$

The constant λ of the material and the function $T(r, \theta)$ have the form

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[1] See G. Kauderer [1], p. 116.

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$$\lambda = \frac{Kg_2}{(3K+G)G^2} = \mu\beta^2,$$
$$T(r, \theta) = \frac{9}{2} G^2 t_0^2; \tag{IV.84}$$

$\mu = 1/g_2$ is a small dimensionless value:

$$\beta^2 = \frac{Kg_2^2}{(3K+G)G^2};$$

t_0^2 is given by expression (IV.79), and the components σ_r, σ_θ and $\tau_{r\theta}$ are related to stress function $U(r, \theta)$ by relations (IV.82). The small parameter λ, which characterizes deviation of the nonlinear law of elasticity from Hooke's law, has the dimension m^4/n^2 and the order of magnitude 10^{-15}-10^{-16}.

The equations for the determination of displacement components $u(r, \theta)$ and $v(r, \theta)$ for the nonlinear elasticity law (IV.78), under condition (IV.80), are found by substituting in relation (IV.78), instead of σ_r, σ_θ and $\tau_{r\theta}$, their values from (IV.82), and instead of ε_r, ε_θ and $\varepsilon_{r\theta}$, their values:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\cdot\frac{\partial v}{\partial \theta},$$
$$\varepsilon_{r\theta} = \frac{1}{r}\cdot\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{IV.85}$$

Since we are interested in the effect of physical nonlinearity of a material on stress concentration near holes, we will analyze in the following the first basic problem with the condition that the contour of the hole is free of external forces and the stress state at infinity is given. /376

Method of Solution for Arbitrary Curvilinear Hole. Consider a hole possessing a shape such that the function

$$z^* = R[\zeta + \varepsilon f(\zeta)] \quad (z^* = r^* e^{i\theta}, r^* = Rr, z = re^{i\theta}, \zeta = \rho e^{i\theta}) \tag{IV.86}$$

accomplishes conformal mapping of an infinite plate with a round hole of unit radius onto an infinite plate with a hole of the above described shape. In the function (IV.86) R is a real constant, characterizing the dimensions of the hole; the function $f(\zeta)$ depends on the shape of the hole; ε is a small parameter, a real value, which satisfies the condition $|\varepsilon| \ll 1$, and the roots of the equation $1 + \varepsilon f'(\zeta) = 0$ should lie within the circle of unit radius in the plane ζ.

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We will represent the solution of equation (IV.81) and of the systems (IV.78), (IV.82), (IV.85) in a form expanded with respect to small parameters μ and ε (IV.84)

$$U(r,\theta;\mu;\varepsilon) = H_0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j U^{(k,j)}(r,\theta); \tag{IV.87}$$

$$u(r,\theta;\mu;\varepsilon) = H_0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j U^{(k,j)}(r,\theta),$$

$$v(r,\theta;\mu;\varepsilon) = H_0 \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j v^{(k,j)}(r,\theta), \tag{IV.88}$$

where the constant H_0 is selected from the condition $\frac{H_0^2\beta^2}{R^4} = 1$. Hence

$$H_0 = \frac{R^2}{\beta} = \frac{GR^2}{g_2}\sqrt{3+\frac{G}{K}}. \tag{IV.89}$$

The components of the stress and deformation states in the coordinate system (ρ, ϑ) are represented in the form of series with respect to μ and ε:

$$\sigma_\rho = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j \sigma_\rho^{(k,j)}, \qquad \sigma_\vartheta = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j \sigma_\vartheta^{(k,j)},$$

$$\tau_{\rho\vartheta} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j \tau_{\rho\vartheta}^{(k,j)}; \tag{IV.90}$$

$$u_\rho = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j u_\rho^{(k,j)}, \qquad u_\vartheta = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j u_\vartheta^{(k,j)}. \tag{IV.91}$$

By substituting the function $U(r, \theta; \mu; \varepsilon)$ in (IV.87) into basic equation (IV.81) and equating to zero the coefficients for identical powers of $\mu^k\varepsilon^j$, we obtain an equation for the determination of the function $U^{(k,j)}$ in the form

$$\Delta\Delta U^{(k,j)}(r,\theta) = L_{(k,j)}(U^{(0,0)}, \ldots, U^{(k-1,j-1)}). \tag{IV.92}$$

We introduce the explicit expressions for the right hand sides of equation (IV.92) for certain values of k and j: /377

for $k = 0$, $j = 0, 1, 2, \ldots$

$$L_{(0,j)} \equiv 0; \tag{IV.93}$$

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for k = 1, j = 0

$$L_{(1,0)} = -\frac{2}{r^4} T^{(0,0)}_{,0} U^{(0,0)}_{,0} + \frac{2}{r^3}\left(T^{(0,0)}_{,0} U^{(0,0)}_{,r0} + T^{(0,0)}_{,r0} U^{(0,0)}_{,0}\right) +$$
$$+ \frac{1}{r^2}\left(T^{(0,0)}_{,rr} U^{(0,0)}_{,00} + T^{(0,0)}_{,00} U^{(0,0)}_{,rr} - 2T^{(0,0)}_{,r0} U^{(0,0)}_{,r0}\right) + \qquad \text{(IV.94)}$$
$$+ \frac{1}{r}\left(T^{(0,0)}_{,rr} U^{(0,0)}_{,r} + T^{(0,0)}_{,r} U^{(0,0)}_{,rr}\right) - \frac{2}{3}\Delta\left(T^{(0,0)} \Delta U^{(0,0)}\right);$$

for k = 2, j = 0

$$L_{(2,0)} = \frac{2}{3}\Delta\left(T^{(0,0)}\Delta U^{(0,1)} + T^{(0,1)}\Delta U^{(0,0)}\right) - \left[T^{(0,0)}_{,rr}\left(\frac{1}{r^2} U^{(0,1)}_{,00} + \frac{1}{r} U^{(0,1)}_{,r}\right) +\right.$$
$$+ T^{(0,1)}_{,rr}\left(\frac{1}{r^2} U^{(0,0)}_{,00} + \frac{1}{r} U^{(0,0)}_{,r}\right) + U^{(0,1)}_{,rr}\left(\frac{1}{r^2} T^{(0,0)}_{,00} + \frac{1}{r} T^{(0,0)}_{,r}\right) +$$
$$\left. + U^{(0,0)}_{,rr}\left(\frac{1}{r^2} T^{(0,1)}_{,00} + \frac{1}{r} T^{(0,1)}_{,r}\right)\right] + 2\left[\left(\frac{1}{r} T^{(0,0)}_{,r0} - \frac{1}{r^2} T^{(0,0)}_{,0}\right) \times \right. \qquad \text{(IV.95)}$$
$$\left. \times \left(\frac{1}{r} U^{(0,1)}_{,r0} - \frac{1}{r^2} U^{(0,1)}_{,0}\right) + \left(\frac{1}{r} T^{(0,1)}_{,r0} - \frac{1}{r^2} T^{(0,1)}_{,0}\right)\left(\frac{1}{r} U^{(0,0)}_{,r0} - \frac{1}{r^2} U^{(0,0)}_{,0}\right)\right];$$

for k = 1, j = 1

$$L_{(1,1)}\left(U^{(0,0)}U^{(0,1)}\right) = T^{(0,0)}_{,rr}\left(\frac{1}{r^2} U^{(0,1)}_{,00} + \frac{1}{r} U^{(0,1)}_{,r}\right) + T^{(0,1)}_{,rr}\left(\frac{1}{r^2} U^{(0,0)}_{,00} + \frac{1}{r} U^{(0,0)}_{,r}\right) +$$
$$+ U^{(0,1)}_{,rr}\left(\frac{1}{r^2} T^{(0,0)}_{,00} + \frac{1}{r} T^{(0,0)}_{,r}\right) + U^{(0,0)}_{,rr}\left(\frac{1}{r^2} T^{(0,1)}_{,00} + \frac{1}{r} T^{(0,1)}_{,r}\right) -$$
$$- 2\left[\left(\frac{1}{r} T^{(0,0)}_{,r0} - \frac{1}{r^2} T^{(0,0)}_{,0}\right)\left(\frac{1}{r} U^{(0,1)}_{,r0} - \frac{1}{r^2} U^{(0,1)}_{,0}\right) + \left(\frac{1}{r} U^{(0,0)}_{,r0} - \frac{1}{r^2} U^{(0,0)}_{,0}\right) \times\right. \qquad \text{(IV.96)}$$
$$\left. \times \left(\frac{1}{r} T^{(0,1)}_{,r0} - \frac{1}{r^2} T^{(0,1)}_{,0}\right)\right] - \frac{2}{3}\Delta\left(T^{(0,1)}\Delta U^{(0,0)} + T^{(0,0)}\Delta U^{(0,1)}\right),$$

where $$T^{(0,0)}_{(r,0)} = \left(U^{(0,0)}_{,rr}\right)^2 - \frac{1}{r} U^{(0,0)}_{,rr} U^{(0,0)}_{,r} + \frac{1}{r^2}\left[\left(U^{(0,0)}_{,r}\right)^2 - U^{(0,0)}_{,rr} U^{(0,0)}_{,00} + 3\left(U^{(0,0)}_{,r0}\right)^2\right] +$$
$$+ \frac{2}{r^3}\left(U^{(0,0)}_{,r} U^{(0,0)}_{,00} - 3U^{(0,0)}_{,r0} U^{(0,0)}_{,0}\right) + \frac{1}{r^4}\left[\left(U^{(0,0)}_{,00}\right)^2 + 3\left(U^{(0,0)}_{,0}\right)^2\right];$$
$$T^{(0,1)}_{(r0)} = 2U^{(0,0)}_{,rr} U^{(0,1)}_{,rr} - \frac{1}{r}\left(U^{(0,0)}_{,rr} U^{(0,1)}_{,r} + U^{(0,1)}_{,rr} U^{(0,0)}_{,r}\right) + \qquad \text{/378}$$
$$+ \frac{1}{r^2}\left(2U^{(0,0)}_{,r} U^{(0,1)}_{,r} - U^{(0,0)}_{,rr} U^{(0,1)}_{,00} - U^{(0,1)}_{,rr} U^{(0,0)}_{,00} + 6U^{(0,0)}_{,r0} U^{(0,1)}_{,r0}\right) + \qquad \text{(IV.96a)}$$
$$+ \frac{2}{r^3}\left(U^{(0,0)}_{,r} U^{(0,1)}_{,00} + U^{(0,1)}_{,r} U^{(0,0)}_{,00} - 3U^{(0,0)}_{,r0} U^{(0,1)}_{,0} - 3U^{(0,1)}_{,r0} U^{(0,0)}_{,0}\right) +$$
$$+ \frac{1}{r^4}\left(2U^{(0,0)}_{,00} U^{(0,1)}_{,00} + 6U^{(0,0)}_{,0} U^{(0,1)}_{,0}\right).$$

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Proceeding as before, we can write the explicit expressions for operators $L_{(k,j)}$ for all ($k > 1$, $j > 1$) values of k and j.

The solution of equation (IV.92) is found in the form of the Fourier series:

$$U^{(k,j)}_{(r,\theta)}(r,\theta) = \sum_{m=0}^{\infty} \{f^{(m)}_{(k,j)}(r)\cos m\theta + g^{(m)}_{(k,j)}(r)\sin m\theta\}. \tag{IV.97}$$

To determine stress components σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ and displacement components u_ρ and u_ϑ in curvilinear orthogonal coordinate system[1] (ρ, ϑ), as given by the function (IV.86), we will use the corresponding formulas[2] of transition from polar coordinates (r, θ) to the curvilinear orthogonal coordinate system (ρ, ϑ), related by mapping function (IV.86).

After combining the expressions thus found for stress components σ_ρ, σ_ϑ and $\tau_{\rho\vartheta}$ and displacement components u_ρ and u_ϑ into series with respect to μ and ε, recalling the form of functions (IV.86), we obtain

$$\begin{aligned}
\sigma^{(k,j)}_{\varrho} &= H_0\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right)U^{(k,1)}(\varrho,\vartheta) + H_0\sum_{m=0}^{j-1}\left[L_1^{(j-m)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right) + \right.\\
&\left. + L_2^{(j-m)}\left(\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} - \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right) - L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho}\right]U^{(k,m)}(\varrho,\vartheta),\\
\sigma^{(k,j)}_{\vartheta} &= H_0\frac{\partial^2}{\partial\varrho^2}U^{(k,j)}(\varrho,\vartheta) + H_0\sum_{m=0}^{j-1}\left[L_1^{(j-m)}\frac{\partial^2}{\partial\varrho^2} + \right.\\
&\left. + L_2^{(j-m)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2} - \frac{\partial^2}{\partial\varrho^2}\right) + L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho}\right]U^{(k,m)}(\varrho,\vartheta),\\
\tau^{(k,j)}_{\varrho\vartheta} &= -H_0\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho}U^{(k,j)}(\varrho,\vartheta) - H_0\sum_{m=0}^{j-1}\left[(L_1^{(j-m)} - 2L_2^{(j-m)})\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho} + \right.\\
&\left. + \frac{1}{2}L_3^{(j-m)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2} - \frac{\partial^2}{\partial\varrho^2}\right)\right]U^{(k,m)}(\varrho,\vartheta);
\end{aligned} \tag{IV.98}$$

---

[1] The coordinate line $\rho = 1$ coincides with the contour of the given hole.

[2] See Chapter VI and X, and also the work of G. N. Savin, A. N. Guz' [1], where the analogous transformations are made in the problem of stress concentration around holes in the case of the plane problem of moment elasticity theory, as well as in shells.

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$$u_\rho^{(k,j)} = u^{(k,j)} + \sum_{m=0}^{j-1} [L_5^{(j-m)} u^{(k,m)} + L_6^{(j-m)} v^{(k,m)}],$$

$$u_\vartheta^{(k,j)} = v^{(k,j)} + \sum_{m=0}^{j-1} [L_5^{(j-m)} v^{(k,m)} - L_6^{(j-m)} u^{(k,m)}]. \qquad \text{(IV.99)} \quad /379$$

By substituting functions $U(r, \theta, \mu, \varepsilon)$ from (IV.87) and $u(r, \theta)$ from (IV.88) into (IV.85) and equating to zero the coefficients for identical powers of μ^k, ε^j, we find the equation system for the determination of $u^{(k,j)}(r, \theta)$, $v^{(k,j)}(r, \theta)$, which are included in (IV.99).

Functions $U^{(k,j)}(\rho, \vartheta)$ in (IV.98) represent the solutions of equations (IV.92) in the form (IV.97), in which the variables r and θ are formally[1] replaced by ρ and ϑ, respectively. Hence the arbitrary constants in $f^{(m)}_{(k,j)}(r)$ and $g^{(m)}_{(k,j)}(r)$ of (IV.97) are found from the corresponding boundary conditions for $U^{(k,j)}(\rho, \vartheta)$, which are obtained from the expansions into double series with respect to μ and ε, analogous to the expansions of (IV.87) and (IV.88) of the given conditions on the given contour.

The components of the stress and deformation states on the contour of the given hole are found from (IV.90) and (IV.97) for $\rho = 1$.

Consider the problem where, on contour Γ of the hole, we have the stresses

$$\sigma_\rho|_\Gamma = \psi_1(r, \theta, \mu), \quad \tau_{\rho\vartheta}|_\Gamma = \psi_2(r, \theta, \mu). \qquad \text{(IV.100)}$$

For generality we will assume that in (IV.100) ψ_1 and ψ_2 depend on μ. The equation of the contour of the hole is defined in parametric form by the function (IV.86) and can be represented in the form

$$r = r(\rho, \vartheta), \quad \theta = \theta(\rho, \vartheta) \quad \text{for} \quad \rho = 1. \qquad \text{(IV.101)}$$

Making use of (IV.86) and (IV.101), we represent the right hand sides of (IV.100) in the form of double series with respect to μ and ε:

$$\sigma_\rho|_\Gamma = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j \psi_1^{(k,j)}(\vartheta), \quad \tau_{\rho\vartheta}|_\Gamma = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mu^k \varepsilon^j \psi_2^{(k,j)}(\vartheta). \qquad \text{(IV.102)}$$

---

[1] In other words, in the function $U(r, \theta)$ found, r and θ are substituted by the variables ρ and ϑ.

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After substituting expressions (IV.90) into (IV.102), assuming that $\rho = 1$ in the latter, and comparing the coefficients for identical powers, we obtain the relations

$$\sigma_{\varrho}^{(k,j)}|_{\Gamma} = \psi_1^{(k,j)}(\vartheta), \quad \tau_{\varrho\vartheta}^{(k,j)}|_{\Gamma} = \psi_2^{(k,j)}(\vartheta). \tag{IV.103}$$

From (IV.98), recalling (IV.103), we obtain the boundary conditions for the determination of the function $U^{(k,j)}_{(r,\theta)}$ in the form /380

$$\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right)U^{(k,j)}(\varrho,\vartheta)|_{\varrho=1} = \frac{1}{H_0}\psi_1^{(k,j)}(\vartheta)|_{\varrho=1} -$$

$$-\left\{\sum_{m=0}^{j-1}\left[L_1^{(j-m)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right) + L_2^{(j-m)}\left(\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} - \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2}\right) - \right.\right.$$

$$\left.\left. - L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho}\right]U^{(k,m)}(\varrho,\vartheta)\right\}\Bigg|_{\varrho=1}; \tag{IV.104}$$

$$\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho}U^{(k,j)}(\varrho,\vartheta)|_{\varrho=1} = -\frac{1}{H_0}\psi_2^{(k,j)}(\vartheta)|_{\varrho=1} - \left\{\sum_{m=0}^{j-1}\left[(L_1^{(j-m)} - 2L_2^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\vartheta}\times\right.\right.$$

$$\left.\left.\times\frac{1}{\varrho} + \frac{1}{2}L_3^{(j-m)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta^2} - \frac{\partial^2}{\partial\varrho^2}\right)\right]U^{(k,m)}(\varrho,\vartheta)\right\}\Bigg|_{\varrho=1}.$$

According to (IV.87), as the solution obtained in the n-th approximation we will use the function

$$U_n(r,\vartheta,\mu,\varepsilon) = H_0\sum_{k,j=0}^{k+j=n-1}\mu^k\varepsilon^j U^{(k,j)}(r,\theta). \tag{IV.105}$$

Functions $U^{(k,j)}(r,\theta)$ are the solutions of equations (IV.92) in the form of Fourier series (IV.97). However, functions $U^{(k,j)}(\rho,\vartheta)$ are the solutions of these equations in which the variables r and θ are *formally* substituted by ρ and ϑ, respectively.

In (IV.98) and (IV.99), $L_1^{(j-m)}$, $L_2^{(j-m)}$, ..., $L_6^{(j-m)}$ are differential operators, the form of which depends on the function $f(\zeta)$ of (IV.86).

The expanded expressions of these operators for the first through third approximations are presented below:

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$$L_1^{(0)} = L_2^{(0)} = L_3^{(0)} = L_5^{(0)} = L_6^{(0)} \equiv 0, \quad L_1^{(1)} = P_1 \frac{\partial}{\partial \varrho} + P_2 \frac{\partial}{\partial \vartheta},$$

$$L_1^{(2)} = P_3 \frac{\partial^3}{\partial \varrho^2} + P_4 \frac{\partial^2}{\partial \varrho \partial \vartheta} + P_5 \frac{\partial^2}{\partial \vartheta^2} + P_6 \frac{\partial}{\partial \varrho} + P_7 \frac{\partial}{\partial \vartheta}, \quad L_2^{(1)} \equiv 0,$$

$$L_3^{(1)} = -i[f'(\zeta) - \overline{f'(\zeta)}] - \frac{i}{\zeta\bar{\zeta}}[\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)]. \tag{IV.106}$$

$$L_3^{(2)} = -\frac{i}{2\zeta\bar{\zeta}}[\bar{\zeta}^2 f^2(\zeta) - \zeta^2\overline{f^2(\zeta)}] + \zeta^2\bar{\zeta}^2[\overline{f'^2(\zeta)} - f'^2(\zeta)] + P_8\frac{\partial}{\partial \varrho} + P_9\frac{\partial}{\partial \vartheta},$$

$$L_5^{(1)} = L_1^{(1)}, \quad L_5^{(2)} = L_1^{(2)} - \frac{1}{2}L_2^{(2)}, \quad L_6^{(1)} = \frac{1}{2}L_3^{(1)}, \quad L_6^{(2)} = \frac{1}{2}L_3^{(2)},$$

where the following definitions are given /381

$$P_1 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{2\varrho}, \quad P_3 = \frac{[\bar{\zeta}f(\zeta) + \zeta f(\zeta)]^2}{8\varrho^2},$$

$$P_2 = \frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}\sin\vartheta,$$

$$P_4 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{\varrho^2}\left[\frac{f(\zeta) - \overline{f(\zeta)}}{4i}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{4}\sin\vartheta\right],$$

$$P_5 = \frac{1}{8i\varrho^2}\{2if(\zeta)\overline{f(\zeta)} - i[f^2(\zeta) + \overline{f^2(\zeta)}]\cos\vartheta - [f^2(\zeta) - \overline{f^2(\zeta)}]\sin 2\vartheta\},$$

$$P_6 = -\frac{1}{8\varrho^3}[\bar{\zeta}f(\zeta) - \zeta\overline{f(\zeta)}]^2,$$

$$P_7 = \frac{1}{4i\varrho^2}\{[f^2(\zeta) + \overline{f^2(\zeta)}]i\sin 2\vartheta - [f^2(\zeta) - \overline{f^2(\zeta)}]\cos 2\vartheta\},$$

$$P_8 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{2i\varrho}[f'(\zeta) - \overline{f'(\zeta)} + \frac{\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)}{\zeta\bar{\zeta}}],$$

$$P_9 = \left[\frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}\sin\vartheta\right]\left[\frac{f'(\zeta) - \overline{f'(\zeta)}}{i} + \frac{\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)}{i\zeta\bar{\zeta}}\right].$$

Stress Distribution Near Round Hole[1]. The solution of the problem of elastic equilibrium of a physically nonlinear elastic unbounded plate with a round hole of radius R, along the contour of which are given external forces, and "at infinity," any stress field, reduces to the integration, in the first approximation, of biharmonic equation (IV.92) (under condition (IV.93)) under the given boundary conditions on the contour of the hole and at "infinity;" in subsequent approximations[2], to integration of the heterogeneous differential

[1]Stress concentration near a round hole under elasticity law (IV.6) for the case of simple tension of a plate was first analyzed by F. Jindra [1]; by I. A. Tsurpal [1-12] for other cases of stress state "at infinity."

[2]Since the problem is nonlinear, the principle of superposing does not apply.

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equation (IV.92) corresponding to the given approximation for the right hand sides of (IV.94)-(IV.96) and for homogeneous boundary conditions on the contour of the hole

$$\frac{1}{r^2}U^{(i,0)}_{,\theta\theta}+\frac{1}{r}U^{(i,0)}_{,r}=0,\quad \frac{1}{r^2}U^{(i,0)}_{,\theta}-\frac{1}{r}U^{(i,0)}_{,r\theta}=0 \text{ for } r=R \tag{IV.107}$$

and conditions "at infinity"

$$\begin{gathered}\frac{1}{r^2}U^{(i,0)}_{,\theta\theta}+\frac{1}{r}U^{(i,0)}_{,r}=0 \text{ for } r=\infty;\\ U^{(i,0)}_{,rr}=0,\quad \frac{1}{r^2}U^{(i,0)}_{,\theta}-\frac{1}{r}U^{(i,0)}_{,r\theta}=0.\end{gathered} \tag{IV.108}$$

We will analyze certain partial cases. /382

Multifold Uniform Tension:

$$\sigma_r^{(\infty)}=\sigma_\theta^{(\infty)}=p,\quad \tau_{r\theta}^{(\infty)}=0. \tag{IV.109}$$

The stress function of the first approximation is given by linear classical theory:

$$U^{(0,0)}(r)=\frac{p}{2H_0}(r^2-2\ln r), \tag{IV.110}$$

where H_0 is defined by formula (IV.89).

The desired complete stress components must satisfy conditions

$$(\sigma_r)_{r=1}=0,\quad (\sigma_r)_{r=\infty}=(\sigma_\theta)_{r=\infty}=p. \tag{IV.111}$$

On the basis of the known stress function (IV.110), we find the right hand side of equation (IV.92), thus determining the equation for the function $U^{(1,0)}$:

$$\Delta\Delta U^{(1,0)}=-4\frac{p^2}{H_0^3}\left(\frac{4}{r^6}+\frac{18}{r^8}\right), \tag{IV.112}$$

which must be integrated under conditions (IV.107) and (IV.108).

The solution of equation (IV.112) which satisfies conditions (IV.107) and (IV.108) is

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$$U^{(1,0)}(r) = -\frac{p^3}{H_0^3}\left[\frac{1}{4}\left(\frac{1}{r^2} + \frac{1}{2}\cdot\frac{1}{r^4}\right) + \ln r\right]. \qquad \text{(IV.113)}$$

Introducing functions $U^{(0,0)}$ (IV.110) and $U^{(1,0)}$ (IV.113) into (IV.105, we find the stress function in the second approximation[1]:

$$U_2(r, \mu) = H_0[U^{(0,0)}(r) + \mu U^{(1,0)}(r)]. \qquad \text{(IV.114)}$$

From (IV.82) and (IV.114) it is easy to find (in the second approximation) the formulas for stress components σ_r, σ_θ and $\tau_{r\theta}$.

The stress concentration factor for the given problem, according to linear theory, reaches its greatest value on the contour of the hole and is equal to two. In our case, i.e. with consideration of physical nonlinearity of the material, the concentration factor will depend on the magnitude of external load and mechanical properties of the material in the second approximation[2] as well, as seen from (IV.82) and (IV.105):

$$\overset{(2)}{k} = \left(\frac{\sigma_\theta}{p}\right)_{r=1} = \left(\frac{\sigma_\theta^{(0,0)}}{p}\right)_{r=1} + \lambda\left(\frac{\sigma_\theta^{(1,0)}}{p}\right)_{r=1} = 2(1 - 1{,}5\lambda p^2), \qquad \text{(IV.115)}$$

and in the third approximation /383

$$\overset{(3)}{k} = \left(\frac{\sigma_\theta}{p}\right)_{r=1} = 2(1 - 1{,}500\lambda p^2 + 10{,}605\lambda^2 p^4). \qquad \text{(IV.116)}$$

The graphs of stress concentration factor $\overset{(2)}{k}$ of (IV.115) as a function of external load and elastic properties of the material are presented in Figure IV.2:

Curve 1 corresponds to copper with the following elastic properties

$$\begin{aligned} K &= 1{,}305\cdot 9{,}81\cdot 10^{10}\ \text{n/m}^2, \\ G &= 0{,}461\cdot 9{,}81\cdot 10^{10}\ \text{n/m}^2 \\ g_2 &= 7{,}260\cdot 10^6, \quad \lambda = \lambda_1 = \frac{1{,}019}{9{,}81^2}\,10^{-13}\,\frac{1}{(\text{n/m}^2)^2} \end{aligned} \qquad \text{(IV.117)}$$

---

[1] Indeed, the function $U_2(r, \theta, \mu, \varepsilon)$ (IV.105) for any curvilinear hole will have the form $U_2(r, \theta, \mu, \varepsilon) = H_0(U^{(0,0)} + \mu U^{(1,0)} + \varepsilon U^{(0,1)} + \mu\varepsilon U^{(1,1)})$. However, for a round hole $\varepsilon = 0$, and we will arrive at $U_2(r, \mu)$ (IV.114).

[2] In formulas (IV.115) and (IV.116) the superscript indicates the order of approximation.

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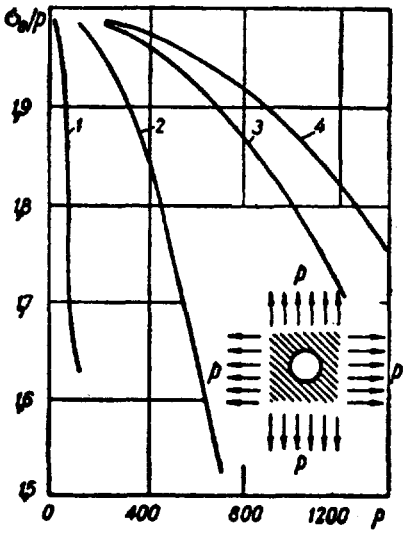


Figure IV.2.

Curve 2 corresponds to the copper with the following properties

$$K = 1{,}343 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,$$
$$G = 0{,}451 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,$$
$$g_2 = 0{,}180 \cdot 10^6, \qquad \text{(IV.118)}$$
$$\lambda = \lambda_2 = \frac{0{,}226}{9{,}81^2}\, 10^{-14}\, \frac{1}{(\text{n/m}^2)^2};$$

Curve 3 corresponds to an aluminum-bronze alloy, for which

$$K = 1{,}324 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,\ G = 0{,}468 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,$$
$$g_3 = 0{,}040 \cdot 10^6,\ \lambda = \lambda_3 = \frac{0{,}055}{9{,}81^2}\, 10^{-14}\, \frac{1}{(\text{n/m}^2)^2}; \qquad \text{IV.119)}$$

Curve 4 corresponds to open-hearth steel, for which

$$K = 1{,}786 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,\ G = 0{,}853 \cdot 9{,}81 \cdot 10^{10}\ \text{n/m}^2,$$
$$g_4 = 0{,}085 \cdot 10^6,\ \lambda = \lambda_4 = \frac{0{,}033}{9{,}81^2}\, 10^{-14}\, \frac{1}{(\text{n/m}^2)^2}. \qquad \text{(IV.120)}$$

Data for a material with elastic properties (IV.117) are presented in Table IV.1, which shows the effect of the magnitude of external load on the stress concentration factor found in the first, second and third approximations, and also on the magnitude of corrections to the first and second approximations, introduced into stress functions (IV.105) by the following terms.

TABLE IV.1

$p/9{,}81 \cdot 10^4$ n/m^2	Stress concentration factor ($k=\sigma_\theta/p$) for $\lambda_1=1{,}019 \cdot 10^{-5}$				
	First approximation (linear theory)	Concentration to first approximation	Second approximation	Concentration to second approximation	Third approximation
40	2,000	—0,047	1,953	0,006	1,959
60	2,000	—0,105	1,895	0,023	1,918
80	2,000	—0,188	1,811	0,078	1,889

Tr. Note: Commas indicate decimal points.

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The calculations of stresses for the above materials with elastic properties (IV.117)-(IV.120) show that for axisymmetrical problems[1], in the case of slight deviation of the law of elasticity from Hooke's law, we may limit ourselves for engineering calculations to the second approximation. /384

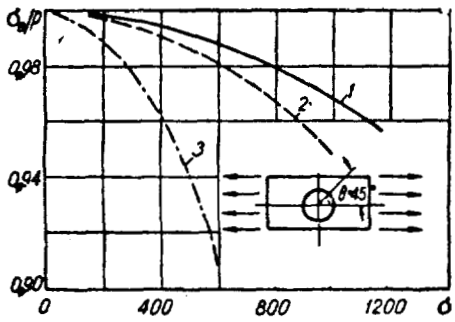


Figure IV.3.

Uniaxial Tension: $\sigma_x^{(\infty)}=p,\ \sigma_y^{(\infty)}=\tau_{xy}^{(\infty)}=0$ (Figure IV.3, where curve 1 corresponds to open-hearth steel, curve 2, to aluminum-bronze alloy, curve 3, to pure copper). In the first approximation the stress function, according to linear theory, is

$$U^{(0,0)}(r,\theta)=\frac{P}{4H_0}\left[r^2-2\ln r+\left(2-r^2-\frac{1}{r^2}\right)\cos 2\theta\right]. \qquad \text{(IV.121)}$$

By knowing the function $U^{(0,0)}$ from (IV.93)-(IV.97), we can find the explicit expressions for the right hand sides of equation (IV.92), which, together with boundary conditions (IV.107) and conditions "at infinity" (IV.108), defines the functions $U^{(k,j)}$ (k, j = 0, 1, 2 ...) which go into stress function $U_n(r, \theta)$ of (IV.105) in the n-th approximation. After determining the function $U_n(r, \theta)$, we can find the stress components by formulas (IV.82).

The function $U^{(0,0)}$ has the form[2]

$$U^{(1,0)}(r,\theta)=\frac{p^3}{H_0^3}\left[-0{,}5292\ln r-0{,}0312\frac{1}{r^2}-0{,}1510\frac{1}{r^4}+0{,}1354\frac{1}{r^6}-\right.$$
$$-0{,}0844\frac{1}{r^8}+\left(0{,}5932+0{,}1257\frac{1}{r^2}-1{,}8750\frac{\ln r}{r^2}-1{,}1042\frac{1}{r^4}+0{,}4562\frac{1}{r^6}-\right. \qquad \text{(IV.122)}$$
$$\left.-0{,}1312\frac{1}{r^8}+0{,}0603\frac{1}{r^{10}}\right)\cos 2\theta+\left(-0{,}1094-0{,}7589\frac{1}{r^2}+0{,}7500\frac{\ln r}{r^2}+\right. \qquad /385$$
$$+0{,}8884\frac{1}{r^4}+1{,}1250\frac{\ln r}{r^4}-0{,}0201\frac{1}{r^8}\Big)\cos 4\theta+$$
$$\left.+\left(-0{,}0312+0{,}1906\frac{1}{r^2}-0{,}2875\frac{1}{r^4}+0{,}1281\frac{1}{r^6}\right)\cos 6\theta\right].$$

[1]See I. A. Tsurpal's work [2], where the axisymmetrical problem of stresses in a thin-wall tube is examined (the Lame problem).
[2]See F. Jindra [1] or G. Kauderer [1], p. 126, 127.

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Analogously, we can calculate the functions $U^{(1,1)}$, $U^{(2,0)}$. After finding stress function $U_n(r, \theta, \mu, \varepsilon)$ in the n-th approximation, the stress components $\sigma_r^{(n)}$, $\sigma_\theta^{(n)}$ and $\tau_{r\theta}^{(n)}$ (in the same approximation) can be found from formulas (IV.82).

Since we are interested in the effect of the physical nonlinearity of the material of the plate on the stress concentration factor, we will introduce the formula for the calculation of stresses σ_θ along the contour of a hole:

$$k^{(2)} = \left(\frac{\sigma_\theta}{p}\right)_{r=1} = 1 - (2 - 4{,}388\lambda p^2)\cos 2\theta - \\ - \lambda p^2 (3{,}066 + 2{,}107\cos 4\theta - 0{,}775\cos 6\theta). \qquad \text{(IV.123)}$$

The graphs of change of the stress concentration factor $k^{(2)}$ (IV.123) as a function of the magnitude of external load p are presented in Figure IV.4 (symbols are the same as in Figure IV.3) for materials with elastic properties (IV.117)-(IV.120) at points of the contour $\theta = \pi/4$ and $\theta = \pi/2$.

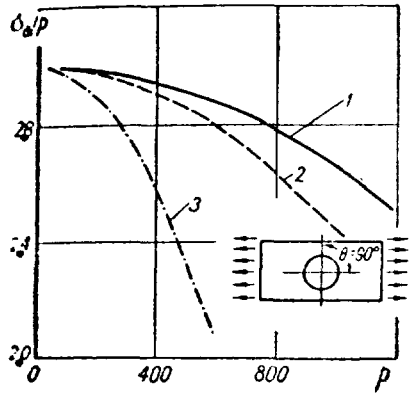


Figure IV.4.

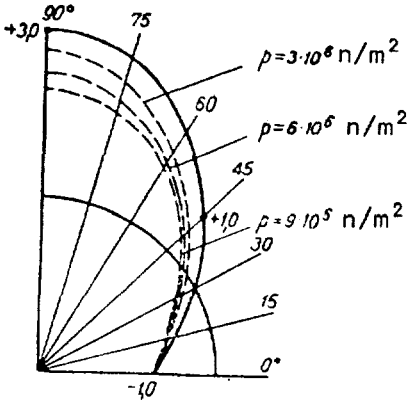


Figure IV.5.

Figures IV.5 and IV.6 show the graphs of change of the stress concentration factor $k^{(2)}$ (IV.123) along the contour of a hole for the same materials (solid lines correspond to linear theory, broken lines -- to nonlinear).

Pure Displacement[1]: $\tau_{xy} = \tau_{yx} = \tau$ (Figure IV.7). In the first approximation /386 the stress function according to linear theory is

[1]See I. A. Tsurpal [4] for greater detail.

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$$U^{(0,0)}(r, \theta) = \frac{\tau}{2H_0}\left(2 - r^2 - \frac{1}{r^2}\right)\sin 2\theta. \tag{IV.124}$$

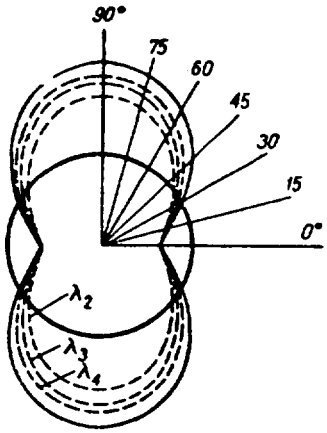


Figure IV.6.

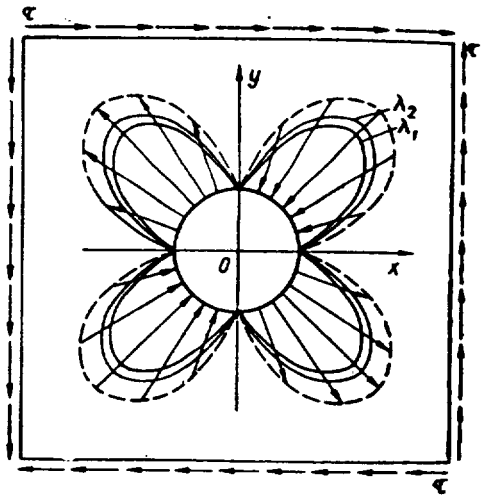


Figure IV.7.

We will determine the function $U^{(1,0)}$ and the following in accordance with the method described above:

$$U^{(1,0)}(r, \theta) = \frac{\tau^3}{H_0^3}\left[\left(\frac{333}{140} + 3{,}64\frac{1}{r^2} - 12\frac{1}{r^2}\ln r - \frac{17}{2}\cdot\frac{1}{r^4} + \frac{61}{20}\cdot\frac{1}{r^6} - \right.\right.$$
$$\left.\left. - \frac{21}{20}\cdot\frac{1}{r^8} + \frac{27}{56}\cdot\frac{1}{r^{10}}\right)\sin 2\theta + \left(\frac{1}{4} - \frac{61}{40}\cdot\frac{1}{r^2} + \frac{23}{10}\cdot\frac{1}{r^4} - \frac{41}{40}\cdot\frac{1}{r^6}\right)\sin 6\theta. \right. \tag{IV.125}$$

The stress concentration factor along the contour of a hole in the second approximation is

$$k^{(2)} = \left(\frac{\sigma_\theta}{\tau}\right)_{r=1} = -(4 - 17{,}380\lambda\tau^2)\sin 2\theta - 6{,}200\lambda\tau^2\sin 6\theta. \tag{IV.126}$$

The change of the concentration factor k (IV.126) is illustrated in Figure IV.7 for two cases: $\lambda = \lambda_1$, $\tau = 70 \cdot 9.81 \cdot 10^4$ n/m^2 and $\lambda = \lambda_3$, $\tau = 700 \cdot 9.81 \cdot 10^4$ n/m^2. The solid lines represent data of nonlinear theory and the broken lines, data of linear theory.

Pure Deflection of Rod by Pairs M. The stress function according to linear theory for the first approximation is

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$$U^{(0,0)} = \frac{M}{8JH_0}\left[\left(-\frac{1}{r} - r^3\right)\sin\theta + \left(-\frac{1}{r} + \frac{1}{3}r^3 + \frac{2}{3}\cdot\frac{1}{r^3}\right)\sin 3\theta\right]. \qquad \text{(IV.127)}$$

By substituting the function (IV.127) and its derivatives into (IV.94), we find /387 the differential equation for functions $U^{(1,0)}$, the integration of which yields[1]

$$\begin{aligned} U^{(1,0)}(r,\theta) = \frac{M^3}{8JH_0}\Bigg[&\frac{1}{28}\cdot\frac{1}{r^{11}} - \frac{19}{320}\cdot\frac{1}{r^9} + \frac{371}{1920}\cdot\frac{1}{r^7} - \\ &- \frac{1}{3}\cdot\frac{1}{r^5} + \frac{3}{8}\cdot\frac{1}{r^3} - \frac{613}{2240}\cdot\frac{1}{r} + \frac{r^5}{12}\Big)\sin\theta + \\ &+ \frac{1}{8}\Big(-\frac{1}{3}\cdot\frac{1}{r^{13}} + \frac{23}{224}\cdot\frac{1}{r^{11}} - \frac{43}{56}\cdot\frac{1}{r^9} - \frac{71}{40}\cdot\frac{1}{r^7} + \frac{7}{r^5} - \frac{391}{70}\cdot\frac{1}{r^3} - \\ &- \frac{1247}{840}\cdot\frac{1}{r} + \frac{17}{8}r\Big)\sin 3\theta + \frac{1}{8}\Big(\frac{1}{12}\cdot\frac{1}{r^{11}} - \frac{5}{56}\cdot\frac{1}{r^9} + \frac{1}{56}\cdot\frac{1}{r^7} + \frac{1}{42}\cdot\frac{1}{r^5} - \\ &- \frac{19}{42}\cdot\frac{1}{r^3} + \frac{17}{24}\cdot\frac{1}{r} - \frac{7}{24}r\Big)\sin 5\theta + \frac{1}{8}\Big(-\frac{2401}{3360}\cdot\frac{1}{r^7} + \\ &+ \frac{139}{120}\cdot\frac{1}{r^5} - \frac{1}{10}\cdot\frac{1}{r^3} - \frac{5}{12}\cdot\frac{1}{r} + \frac{7}{96}r\Big)\sin 7\theta + \frac{1}{8}\Big(\frac{51}{280}\cdot\frac{1}{r^9} - \\ &- \frac{41}{224}\cdot\frac{1}{r^7} - \frac{13}{56}\cdot\frac{1}{r^5} + \frac{1}{7}\cdot\frac{1}{r^3} + \frac{1}{8}\cdot\frac{1}{r} - \frac{9}{160}r\Big)\sin 9\theta\Bigg]. \end{aligned} \qquad \text{(IV.128)}$$

The stress concentration factor on the contour of a round hole is

$$\begin{aligned} k = \left(\frac{\sigma_\theta}{M/J}\right)_{r=1} = &-\sin\theta + \sin 3\theta + \lambda\frac{M^2}{J^2}(1{,}453\sin\theta - \\ &- 2{,}363\sin 3\theta + 0{,}021\sin 5\theta - 0{,}228\sin 7\theta + 0{,}096\sin 9\theta). \end{aligned} \qquad \text{(IV.129)}$$

Elliptical Hole. Multifold Uniform Tension-Compression. Let us examine the simple case of multifold uniform tension by forces p of an infinite isotropic plate made of a physically nonlinear material (Figure IV.8), subordinate to elasticity law (IV.78) under conditions (IV.80), with an elliptical hole.

The function that maps the exterior of the elliptical hole (IV.86) on the exterior of the unit circle for this case, as we know, has the form

$$z^* = R\left(\zeta + \frac{e}{\zeta}\right), \qquad \text{(IV.130)}$$

[1]See I. A. Tsurpal [12] and his other works [6-9] in which the effect of a round hole reinforced by an elastic (linear or nonlinear) ring is analyzed.

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where

$$R = \frac{a+b}{2}, \quad e = \frac{a-b}{a+b}, \quad \zeta = \varrho e^{i\phi}, \quad z^* = Rre^{i\theta}; \tag{IV.131}$$

a and b are the semiaxes of the ellipse (see Figure IV.8). The function $f(\zeta)$ in (IV.86), as we can see, is $f(\zeta) = 1/\zeta$ in the given case[1].

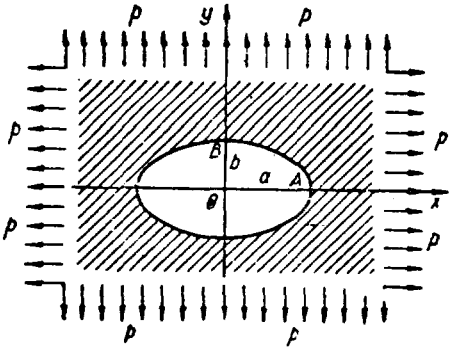


Figure IV.8.

The approximate solution of this problem with consideration of three approximations reduces to series integration of equations (IV.92) with consideration of the form of operators (IV.93)-(IV.96). The stress functions of the first through third approximations for multifold tension of a physically nonlinear elastic plate with a round hole are known[2]: /388

$$U^{(0,0)} = \frac{p}{2H_0}(r^2 - 2\ln r); \tag{IV.132}$$

$$U^{(1,0)} = -\frac{p^2}{H_0^3} \times$$

$$\times \left[\frac{1}{4}\left(\frac{1}{r^2} + \frac{1}{2}\cdot\frac{1}{r^4}\right) + \ln r\right]; \tag{IV.133}$$

$$U^{(2,0)} = \frac{p^5}{H_0^5}\left[\frac{13}{5}\ln r + \frac{1}{4}\left(-\frac{1}{r^2} - \frac{1}{3}\times \quad \times\frac{1}{r^4} + \frac{47}{36}\cdot\frac{1}{r^6} + \frac{51}{80}\cdot\frac{1}{r^8}\right)\right]. \tag{IV.134}$$

The stress function for a linear-elastic plate with an elliptical hole[3] is

$$U^{(0,1)}(r,\theta) = \frac{p}{H_0}\left(\frac{1}{r^2} - 1\right)\cos 2\theta,$$
$$U^{(0,2)}(r,\theta) = \frac{p}{H_0}\left[\left(\frac{3}{2}\cdot\frac{1}{r^4} - \frac{1}{r^2}\right)\cos 4\theta - \ln r\right]. \tag{IV.135}$$

By substituting functions (IV.132) into (IV.96a), we find the function $T^{(0,0)}(r, \theta)$ in the form

$$T^{(0,0)} = \frac{p^2}{H_0^2}\left(1 + 3\frac{1}{r^4}\right). \tag{IV.136}$$

---

[1]For square and triangular holes with rounded corners, the functions $f(\zeta)$ are $f(\zeta) = 1/\zeta^3$ and $f(\zeta) = 1/\zeta^2$, respectively (see Chapter I).
[2]See I. A. Tsurpal [9].
[3]See A. N. Guz' [2].

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If we know functions (IV.132) and (IV.135), we can find, from (IV.96a), the function

$$T^{(0,1)}(r,\theta)=4\frac{p^2}{H_0^2}\cdot\frac{1}{r^2}\left(1-3\frac{1}{r^2}+9\frac{1}{r^4}\right)\cos 2\theta. \tag{IV.137}$$

After substituting functions (IV.132), (IV.135)-(IV.137) and their derivatives into (19), we find an expanded expression for operator $L_{(1,1)}$.

Differential equation (IV.60) for the function $U^{(1,1)}(r,\theta)$ will have the form

$$\Delta\Delta U^{(1,1)}+64\frac{p^3}{H_0^3}\left(\frac{1}{r^8}+36\frac{1}{r^{10}}\right)\cos 2\theta=0. \tag{IV.138}$$

The partial integral of equation (IV.138) is

$$U^{(1,1)}_{\text{part}}(r,\ \theta)=-\frac{p^3}{H_0^3}\left(\frac{1}{6}\cdot\frac{1}{r^4}+\frac{6}{5}\cdot\frac{1}{r^6}\right)\cos 2\theta. \tag{IV.139}$$

The general integral of homogeneous equation (IV.138) is /389

$$U^{(1,1)}_{\text{hom}}(r,\ \theta)=\sum_{m=2}^{\infty}(C_{m3}r^{-m+2}+C_{m4}r^{-m})\cos m\theta. \tag{IV.140}$$

Integration constants C_{m3} and C_{m4} in (IV.140) are determined from boundary conditions (IV.104):

$$\begin{aligned}&\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\vartheta}\right)U^{(1,1)}(\varrho,\vartheta)|_{\varrho=1}+R\left[\cos 2\vartheta\frac{\partial}{\partial\varrho}\cdot\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right]U^{(1,0)}(\varrho,\vartheta)|_{\varrho=1}=0,\\&\frac{\partial^2}{\partial\varrho\partial\vartheta\varrho}\cdot\frac{1}{\varrho}U^{(1,1)}(\varrho,\vartheta)\Big|_{\varrho=1}-R\left[2\sin 2\vartheta\left(\frac{\partial^2}{\partial\varrho^2}-\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right)\right]U^{(1,0)}(\varrho,\vartheta)\Big|_{\varrho=1}=0.\end{aligned} \tag{IV.141}$$

We will omit the intermediate calculations and present the final expression for the function $U^{(1,1)}(r,\theta)$:

$$U^{(1,1)}(r,\theta)=-\frac{1}{30}\cdot\frac{p^3}{H_0^3}\left(32-73\frac{1}{r^2}+5\frac{1}{r^4}+36\frac{1}{r^6}\right)\cos 2\theta. \tag{IV.142}$$

Stress function (IV.105) in the third approximation is

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$$U_3(r, \theta, \mu, \varepsilon) = H_0[U^{(0,0)} + \mu U^{(1,0)} + \mu^2 U^{(2,0)} + \varepsilon U^{(0,1)} + \varepsilon^2 U^{(0,2)} + \mu\varepsilon U^{(1,1)}]. \quad \text{(IV.143)}$$

According to formulas (IV.90), recalling the values of stress components (IV.98), and also of functions (IV.132)-(IV.137), (IV.142) and (IV.143), we can find the stress state in a physically nonlinear thin plate weakened by an elliptical hole, in the third approximation. The stress concentration factor on the contour of the hole will be

$$\overset{(3)}{k} = \left(\frac{\sigma_\vartheta}{p}\right)_{\varrho=1} = 2[1 - 1{,}500\lambda p^2 + 10{,}605\lambda^2 p^4 + 2\varepsilon\cos 2\vartheta + 2\varepsilon^2\cos 4\vartheta - 10{,}660\lambda\varepsilon p^2\cos 2\vartheta]. \quad \text{(IV.144)}$$

We see from (IV.144) that the stress concentration factor with consideration of the physical nonlinearity of the material, satisfying elasticity relations (IV.78) under conditions (IV.80), depends nonlinearly both on the magnitude of the forces of tension p (see Figure IV.8), parameter λ, which characterizes the mechanical properties of the plate, as well as on the ellipticity of the hole, characterized by parameter ε (IV.131). Assuming in (IV.144) that $\varepsilon = 0$, we obtain the values of k for a round hole. Assuming in (IV.144) that $\lambda = 0$, we find the values of k found by A. N. Guz' in [2], by the approximation method described above for an elliptical hole for the case where the material from which the plate is made obeys Hooke's law. However, for the latter case there is a rigorous solution[1]. By comparing the appropriate values of k found from the accurate solution with the approximate solutions given above, we can find a clear representation of the rate of convergence of the approximate solution of problems of stress concentration near curvilinear holes for which there are no rigorous solutions. Such a comparison is presented in the first two rows of Tables IV.2 and IV.3. The third approximation, even for a strongly elongated ellipse a/b = 1.6, yields for k (IV.144) very good agreement with the precise value (the difference does not exceed 2.5-3.0%). The values of k (IV.144) calculated for two two points A and B (see Figure IV.8) of the contour of the hole are presented in the tables. The values of k at the point A($\vartheta = 0$) are in the numerator, and the values of k at the point B($\vartheta = \pi/2$) are in the denominator for various values of a/b, p and λ. The values λ_1(IV.117), λ_3 (IV.119) and λ_4 (IV.120) were used for λ. These data indicate the following: /390

a) the ellipticity of a hole, as in the classical case, i.e. when the material of the plate obeys Hooke's law, has a considerable effect on the magnitude of the stress concentration factor k; b) as the forces of tension p are increased (see Figure IV.8) the magnitude of factor k at point A decreases, but increases at point B. Hence as the values of parameters p and λ increase, the "soft" (see Figure IV.1) physical nonlinearity, as a rule, results in a more uniform distribution of stresses around the contour of the holes.

[1]See §2, Chapter II.

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TABLE IV.2

Theory	a/b	1,00	1,05	1,10	1,20	1,30	1,50	1,60
Linear λ	Rigorous solution	2 2	2,101 1,904	2,212 1,818	2,444 1,666	2,616 1,538	3,000 1,333	3,200 1,250
	Approximate solution	2 2	2,097 1,904	2,198 1,818	2,435 1,669	2,587 1,546	2,960 1,360	3,136 1,289
p		In linear theory k is independent of p						
Nonlinear λ_1	60	1,920 1,920	2,002 1,843	2,084 1,775	2,248 1,658	2,412 1,565	2,730 1,431	2,709 1,557
	70	1,904 1,904	1,980 1,834	2,055 1,772	2,208 1,667	2,361 1,584	2,660 1,469	2,568 1,667
	80	1,895 1,895	1,962 1,832	2,301 1,777	2,170 1,686	2,310 1,616	2,587 1,522	2,414 1,802
	90	1,868 1,868	1,954 1,841	2,014 1,794	2,138 1,718	2,264 1,662	2,517 1,593	2,250 1,966

Tr. Note: Commas indicate decimal points.

The approximation method proposed above for the solution of problems of an arbitrary curvilinear hole was based on the formal expansion of the desired functions into double series with respect to small parameters μ and ε without consideration of the convergence of the examined series. An idea of the rate of convergence of the proposed method can be obtained in the general case by calculating the magnitudes of concentration factors $k^{(n-2)}$, $k^{(n-1)}$, $k^{(n)}$ on the basis of formulas analogous to (IV.144), corresponding to stress functions U_{n-2}, U_{n-1}, U_n for the preceding approximations.

Uniaxial Tension. The stress functions of the first and second approximations in the case of the nonlinear problem for a round hole were found on the basis of formulas (IV.121) and (IV.122). /391

The functions $U^{(0,1)}(r, \theta)$ and $U^{(0,2)}(r, \theta)$, however, have the form[1]

$$U^{(0,1)}(r,\theta)=\frac{p}{2H_0}\left[2\ln r-\left(1-\frac{1}{r^2}\right)\cos 2\theta+\left(\frac{1}{r^2}-\frac{1}{r^4}\right)\cos 4\theta\right];$$

$$U^{(0,2)}(r,\theta)=\frac{p}{2H_0}\left[-\ln r+\left(1-\frac{11}{6}\cdot\frac{1}{r^2}\right)\cos 2\theta+\right. \quad \text{(IV.145)}$$

$$\left.+\left(-\frac{1}{r^2}+\frac{3}{2}\cdot\frac{1}{r^4}\right)\cos 4\theta+\left(\frac{9}{5}\cdot\frac{1}{r^4}-\frac{33}{14}\cdot\frac{1}{r^6}\right)\cos 6\theta\right]. \quad \text{(IV.146)}$$

[1]See A. N. Guz' [2].

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TABLE IV.3

Theory	a/b		1,00	1,05	1,10	1,20	1,30	1,50	1,60
Linear	Rigorous solution		2,0000 / 2,0000	2,1010 / 1,9047	2,2120 / 1,8182	2,4440 / 1,6667	2,6160 / 1,5387	3,0000 / 1,3333	3,2006 / 1,2509
	Approximate solution		2,0000 / 2,0000	2,0970 / 1,9048	2,1980 / 1,8188	2,4350 / 1,6696	2,5870 / 1,5460	2,9600 / 1,3600	3,1360 / 1,2896
p			In linear theory k is independent of p						
Nonlinear	600	λ_3	1,9606 / 1,9606	2,0406 / 1,8654	2,1308 / 1,7886	2,3102 / 1,6570	2,4874 / 1,5500	2,8298 / 1,3914	2,8987 / 1,4278
		λ_4	1,9682 / 1,9682	2,0622 / 1,8790	2,1562 / 1,7984	2,3424 / 1,6600	2,5260 / 1,5466	2,8790 / 1,3774	2,9909 / 1,3712
	800	λ_3	1,9226 / 1,9226	2,0050 / 1,8450	2,0878 / 1,7756	2,2534 / 1,6578	2,4196 / 1,5616	2,7380 / 1,4272	2,7248 / 1,5460
		λ_4	1,9474 / 1,9474	2,0368 / 1,8628	2,1262 / 1,7868	2,3044 / 1,6564	2,4914 / 1,5504	2,8200 / 1,3948	2,8810 / 1,4386
	1000	λ_3	1,9006 / 1,9006	1,9730 / 1,8330	2,0466 / 1,7730	2,1944 / 1,6728	2,3430 / 1,5944	2,6346 / 1,4866	2,5150 / 1,7118
		λ_4	1,9258 / 1,9258	2,0092 / 1,8472	2,0930 / 1,7768	2,2604 / 1,6572	2,4266 / 1,5612	2,7494 / 1,4222	2,7468 / 1,5302
	1200	λ_3	1,8946 / 1,8946	1,9550 / 1,8390	2,0168 / 1,7906	2,1432 / 1,7120	2,2722 / 1,6532	2,5292 / 1,5800	2,2279 / 1,9353
		λ_4	1,9068 / 1,9068	1,9828 / 1,8356	2,0596 / 1,7722	2,2142 / 1,6654	2,3684 / 1,5814	2,6704 / 1,4632	2,5892 / 1,6496

Tr Note: Commas indicate decimal points.

By substituting functions (IV.121), (IV.122), (IV.145), (IV.146) and the derivatives corresponding to them into equation (IV.92) for k = 1 and f = 1, we obtain the equation for the determination of function $U^{(1,1)}$, the integration of which, under conditions (IV.104), yields[1] /392

[1]See I. A. Tsurpal [14].

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$$U^{(1,1)}(r,\theta)=\frac{p^3}{H_0^3}\Big[0{,}9691\ln r-0{,}0312\,\frac{1}{r^2}+\frac{1{,}3542}{r^4}-\frac{2{,}1146}{r^6}+$$
$$+\frac{1{,}9375}{r^8}-\frac{0{,}9375}{r^{10}}+\frac{0{,}4018}{r^{12}}+\Big(-0{,}6057+\frac{0{,}3983}{r^2}+\frac{2{,}6250}{r^2}\ln r+$$
$$+\frac{2{,}3187}{r^6}+\frac{1{,}9479}{r^8}-\frac{1{,}1919}{r^{10}}\Big)\cos 2\theta+\Big(-0{,}0625+\frac{3{,}6048}{r^2}-$$
$$-\frac{1{,}5000}{r^2}\ln r+\frac{1{,}1693}{r^4}-\frac{12{,}3750}{r^4}\ln r-\frac{7{,}2250}{r^6}+\frac{2{,}9509}{r^8}-$$
$$-\frac{0{,}9062}{r^{10}}+\frac{0{,}4687}{r^{12}}\Big)\cos 4\theta+\Big(-0.0312-\frac{0{,}2375}{r^2}- \qquad \text{(IV.147)}$$
$$-\frac{5{,}2214}{r^4}+\frac{4{,}5000}{r^4}\ln r+\frac{5{,}6509}{r^6}+\frac{7{,}5000}{r^6}\ln r-\frac{0{,}0045}{r^8}-$$
$$-\frac{0{,}1562}{r^{10}}\Big)\cos 6\theta+\Big(-0{,}1875\,\frac{1}{r^2}+\frac{1{,}2321}{r^4}-\frac{2{,}0955}{r^6}+$$
$$+\frac{1{,}0509}{r^8}\Big)\cos 8\theta.$$

By substituting the functions found (IV.121), (IV.122), (IV.145)-(IV.147), in which the variables r and θ should be substituted by ρ and ϑ, respectively, into formulas (IV.98), we find the stress components $\sigma_\rho^{(k,j)}$, $\sigma_\vartheta^{(k,j)}$ and $\tau_{\rho\vartheta}^{(k,j)}$ (k, j = 0, 1, 2). In this case the function $f(\zeta)$ in differential operators $L_1^{(j-m)}$, $L_2^{(j-m)}$, ..., $L_6^{(j-m)}$ (IV.106) should be taken from (IV.86), which, in the case of an elliptical hole, will have the form $f(\zeta)=1/\zeta$.

We will write the formula for the calculation of the stress concentration factor along the contour of an elliptical hole in the third approximation:

$$\overset{(3)}{k}=\left(\frac{\sigma_\vartheta}{p}\right)_{\rho=1}=1-2\cos 2\vartheta+2\varepsilon(\cos 2\vartheta-\cos 4\vartheta)+$$
$$+\varepsilon^2(2\cos 4\vartheta-\cos 2\vartheta-\cos 6\vartheta)+\lambda p^2(-3{,}0680+$$
$$+4{,}2922\cos 2\vartheta-2{,}1076\cos 4\vartheta+0{,}7738\cos 6\vartheta)+ \qquad \text{(IV.148)}$$
$$+\lambda p^2\varepsilon(1{,}2977-11{,}5517\cos 2\vartheta+8{,}9039\cos 4\vartheta-$$
$$-5{,}3963\cos 6\vartheta+2{,}2834\cos 8\vartheta).$$

The concentration factors k (IV.148) for various values of a/b and external load p are presented in Tables IV.4 and IV.5 for a copper plate $\lambda=\lambda_1$ (IV.117) /393 and open-hearth steel $\lambda=\lambda_4$, respectively. For comparison, the corresponding data of linear theory are also presented.

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TABLE IV.4

Theory	p	a/b							
		1,00	1,10	1,30	1,50	$\frac{1}{1,10}$	$\frac{1}{1,20}$	$\frac{1}{1,30}$	$\frac{1}{1,50}$
Nonlinear λ_1	40	2,836	2,676	2,445	2,290	3,015	3,190	3,362	3,702
	50	2,754	2,605	2,393	2,255	2,921	3,087	3,250	3,573
	60	2,642	2,508	2,323	2,207	2,793	2,945	3,095	3,395
	70	2,508	2,393	2,240	2,151	2,642	2,777	2,912	3,186
Linear		3,000	2,817	2,547	2,360	3,201	3,397	3,587	3,960

Tr. Note: Commas indicate decimal points.

TABLE IV.5

Theory	p	a/b						
		1,000	1,20	1,40	1,50	$\frac{1}{1,20}$	$\frac{1}{1,40}$	$\frac{1}{1,50}$
Nonlinear λ_4	500	2,918	2,609	2,403	2,326	3,294	3,653	3,831
	600	2,882	2,582	2,384	2,309	3,248	3,600	3,774
	700	2,839	2,550	2,362	2,291	3,194	3,536	3,707
	800	2,790	2,514	2,336	2,271	3,313	3,464	3,629
	900	2,735	2,473	2,307	2,247	3,062	3,382	3,542
	1000	2,672	2,427	2,274	2,221	2,984	3,290	3,441
	1100	2,604	2,376	2,238	2,171	2,897	3,188	3,556
	1200	2,528	2,321	2,199	2,159	2,801	3,076	3,216
Linear		3,000	2,669	2,446	2,360	3,397	3,774	3,960

Tr. Note: Commas indicate decimal points.

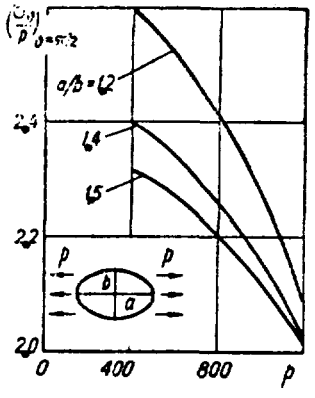


Figure IV.9.

Figure IV.9 shows the change of concentration factor k (IV.148) at the point of contour $\vartheta = \pi/2$ as a function of the magnitude of external load p and ratio a/b for a copper plate $\lambda = \lambda_1$ (IV.117) and a plate made of open-hearth steel $\lambda = \lambda_4$ (IV.120), and Figures IV.10 and IV.11 show the graphs characterizing the change of concentration factor k (IV.148) as a function of the magnitude of the force of tension p for various physically nonlinear materials with elastic characteristics (IV.117)-(IV.120) for a/b = 1.5 at the same point of the contour of the hole $\vartheta = \pi/2$.

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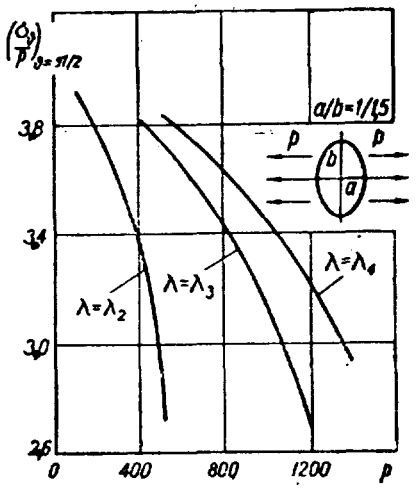


Figure IV.10.

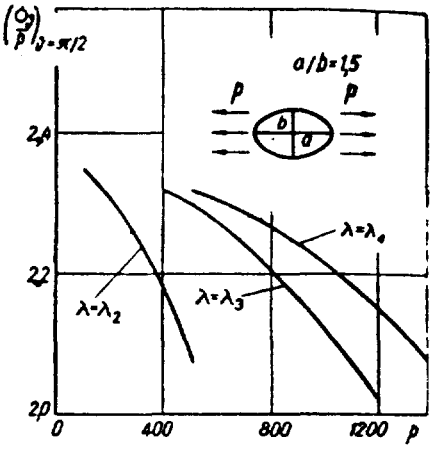


Figure IV.11.

Square Hole with Rounded Corners. We will examine the stress state near a square hole with rounded corners[1], the contour of which is defined by the function

$$\omega(\zeta) = R\left(\zeta + \frac{1}{9}\cdot\frac{1}{\zeta^3}\right) \tag{IV.149}$$

where $\rho = 1$, ($\zeta = \rho^{ei\vartheta}$, $\varepsilon = 1/9$) in the case of multifold tension of a physically nonlinear plate by forces p at infinity, i.e. for

$$\sigma_r^{(\infty)} = \sigma_\theta^{(\infty)} = p, \quad \tau_{r\theta}^{(\infty)} = 0. \tag{IV.150}$$

The approximate solution of this problem will be found with consideration of three approximations, i.e. we will search for stress function (IV.105) in the form

$$U_3(r, \theta, \mu, \varepsilon) = H_0[U^{(0,0)} + \varepsilon U^{(0,1)} + \varepsilon^2 U^{(0,2)} + \mu U^{(1,0)} + \mu^2 U^{(2,0)} + \mu\varepsilon U^{(1,1)}]. \tag{IV.151}$$

[1]For a square hole with sharper corners, it is necessary to retain a larger number of terms in the function $f(\zeta)$ (IV.86) (see §2, Chapter II).

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The functions $U^{(0,0)}$, $U^{(1,0)}$ and $U^{(2,0)}$ in (IV.151) were found from formulas (IV.132)-(IV.134). They yield the solution for a physically nonlinear plate with a round hole. /394

Stress functions $U^{(0,1)}$ and $U^{(0,2)}$ in (IV.151), which correspond to a linear-elastic plate with a square hole (IV.149), were determined[1] with an accuracy up to ε and ε^2, and have the form

$$U^{(0,1)}(r, \theta) = -\frac{p}{H_0}\left(\frac{1}{r^2} - \frac{1}{r^4}\right)\cos 4\theta; \tag{IV.152}$$

$$U^{(0,2)}(r, \theta) = \frac{p}{H_0}\left[\left(\frac{7}{2}\cdot\frac{1}{r^8} - 3\frac{1}{r^6}\right)\cos 8\theta - 3\ln r\right]. \tag{IV.153}$$

By substituting the corresponding functions from (IV.132)-(IV.134) and (IV.152) into the right hand side of equation (IV.92), we find for the function $U^{(1,1)}$, the equation /395

$$\Delta\Delta U^{(1,1)} = 96\frac{p^3}{H_0^3}\left(28\frac{1}{r^{10}} - 135\frac{1}{r^{12}}\right)\cos 4\theta. \tag{IV.154}$$

By integrating equation (IV.154) under boundary conditions (IV.104), we obtain[2]

$$U^{(1,1)}(r, \theta) = \frac{p^3}{H_0^3}\left(-\frac{149}{70}\cdot\frac{1}{r^2} + \frac{89}{35}\cdot\frac{1}{r^4} + \frac{14}{5}\cdot\frac{1}{r^6} - \frac{45}{14}\cdot\frac{1}{r^8}\right)\cos 4\theta. \tag{IV.155}$$

By substituting in functions $U^{(0,0)}$, $U^{(0,1)}$, $U^{(0,2)}$, $U^{(1,0)}$, $U^{(2,0)}$ and $U^{(1,1)}$ (found in the function U_3 (IV.151)) r and θ by ρ and ϑ, respectively, and by substituting them into formulas (IV.98), recalling the form of function (IV.149) for the established form of operators $L_1^{(j-m)}$, $L_2^{(j-m)}$, ..., $L_6^{(j-m)}$ (IV.106), we find the expressions for stress components $\sigma_\rho^{(k,j)}$, $\sigma_\vartheta^{(k,j)}$ and $\tau_{\rho\vartheta}^{(k,j)}$ (k, j = 0, 1, 2). By substituting them into formulas (IV.90), we obtain the final (with the given degree of accuracy) expressions for the desired stress components in a curvilinear orthogonal coordinate system described by mapping function (IV.149). We will not write these expressions for σ_ρ and σ_ϑ due to their awkwardness, but will present only the formula for the concentration factor along the contour of the hole:

---

[1] See G. N. Savin and A. N. Guz' [1].
[2] For more detail see I. A. Tsurpal [11].

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$$\overset{(3)}{k} = \left(\frac{\sigma_\vartheta}{p}\right)_{\varrho=1} = 2(1 + 0{,}666\cos 4\vartheta + 0{,}197\cos 8\vartheta - $$
$$- 1{,}500\lambda p^2 - 3{,}152\lambda p^2\cos 4\vartheta + 10{,}608\lambda^2 p^4). \qquad \text{(IV.156)}$$

The values of $\overset{(3)}{k}$ (IV.156) calculated for aluminum bronze alloy $\lambda = \lambda_3$ (IV.119) for $p = 11 \cdot 9.81 \cdot 10^6$ n/m^2, and for pure copper $\lambda = \lambda_2$ (IV.118) when $p = 6 \cdot 9.81 \cdot 10^6$ n/m^2, are presented in Figure IV.12. The solid lines correspond to linear theory, and the broken lines correspond to the mentioned physically nonlinear materials.

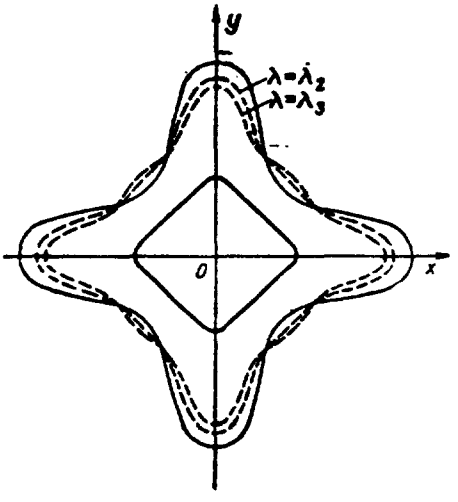


Figure IV.12.

The values of $\overset{(3)}{k}$ (IV.156) for copper $\lambda = \lambda_4$ (IV.117) at the points $\vartheta = 0$ and $\vartheta = \pi/4$ of the contour of the hole are presented in Table IV.6.

Concluding Comments. By comparing the solutions derived in §3 with the corresponding results of the linear theory, we can make the following additional conclusions: /396

a) if in classical linear elasticity theory the stress concentration factor (in these problems) is independent of the magnitude of the external load and elastic properties of the material, then, in consideration of the physical nonlinearity of the material (IV.1) (under the condition that deformations are small, i.e. such that the coordinates of the deformed and nondeformed state of the body can be assumed to be equal) this stress concentration factor actually (nonlinearly) depends on the elastic properties of the material and the magnitude of external load;

TABLE IV.6

ϑ°	Linear theory	Nonlinear theory					
		200	400	600	800	1000	1100
0	3,726	3,705	3,645	3,550	3,424	3,278	3,000
45	1,062	1,069	1,092	1,135	1,204	1,307	1,375

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b) consideration of the physical nonlinearity of a material in the form (IV.1) leads to a decrease[1] in the concentration factor in the most stressed points of the contour of the hole and to a more uniform distribution of stresses around the hole in comparison with linear theory;

c) the ellipticity of the hole, as in classical theory, has a considerable effect on the magnitude of the stress concentration factor around the hole;

d) the nonlinearity of the material, even in the case of great deviation from Hooke's law, can reduce considerably the peaks of stresses in the weak point around the hole.

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[1]As will be shown in §3 of Chapter IX, this phenomenon occurs only in the case of the so-called "soft" nonlinearity, and the converse is true for "hard" nonlinearity.

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CHAPTER V. FORMATION OF PLASTIC ZONES OR EQUILIBRIUM CRACKS AROUND HOLES

Abstract. This chapter investigates stresses beyond the elastic limit. Problems of the determination of plastic strain zones arising near holes, lines of sliding and of critical loads at the cracks are considered. Solutions are given of the elasto-plastic problems for a plane with a circular hole in the case of normal and tangential forces applied at the hole contour; an effect is shown for the nonuniform stressed field on the size and configuration of the plastic strain zone and also the distribution of the plastic zones in a plate with an infinite series of identical circular holes.

§1. Statement of Problem

The results of the analyses presented in the preceding chapters show that /399
a zone of increased stresses, the so-called "stress concentration" zone, is formed around a hole. It is clear that if this zone achieves a certain size, the given material can no longer be elastic near the hole and will pass into a state which we will call the "state beyond the limit of elasticity." We will assume that this state will occur in some range near the examined hole[1]. The stress and deformation states beyond the limit of elasticity are defined by the relations of plasticity theory.

Two types of materials are distinguished in plasticity theory: "ideally plastic" and "reinforcing" materials.

An ideally plastic material is characterized by the fact that it cannot endure stresses exceeding a definite yield point. In other words, for ideally plastic materials there is a completely defined function

$$f(\sigma_{ij}) = 0, \tag{V.1}$$

called the condition of plasticity or yield function.

When equation (V.1) is satisfied at a given point of the body, plastic deformations can occur. In the elastic range

$$f(\sigma_{ij}) < 0,$$

[1] In the case of the heterogeneous stress state, we will limit the dimensions of the plate such that the stress state beyond the "concentration zone" will be elastic.

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In contrast to an ideally plastic material, in order to obtain increments of plastic deformations in a reinforcing material it is necessary to increase the stresses beyond the limit of elasticity.

One feature of the relations of plasticity theory is their nonholonomic (do not yield to integration) character. The plastic state of a material, in contrast to the elastic state, depends not only on the final magnitudes of loads, but also on the character of loading of the body (history of the load).

The increments of the total deformation in the plastic range are composed respectively of the increments of elastic and plastic deformations:

$$de_{ij} = de^{e}_{ij} + de^{p}_{ij}. \tag{V.2}$$

It is assumed that elastic deformations in the plastic range are related to the stresses by Hooke's law /400

$$e^{e}_{ij} = \frac{1}{2G}(\sigma_{ij} - \delta_{ij}\sigma) + (1 - 2\nu)\,\delta_{ij}\frac{\sigma}{E}, \tag{V.3}$$

where E, G, ν are, respectively, Young's modulus, shear modulus, and Poisson's ratio; δ_{ij} is Kronecker's symbol; σ is average pressure.

Increments of plastic deformations are defined by the relations of the associated yield law. For an ideally plastic material,

$$de^{p}_{ij} = d\mu\,\frac{\partial f}{\partial \sigma_{ij}}, \tag{V.4}$$

where $d\mu = 0$ if $f < 0$.

The plane problem of ideal plasticity theory has the following feature: three equations -- two equilibrium equations plus plasticity equation (V.1) -- relative to three stress components σ_x, σ_y, τ_{xy} define a closed equation system. Therefore the plane problem of ideal plasticity theory is often called "statically defined," which refers to the closure of the equation system for stresses. However, only those problems whose boundary conditions are also given only in stresses (contour of a hole is free of loads or is loaded by given forces) can be related to statistically defined problems.

The condition of plasticity for an anisotropic (metallic) body has the form

$$f(\Sigma_2, \Sigma_3) = 0, \tag{V.5}$$

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where Σ_2, Σ_3 are the second and third invariants, respectively, of the stress deviator (volumetric deformation in metals is elastic within sufficient limits, and therefore plasticity condition (V.5) is independent of the first invariant of tensor of stresses σ).

If for plane deformation we have from the condition $e_z = 0$

$$\sigma_z = \frac{1}{2}(\sigma_x + \sigma_y), \tag{V.6}$$

then the third invariant of the deviator of stresses Σ_3 is equal to zero, and plasticity condition (V.5) reduces to the form

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2 \quad k = \text{const.} \tag{V.7}$$

For the Mises plasticity condition (Σ_2 = const), relation (V.6) will occur only in the case where the material is incompressible ($\nu = 1/2$). Consideration of elastic compressibility of a material under the Mises plasticity condition requires the joint analysis of stress fields and deformation rate. In other words, the problem in this case is not statically defined. This fact is the outcome of examination of the Mises plasticity condition and relations (V.2)-(V.4) under the condition $e_z = e_{xz} = e_{yz} = 0$.

For the Tresca plasticity condition (condition of maximum tangential stress), plasticity condition (V.7) can also be valid when $\nu \neq 1/2$. However, certain limitations[1] are involved here when using plasticity condition (V.5). /401

Generally speaking, in solving statically defined elasto-plastic problems it is possible to limit the analysis to the assumption of incompressibility of the material required for the determination of displacements. As the solutions of problems show[2], this assumption of incompressibility of a material throughout the elastic and plastic zones has a considerable effect on the magnitude of component σ_z. However, the latter is usually unimportant.

The equation system of the plane deformation state of an ideally plastic body, according to (V.2)-(V.4) and (V.7) when $\nu = 1/2$, can be represented in the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$
$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2; \tag{V.8}$$

[1]See V. V. Sokolovskiy [1].
[2]See L. M. Kachanov [1].

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$$\frac{2de_x^p}{\sigma_x-\sigma_y}=\frac{2de_y^p}{\sigma_y-\sigma_x}=\frac{de_{xy}^p}{2\tau_{xy}},$$

$$e_x=e_x^p+\frac{\sigma_x-\sigma_y}{4G},\quad e_y=e_y^p+\frac{\sigma_y-\sigma_x}{4G},\quad e_{xy}=e_{xy}^p+\frac{\tau_{xy}}{G},\tag{V.9}$$

$$e_x=\frac{\partial u_x}{\partial x},\quad e_y=\frac{\partial u_y}{dy},\quad e_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),$$

where u_x, u_y are displacement components along the x and y axes.

To the contour of the given hole, let normal and tangential forces

$$\sigma_n=f_1(s,\lambda),\qquad \tau_{nt}=f_2(s,\lambda),\tag{V.10}$$

be applied, and at infinity, the stresses in the elastic state

$$\sigma_x^\infty=P_1(x,y,\lambda),\quad \sigma_y^\infty=P_2(x,y,\lambda),\quad \tau_{xy}^\infty=-P_3(x,y,\lambda),$$

where λ is the stress parameter.

We will assume that stress function $U_1(x, y, \lambda)$ in the plastic range satisfies some hyperbolic equation

$$f\left(x,y,\frac{\partial^2U_1}{\partial x^2},\ \frac{\partial^2U_1}{\partial y^2},\ldots,\ \frac{\partial U_1}{\partial x},\ \frac{\partial U_1}{\partial y}\right)=0,\tag{V.11}$$

i.e., satisfies the condition of plasticity and boundary conditions (V.10).

The problem of determining the stresses boils down to the determination of biharmonic function $U_2(x, y, \lambda)$ outside of some unknown contour L which encompasses the hole, under the condition that the following relations are valid on contour L: /402

$$\frac{\partial^2U_1}{\partial x^2}=\frac{\partial^2U_2}{\partial x^2},\quad \frac{\partial^2U_1}{\partial y^2}=\frac{\partial^2U_2}{\partial y^2},\quad \frac{\partial^2U_1}{\partial x\partial y}=\frac{\partial^2U_2}{\partial x\partial y},\tag{V.12}$$

and if $x\to\infty$ and $y\to\infty$

$$\frac{\partial^2U_2}{\partial x^2}=P_2(x,y,\lambda),\quad \frac{\partial^2U_2}{\partial y^2}=P_1(x,y,\lambda),\quad \frac{\partial^2U_2}{\partial x\partial y}=P_3(x,y,\lambda).\tag{V.13}$$

The basic difficulty of this problem is the determination of contour L which separates the "plastic" zone from the elastic zone. If this contour is known,

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the determination of function $U_2(x, y)$ is equivalent to the solution of the elastic problem for a plane with hole L under known conditions, both on this contour L, and at infinity. So far, no general method for solving the formulated problem has been found. However, when the stress components in the plastic range are known

$$\frac{\partial^2 U_1}{\partial y^2} = \sigma_x^{(1)}, \quad \frac{\partial^2 U_1}{\partial x^2} = \sigma_y^{(1)}, \quad \frac{\partial^2 U_1}{\partial x \partial y} = -\tau_{xy}^{(1)}, \tag{V.14}$$

the solution of the stated problem can be found in some cases by a comparatively simple method.

The problem of determining displacements in the plastic range is solved after the stresses are found, and reduces to the solution of equation system (V.9), of the hyperbolic type. Displacements on contour L, known from the solution of the problem in the elastic range, play the part of the initial Cauchy data for equation system (V.9).

In solving elasto-plastic problems of stress concentration near holes, it is essential to define the range of possible change of external stresses such that relaxation zones will not occur in the plastic ranges under stress. Equation systems (V.8), (V.9) are no longer valid in the case of relaxation.

§2. Elasto-Plastic Problem for Infinite Plate with Round Hole (Plane Deformation)

Case of Normal Pressure on Contour of Round Hole[1]. The origin of the coordinate system is placed at the center of a round hole of radius R (Figure V.1).

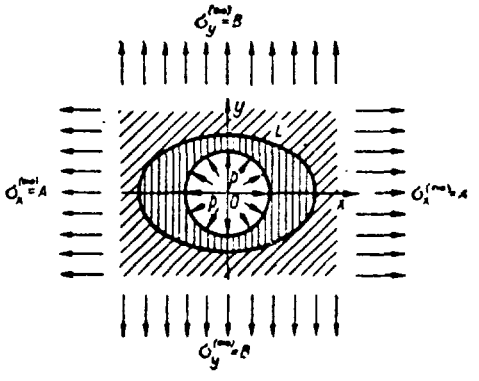


Figure V.1.

Equation (V.3), which defines stresses in the plastic range, i.e. the "plasticity condition," is represented[2] in the form

$$\left(\frac{\partial^2 U_1}{\partial x^2} - \frac{\partial^2 U_1}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 U_1}{\partial x \partial y}\right)^2 = 4k^2, \tag{V.15}$$

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where k is a constant of the material, equal to:

a) according to the theory of greatest tangential stresses

[1]The solution of this problem was first found by L. A. Galin [1], and later by O. S. Parasyuk [1] by a different method.
[2]See A. Nadai [1].

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$$k = \frac{\sigma_s}{2};$$

b) according to the theory of octrahedral tangential stresses

$$k = \frac{\sigma_s}{\sqrt{3}}.$$

where σ_s is the yield point of the material under unilateral tension.

Let only one normal force be applied to the contour of the hole, i.e. when r = R:

$$\sigma_r = -p, \quad \tau_{r\theta} = 0, \tag{V.16}$$

and let stresses

$$\sigma_x^{(\infty)} = A; \qquad \sigma_y^{(\infty)} = B.$$

be given at infinity.

The solution of equation (V.15) under boundary condition (V.16) has the form

$$U_1(x, y) = kr^2 \ln \frac{r}{R} - \frac{p+k}{2} r^2. \tag{V.17}$$

On the basis of stress function (V.17) we find the stress components $\sigma_x^{(1)}$, $\sigma_y^{(1)}$, $\tau_{xy}^{(1)}$, which are conveniently represented in the following combinations:

$$\begin{aligned} \sigma_x^{(1)} + \sigma_y^{(1)} &= 2k - 2p + 2k \ln \frac{z\bar{z}}{R^2}, \\ \sigma_y^{(1)} - \sigma_x^{(1)} + 2i\tau_{xy}^{(1)} &= 2k \frac{\bar{z}}{z}. \end{aligned} \tag{V.18}$$

The problem boils down to the determination of the function $U_2(x, y)$ and contour L that separates the plastic and elastic zones, under the following conditions:

$$\frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2} = \begin{cases} 2k - 2p + 2k \ln \frac{z\bar{z}}{R^2} \text{ on L.} \\ A + B \text{ for } z \to \infty; \end{cases} \tag{V.19}$$

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$$\frac{\partial^2 U_2}{\partial x^2} - \frac{\partial^2 U_2}{\partial y^2} - 2i\frac{\partial^2 U_2}{\partial x \partial y} = \begin{cases} 2k\frac{\bar{z}}{z} & \text{on L,} \\ B - A & \text{for } z \to \infty. \end{cases} \tag{V.20}$$

For any biharmonic function $U_2(x, y)$ in range S (beyond contour L), there are always two holomorphic Muskhelishvili [1] functions within it, namely $\Phi_2^*(z)$ and $\Psi_2^*(z)$, such that /404

$$\begin{aligned} \frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2} &= 4\,\mathrm{Re}\,[\Phi_2^*(z)], \\ \frac{\partial^2 U_2}{\partial x^2} - \frac{\partial^2 U_2}{\partial y^2} - 2i\frac{\partial^2 U_2}{\partial x \partial y} &= 2[\bar{z}\Phi_2^{*\prime}(z) + \Psi_2^*(z)]. \end{aligned} \tag{V.21}$$

Thus, the problem can be formulated as follows: to find contour L that bounds the plastic range, and two functions $\Phi_2^*(z)$ and $\Psi_2^*(z)$ that are holomorphic within the entire plate beyond contour L and continuous all the way to L, according to the following conditions:

$$4\,\mathrm{Re}\,\Phi_2^*(z) = \begin{cases} 2k - 2p + 2k\ln\frac{z\bar{z}}{R^2} & \text{on L,} \\ A + B & \text{for } z \to \infty; \end{cases} \tag{V.22}$$

$$2[\bar{z}\Phi_2^{*\prime}(z) + \Psi_2^*(z)] = \begin{cases} 2k\frac{\bar{z}}{z} & \text{on L,} \\ B - A & \text{for } z \to \infty. \end{cases} \tag{V.23}$$

To solve this problem we will map the exterior of contour L onto the exterior of unit circle γ of plane ζ using the function

$$z = \omega(\zeta) = c\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots + \frac{c_n}{\zeta^n}. \tag{V.24}$$

Then, assuming

$$\Phi_2^*[\omega(\zeta)] = \Phi_2(\zeta), \quad \Psi_2^*[\omega(\zeta)] = \Psi_2(\zeta), \tag{V.25}$$

conditions (V.22) and (V.23) acquire the form

$$4\,\mathrm{Re}\,\Phi_2(\zeta) = \begin{cases} 2k - 2p + 2k\ln\frac{\omega(\zeta)\overline{\omega(\zeta)}}{R^2} & \text{on } \gamma, \\ A + B & \text{for } \zeta \to \infty; \end{cases} \tag{V.26}$$

$$2\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_2'(\zeta) + \Psi_2(\zeta)\right] = \begin{cases} 2k\frac{\overline{\omega(\zeta)}}{\omega(\zeta)} & \text{on } \gamma, \\ B - A & \text{for } \zeta \to \infty. \end{cases} \tag{V.27}$$

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The functions $\Phi_2(\zeta)$ and $\Psi_2(\zeta)$ are sought in the form

$$\Phi_2(\zeta) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \ldots + \frac{a_n}{\zeta^n} + \ldots.$$
$$\Psi_2(\zeta) = b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \ldots + \frac{b_n}{\zeta^n} + \ldots. \qquad (V.28)$$

We will assume that the function $z = \omega(\zeta)$ is of the form

$$z = c\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2}. \qquad (V.29)$$

Recalling that on the contour of unit circle γ /405

$$\bar{\zeta} = \bar{\sigma} = \frac{1}{\sigma} = \frac{1}{\zeta},$$

we rewrite the first condition (V.27) in the form

$$2\left[\frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{\omega'(\sigma)}\Phi_2'(\sigma) + \Psi_2(\sigma)\right] = 2k\,\frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{\omega(\sigma)}. \qquad (V.30)$$

Simple calculation shows that

$$\frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{\omega(\sigma)} = \frac{\bar{c}\frac{1}{\sigma} + \bar{c}_1\sigma + \bar{c}_2\sigma^2}{c\sigma + \frac{c_1}{\sigma} + \frac{c_2}{\sigma^2}} = \frac{\bar{c}_2}{c}\sigma + \frac{\bar{c}_1}{c} + M(\sigma), \qquad (V.31)$$

where $M(\sigma)$ is a function that is holomorphic outside of γ and $M(\infty) = 0$.

Analogously,

$$\frac{\omega\left(\frac{1}{\sigma}\right)}{\omega'(\sigma)}\Phi_2'(\sigma) = -\frac{\bar{c}_2}{c}a_1 + N(\sigma), \qquad (V.32)$$

where $N(\sigma)$ is a function that is holomorphic outside of γ and $N(\infty) = 0$.

We multiply equation (V.30) by $1/2\pi i \cdot d\sigma/\sigma - \zeta$, where ζ is a point within γ, and then we integrate with respect to γ:

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$$\frac{1}{2\pi i}\int_{\gamma}\frac{2\left[\frac{\overline{\omega}\left(\frac{1}{\sigma}\right)}{\overline{\omega'(\sigma)}}\Phi_2'(\sigma)+\Psi_2(\sigma)\right]}{\sigma-\zeta}d\sigma=\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega}\left(\frac{1}{\sigma}\right)}{\frac{2k}{\omega(\sigma)}}d\sigma. \tag{V.33}$$

According to Garnak's theorem[1], (V.33) is the outcome of condition (V.30). From (V.33), because of the known properties of the Cauchy integral, recalling relations (V.31) and (V.32) and the second condition of (V.27), we obtain

$$-2\frac{\overline{c_2}}{c}a_1+B-A=2k\frac{c_2}{c}\zeta+2k\frac{\overline{c_1}}{c}.$$

Hence, $\overline{c_2}=0$, $B-A=2k\frac{c_1}{c}$, or

$$\overline{c_1}=\frac{B-A}{2k}c=c_1=c\beta,$$

where

$$\beta=\frac{B-A}{2k}.$$

Thus, the function $\omega(\zeta)$ (V.29) should have the form /406

$$z=\omega(\zeta)=c\left(\zeta+\frac{\beta}{\zeta}\right), \tag{V.34}$$

where c is as yet an undefined constant, which can be found from conditions (V.26).

The first of conditions (V.26) will be satisfied if

$$\begin{aligned}\Phi_2(\zeta)&=k\ln\frac{\omega(\zeta)}{R}+\frac{k-p}{2}-k\ln\zeta=\\&=k\ln c-k\ln R+k\ln\left(\zeta+\frac{\beta}{\zeta}\right)+\frac{k-p}{2}-k\ln\zeta.\end{aligned} \tag{V.35}$$

The second condition of (V.26) will be satisfied if the constant is determined from equation

$$A+B=4\left(k\ln c-k\ln R+\frac{k-p}{2}\right).$$

[1]See N. I. Muskhelishvili [1].

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Hence

$$c = Re^{\frac{1}{2k}\left(\frac{A+B}{2}+p-k\right)}. \tag{V.36}$$

After we have found functions (V.34) and (V.35), we find the function $\Psi_2(\zeta)$ from the first condition of (V.27):

$$\Psi_2(\zeta) = k\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega(\zeta)} - \frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\Phi_2'(\zeta) = \\ = k\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega(\zeta)} - k\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\left[\frac{\omega'(\zeta)}{\omega(\zeta)} - \frac{1}{\zeta}\right] = k\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\cdot\frac{1}{\zeta}. \tag{V.37}$$

Thus, the stated problem is solved. Contour L, i.e., the boundary of the plastic range, will be an ellipse with the semiaxes $a = c(1 + \beta)$, $b = c(1 - \beta)$, where $\beta = B - A/2k$, and the constant c is defined by formula (V.36). If $A = B \neq 0$, then, as follows from (V.34) and (V.36), the boundary of the plastic range will be a circle of radius

$$c = Re^{\frac{1}{2k}(A+p-k)}. \tag{V.38}$$

If, in the partial case A = B = 0, i.e. there are no stresses at infinity, then

$$c = Re^{\frac{p-k}{2k}}. \tag{V.39}$$

If the contour of the round hole R is free of external stresses, and the stresses at infinity are $A = B \neq 0$, the boundary of the plastic range will also be a circle of radius

$$c = Re^{\frac{A-k}{2k}}. \tag{V.40}$$

By comparing (V.39) and (V.40), we see that if A = p, then, a plastic zone /407
of the same magnitude is formed near the round hole.

The stresses in the plastic zone are defined by formulas (I.27) if we substitute in them the values we have found for the functions $\omega(\zeta)$ (V.34), $\Phi_2(\zeta)$ (V.35) and $\Psi_2(\zeta)$ (V.37). The stresses in the plastic range are defined, however, by stress function (V.17).

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From the solution we have obtained, we may derive an important practical conclusion: since the form of functions $\Phi_2(\zeta)$ and $\Psi_2(\zeta)$, which satisfy only the first condition of (V.26) and the first condition of (V.27), is independent of the special selection of $\omega(\zeta)$, and the function $z = \omega(\zeta)$ (V.34), which defines the boundary of the plastic zone, is completely defined from the second condition of (V.26) and the second condition of (V.27), then we may use any function $\omega(\zeta)$, i.e. define the boundary of the plastic zone in advance, and proceeding in the opposite manner, we can find the stress state at infinity for which the function $\omega(\zeta)$ if $|\zeta| = 1$, will define the boundary of the plastic zone, i.e. the contour L. Let us take, for instance, the function

$$z = \omega(\zeta) = c\left(\zeta + \frac{\beta}{\zeta} + \frac{\delta}{\zeta^2}\right), \tag{V.41}$$

where c, β and δ are constants.

Then, on the basis of (V.35),

$$\Phi_2(\zeta) = k \ln c - k \ln R + k \ln\left(\zeta + \frac{\beta}{\zeta} + \frac{\delta}{\zeta^2}\right) + \frac{k-p}{2} - k \ln \zeta, \tag{V.42}$$

and on the basis of (V.37)

$$\Psi_2(\zeta) = k \frac{\frac{1}{\zeta} + \beta\zeta + \delta\zeta^2}{1 - \frac{\beta}{\zeta^2} - \frac{2\delta}{\zeta^3}} \frac{1}{\zeta}. \tag{V.43}$$

Using relations (V.22), (V.23) and formulas (V.21), we find the stresses at infinity:

$$\begin{aligned} \sigma_x^{(\infty)} + \sigma_y^{(\infty)} &= 4\left(k \ln c - k \ln R + \frac{k-p}{2}\right), \\ \sigma_y^{(\infty)} - \sigma_x^{(\infty)} + 2i\tau_{xy}^{(\infty)} &= 2k\delta(x + iy) + 2k\beta. \end{aligned} \tag{V.44}$$

Thus, if in sufficiently remote points of the plane (at infinity), we know stresses (V.44), and if we know forces (V.16) on the contour of a round hole of radius R, the boundary of the plastic zone that surrounds the hole will be contour L, the equation of which is given by the function $\omega(\zeta)$ (V.41) if $|\zeta| = 1$.

We will now the discuss the limitations imposed on stress. In order to avoid relaxation, which is intolerable under the conditions of the given problem, it is essential that the plastic zone, at any moment of stress, completely encompass the plastic zone at any preceding moment of stress.

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The boundary of the plastic range is an ellipse with semiaxes /408
$a = c(1 + \beta)$, $b = c(1 - \beta)$. We will denote the values in the preceding moment of stress through the subscript 1, and in the following moment of stress, through the subscript 2:

$$c_2(1+\beta_2) \geqslant c_1(1+\beta_1), \quad c_2(1-\beta_2) \geqslant c_1(1-\beta_1). \tag{V.45}$$

The values c, β are defined by the magnitudes of loads $A(\lambda)$, $B(\lambda)$, $P(\lambda)$. Therefore inequalities (V.45) give us the possible boundaries of change of the loads.

It should also be noted that the solution obtained is valid only in the case where any point in the plastic range can be connected to the contour of the hole by two slip lines that lie entirely within the plastic range. For this purpose it is necessary that the semiaxes of the ellipse that defines the boundary of the plastic zone not exceed $\sqrt{2}$ ($\beta \leqslant 0.171$). Otherwise the problem becomes statically undefinable.

Let us examine the equations for the solution of displacements (V.23). In the polar coordinate system they acquire the form

$$\frac{2de_r^p}{\sigma_r-\sigma_\theta} = \frac{2de_\theta^p}{\sigma_\theta-\sigma_r} = \frac{de_{r\theta}^p}{2\tau_{r\theta}};$$

$$e_r = e_r^p + \frac{\sigma_r-\sigma_\theta}{4G}, \quad e_\theta = e_\theta^p + \frac{\sigma_\theta-\sigma_r}{4G}, \quad e_{r\theta} = e_{r\theta}^p + \frac{\tau_{r\theta}}{G}, \tag{V.46}$$

$$e_r = \frac{\partial u_r}{\partial r}, \quad e_\theta = \frac{u_r}{r} + \frac{1}{r}\cdot\frac{\partial u_\theta}{\partial \theta},$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\cdot\frac{\partial u_r}{\partial \theta}\right), \tag{V.47}$$

where u_r, u_θ are displacement components along the r, θ axes.

From the assumption of incompressibility of the material we have

$$e_r + e_\theta = 0. \tag{V.48}$$

In the plastic range $\tau_{r\theta} = 0$ and therefore from (V.47) it follows that $e_{r\theta} = e^p_{r\theta}$. Further, from (V.46) we know that in the plastic range

$$de_{r\theta} = 0. \tag{V.49}$$

Equation (V.49) defines the increment of deformation at a point of the body as the load parameter changes, and therefore integration of relation (V.49) yields

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$$e_{r\theta} = f(r, \theta). \tag{V.50}$$

We will show that the constant in the right hand side of equation (V.50) is equal to zero at any point. We will analyze the deformation process. All stress components and deformations on the boundary of plastic zone L are continuous, and therefore, from condition $\tau_{r,\theta} = 0$ and Hooke's law, we know that $e_{r,\theta} = 0$ on contour L. Consequently, after the subsequent extension of the plastic zone to any fixed point at the moment the boundary of the plastic range passes through it, $e_{r\theta} = 0$. And since at any subsequent moment in the plastic zone relation (V.49) is valid, then, consequently, deformation $e_{r,\theta}$ in the plastic range will always be equal to zero, i.e. $f(r, \theta) = 0$.

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Therefore, for the determination of displacements in the plastic range, we have the equations

$$e_r + e_\theta = 0, \quad e_{r.\theta} = 0 \tag{V.51}$$

or in the components of displacements:

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} = 0, \qquad \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial \theta} = 0. \tag{V.52}$$

Equation system (V.52) is of the hyperbolic type, and its characteristics coincide with those of the equation system for stresses. Equations (V.52) can be solved numerically. The solution of equation system (V.52) by the small parameter method is presented in §5 for the examined problem.

The solution of the problem presented in this section was extended by V. L. Fomin [1, 2] to the case of a stationary thermal field for $p = 0$, $\tau_{r\theta} = 0$ (no external forces are applied to the contour of a round hole). It is assumed that the field is thermally insulated and that the plastic zone completely encompasses the round hole.

The boundary L that separates the plastic zone (which encompasses the hole) from the elastic zone is also an ellipse in this case, although its center is displaced in relation to the center of the hole; the effect of the above displacement is determined by the temperature field. Here, instead of mapping functions (V.34), we will use the function

$$z = \omega(\zeta) = \frac{\gamma}{k} + c\left(\zeta + \frac{\beta}{\zeta}\right),$$

where γ is a constant, defined by the temperature field;

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$$c = Re^{\left(\frac{A+B}{4k} - \frac{1}{2}\right)}; \qquad \beta = \frac{A-B}{2k};$$

A and B are the values of stresses σ_x and σ_y at infinity. The function $\omega(\zeta)$ maps the exterior of the unit circle onto the exterior of the ellipse with the center at the point $\gamma/k = x_0 + iy_0$ and with the semiaxes $a = c(1 + \beta)$, $b = c(1 - \beta)$.

The case where k is a function of radius r, i.e. $k = k(r)$, is analyzed by A. I. Kuznetsov [1].

The analysis of the elasto-plastic problem for a material with linear reinforcement in the particular case where a single normal force is given on the contour of the hole, with no stresses at infinity, is given by K. N. Shevchenko [1]. The analysis of this problem for a material with gradual reinforcement is presented by O. I. Mangasarian [1]. It is shown in the latter work that the difference in the results obtained by the theory of small elasto- /410 plastic deformations and by the theory of flow is slight.

Case of Normal and Tangential Forces Applied to Contour of Round Hole[1]. We will examine the general case where both normal and tangential forces are applied to the contour of a round hole of radius R, i.e. when $r = R$

$$\sigma_r = p, \quad \tau_{r\theta} = \tau, \tag{V.53}$$

and at infinity, as in the preceding case, the stresses

$$\sigma_x^{(\infty)} = A, \qquad \sigma_y^{(\infty)} = B. \tag{V.54}$$

The solution of equation (V.15) under condition (V.53) was found by S. G. Mikhlin [1]

$$\begin{aligned}
\sigma_r^{(1)} &= \varepsilon k\left[2\ln\left(\sqrt{r^2 - C} + \sqrt{r^2 + C}\right) - \frac{\sqrt{r^4 - C^2}}{r^2}\right] + kD,\\
\sigma_\theta^{(1)} &= \varepsilon k\left[2\ln\left(\sqrt{r^2 - C} + \sqrt{r^2 + C}\right) + \frac{\sqrt{r^4 - C^2}}{r^2}\right] + kD,\\
\tau_{r\theta}^{(1)} &= \frac{kC}{r^2},
\end{aligned} \tag{V.55}$$

where

$$\varepsilon = \pm 1, \quad C = \frac{\tau R^2}{k};$$

---

[1]The solution is given by O. S. Parasyuk [1].

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$$D = \frac{p}{k} - \varepsilon\left[2\ln\left(\sqrt{R^2+C}+\sqrt{R^2-C}\right) - \frac{\sqrt{R^4-C^2}}{R^2}\right];$$

r, θ are polar coordinates.

For our purposes it will be convenient to change to Cartesian coordinates and write the stresses in the following combinations:

$$\begin{aligned}\sigma_x^{(1)}+\sigma_y^{(1)} &= 2\varepsilon k\left[2\ln\left(\sqrt{r^2-C}+\sqrt{r^2+C}\right)\right]+2kD,\\ \sigma_y^{(1)}-\sigma_x^{(1)}+2i\tau_{xy}^{(1)} &= \left(\sigma_\theta^{(1)}-\sigma_r^{(1)}+2i\tau_{xy}^{(1)}\right)e^{-2i\theta} =\\ &= 2ke^{-2i\theta}\left[\frac{\varepsilon\sqrt{r^4-C^2}}{r^2}+i\frac{C}{r^2}\right].\end{aligned} \tag{V.56}$$

The problem involves the determination of contour L and such functions $\Phi_2^*(z)$ and $\Psi_2^*(z)$, holomorphic beyond L, that satisfy the following relations:

$$4\,\mathrm{Re}\,\Phi_2^*(z) = \begin{cases}2\varepsilon k\left[2\ln\left(\sqrt{r^2-C}+\sqrt{r^2+C}\right)\right]+2kD & \text{on L,}\\ A+B & \text{for } z\to\infty;\end{cases} \tag{V.57}$$

$$2\left[\bar{z}\Phi_2^{*\prime}(z)+\Psi_2^*(z)\right] = \begin{cases}2k\left[\frac{\varepsilon\sqrt{r^4-C^2}}{r^2}+i\frac{C}{r^2}\right]e^{-2i\theta} & \text{on L,}\\ B-A & \text{for } z\to\infty.\end{cases} \tag{V.58}$$

The rigorous solution is very difficult. However, considering that the plastic zone will be quite large, we may disregard in the first boundary condition (V.58) the value C/r^2 and replace it with the following: /411

$$2\left[\bar{z}\Phi_2^{*\prime}(z)+\Psi_2^*(z)\right] = 2\varepsilon k e^{-2i\theta}. \tag{V.59}$$

We will map the exterior of contour L onto the exterior of unit circle γ of plane ζ using the function

$$z = \omega(\zeta) = c\zeta + \sum_{n=1}^{\infty} c_n\zeta^{-n} \tag{V.60}$$

and assume, as before,

$$\Phi_2^*[\omega(\zeta)] = \Phi_2(\zeta);\quad \Psi_2^*[\omega(\zeta)] = \Psi_2(\zeta).$$

We will substitute relations (V.57)-(V.59) with the following

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$$\operatorname{Re}\Phi_2(\zeta) = \begin{cases} 2\varepsilon k\left[2\ln\left(\sqrt{\omega(z)\,\overline{\omega(\zeta)} - C} + \sqrt{\omega(\zeta)\,\overline{\omega(\zeta)} + C}\right)\right] + 2kD \text{ on } \gamma, \\ A + B \quad \text{for} \quad \zeta \to \infty; \end{cases} \tag{V.61}$$

$$2\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_2'(\zeta) + \Psi_2(\zeta)\right] = \begin{cases} 2\varepsilon k\,\frac{\overline{\omega(\zeta)}}{\omega(\zeta)} \text{ on } \gamma, \\ B - A \text{ for } \zeta \to \infty. \end{cases} \tag{V.62}$$

From here the analysis is the same as in the preceding case. Taking

$$z = \omega(\zeta) = c\zeta + \frac{c_1}{\zeta}, \tag{V.63}$$

we multiply the first condition (V.62) by $1/2\pi i \cdot d\sigma/\sigma - \zeta$, where ζ is a point within γ, and integrate with respect to γ:

$$B - A = \frac{\bar{c}_1}{c} 2k\varepsilon \tag{V.64}$$

or

$$\bar{c}_1 = \frac{B - A}{2k\varepsilon} c = \beta c = c_1, \tag{V.65}$$

where

$$\beta = \frac{B - A}{2k\varepsilon}, \quad \varepsilon = +1.$$

Thus, from the first condition of (V.62), the function is found in the form

$$z = \omega(\zeta) = c\left(\zeta + \frac{\beta}{\zeta}\right). \tag{V.66}$$

We will determine the constant c from the second condition of (V.61). If /412
$A = B \neq 0$ or $A = B = 0$, then it is easy to see from (V.65) and (V.66) that $\beta = 0$ and the boundary of the plastic zone will be a circle.

We will examine several examples, the solutions of which are presented in the table.

Comparison of examples 1 and 2 shows that with the addition of tangential stress $\tau_{r\theta}$ = const, the plastic zone increases. The graphs of contours L that separate the plastic zone from the elastic zone for examples 1 and 2 (curves 1 and 2, respectively) are shown in Figure V.2 by way of graphical illustration. Comparison of examples 3 and 4, 5 and 6 shows how normal pressure and tangential stress, when applied to the contour of the hole, affect the plastic zone,

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and specifically, from the point of view of the magnitude of the plastic zone, the application of normal pressure of a certain magnitude is equivalent to a decrease in stresses at infinity by the same magnitude, and the application of tangential stress is also equivalent to some decrease in stresses at infinity.

No. of example	Stresses at infinity		Stresses on contour of hole		c	β	R
	$\frac{\sigma_x^{(\infty)}}{k}$	$\frac{\sigma_y^{(\infty)}}{k}$	$\frac{\tau_{r\theta}}{k}$	$\frac{\sigma_r}{k}$			
1	—1,23769	—0,57102	0,5	—4	3	1/3	1
2	—1,23769	—0,57102	0	—4	2,8513	1/3	1
3	1,49650	1,89650	0,5	0	1,5	0,2	1
4	0,49650	1,89650	0,5	—1	1,5	0,2	1
5	0,45299	1,11965	0,5	—1	1,5	1/3	1
6	0,47759	1,14425	0	—1	1,5	1/3	1

Tr. Note: Commas indicate decimal points.

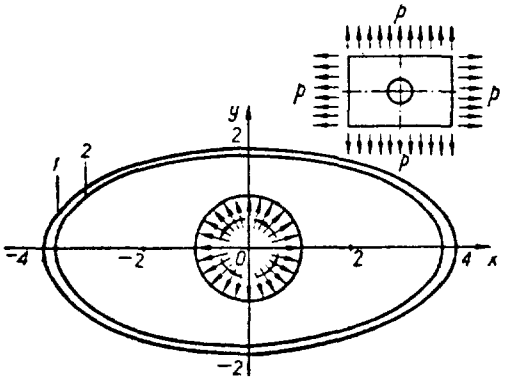


Figure V.2.

§3. Effect of Heterogeneity of Stress Field on Plastic Zone Near Round Hole[1]

Suppose forces

$$\sigma_r = p, \quad \tau_{r\theta} = 0. \qquad \text{V.67)}$$

are applied to the contour of a round hole of radius R.

If the material around the hole is converted to the plastic state, the stress function $U_1(x, y)$ will acquire the form (V.17): /413

$$U_1(r) = kr^2 \ln \frac{r}{R} + \frac{p-k}{2} r^2. \qquad \text{(V.68)}$$

Function (V.68) possesses and important property, such that it satisfies simultaneously two equations: plasticity condition (V.15) and biharmonic equation (V.1). This fact, first proposed by L. A. Galin [1], makes it possible to solve the elasto-plastic problem for more general conditions at infinity than was done in preceding sections, where only tension was analyzed.

We will examine the elasto-plastic problem for the case where the stresses at infinity are expressed by polynomials.

[1]The case of pure deflection is analyzed by L. A. Galin [1]. G. N. Savin and O. S. Parasyuk [1, 2] give the solutions for more complex cases of the basic stress state.

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We will find the biharmonic function $U_2(x, y)$ outside of some unknown contour L which bounds the plastic range, under the following conditions:

1) at infinity (when $z \to \infty$)

$$\frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2} = 4 \operatorname{Re}[\Phi_2^*(z)],$$
$$\frac{\partial^2 U_2}{\partial x^2} - \frac{\partial^2 U_2}{\partial y^2} - 2i \frac{\partial^2 U_2}{\partial x \partial y} = 2[\bar{z}\Phi_2^{*\prime}(z) + \Psi_2^*(z)], \tag{V.69}$$

where

$$\Phi_2^*(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m; \tag{V.70}$$
$$\Psi_2^*(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m, \tag{V.71}$$

and m is a whole real number;

2) along unknown smooth contour L, bounding the plastic range,

$$\frac{\partial^2 U_1}{\partial x^2} = \frac{\partial^2 U_2}{\partial x^2}, \quad \frac{\partial^2 U_1}{\partial y^2} = \frac{\partial^2 U_2}{\partial y^2}, \quad \frac{\partial^2 U_1}{\partial x \partial y} = \frac{\partial^2 U_2}{\partial x \partial y}, \tag{V.72}$$

where $U_1(x, y)$ is function (V.68).

Considering that $U_1(x, y)$ and $U_2(x, y)$ are biharmonic functions, we introduce the biharmonic function

$$U_3(x, y) = U_2(x, y) - U_1(x, y) \tag{V.73}$$

and represent it through two holomorphic functions $\phi_3^*(z)$ and $\chi_3^*(z)$ according to the formula of E. Gurs:

$$U_3(x, y) = \operatorname{Re}[\bar{z}\varphi_3^*(z) + \chi_3^*(z)]. \tag{V.74}$$

We introduce the definitions $\varphi_3^{*\prime}(z) = \Phi_3^*(z)$, $\chi_3^{*\prime}(z) = \Psi_3^*(z)$.

Now the problem can be formulated as follows: to find two functions $\Phi_3^*(z)$ and $\Psi_3^*(z)$, holomorphic outside of contour L, under conditions /414

$$\operatorname{Re}[\Phi_3^*(z)] = \begin{cases} 0 & \text{on L,} \\ \operatorname{Re}[\Phi_2^*(z)] - \frac{p+k}{2} - k \ln \frac{r}{R} & \text{for } z \to \infty. \end{cases} \tag{V.75}$$

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$$\bar{z}\Phi_3^{*\prime}(z) + \Psi_3^*(z) = \begin{cases} 0 & \text{on } L, \\ \bar{z}\Phi_2^{*\prime}(z) + \Psi_2^*(z) - e^{-2i\theta} & \text{for } z \to \infty, \end{cases} \tag{V.76}$$

where $\theta = \arg z$, L is an unknown contour separating the elastic range from the plastic range.

We will map region S of plane xOy, which is external in relation to L, onto the exterior of unit circle γ of plane $\xi O\eta$ using the function

$$z = \omega(\zeta) = c\zeta + \sum_{n=1}^{\infty} c_n \zeta^{-n}. \tag{V.77}$$

We introduce the definitions

$$\Phi_3^*[\omega(\zeta)] = \Phi_3(\zeta), \quad \Psi_3^*[\omega(\zeta)] = \Psi_3(\zeta).$$

On the basis of (V.75) and (V.76)

$$\operatorname{Re}[\Phi_3(\zeta)] = \begin{cases} 0 & \text{on } \gamma, \\ \operatorname{Re} a_0 + \operatorname{Re}[a_1 c\zeta + a_2 c^2 \zeta^2 + \ldots + a_m c^m \zeta^m] + \\ + k \ln R - \frac{k+p}{2} - k \ln|\zeta| - k \ln c & \text{for } \zeta \to \infty; \end{cases} \tag{V.78}$$

$$\left[\overline{\frac{\omega(\zeta)}{\omega'(\zeta)}}\,\Phi_3'(\zeta) + \Psi_3(\zeta)\right] = \begin{cases} 0 & \text{on } \gamma, \\ \bar{c}\bar{\zeta}\Phi_2'(\zeta) + \Psi_2(c\zeta) - ke^{-2i\theta_1} & \text{for } \zeta \to \infty, \end{cases} \tag{V.79}$$

where $\theta_1 = \arg \zeta$; $a_0, a_1, \ldots, a_m$ are coefficients of functions $\Phi_2^*(z)$. From relations (V.78) and (V.79) we must determine functions $\Phi_3(\zeta)$, $\Psi_3(\zeta)$, $\omega(\zeta)$. Obviously, conditions (V.78) will be satisfied if

$$\Phi_3(\zeta) = a_1 c\zeta - \bar{a}_1 c \frac{1}{\zeta} + a_2 c^2 \zeta^2 - \bar{a}_2 c^2 \frac{1}{\zeta^2} + \ldots + a_m c^m \zeta^m - \bar{a}_m c^m \frac{1}{\zeta^m} - k \ln \zeta; \tag{V.80}$$

whereupon the constant c is defined such that

$$\operatorname{Re} a_0 + k \ln R - \frac{k+p}{2} - k \ln c = 0$$

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or

$$c = Re^{\frac{1}{k}\left(\operatorname{Re} a_0 - \frac{k+p}{2}\right)}. \tag{V.81}$$

Since when $\zeta \to \infty$,

$$\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_3'(\zeta) = c\zeta\Phi_2^{\cdot\cdot}(c\zeta) - ke^{-2i\theta_1},$$

then on the basis of the second condition of (V.79), /415

$$\Psi_3(\zeta) = \Psi_2^*(c\zeta) + M(\zeta), \tag{V.82}$$

where $M(\zeta)$ is a function that is regular outside of γ and $M(\infty) = 0$.

The first condition of (V.79) can be represented in the form

$$\overline{\omega(\zeta)}\,\Phi_3'(\zeta) = -\omega'(\zeta)\,\Psi_3(\zeta) \text{ on } \gamma. \tag{V.83}$$

This is the basic functional equation by which we determine the function $\omega(\zeta)$.

Assuming that it is possible to expand $\omega(\zeta)$ into a series of the form (V.77), we obtain, on the basis of (V.83),

$$\begin{aligned}&\left(\frac{c}{\zeta} + \bar{c}_1\zeta + \bar{c}_2\zeta^2 + \ldots + \bar{c}_n\zeta^n + \ldots\right)\Phi_3'(\zeta) = \\ &= -\left(c - \frac{c_1}{\zeta^2} - \frac{2c_2}{\zeta^3} - \ldots - \frac{nc_n}{\zeta^{n+1}} - \ldots\right)\Psi_3(\zeta) \text{ on } \gamma.\end{aligned} \tag{V.84}$$

The function $\Psi_3(\zeta)$ is known with an accuracy up to the regular part outside of γ, and therefore, to determine the coefficients of the function $\omega(\zeta)$ of (V.77) from relation (V.84), it is sufficient to compare the terms with the positive powers of ζ. Consequently, we obtain an infinite equation system from which it is necessary also to determine the values of the coefficients of function $\omega(\zeta)$, such that series (V.77) converges everywhere outside of γ. By selecting polynomials (V.70) and (V.71) of a certain form, we obtain the solutions of several important practical problems.

Pure Deflection of Rod (Beam). In this case, as we know[1], polynomials (V.70) and (V.71) acquire the form

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[1]See Formulas (II.95) and (II.96).

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$$\Phi_2^*(z) = i\frac{M}{4J}z, \quad \Psi_2^*(z) = -i\frac{M}{4J}z,$$

where M is the magnitude of the deflecting moment; J is the moment of inertia of the cross section of the beam.

On the basis of (V.80) and (V.82),

$$\Phi_3(\zeta) = i\frac{Mc}{4J}\zeta + i\frac{Mc}{4J}\cdot\frac{1}{\zeta} - k\ln\zeta,$$

$$\Phi_3'(\zeta) = i\frac{Mc}{4J} - i\frac{Mc}{4J}\cdot\frac{1}{\zeta^2} - k\frac{1}{\zeta},$$

$$\Psi_3(\zeta) = -i\frac{Mc}{4J}\zeta + N(\zeta),$$

where $N(\zeta)$ is a function that is holomorphic outside of γ, and $N(\infty) = 0$.

The basic relation (V.84) acquires the form /416

$$\left[\left(c\frac{1}{\zeta} + \bar{c}_1\zeta + \bar{c}_2\zeta^2 + \ldots + \bar{c}_n\zeta^n + \ldots\right)\left(i\frac{Mc}{4J} - i\frac{Mc}{4J}\cdot\frac{1}{\zeta^2} - \frac{k}{\zeta}\right)\right] =$$
$$= \left[\left(c - c_1\frac{1}{\zeta^2} - \frac{2c_2}{\zeta^3} - \ldots - \frac{nc_n}{\zeta^{n+1}} - \ldots\right)\left(i\frac{Mc}{4J}\zeta - N(\zeta)\right)\right].$$

Introducing the definition $\alpha = i\frac{Mc}{4J}$ and comparing the coefficients for positive powers of ζ, we obtain the following system of equations for the determination of the coefficients c_k of the function $\omega(\zeta)$:

$$\begin{aligned}
k\bar{c}_1 + \alpha\bar{c}_2 &= 0,\\
\alpha\bar{c}_1 - k\bar{c}_2 - \alpha\bar{c}_3 &= c\alpha,\\
\alpha\bar{c}_2 - k\bar{c}_3 - \alpha\bar{c}_4 &= 0,\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
\alpha\bar{c}_n - k\bar{c}_{n+1} - \alpha\bar{c}_{n+2} &= 0 \quad (n = 2, 3, 4, \ldots).
\end{aligned}$$

For the solution of this system we will note that, beginning with the third equation, we will have the difference equation

$$\bar{c}_{n+2} + \frac{k}{\alpha}\bar{c}_{n+1} - \bar{c}_n = 0,$$

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the general solution of which, as we know, is of the form

$$\bar{c}_n = A\nu_1^n + B\nu_2^n,$$

where ν_1 and ν_2 are the roots of the characteristic equation

$$\nu^2 + \frac{k}{a}\nu - 1 = 0$$

respectively,

$$\nu_1 = -\frac{k}{2a} + \sqrt{\frac{k^2}{4a^2} + 1}, \quad \nu_2 = -\frac{k}{2a} - \sqrt{\frac{k^2}{4a^2} + 1}.$$

The root of ν_2 with respect to the modulus is greater than unity, and therefore we discard it; then $\bar{c}_n = A\nu_1^n$ when $n \geqslant 2$.

From the first two equations of our system we determine $\bar{c}_1$ and A:

$$\bar{c}_1 = \frac{ca^2}{a^2 + k^2 + ak\nu_1}, \quad A = -\frac{kac}{(a^2+k^2)\nu_1^2 + ka\nu_1^3}.$$

Finally, by summing the series for $\omega(\zeta)$, we find

$$z = \omega(\xi) = c\zeta + \frac{c_1}{\zeta} + \frac{\overline{A\nu_1^2}}{\zeta(\zeta - \overline{\nu_1})}. \qquad (V.85)$$

Thus, the problem of determining the boundary of the plastic range in the case of pure deflection can be considered as solved. /417

Let M/J = 28.8 kg/cm^3, R = 5 cm, 2h = 100 cm, p = 3k be given. Function (V.85) is

$$z = 13{,}6\zeta - \frac{0{,}133}{\zeta} + \frac{1{,}357i}{\zeta(\zeta + 0{,}1i)}.$$

In Figure V.3 curve 1, a circle of radius $c_0 = Re^{\frac{p-k}{2k}}$, is the boundary of the plastic zone around a round hole of radius R = 5 cm, on the contour of which is applied force p = 3k in the case where the stresses at infinity are equal to zero; curve 2 is the boundary of the plastic zone around the same hole, on the

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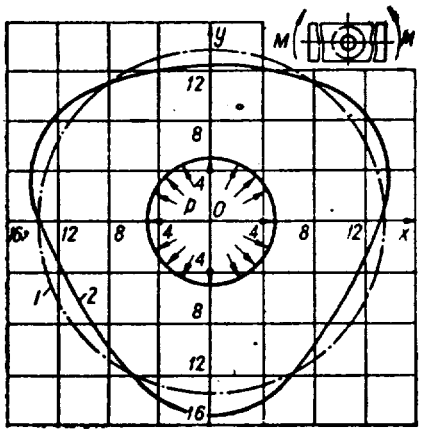


Figure V.3.

contour of which is applied the force[1] p = 3k for the stress state at infinity corresponding to pure deflection by moment M.

Deflection of Rod under Constant Shear Force. If polynomials (V.70) and (V.71) are represented in the form

$$\Phi_2^*(z) = i\frac{Q}{8J}z^2 - i\frac{Q(l-a)}{4J}z,$$

$$\Psi_2^*(z) = -i\frac{Q}{4J}z^2 + i\frac{Q(l-a)}{4J}z - i\frac{Qh^2}{2J},$$

then the stress state in the parts of the rod that are quite distant from the hole will correspond to deflection of a cantilever by constant shear force Q.

We introduce the definitions:

$$\alpha = i\frac{Qh^2}{2J}, \qquad \beta_1 = -\frac{Q(l-a)c}{4J}, \qquad \delta = i\frac{Qc^2}{4J};$$

whereupon the equation system obtained from (V.84) will acquire the form

$$\begin{aligned}
c\delta - k\bar{c}_1 - \beta\bar{c}_2 - \delta\bar{c}_3 &= \alpha c - c_1\delta,\\
\beta\bar{c}_1 - k\bar{c}_2 - \beta\bar{c}_3 - \delta\bar{c}_4 &= \beta c,\\
\delta\bar{c}_1 + \beta\bar{c}_2 - k\bar{c}_3 - \beta\bar{c}_4 - \delta\bar{c}_5 &= \delta c,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta\bar{c}_n + \beta\bar{c}_{n+1} - k\bar{c}_{n+2} - \beta\bar{c}_{n+3} - \delta\bar{c}_{n+4} &= 0.\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \qquad \text{(V.86)}$$

To solve equation system (V.86) we will examine the equation in finite differences /418

$$\delta\bar{c}_{n+4} + \beta\bar{c}_{n+3} + k\bar{c}_{n+2} - \beta\bar{c}_{n+1} - \delta\bar{c}_n = 0 \qquad (n = 2, 3, 4, \ldots),$$

[1]Since it is necessary to determine the effect deflecting moment M on the size of the plastic zone near the round hole, we will select the value of p at which the plastic zone will occur around the hole.

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the general solution of which is of the form

$$\bar{c}_n = k_1 \nu_1^n + k_2 \nu_2^n + k_3 \nu_3^n + k_4 \nu_4^n , \tag{V.87}$$

where k_1, k_2, k_3, k_4 are arbitrary constants; ν_1, ν_2, ν_3, ν_4 are the roots of the characteristic equation

$$\delta\nu^4 + \beta\nu^3 + k\nu^2 - \beta\nu - \delta = 0. \tag{V.88}$$

It can be proved that under certain conditions imposed on the coefficients of equation (V.88), the latter will have roots that satisfy the following inequalities:

$$|\nu_1| < 1; \quad |\nu_2| < 1;$$
$$|\nu_3| > 1; \quad |\nu_4| > 1.$$

Indeed, we write equation (V.88) in the form

$$\nu^4 + \frac{\beta}{\delta}\nu^3 + \frac{k}{\delta}\nu^2 - \frac{\beta}{\delta}\nu - 1 = 0$$

and assume $\beta/\delta = a$, $k/\delta = ib$. We obtain

$$\nu^4 + a\nu^3 + ib\nu^2 - a\nu - 1 = 0.$$

We write the expansion

$$\nu^4 + a\nu^3 + ib\nu^2 - a\nu - 1 = (\nu^2 + p\nu + q)(\nu^2 + p_1\nu + q_1).$$

Comparing the coefficients, we obtain

$$p + p_1 = a, \qquad q_1 p + q p_1 = -a,$$
$$q q_1 = -1, \qquad q_1 + q + p p_1 = ib.$$

Hence we obtain the following equation for q:

$$\left(q - \frac{1}{q}\right)\left(q^2 + 2 + \frac{1}{q^2}\right) - a^2\left(q - \frac{1}{q}\right) = ib\left(q^2 + 2 + \frac{1}{q^2}\right).$$

Assuming now that

$$q - \frac{1}{q} = iy,$$

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we obtain

$$-y^3+by^2+(4-a^2)y-4b=0.$$

By denoting the real root of the last equation through y_0, we readily see that /419 roots of equation (V.88) are

$$\nu_{1,2}=-\frac{a\left[1+i\frac{y_0}{2}+\sqrt{1-\left(\frac{y_0}{2}\right)^2}\right]}{4\sqrt{1-\left(\frac{y_0}{2}\right)^2}}\pm$$

$$\pm\sqrt{\frac{a^2\left(1+i\frac{y_0}{2}+\sqrt{1-\left(\frac{y_0}{2}\right)^2}\right)}{16\left[1-\left(\frac{y_0}{2}\right)^2\right]}-i\frac{y_0}{2}-\sqrt{1-\left(\frac{y_0}{2}\right)^2}};$$

$$\nu_{3,4}=\frac{a\left(1+i\frac{y_0}{2}-\sqrt{1-\left(\frac{y_0}{2}\right)^2}\right)}{4\sqrt{1-\left(\frac{y_0}{2}\right)^2}}\pm$$

$$\pm\sqrt{\frac{a^2\left(1+i\frac{y_0}{2}-\sqrt{1-\left(\frac{y_0}{2}\right)^2}\right)}{16\left[1-\left(\frac{y_0}{2}\right)^2\right]}-i\frac{y_0}{2}+\sqrt{1-\left(\frac{y_0}{2}\right)^2}},$$

but since y_0 can be quite large, we readily see that ν_1 and ν_2 fall within, while ν_3 and ν_4 fall outside of the unit circle.

Assuming in (V.87) that $k_3 = k_4 = 0$ (otherwise the series for $\omega(\zeta)$ in (V.77) would be divergent) we will have

$$\begin{aligned}\bar{c}_2&=k_1\nu_1^2+k_2\nu_2^2,\\ \bar{c}_3&=k_1\nu_1^3+k_2\nu_2^3,\\ &\cdots\cdots\cdots\\ &\cdots\cdots\cdots\\ \bar{c}_n&=k_1\nu_1^n+k_2\nu_2^n,\\ &\cdots\cdots\cdots\end{aligned}\qquad(V.89)$$

By substituting (V.89) into the second and third conditions of (V.86), we obtain

$$k_1=-\frac{\bar{c}_1-c}{\nu_2-\nu_1},$$

$$k_2=\frac{\bar{c}_1-c}{\nu_2-\nu_1}.\qquad(V.90)$$

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Then, by substituting (V.89) and (V.90) into the first condition of (V.86) we obtain the equation for the determination of c_1:

$$c_1\delta - \bar{c}_1[k + \beta(\nu_1 + \nu_2) + \delta(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)] = \\ = c\{\alpha - \delta - [\beta(\nu_1 + \nu_2) + \delta(\nu_1^2 + \nu_1\nu_2 + \nu_2^2)]\}.$$

Consequently, the function is /420

$$z = \omega(\zeta) = c\zeta + \frac{c_1}{\zeta} + \frac{\bar{k}_1\bar{\nu}_1^2}{\zeta(\zeta - \bar{\nu}_1)} + \frac{\bar{k}_2\bar{\nu}_2^2}{\zeta(\zeta - \bar{\nu}_2)}. \qquad (V.91)$$

We will examine deflection of a rod (beam) by constant shear force Q. Let l - a = 300 cm, 2h = 100 cm, p = 3k, R = 5 cm.

When Q = 400 kg the equation of the contour separating the plastic zone from the elastic zone will have the form

$$x \approx 13{,}56\cos\theta + 3{,}26\sin\theta - \frac{0{,}16\alpha + 0{,}68\beta}{\alpha^2 + \beta^2},$$

$$y \approx 13{,}64\sin\theta + 3{,}26\cos\theta - \frac{0{,}68\alpha - 0{,}16\beta}{\alpha^2 + \beta^2},$$

where

$$\alpha = \cos 2\theta + 0{,}05\sin\theta; \qquad \beta = \sin 2\theta - 0{,}05\cos\theta.$$

When Q = 600 kg the equation of contour L will have the form

$$x \approx 13{,}45\cos\theta + 5{,}23\sin\theta - \frac{0{,}73\alpha + 1{,}92\beta}{\alpha^2 + \beta^2},$$

$$y \approx 13{,}75\sin\theta + 5{,}23\cos\theta - \frac{1{,}92\alpha - 0{,}73\beta}{\alpha^2 + \beta^2},$$

where

$$\alpha = \cos 2\theta + 0{,}14\sin\theta; \qquad \beta = \sin\theta - 0{,}14\cos\theta.$$

In Figure V.4 curve 1, a circle of radius $c_0 = Re^{\frac{p-k}{2k}}$, is a boundary of the plastic zone near a round hole, upon the contour of which is applied force p = 3k and the stresses at infinity are equal to zero; curves 2 and 3 correspond to the boundaries of the plastic zone near a round hole, to the contour of which is applied force p = 3k and the stresses at infinity correspond to deflection of a cantilever beam (plane deformation) by constant shear force Q, equal to 4,000 and 6,000 kg, respectively.

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The case of normal and tangential forces applied to the contour of a round hole for given complex potentials (V.70) and (V.71) is presented here by the method described by I. Yu. Khoma [1].

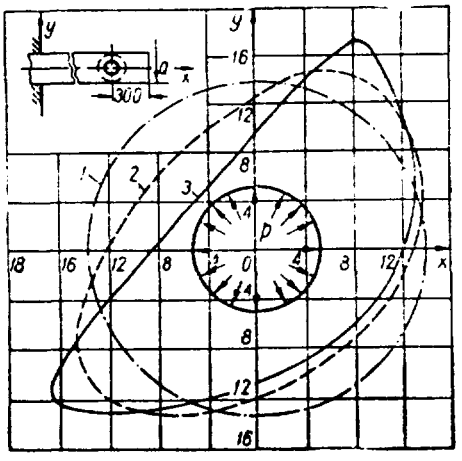


Figure V.4.

Inverse Elasto-Plastic Problem[1]. We will assume that an infinite plane under the conditions of plane deformation is weakened by a hole on the contour L of which are applied constant normal and tangential forces, and the stresses at infinity are nonlinear functions of the coordinates. The problem consists of the determination of the contour of the hole L under the condition that the plastic zone at the moment of origin encompasses immediately the entire hole L without penetrating into the interior of the body, i.e. only the contour of hole L is in the plastic state.

On an unknown contour L let normal and tangential forces

$$\sigma_n = p,\ \tau_{tn} = \tau, \tag{V.92}$$

where t and n are the directions of the lines tangent and normal to L.

We will assume that in the plastic zone the plasticity condition[2] /421

$$(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2 = 4k^2, \tag{V.93}$$

which can be written as

$$|\sigma_y - \sigma_x + 2i\tau_{xy}| = 2k. \tag{V.94}$$

is valid. Since the magnitude of the left hand side of (V.94) is invariant relative to the coordinate axes at the given point of the body, from condition (V.93), with consideration of (V.92), we have the relation

$$\sigma_t = p \pm 2\sqrt{k^2 - \tau^2}, \tag{V.95}$$

which is valid for all points of the contour L. The "plus" or "minus" sign in front of the radical in (V.94) is selected from physical considerations.

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[1]The solution is given by G. P. Cherepanov [1].
[2]See L. M. Kachanov [1].

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The stress components in elastic region S (outside of contour L) are expressed through two analytical functions of N. I. Muskhelishvili $\Phi^*(z)$, $\Psi^*(z)$ according to formulas

$$\sigma_x^{(2)} + \sigma_y^{(2)} = 4\mathrm{Re}\,\Phi^*(z),$$

$$\sigma_y^{(2)} - \sigma_x^{(2)} + 2i\tau_{xy}^{(2)} = 2\left[\bar{z}\Phi^{*\prime}(z) + \Psi^*(z)\right]. \tag{V.96}$$

The functions at infinity are

$$\begin{aligned} \Phi^*(z) &= a_0 + a_1 z, \\ \Psi^*(z) &= b_0 + b_1 z, \end{aligned} \tag{V.97}$$

where a_0, a_1, b_0, b_1 are known complex constants.

We will map the interior of contour L onto the exterior of unit circle γ of plane ζ with the aid of the function

$$z = \omega(\zeta) = c\zeta + \sum_{k=1}^{\infty} c_k \zeta^{-k}. \tag{V.98}$$

If we assume that

$$\Phi^*[\omega(\zeta)] = \Phi(\zeta), \qquad \Psi^*[\omega(\zeta)] = \Psi(\zeta),$$

then function (V.97), with consideration of (V.98) when $\zeta \to \infty$ acquire the form

$$\Phi(\zeta) = a_0 + a_1 c\zeta; \tag{V.99}$$

$$\Psi(\zeta) = b_0 + b_1 c\zeta. \tag{V.100}$$

We will make use of the following formulas: /422

$$\begin{aligned} \sigma_t - \sigma_n + 2i\tau_{nt} &= \frac{\zeta^2}{\rho^2}\cdot\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\left[\sigma_y - \sigma_x + 2i\tau_{xy}\right], \\ \sigma_t + \sigma_n &= \sigma_x + \sigma_y. \end{aligned} \tag{V.101}$$

From relation (V.101) with consideration of conditions (V.92) and (V.96), we obtain the boundary conditions on the contour of the unit circle:

$$\mathrm{Re}\,\Phi(\zeta) = a \text{ for } |\zeta| = 1; \tag{V.102}$$

$$\frac{\zeta^2\omega'(\zeta)}{\overline{\omega'(\zeta)}}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi'(\zeta) + \Psi(\zeta)\right] = b \text{ for } |\zeta| = 1, \tag{V.103}$$

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where

$$2a = p \pm \sqrt{k^2 - \tau^2}, \qquad b = \pm \sqrt{k^2 - \tau^2} + i\tau. \tag{V.104}$$

Thus, the solution of the problem consists in the determination of functions $\Phi(\zeta)$ and $\Psi(\zeta)$ on the basis of boundary conditions (V.102) and (V.103) with consideration of relations (V.99) and (V.100).

Conditions (V.102) and (V.99) will be satisfied if the function is

$$\Phi(\zeta) = a + a_1 c\zeta - \bar{a}c\frac{1}{\zeta}. \tag{V.105}$$

Then the equality

$$a_0 = a \tag{V.106}$$

will be valid.

Consequently, the forces on the contour of the hole and at infinity cannot be assigned arbitrarily; they must be related by relations (V.106).

Recalling that on the contour of the unit circle

$$\bar{\zeta} = \bar{\sigma} = \frac{1}{\sigma} + \frac{1}{\zeta},$$

we rewrite condition (V.103) as follows:

$$b\bar{\omega}'\left(\frac{1}{\zeta}\right) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) = \zeta^2\omega'(\zeta)\Psi(\zeta) \quad \text{on } \gamma. \tag{V.107}$$

We denote the left hand side of equation (V.107) through

$$F^+(\zeta) = b\bar{\omega}'\left(\frac{1}{\zeta}\right) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta). \tag{V.108}$$

Then, on the basis of formulas (V.98) and (V.105), it is easy to see that the function $F^+(\zeta)$ will be analytic inside of the circle $|\zeta| < 1$ except for the point $\zeta = 0$, at which it will have a pole with the principal part $-a_1c^2\zeta^{-1}$.

If the function $\Psi(\zeta)$ is represented in the form

$$\Psi(\zeta) = b_1 c\zeta + b_0 + \sum_{k=1}^{\infty} b_{-k}\zeta^{-k}, \tag{V.109}$$

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in accordance with condition (V.100), and if we denote the right hand side of equation (V.107) through /423

$$F^{-}(\zeta) = \zeta^2 \omega'(\zeta)\, \Psi(\zeta), \tag{V.110}$$

then the function $F^{-}(\zeta)$ will be analytic outside of the circle $|\zeta| > 1$, except at an infinitely remote point, at which

$$F^{-}(\zeta) = b_1 c^2 \zeta^3 + b_0 c \zeta^2 + c(b_{-1} - b_1 c_1)\zeta + c b_{-2} - 2 b_1 c c_2 - b_0 c_1 + 0(\zeta^{-1}). \tag{V.111}$$

Thus the funcitons $F^{+}(\zeta)$ and $F^{-}(\zeta)$, on the basis of boundary condition (V.107), are the analytical extension of each other through the unit circle:

$$F^{+}(\zeta) = F^{-}(\zeta) \text{ on } \gamma. \tag{V.112}$$

Hence according to J. Liouville's theorem, we have the single analytical function

$$F(\zeta) = -\bar{a}_1 c^2 \zeta^{-1} + c b_{-2} - 2 b_1 c c_2 - b_0 c_1 + c(b_{-1} - b_1 c_1)\zeta + b_0 c \zeta^2 + b_1 c^2 \zeta^3. \tag{V.113}$$

From condition (V.110), according to (V.113), we obtain the function

$$\Psi(\zeta) = \frac{F(\zeta)}{\zeta^2 \omega'(\zeta)}. \tag{V.114}$$

To determine the function $\omega(\zeta)$, we expand relation (V.108) with consideration of (V.113) into a series and compare the coefficients for identical positive powers of ζ. We obtain an infinite system of linear algebraic equations relative to coefficients c_k:

$$\begin{aligned}
b c &= c b_{-2} - 2 b_1 c c_2 - b_0 c_1, \\
a_1 c + \bar{a}_1 \bar{c}_1 &= -b_{-1} + b_1 c_1, \\
b \bar{c}_1 + \bar{a}_1 c \bar{c}_2 &= -b_0 c, \\
2 b \bar{c}_2 + a_1 c \bar{c}_1 + \bar{a}_1 c \bar{c}_3 &= -b_1 c^2, \\
a_1 c \bar{c}_4 + 3 b \bar{c}_3 + a_1 c \bar{c}_2 &= 0, \\
a_1 c \bar{c}_5 + 4 b \bar{c}_4 + a_1 c \bar{c}_3 &= 0, \\
&\cdots\cdots\cdots \\
a_1 c c_{n+2} + (n+1) b \bar{c}_{n+1} + a_1 c \bar{c}_n &= 0, \\
&\cdots\cdots\cdots
\end{aligned} \tag{V.115}$$

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In the particular case, where constant forces are given at infinity, i.e. /424
$a_1 = b_1 = 0$, the solution of system (V.115) will be of the form

$$c_1 = -\frac{\bar{b}_0}{\bar{b}}c, \quad c_2 = c_3 = \ldots = 0, \quad b_{-1} = 0, \quad b_{-2} = \frac{b\bar{b} - b_0\bar{b}_0}{\bar{b}}, \tag{V.116}$$

where the constant c is arbitrary. Hence, according to (V.98), (V.105) and (V.114),

$$\omega(\zeta) = c\left(\zeta - \frac{\bar{b}_0}{b}\cdot\frac{1}{\zeta}\right), \quad \Phi(\zeta) = a,$$
$$\Psi(\zeta) = \frac{b_0\bar{b}\zeta^2 + b\bar{b}}{\bar{b}\zeta^2 + \bar{b}_0}, \tag{V.117}$$

where $a = a_0$. As seen from (V.117), the desired contours of the hole represent a set of ellipses.

In the case where $a_1 \neq 0$ or $b_1 \neq 0$, the solution of infinite equation system (V.115) is equivalent to the solution of the equation in finite differences[1]. As shown by G. P. Cherepanov [1], the contours of the hole in this case too will represent a set of ellipses.

In the usual direct statement, the solutions of elasto-plastic problems (see §1-3) boil down to the determination of the boundary of the plastic zone and the stress component in the elastic range on the basis of known forces on the contour of the hole and at infinity: $\sigma_x^{(\infty)} = A$, $\sigma_y^{(\infty)} = B$. By altering these values in a certain manner, it is possible to find two parametric sets of contours L, separating the plastic and elastic zones. Thus, for two, generally speaking, different points on the exterior of the hole, there are stresses at infinity such that boundary L defined by them will pass through these points. Noting this, P. I. Perlin [1, 2] proposed an approximation method for the solution of elasto-plastic problems in such a round about statement, the essence of which is that the position of some two points of boundary L are assumed to be known, and that the stresses at infinity and contour L itself are determined during the process of the solution of the problem. On the basis of this procedure Yu. I. Solodilov [1] analyzed the elasto-plastic problem for a plate with an elliptical hole. V. S. Sazhin [1, 2], using this method, attempted to determine the boundaries of the plastic zone occurring around a square (with rounded corners), oval and arch-shaped holes.

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[1]See A. O. Gel'fand [1].

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§4. Plastic Zones near Round Hole in Plate under Biaxial Tension (Rigorous Solution[1])

We will examine an infinite thin plate in a plane stress state, with a round hole of radius R. We will place the origin of coordinate system xOy (Figure V.5) at the center of the round hole, using the middle surface of the plate as the xOy plane.

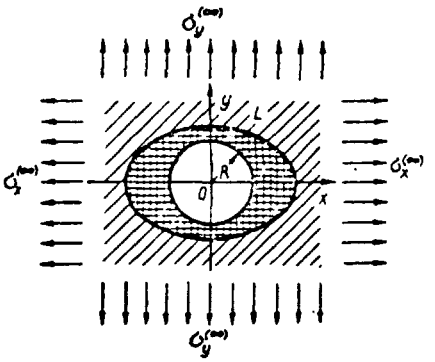


Figure V.5.

Let a constant normal load be appled to the contour of the hole, i.e. for r = R

$$\sigma_r = p, \qquad \tau_{r\theta} = 0, \tag{V.118}$$

and let the stresses

$$\sigma_x^{(\infty)} = A, \qquad \sigma_y^{(\infty)} = B. \tag{V.119}$$

be known at infinity. It is assumed that in the given system of external forces (V.118) and (V.119), the plastic zone encompasses the entire hole, and the stresses in the plastic range are determined by the shape of the contour of the hole and the boundary of the load, and do not depend on the stress state in the elastic range. Then, the stress components in the plastic range are found from equilibrium equations

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\cdot\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\cdot\frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0 \end{aligned} \tag{V.120}$$

and from the plasticity condition[2]. As the plasticity condition we will use

$$\sigma_\theta = \sigma_s, \tag{V.121}$$

where σ_s is the yield point of the material under simple tension.

We further assume that the inequality $\sigma_\theta^{(1)} \geqslant \sigma_r^{(1)} > 0$ is in effect in the plastic zone.

It is easy to check that equilibrium equations (V.120) and plasticity condition (V.121) under boundary conditions (V.118) will be satisfied if the stress components are taken in the form

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[1]The solution of the problem is given by G. P. Cherepanov [2].
[2]See V. V. Sokolovskiy [1].

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$$\sigma_r^{(1)} = \sigma_s + (p - \sigma_s)\frac{R}{r}, \qquad \sigma_\theta^{(1)} = \sigma_s, \qquad \tau_{r\theta}^{(1)} = 0. \tag{V.122}$$

It is clear from the solution of (V.122) that in order for the inequality /426
$\sigma_\theta^{(1)} \geqslant \sigma_r^{(1)} > 0$ to be satisfied, load p must satisfy condition $p \leqslant \sigma_s$.

For convenience we will change from polar coordinates to Cartesian coordinates, writing the stress components (V.122) in the following combinations:

$$\begin{aligned} \sigma_x^{(1)} + \sigma_y^{(1)} &= 2\sigma_s + \frac{(p - \sigma_s)R}{r}, \\ \sigma_y^{(1)} - \sigma_x^{(1)} + 2i\tau_{xy}^{(1)} &= \frac{(\sigma_s - p)R}{r} e^{-2i\theta}. \end{aligned} \tag{V.123}$$

The stress components in the elastic range are expressed through two holomorphic functions of N. I. Muskhelishvili, $\Phi^*(z)$ and $\Psi(z)$, on the basis of formulas (V.96). Then, on contour L, which divides the elastic and plastic zones, we will have the following boundary conditions:

$$\begin{aligned} 4\,\mathrm{Re}\,\Phi^*(z) &= 2\sigma_s + \frac{(p - \sigma_s)R}{r}, \\ 2[\bar{z}\Phi^{*\prime}(z) + \Psi^*(z)] &= \frac{(\sigma_s - p)R}{r} e^{-2i\theta}. \end{aligned} \tag{V.124}$$

We will map the exterior of contour L onto the exterior of unit circle γ of plane ζ with the aid of function $\omega(\zeta)$ of (V.129). After introducing the definitions

$$\Phi^*[\omega(\zeta)] = \Phi(\zeta), \qquad \Psi^*[\omega(\zeta)] = \Psi(\zeta),$$

we represent boundary conditions (V.124) in the following form:

$$4\,\mathrm{Re}\,\Phi(\zeta) = 2\sigma_s + \frac{(p - \sigma_s)R}{|\omega(\zeta)|} \quad \text{on } \gamma; \tag{V.125}$$

$$2\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi'(\zeta) + \Psi(\zeta)\right] = \frac{(\sigma_s - p)R\overline{\omega(\zeta)}}{\omega(\zeta)|\omega(\zeta)|} \quad \text{on } \gamma. \tag{V.126}$$

In accordance with (V.119) the functions at infinity are

$$\Phi(\zeta) = \frac{1}{4}(A + B) + 0(\zeta^{-2}), \qquad \Psi(\zeta) = \frac{1}{2}(B - A) + 0(\zeta^{-2}). \tag{V.127}$$

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Thus, the problem reduces to the determination, on the basis of boundary conditions (V.125) and (V.126) with consideration of relations (V.127), of functions $\Phi(\zeta)$, $\Psi(\zeta)$ and $\omega(\zeta)$.

From conditions (V.126) it follows that the solution of the problem requires analysis of auxiliary functional equation

$$\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\Phi'(\zeta)+\Psi(\zeta)=\frac{(\sigma_s-p)R\overline{\omega}\left(\frac{1}{\zeta}\right)}{2\omega(\zeta)\sqrt{\omega(\zeta)\overline{\omega}\left(\frac{1}{\zeta}\right)}}\quad(|\zeta|>1), \tag{V.128}$$

the solution of which (relative to function $\omega(\zeta)$) will be found in the form /427

$$\omega(\zeta)=c\zeta+\overline{p}_n\left(\frac{1}{\zeta}\right), \tag{V.129}$$

where $\overline{p}_n$ is a polynomial of the n-th power with as yet undefined coefficients. By substituting the solution of (V.129) into equation (V.128) and combining all functions into a series in the vicinity of an infinitely remote point, we see that n = 3. Consequently, the function is

$$\omega(\zeta)=c\zeta+\frac{c_1}{\zeta}+\frac{c_2}{\zeta^2}+\frac{c_3}{\zeta^3}. \tag{V.130}$$

From the conditions of symmetry we know that the constant c_2 is equal to zero, and the other constants are real. Consequently, the functions are

$$\omega(\zeta)=\zeta\left(c+c_1\frac{1}{\zeta^2}+c_3\frac{1}{\zeta^4}\right),$$
$$\overline{\omega}\left(\frac{1}{\zeta}\right)=\frac{1}{\zeta}(c+c_1\zeta^2+c_3\zeta^4). \tag{V.131}$$

Obviously, the function $\omega(\zeta)$ has four zeros, located within the unit circle, whereas the function $\overline{\omega}(1/\zeta)$ has four zeros located outside of the unit circle. The constants c, c_1 and c_3 can be selected such that the right hand side of the functional equation (V.128) will be analytical outside of the unit circle, and for this purpose it is necessary and sufficient to require that the zeros of function $\overline{\omega}(1/\zeta)$ coincide in pairs. For the latter condition it is sufficient that the discriminant of the biquadratic equation be equal to zero:

$$c_3\zeta^4+c_1\zeta^2+c=0.$$

Hence

$$c_1^2=4cc_3. \tag{V.132}$$

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Then functions (V.131), under condition (V.132), can be written in the form

$$\omega(\zeta) = \frac{c}{\zeta^3}\left(\zeta^2 + \frac{c_1}{2c}\right)^2, \qquad \overline{\omega}\left(\frac{1}{\zeta}\right) = \frac{c_3}{\zeta}\left(\zeta^2 + \frac{c_1}{2c_3}\right)^2. \tag{V.133}$$

On the basis of functions (V.133), boundary condition (V.125) is

$$4\mathrm{Re}\,\Phi(\zeta) = 2\sigma_s + \frac{(p-\sigma_s)R\zeta^2}{\sqrt{cc_3}\left(\zeta^2 + \frac{c_1}{2c_3}\right)\left(\zeta^2 + \frac{c_1}{2c}\right)}. \tag{V.134}$$

We will introduce the function

$$F^+(\zeta) = -2\overline{\Phi}\left(\frac{1}{\zeta}\right) + \frac{2R(p-\sigma_s)\sqrt{cc_3}}{c_1(c_3-c)} \cdot \frac{\zeta^2}{\zeta^2 + \frac{c_1}{2c_3}}, \tag{V.135}$$

which, as is obvious, will be analytic within unit circle $|\zeta| < 1$; however, the /428 function

$$F^-(\zeta) = 2\Phi(\zeta) - 2\sigma_s + \frac{2R(p-\sigma_s)\sqrt{cc_3}}{c_1(c_3-c)} \cdot \frac{\zeta^2}{\zeta^2 + \frac{c_1}{2c}} \tag{V.136}$$

will be analytic universally outside of unit circle $|\zeta| > 1$.

It is not difficult to see that boundary condition (V.134) with consideration of relations (V.135) and (V.136) can be represented in the form

$$F^+(\zeta) = F^-(\zeta) \quad \text{on } \gamma. \tag{V.137}$$

Thus, functions $F^+(\zeta)$ and $F^-(\zeta)$ are analytic extensions of each other through the unit circle. According to J. Liouville's theory they are identically equal to the same constant. Hence, and also from the conditions at infinity (V.127),

$$\Phi(\zeta) = \sigma_s - \frac{A+B}{4} - \frac{(p-\sigma_s)R\sqrt{cc_3}}{c_1(c_3-c)} \cdot \frac{\zeta^2}{\zeta^2 + \frac{c_1}{2c}}. \tag{V.138}$$

So that conditions (V.127) that go into (V.138) will be satisfied, the constants must satisfy the relation

$$A + B - 2\sigma_s + \frac{2(p-\sigma_s)R\sqrt{cc_3}}{c(c_3-c)} = 0. \tag{V.139}$$

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From functional equation (V.128) we find the function

$$\Psi(\zeta) = \frac{(\sigma_s - p) R\overline{\omega}\left(\frac{1}{\zeta}\right)}{2\omega(\zeta)\sqrt{\omega(\zeta)\,\overline{\omega}\left(\frac{1}{\zeta}\right)}} - \frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\Phi'(\zeta). \tag{V.140}$$

By substituting into (V.140) the values of the functions (V.133) and $\Phi(\zeta)$ (V.138) and also considering the conditions at infinity (V.127), we find the following relationship between the coefficients

$$B - A = \frac{(\sigma_s - p)R\sqrt{c_3}}{c\sqrt{c}} \cdot \frac{c + c_3}{c - c_3}. \tag{V.141}$$

Thus, coefficients c, c_1, c_3 are found from a system of three equations, (V.132), (V.139) and (V.141), and functions $\Phi(\zeta)$, $\Psi(\zeta)$ and $\omega(\zeta)$ are defined by formulas (V.138), (V.140) and (V.133), respectively.

The solution of equation system (V.132), (V.139), (V.141) can be represented in the form

$$c = \frac{4R}{\alpha(a^2 - 4)}, \qquad c_1 = \frac{4aR}{\alpha(a^2 - 4)}, \qquad c_3 = \frac{a^2 R}{\alpha(a^2 - 4)}, \tag{V.142}$$

where $\alpha = \frac{A + B - 2\sigma_s}{\sigma_s - p}$ a is real root of cubic equation /429

$$a^3 + 4a + \beta = 0. \tag{V.143}$$

Here

$$\beta = \frac{8(B - A)}{A + B - 2\sigma_s}.$$

Thus, considering (V.142), we find the functions

$$\begin{aligned} \omega(\zeta) &= \frac{R(2\zeta^2 + a)^2}{\alpha(a^2 - 4)\zeta^3}, \\ \Phi(\zeta) &= \sigma_s - \frac{A + B}{4} - \frac{\alpha(p - \sigma_s)\zeta^2}{2\zeta^2 + a}, \\ \Psi(\zeta) &= \alpha(p - \sigma_s)\zeta^4 \frac{(a\zeta^2 + a)[2(a^2 + 4)\zeta^2 - a(4 - 3a^2)]}{2(2\zeta^2 + a)^2(2\zeta^2 + a)(2\zeta^2 - 3a)} \end{aligned} \tag{V.144}$$

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The solution of the problem can be regarded as complete. However, for thoroughness, we will determine the intervals of the known external forces for which the solutions we have obtained are valid. For definition, we will assume that $B \geqslant A \leqslant 0$ and moreover, that $B \leqslant \sigma_s$ must be satisfied. Hence

$$\alpha < 0, \qquad \beta < 0, \qquad \alpha > 0.$$

Tangency of the hole by the plastic zone will occur at the points +i and -i. Thus, for the total enclosure of the round hole by the plastic zone, it is necessary that inequality $iR < \omega(i)$ be satisfied. Therefore parameter a should satisfy the inequality

$$a < 2\frac{1+\alpha}{1-\alpha}.$$

Then

$$0 < a < 2\frac{1+\alpha}{1-\alpha}, \qquad -1 < \alpha < 0 \qquad (0 < a < 2). \tag{V.145}$$

In order that the function $\omega(\zeta)$ map conformally the exterior of contour L onto the exterior of unit circle γ, it is necessary that the derivative

$$\omega'(\zeta) = \frac{R(2\zeta^2 + a)(2\zeta^2 - 3a)}{a(a^2 - 4)\zeta^4} \tag{V.146}$$

not be equal to zero anywhere. Consequently, as seen from (V.146), the parameter a should satisfy inequality $0 < a < 2/3$. Hence, and from (V.146), we obtain $-1 < \alpha < -1/2$. In the particular case, where A = B, the analysis of this elasto-plastic problem is given by G. Yu. Dzhanelidze [1] for two cases: when either the Mises or Saint Venant plasticity conditions occur in the plastic zone. /430

On the basis of V. V. Sokolovskiy's elementary solution [1] for an axisymmetric infinite plate with a round hole, V. Panc [1] analyzes the distribution of stresses and deformations for various stages of loading.

§5. Plastic Zones near Curvilinear Holes (Approximate Solution)

As we have learned from preceding sections, the solutions of plane elasto-plastic problems in closed form pertain basically to ranges with a round hole. As concerns ranges with nonround holes, no general methods for the solution of problems in closed form have yet been developed. There are several approximation methods, one of which is the small parameter method[1].

[1] See D. D. Ivlev [1-4].

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We will represent stress components and static boundary conditions in the form of series by degrees of a small parameter:

$$\sigma_r = \sum_{k=0}^{\infty} \varepsilon^k \sigma_r^{(k)}, \quad \sigma_\theta = \sum_{k=0}^{\infty} \varepsilon^k \sigma_\theta^{(k)}, \quad \tau_{r\theta} = \sum_{k=0}^{\infty} \varepsilon^k \tau_{r\theta}^{(k)}, \tag{V.147}$$

where ε is a dimensionless small parameter characterizing deviation of contour of a hole from round.

Stress components σ_r, σ_θ, $\tau_{r\theta}$ (for convenience we will use polar coordinates) in the entire range, i.e. in both the elastic and in the plastic ranges, should satisfy equilibrium equations (V.120). Moreover, they should satisfy the compatibility condition in the elastic range and the plasticity condition in the plastic range. As the plasticity condition for plane deformation, we will use the condition

$$\frac{1}{4}(\sigma_r - \sigma_\theta)^2 + \tau_{r\theta}^2 = 1. \tag{V.148}$$

Here and in the following stress components pertain to the constant in the right hand side of the plasticity condition. For the plane stress state, the plasticity condition will be written in the form

$$\frac{1}{4}(\sigma_r - \sigma_\theta)^2 + \tau_{r\theta}^2 = 1 \text{ for } \sigma_r\sigma_\theta < \tau_{r\theta}^2; \tag{V.149}$$

$$\frac{1}{4}(\sigma_r - \sigma_\theta)^2 + \tau_{r\theta}^2 = \left(1 - \frac{1}{2}|\sigma_r + \sigma_\theta|\right)^2 \text{ for } \sigma_r\sigma_\theta \geqslant \tau_{r\theta}^2. \tag{V.150}$$

We will consider that the inequality $\sigma_\theta > \sigma_r > 0$ corresponds to condition (V.150) in the plastic range. /431

By substituting the expansions of (V.147) into plasticity condition (V.148) and equating the terms for identical powers of parameter ε for $\tau_{r\theta}^{(0)} = 0$, we obtain

$$\begin{aligned}
\sigma_r^{(0)} - \sigma_\theta^{(0)} &= 2\mu, \\
\sigma_r^{(1)} - \sigma_\theta^{(1)} &= 0, \\
(\sigma_r^{(2)} - \sigma_\theta^{(2)})\mu + \tau_{r\theta}^{(1)2} &= 0, \\
(\sigma_r^{(3)} - \sigma_\theta^{(3)})\mu + 2\tau_{r\theta}^{(1)}\tau_{r\theta}^{(2)} &= 0, \\
(\sigma_r^{(4)} - \sigma_\theta^{(4)})\mu + \frac{1}{4}(\sigma_r^{(2)} - \sigma_\theta^{(2)})^2 + 2\tau_{r\theta}^{(1)}\tau_{r\theta}^{(3)} + \tau_{r\theta}^{(2)2} &= 0, \\
&\cdots\cdots\cdots

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where $\mu = \text{sing}\,(\sigma_r^{(0)} - \sigma_\theta^{(0)})$. For the plane stress state, plasticity condition (V.149) is written in the form (V.151), and condition (V.150) for $\tau_{r\theta}^{(0)} = 0$ acquires the following form:

$$\begin{aligned}
\sigma_\theta^{(0)} = 1, \quad \sigma_\theta^{(1)} = 0,\\
(1 - \sigma_r^{(0)})\,\sigma_\theta^{(2)} + \tau_{r\theta}^{(1)2} = 0,\\
(1 - \sigma_r^{(0)})\,\sigma_\theta^{(3)} - \sigma_r^{(1)}\sigma_\theta^{(2)} + 2\tau_{r\theta}^{(1)}\tau_{r\theta}^{(2)} = 0,\\
(1 - \sigma_r^{(0)})\,\sigma_\theta^{(4)} - \sigma_r^{(1)}\sigma_\theta^{(3)} - \sigma_r^{(2)}\sigma_\theta^{(2)} + 2\tau_{r\theta}^{(1)}\tau_{r\theta}^{(3)} + \tau_{r\theta}^{(2)2} = 0,\\
\cdots\cdots\cdots

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$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\varepsilon^{k+m}\frac{d^m\tau_{tn}^{(k)}}{dr^m}\frac{(r_1+\varepsilon r_2+\ldots)^m}{m!}=\sum_{m=0}^{\infty}\varepsilon^m\frac{d^m\tau}{dr^m}\frac{(r_1+\varepsilon r_2+\ldots)^m}{m!}. \tag{V.158}$$

From condition (V.157), after equating the terms for identical powers of parameter ε, we obtain

$$\begin{aligned}
&\sigma_n^{(1)}+\frac{d\sigma_n^{(0)}}{dr}r_1=\frac{dp}{dr}r_1;\\
&\sigma_n^{(2)}+\frac{d\sigma_n^{(1)}}{dr}r_1+\frac{d^2\sigma_n^{(0)}}{dr^2}\cdot\frac{r_1^2}{2!}+\frac{d\sigma_n^{(0)}}{dr}r_2=\frac{d^2p}{dr^2}\frac{r_1^2}{2}+\frac{dp}{dr}r_2;\\
&\sigma_n^{(3)}+\frac{d\sigma_n^{(2)}}{dr}r_1+\frac{d^2\sigma_n^{(1)}}{dr^2}\cdot\frac{r_1^2}{2!}+\frac{d^3\sigma_n^{(0)}}{dr^3}\frac{r_1^3}{3!}+\frac{d\sigma_n^{(0)}}{dr}r_3+\\
&+\frac{d\sigma_n^{(1)}}{dr}r_2+\frac{d^2\sigma_n^{(0)}}{dr^2}r_1r_2=\frac{d^3p}{dr^3}\frac{r_1^3}{3!}+\frac{d^2p}{dr^2}r_1r_2+\frac{dp}{dr}r_3;
\end{aligned} \tag{V.159}$$

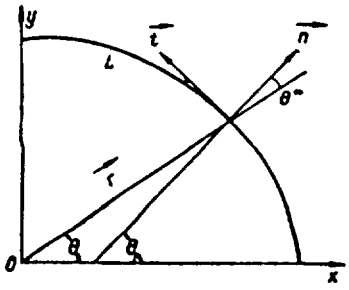


Figure V.6.

Analogously, we find the linearized boundary conditions (V.154) for σ_n and τ_{tn} on the basis of the formulas of elasticity theory

$$\begin{aligned}
\sigma_n&=\sigma_r\cos^2\theta^*+\sigma_\theta\sin^2\theta^*+2\tau_{r\theta}\sin\theta^*\cos\theta^*,\\
\tau_{tn}&=(\sigma_\theta-\sigma_r)\sin\theta^*\cos\theta^*+\tau_{r\theta}(\cos^2\theta^*-\sin^2\theta^*),
\end{aligned} \tag{V.160}$$

where $\theta^* = \theta_1 - \theta$ is the angle shown in Figure V.6.

As a result of transformations, the linearized boundary conditions will acquire the form

$$\begin{aligned}
&\sigma_r^{(1)}+\frac{d\sigma_r^{(0)}}{dr}r_1=\frac{dp}{dr}r_1;\\
&\tau_{r\theta}^{(1)}-(\sigma_\theta^{(0)}-\sigma_r^{(0)})\dot{\varrho}_1=\frac{d\tau}{dr}r_1;\\
&\sigma_r^{(2)}+\frac{d\sigma_r^{(1)}}{dr}r_1+\frac{d^2\sigma_r^{(0)}}{dr^2}\frac{r_1^2}{2!}+\frac{d\sigma_r^{(0)}}{dr}r_2+(\sigma_\theta^{(0)}-\sigma_r^{(0)})\dot{\varrho}_1^2-\\
&-2\tau_{r\theta}^{(1)}\dot{\varrho}_1=\frac{d^2p}{dr^2}\cdot\frac{r_1^2}{2!}+\frac{dp}{dr}r_2;\\
&\tau_{r\theta}^{(2)}-(\sigma_\theta^{(0)}-\sigma_r^{(0)})(\dot{\varrho}_2-\varrho_1\dot{\varrho}_1)-(\sigma_\theta^{(1)}-\sigma_r^{(1)})\dot{\varrho}_1+\frac{d}{dr}[\tau_{r\theta}^{(1)}-\\
&-(\sigma_\theta^{(0)}-\sigma_r^{(0)})\dot{\varrho}_1]r_1=\frac{d^2\tau}{dr^2}\frac{r_1^2}{2!}+\frac{d\tau}{dr}r_2;
\end{aligned}$$

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$$\sigma_r^{(3)} + \frac{d\sigma_r^{(2)}}{dr} r_1 + \frac{d^2\sigma_r^{(1)}}{dr^2} \frac{r_1^2}{2!} + \frac{d^3\sigma_r^{(0)}}{dr^3} \frac{r_1^3}{3!} + 2(\sigma_\theta^{(0)} - \sigma_r^{(0)})(\dot{\varrho}_1\dot{\varrho}_2 - \varrho_1\dot{\varrho}_1^2) +$$

$$+ \frac{d(\sigma_\theta^{(0)} - \sigma_r^{(0)})}{dr} \dot{\varrho}_1^2 r_1 + (\sigma_\theta^{(1)} - \sigma_r^{(1)})\dot{\varrho}_1^2 - 2\tau_{r\theta}^{(1)}(\dot{\varrho}_2 - \varrho_1\dot{\varrho}_1) - 2\frac{d\tau_{r\theta}^{(1)}}{dr} r_1\dot{\varrho}_1 -$$

$$- 2\tau_{r\theta}^{(2)}\dot{\varrho}_1 + \frac{d\sigma_r^{(1)}}{dr} r_2 + \frac{d\sigma_r^{(0)}}{dr} r_1 r_2 + \frac{d\sigma_r^{(0)}}{dr} r_3 = \frac{d^3p}{dr^3} \cdot \frac{r_1^3}{3!} + \frac{d^2p}{dr^2} r_1 r_2 + \frac{dp}{dr} r_3;$$

$$\tau_{r\theta}^{(3)} - 2\tau_{r\theta}^{(1)}\dot{\varrho}_1^2 - (\sigma_\theta^{(0)} - \sigma_r^{(0)})(\dot{\varrho}_3 - \varrho_1\dot{\varrho}_2 + \varrho_1^2\dot{\varrho}_1 - \dot{\varrho}_1\varrho_2 - \dot{\varrho}_1^3) - \qquad \text{(V.161)} \quad \underline{/433}$$

$$- (\sigma_\theta^{(1)} - \sigma_r^{(1)})(\dot{\varrho}_2 - \varrho_1\dot{\varrho}_1) - (\sigma_\theta^{(2)} - \sigma_r^{(2)})\dot{\varrho}_1 + \frac{d}{dr}[\tau_{r\theta}^{(2)} - (\sigma_\theta^{(0)} - \sigma_r^{(0)}) \times$$

$$\times (\dot{\varrho}_2 - \varrho_1\dot{\varrho}_1) - (\sigma_\theta^{(1)} - \sigma_r^{(1)})\dot{\varrho}_1] r_1 + \frac{d^2}{dr^2}[\tau_{r\theta}^{(1)} - (\sigma_\theta^{(0)} - \sigma_r^{(0)})\dot{\varrho}_1] \frac{r_1^2}{2} +$$

$$+ \frac{d}{dr}[\tau_{r\theta}^{(1)} - (\sigma_\theta^{(0)} - \sigma_r^{(0)})\dot{\varrho}_1] r_2 = \frac{d^3\tau}{dr^3} \frac{r_1^3}{3!} + \frac{d^2\tau}{dr^2} r_1 r_2 + \frac{d\tau}{dr} r_3.$$

Here $\rho_i = r_i/r_0$, and the dot overhead, i.e. $\rho = d\rho/d\theta$, denotes differentiation with respect to θ.

From the continuity of stress components on the common boundary L_s, which divides the elastic and plastic zones, we have

$$[\sigma_r] = [\sigma_\theta] = [\tau_{r\theta}] = 0 \text{ on } L_s, \qquad \text{(V.162)}$$

where the increments of the respective components enclosed in the brackets, during transition through boundary L_s, are denoted through [...]. If the contour equation is represented in the form

$$L_S = \sum_{k=0} \varepsilon^k r_{ks}(\theta), \qquad \text{(V.163)}$$

then, to find the conjugation conditions it is necessary to substitute in (V.159), σ_n by $\sigma_r^{(k)}$, $\sigma_\theta^{(k)}$, $\tau_{r\theta}^{(k)}$, respectively, and r_k by r_{ks}, and to enclose them within the brackets and equate them to zero. When $\tau_{r\theta}^{(k)} = 0$, from equilibrium conditions (V.120) and conjugation conditions (V.162), it follows that $\left[\frac{\partial\sigma_r^{(0)}}{dr}\right] = 0$ on L_s. Hence conjugation conditions $\sigma_r^{(k)}$ and $\tau_{r\theta}^{(k)}$ will not contain terms r_{ks} in the k-th approximation. We determine r_{ks} from conjugation condition $\sigma_\theta^{(k)}$, and $\sigma_r^{(k)}$ and $\tau_{r\theta}^{(k)}$ plays a part of boundary conditions for the determination of stresses in the elastic range.

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Equilibrium equations (V.120) will be satisfied if the stress components are expressed through the stress function

$$\sigma_r^{(k)} = \frac{1}{r}\cdot\frac{\partial U^{(k)}}{dr} + \frac{1}{r^2}\cdot\frac{\partial^2 U^{(k)}}{\partial\theta^2}, \quad \sigma_\theta^{(k)} = \frac{\partial^2 U^{(k)}}{\partial r^2}, \qquad \tau_{r\theta}^{(k)} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial U^{(k)}}{\partial\theta}\right). \tag{V.164}$$

By substituting (V.164) into linearized plasticity conditions (V.151) and (V.152), we obtain a series of linear equations which enable us to determine stress function $U^{(k)}$, and consequently, the stress components. /434

The determination of the boundary of plastic zone L_s is very important. Let us consider several examples.

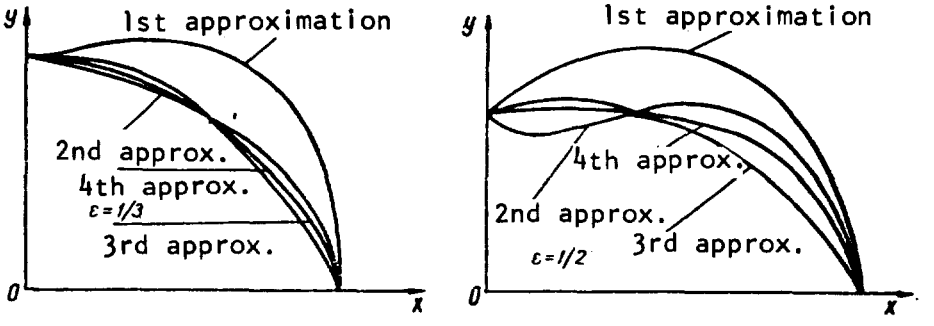


Figure V.7.

Biaxial Tension of Plate Weakened by Round Hole for the Case of Plane Deformation. Let a plane weakened by a round hole of radius R be subjected to tension by constant forces A and B at infinity. Omitting the calculations, we find the formula for the contour of the boundary of the plastic zone

$$r_s = 1 + \varepsilon\cos 2\theta - \frac{3}{4}\varepsilon^2(1-\cos 4\theta) + \frac{5}{8}\varepsilon^3(-\cos 2\theta + \cos 6\theta) + \frac{7}{64}\varepsilon^4(-1 - 4\cos 4\theta + 5\cos 8\theta) + \ldots, \tag{V.165}$$

where

$$\varepsilon = \frac{B-A}{2k}.$$

It is easy to check that expansion (V.165) coincides with the expansion of the equation of an ellipse with semiaxes $1 + \varepsilon$ and $1 - \varepsilon$, which yields the rigorous solution of this problem (see §2).

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Figure V.7 illustrates graphs of the boundary of the plastic zone for two values of the parameter $\varepsilon = 1/3$ and $\varepsilon = 1/2$, from which it is obvious that the obtained approximations are convergent.

The small parameter method gives approximate analytical expressions[1] for displacements. By assigning to the components that pertain to the elastic range, the superscript e, and to components pertaining to the plastic range, the superscript p, we obtain

$$\frac{4Gu_r^e}{ka} = \frac{2}{r} - 2\varepsilon\left(r - \frac{2}{r} - \frac{1}{r^3}\right)\cos 2\theta + \varepsilon^2\left[\frac{2}{r} + 4\left(-\frac{1}{r^3} + \frac{1}{r^5}\right)\cos 4\theta\right] +$$
$$+ \varepsilon^3\left[\frac{2}{r^3}\cos 2\theta - 2\left(\frac{4}{r^5} - \frac{5}{r^7}\right)\cos 6\theta\right] + \ldots$$

$$\frac{4Gu_\theta^e}{ka} = 2\varepsilon\left(r + \frac{1}{r^3}\right)\sin 2\theta + 2\varepsilon^2\left(-\frac{1}{r^3} + \frac{2}{r^5}\right)\sin 4\theta +$$
$$+ \varepsilon^3\left[\frac{2}{r^3}\sin 2\theta + 2\left(-\frac{8}{3r^5} + \frac{5}{r^7}\right)\sin 6\theta\right] + \ldots \qquad /435$$

$$\frac{4Gu_r^p}{ka} = \frac{2}{r} - 4\varepsilon\left[\cos(\sqrt{3}\ln r) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}\ln r)\right]\cos 2\theta + \frac{2\varepsilon^2}{r} -$$
$$- 2\varepsilon^3\left\{\left[\cos(\sqrt{3}\ln r) - \frac{3}{\sqrt{3}}\sin(\sqrt{3}\ln r)\right]\cos 2\theta + \right. \qquad \text{(V.166)}$$
$$\left. + \left[\cos(\sqrt{35}\ln r) - \frac{3}{\sqrt{35}}\sin(\sqrt{35}\ln r)\right]\cos 6\theta\right\} + \ldots$$

$$\frac{4Gu_\theta^p}{ka} = 2\varepsilon\left[2\cos(\sqrt{3}\ln r) + \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)\sin(\sqrt{3}\ln r)\right]\sin 2\theta +$$
$$+ 2\varepsilon^2\left[\cos(\sqrt{15}\ln r) + \frac{1}{\sqrt{15}}\sin(\sqrt{15}\ln r)\right]\sin 4\theta - \varepsilon^3\left\{\left[2\cos(\sqrt{3}\ln r) + \right.\right.$$
$$+ \left(\frac{3}{\sqrt{3}} + \sqrt{3}\right)\sin(\sqrt{3}\ln r)\left.\right]\sin 2\theta + \left[\frac{2}{3}\cos(\sqrt{35}\ln r) + \right.$$
$$\left.\left. + \left(\frac{1}{\sqrt{35}} + \frac{\sqrt{35}}{3}\right)\sin(\sqrt{35}\ln r)\right]\sin 6\theta\right\} + \ldots$$

The approximations thus found yield good convergence. The calculations show that in the case of biaxial tension, the contour of the hole expands, being elongated in the direction of the greatest of the forces acting upon it.

Biaxial Tension of Thin Plate with Round Hole of Radius R by Forces A and B (Plane[2] Stress State). The boundary L_s of the plastic zone is found by the formula

---

[1] See D. D. Ivlev [3, 4].

[2] The solution of this problem was first given by A. P. Sokolov [1]. The solution of the given problem is also found in §4, Chapter IV of G. N. Savin [1].

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$$r_s = 1 + 4\varepsilon^* \cos 2\theta - 8\varepsilon^{*2}(1 - 2\cos 4\theta) - 80\varepsilon^{*3}(\cos 2\theta - \cos 6\theta) + \\ + 32\varepsilon^{*4}(1 - 16\cos 4\theta + 14\cos 8\theta) + \ldots, \qquad (V.167)$$

where

$$\varepsilon^* = \frac{A - B}{2ak}.$$

Here $\alpha = R/r_s^0$ (R is the radius of the hole; r_s^0 is the dimension radius of the plastic zone when $\varepsilon^* = 0$). Figure V.8 (a, b) illustrates the graphs of the boundary of the plastic zone for two values of the parameter ε^*, 0.05 and 0.10.

Biaxial Tension of Thin Plate with Elliptical Hole. Let a thin plate with an elliptical hole be under tension at infinity by mutually perpendicular forces A* = const and B* = const, directed at some angle to the principal axes of the ellipse. It is assumed that $A^* \geqslant B^*$ and force A* is directed at angle θ_0 to the large axis of the ellipse. The forces are

$$A^* = \frac{A + B}{2} + d_2 \frac{A - B}{2}, \quad B^* = \frac{A + B}{2} - d_2 \frac{A - B}{2}, \qquad (V.168)$$

where d_2 is some parameter.

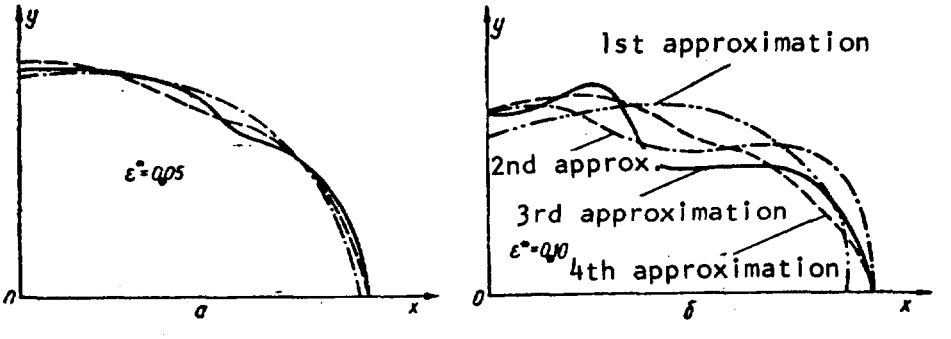


Figure V.8.

We will represent the contour equation of the elliptical hole in the form /436

$$r = \alpha + \varepsilon\alpha d_1 \cos 2\theta - \varepsilon^2 \frac{3\alpha d_1^2}{4}(1 - \cos 4\theta) + \ldots, \qquad (V.169)$$

where $\varepsilon = A - B/2k$, $\alpha = R/r_s^0$ (R is the radius of the original round hole; r_s^0 is the dimension radius of the plastic zone of zero approximation); d_1 is a parameter.

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The equation of contour L_s, i.e. the boundary of the plastic zone, assuming that the contour of the hole is free of external forces, is

$$r_s = 1 + \varepsilon^* [4d_2 \cos(\theta - \theta_0) + 3\alpha d_1 \cos 2\theta] + \varepsilon^{*2} \left\{ d_1^2 \left(\frac{\alpha^2}{4} - 8\alpha^4 \right) - \right.$$
$$- (18 d_1 d_2 \alpha \cos 2\theta_0 + 8 d_2^2) + \left[-d_1^2 \left(\frac{15}{4\alpha^2} - 8\alpha^3 - \frac{3}{4\alpha^4} \right) + \right.$$
$$\left. \left. + (18 d_1 d_2 \alpha \cos 2\theta_0 + 16 d_2^2 \cos 4\theta_0) \right] \cos 4\theta + [18 d_1 d_2 \alpha \sin 2\theta_0 + 16 d_2^2 \sin 4\theta_0] \sin 4\theta \right\} + \ldots . \qquad (V.170)$$

$$\varepsilon^* = \frac{\varepsilon}{\alpha} = \frac{A - B}{2\alpha k}.$$

By changing in (V.170) the parameters d_1, d_2 and θ_0, we obtain the solution for several important particular cases. Thus, when $d_1 = 0$, $d_2 = 1$ and $\theta_0 = 0$, we will have the case of biaxial tension of a plate with a round hole by forces A and B at infinity; when $d_2 = 0$ and $d_1 = 1$, the case of uniform tension of the plate with an elliptical hole $A^* = B^* = A + B/2$ at infinity. In the latter case the boundary of the plastic zone is

$$r_s = 1 + 3\varepsilon \cos 2\theta + \varepsilon^2 \left[\left(\frac{1}{4} - 8\alpha^2 \right) - \left(\frac{15}{4} - 8\alpha - \frac{3}{4}\alpha^2 \right) \cos 4\theta \right] + \ldots . \qquad (V.171)$$

Figure V.9 illustrates the graphs of the boundary of the plastic zone in the case of biaxial tension of a thin plate with an elliptical hole by forces A and B, directed at an angle of 45° to the principal axes of the ellipse, when the values of the parameters are $\varepsilon = 0.20$ and $\varepsilon^* = 0.05$. The boundary of the plastic zone in the case of biaxial tension of a plate with a round hole by the same forces is denoted by the broken line.

The graph of uniform tension of a plate with an elliptical hole by forces $A^* = B^* = (A + B)/2$ when $\varepsilon = 0.166$, $\alpha = 0.500$ is illustrated in Figure V.10.

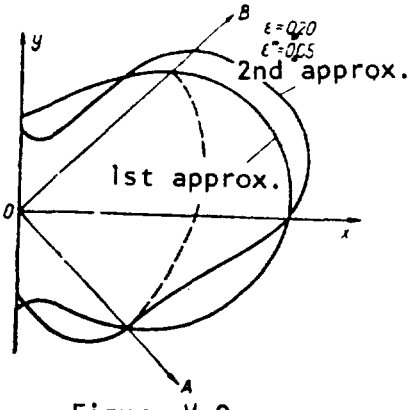


Figure V.9.

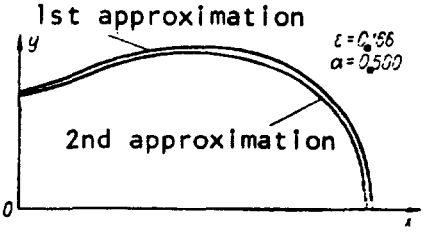


Figure V.10.

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The plastic zone near a round hole in a plate located in the state of plane deformation, when to the contour of the hole is applied normal pressure, distributed by the law of rectilinear equilateral trapezoid, is analyzed by A. P. Sokolov [2]. A. I. Durelli, C. A. Sciamarella [1] determined the distribution of elasto-plastic stresses and deformations near a round hole in a finite aluminum plate during large plastic deformations by an experimental method. I. S. Tuba [1] analyzes stress concentration near a round hole during elasto-plastic deformation in a uniformly stressed plate made of a linearlly reinforced material (also see I. S. Tuba [2]). The plastic zone in a physically nonlinear plate is analyzed by I. Yu. Khoma [3]. /437

§6. Elasto-Plastic Problem for a Plate Weakened by an Infinite Row of Identical Round Holes[1]

We will analyze a plate located in the plane deformation state, with an infinite row of identical round holes, the centers of which are located on the Ox axis at a distance l from each other. We will assume that to the contours of the holes, of radius R (for simplicity we will assume that R = 1) are applied normal forces p, and at infinity are given constant forces A and B (Figure V.11).

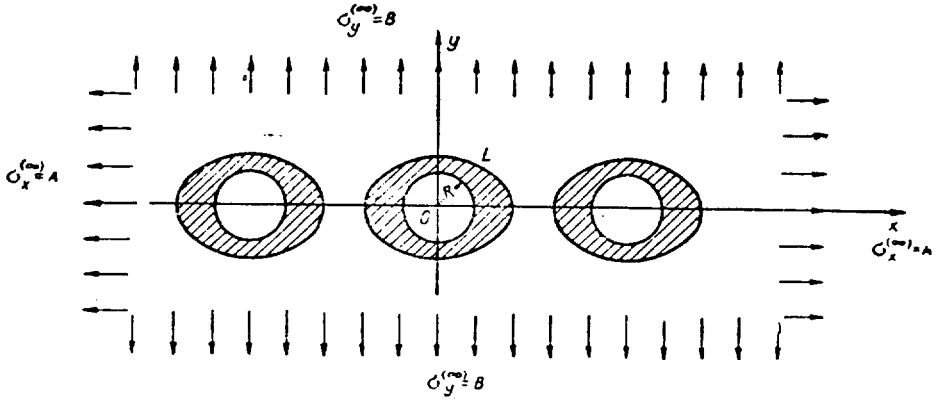


Figure V.11.

Let plastic zones occur under the effect of the given system of external forces near the holes, completely surrounding the holes but not overlapping each other. The problem now amounts to finding the boundary lines between the elastic and plastic zones, and also the stress state of the plate. Through L_s we will denote the boundary of the plastic zone encompassing the round hole, in the center of which the origin of the coordinate system is placed. This hole will be called the basic hole.

[1]The solution of the problem is found by A. S. Kosmodamianskiy [1].

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Due to the geometrical and force symmetry, the shape of lines L_s of the boundary between the zones near each hole will be the same as near the basic hole. Since the stresses in the plastic range are determined only by the formula of the contour of the hole and boundary of the load, they will be the same in each plastic zone as near the basic hole. Consequently, in the given case they will satisfy relations (V.16). These stresses do not depend on the forces at infinity, nor on the effect of the adjacent holes. The latter can have an effect on changes of line L_s of the boundary of the zones and on the stress state in the elastic range. /438

Additional stress components appear in the elastic range due to the presence of holes; we will express these, in accordance with formulas (V.96), through two holomorphic functions $\Phi^*(z)$ and $\Psi^*(z)$. From the conditions of continuity of stress components in the elastic and plastic zones on the common boundary L_s separating these zones, we obtain

$$4\operatorname{Re}\Phi^*(z) = 2(k-p) + 2k\ln\frac{z\bar{z}}{R^2} - (A+B),$$
$$2[\bar{z}\Phi^{*\prime}(z) + \Psi^*(z)] = 2k\frac{\bar{z}}{z} - (B-A). \qquad (V.172)$$

The function $\Phi^*(z)$, which is directly related to stress components, is periodic in the direction of the Ox axis. Using S. G. Mikhlin's approach [2], we introduce the periodic function

$$\tilde{\Psi}^*(z) = \Psi^*(z) + z\Phi^{*\prime}(z). \qquad (V.173)$$

By substituting the value of $\Psi^*(z)$ from (V.173) into (V.172), we obtain the conditions on contour L_s:

$$4\operatorname{Re}\Phi^*(z) = 2(k-p) + 2k\ln\frac{z\bar{z}}{R^2} - (A+B),$$
$$(\bar{z}-z)\Phi^*(z) + \tilde{\Psi}^*(z) = k\frac{\bar{z}}{z} - \frac{B-A}{2}. \qquad (V.174)$$

The analytical functions that vanish at infinity are represented[1] in the form /439

$$\Phi^*(z) = \sum_{k=1}^{\infty}\frac{a_k}{[\zeta(z)]^k} + \sum_{n=-\infty}^{\infty}{}'\sum_{k=1}^{\infty}\frac{a_k}{[\zeta(z-ne)]^k},$$
$$\tilde{\Psi}^*(z) = \sum_{k=1}^{\infty}\frac{b_k}{[\zeta(z)]^k} + \sum_{n=-\infty}^{\infty}{}'\sum_{k=1}^{\infty}\frac{b_k}{[\zeta(z-ne)]^k}, \qquad (V.175)$$

[1]See I. I. Vorovich, A. S. Kosmodamianskiy [1].

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where the * denotes the absence of terms with the number n = 0 in the sums.

The variable ζ is related to z by the relation

$$z - nl = \omega(\zeta) = c\zeta + \sum_{k=1}^{\infty} c_k \zeta^{-k}. \tag{V.176}$$

The first sums in (V.175) are analytical functions outside of contour L_s. We will denote them in the transformed (with the aid of function $\omega(\zeta)$ (V.176)) plane ζ through $\Phi_2(\zeta)$ and $\Psi_2(\zeta)$, i.e.

$$\Phi_2(\zeta) = \sum_{k=1}^{\infty} \frac{a_k}{[\zeta(z)]^k}, \qquad \Psi_2(\zeta) = \sum_{k=1}^{\infty} \frac{b_k}{[\zeta(z)]^k}. \tag{V.177}$$

The second sums in (V.175) will be holomorphic functions within contour L_s. They also depend on small parameter $\varepsilon = 1/l$. By expanding them into a series by degrees of this parameter, and retaining the terms that contain, for instance, ε^2, we obtain

$$\begin{aligned} \tilde{\Phi}_2(\zeta) &= \frac{1}{3} a_2 c^2 \pi^2 \varepsilon^2 - \frac{1}{3} a_1 c^2 \pi^2 \varepsilon^2 \omega(\zeta), \\ \tilde{\Psi}_2(\zeta) &= \frac{1}{3} b_2 c^2 \pi^2 \varepsilon^2 - \frac{1}{3} b_1 c^2 \pi^2 \varepsilon^2 \omega(\zeta). \end{aligned} \tag{V.178}$$

Boundary conditions (V.174), according to (V.177) and (V.178), can be written in the form

$$\begin{aligned} 2\operatorname{Re}\Phi_2(\zeta) &= (k-p) + k \ln \frac{\omega(\zeta)\overline{\omega(\zeta)}}{R^2} - \frac{A+B}{2} + \\ &+ \frac{2}{3} a_1 c^2 \pi^2 \varepsilon^2 [\omega(\zeta) + \overline{\omega(\zeta)}] - \frac{2}{3} a_2 c^2 \pi^2 \varepsilon^2 \text{ on } \gamma; \\ \frac{\overline{\omega(\zeta)} - \omega(\zeta)}{\omega'(\zeta)} \Phi_2'(\zeta) + \Psi_2(\zeta) &= k \frac{\overline{\omega(\zeta)}}{\omega(\zeta)} - \frac{B-A}{2} + \\ + \frac{1}{3} a_1 c^2 \pi^2 \varepsilon^2 [\overline{\omega(\zeta)} - \omega(\zeta)] + \frac{1}{3} b_1 c^2 \pi^2 \varepsilon^2 \omega(\zeta) - \frac{1}{3} b_2 c^2 \pi^2 \varepsilon^2 \text{ on } \gamma. \end{aligned} \tag{V.179}$$

By N. I. Muskhelishvili's method[1], we find from boundary conditions (V.179) /440 the functions

[1]See §1, Chapter I.

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$$\Phi_2(\zeta) = k \ln \frac{\omega(\zeta)}{c\zeta},$$

$$\Psi_2(\zeta) = k\left[1 + \frac{\overline{\omega}\left(\frac{1}{\zeta}\right) - \omega(\zeta)}{\omega'(\zeta)} \frac{1}{\zeta}\right] - \frac{B-A}{2} - \frac{1}{3} b_2 c^2 \pi^2 \varepsilon^2. \tag{V.180}$$

where

$$\omega(\zeta) = c\zeta + \frac{c_1}{\zeta}. \tag{V.181}$$

The constants in formulas (V.180) are found from the following equation system:

$$c = R \exp\left\{\frac{1}{2k}\left(p - k + \frac{A+B}{2} + \frac{2}{3} a_2 c^2 \pi^2 \varepsilon^2\right)\right\},$$

$$c_1 = \frac{c}{k}\left(\frac{B-A}{2} + \frac{1}{3} b_2 c^2 \pi^2 \varepsilon^2\right),$$

$$a_2 = k \frac{c_1}{c}, \tag{V.182}$$

$$b_2 = \frac{2kc + (B-A)c_1 - 4kc_1}{2c\left(1 - \frac{1}{3} cc_1 \pi^2 \varepsilon^2\right)}.$$

After determining the functions $\Phi_2(\zeta)$, $\Psi_2(\zeta)$ and $\tilde{\Phi}_2(\zeta)$, $\tilde{\Psi}_2(\zeta)$, we find the stress components in the elastic range:

$$\sigma_x^{(2)} + \sigma_y^{(2)} = A + B + 4 \operatorname{Re} \Phi(\zeta),$$

$$\sigma_y^{(2)} - \sigma_x^{(2)} + 2i\tau_{xy}^{(2)} = B - A + 2\left[\frac{\overline{\omega}(\zeta^{-1}) - \omega(\zeta)}{\omega'(\zeta)} \Phi'(\zeta) + \Psi(\zeta)\right]. \tag{V.183}$$

Thus, the stated problem is solved.

We will consider the case where p = 3k, A = B = 0, $\varepsilon = 1/7 = 0.1$. From the solution of system (V.182), we find c = 2.85 R, c_1 = 0.499 R. Then lines L_s of the boundary will be ellipses with semiaxes a = 3.35 R, b = 2.35 R. In this case, for a range with one hole (see §2), the boundary of the plastic range will be a circle of radius r = 2.72 R.

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§7. Plastic Zones near Round Hole in Plate under Uniaxial Tension /441

When a plate is under biaxial tension by certain external forces at infinity, a plastic zone is formed near a hole that will partially or completely surround it. With tension, however, along only one coordinate axis, the plastic zone will begin to develop in the vicinity of points A and B (Figure V.12), and may not completely surround the hole, but will acquire the form indicated by the shaded areas. We will assume that a thin plate of thickness 2h and width b, with a round hole of radius R, the edge of which is free of external forces is under tension along the Oy axis by forces

$$p_y^{(\infty)} = p = 2\varepsilon h\sigma_s, \tag{V.184}$$

where ε is a dimensionless parameter ($0 \leqslant \varepsilon \leqslant 1$). We will further assume that the hole is small, i.e. $b/2R \geqslant 4$. Under this assumption, for $\varepsilon \geqslant 1/3$, the plastic zones will occur at points A and B near the hole. The rigorous solution of this problem has not yet been found, and therefore we will introduce the basic results of an approximation method proposed by I. I. Fayerberg [1], based on the theory of elasto-plastic deflection of a curved beam.

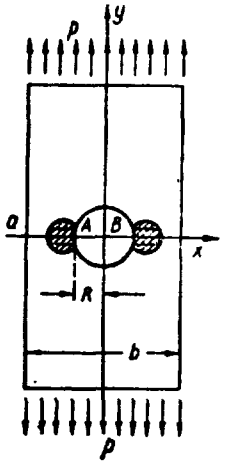


Figure V.12.

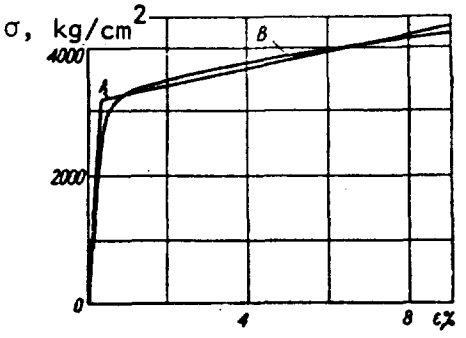


Figure V.13.

I. I. Fayerberg [1] constructed the graphs[1] for a duralumin plate D = 16T with a round hole for n = b/2R = 7 and n = 6. The tension diagram of the material from which the plate is made, i.e. duralumin, is shown in Figure V.13. For the calculations, a diagram consisting of two rectilinear segments OA and AB was used instead of the actual diagram. As shown in Figure V.13, $E_{pl} = E/60$ and $k = E_{pl}/E = 0.0167$ for duralumin. /442

[1]Figures V.13-V.18 are borrowed from the work of I. I. Fayerberg [1].

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Figures V.14 and V.15 show the curves that represent the boundaries between the elastic and plastic ranges for various values of ε. These curves also show that the plastic zone, occurring at the points A and B (see Figure V.12) when ε = 1/3 and for further increase of ε from 1/3 to 0.65, rapidly spreads along the contour of the hole. When ε > 0.65, the plastic zones extend outward in the direction perpendicular to the direction of the forces of tension, decreasing near the hole itself. Figures V.16 and V.17 show the curves of distribution of deformations through cross section aA (see Figure V.12) as a function of n for certain values of ε. As ε increases, or, by the same token, as the force of tension $p = 2h\varepsilon\sigma_s$ increases, deformations near the hole increase much more rapidly than on the outer edges of the plate.

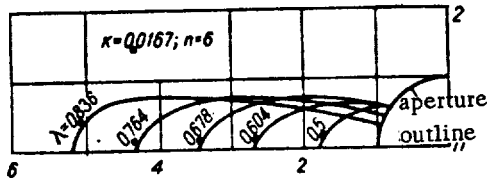


Figure V.14.

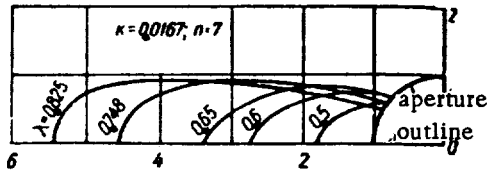


Figure V.15.

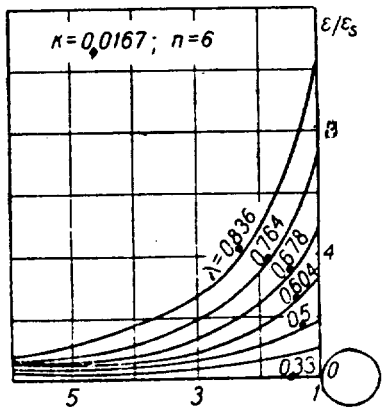


Figure V.16.

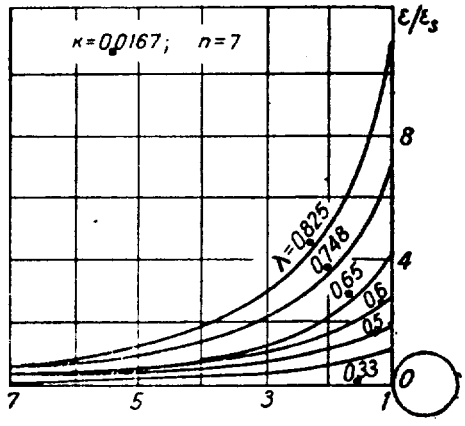


Figure V.17.

The point curve on Figure V.18 represents the relationship between e/e_s (e_s is deformation at the moment of formation of the plastic zone) for point A (see Figure V.12) and the parameter ε found by I. I. Fayerberg [1] experimentally with the aid of the tensometers of Gugenberg with a base of 2 mm. The solid curve, which represents the theoretical relationship between ε and e/e_s /443 is presented in the same figure for comparison. The parameter e was measured at point A in the direction of tension of a duralumin plate measuring 800 × 180 × 5 mm containing a hole of radius R = 15 mm. As we see, the experimental data coincide almost exactly with the theoretical data for ε ≤ 0.5, but

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deviate somewhat for $\varepsilon > 0.5$. Thus, the solution of the stated problem yields good approximation. O. G. Rybakina [1], using this method, generalizes the case of large plastic deformations with consideration of the true law of strengthening.

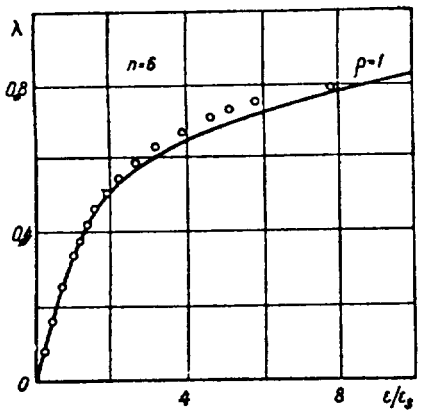


Figure V.18.

Using the variation methods in combination with the method of Reyleigh-Ritz, the investigators B. Budiansky and R. Vidensek [1] found the approximate solution of the problem in the given statement of stresses near a round hole in an infinite plate located under uniaxial tension. R. Yu. Kerimov and L. P. Khoroshun [1] and R. Yu. Kerimov [1] analyzed the elasto-plastic problem for the case of gradual strengthening of the material[1]. E. G. Marcetos [1] analyzed the development of the plastic zone near a round hole in a rod under tension on the basis of the method of optically active coatings.

§8. Slip Zones near Elliptical Hole

So far in this chapter we have examined rather thoroughly the development of plastic deformations that occupy much of the area near a hole. Experiments show that plastic deformations, during the first stages of their development, are often localized in a few thin layers (slip zones), which occupy an insignificant part of the volume of the body in comparison with its elastic part. Such a development of plastic deformations is particularly characteristic of materials possessing (during usual tests) a clearly defined area of yield, and also in the presence of a sufficiently heterogeneous field, i.e. stress field with a rather large stress gradient.

In this section we will analyze the simplest cases of the occurrence and /444
development of layers (zones) of slip in thin plates, which occur around an elliptical hole when $b = 0$ and $a = b$, i.e. in two extreme cases: around a slit, and around a round hole (a and b are the semiaxes of an ellipse.

[1]The problem (plane deformation) of the change of radius, in time, of the plastic zone near a round hole in a plate made of a linearlly strengthening and relaxing material (A. Yu. Ishlinskiy's model [1]) is investigated in the work of M. I. Rozovskiy [1]. It is shown in this work that in the presence of relaxation, strengthening has not only a quantitative, but also a qualitative effect on the process of change of the plastic zone.

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For our analytical investigation of the occurrence and development of slip zones, we can use the following method[1]. In view of the thinness of these zones, plastic deformation is considered to be concentrated along certain lines, i.e. the zone is replaced by its median line. Hence it is necessary to assume the presence of a discontinuity of displacements on such lines. These assumed discontinuities of displacements should not contradict the kinematically possible pattern of plastic flow and should not denote the presence of friction (voids) in the body, if the latter are not formed as the result of plastic deformation, i.e. if the material remains solid. In thin plates, such requirements can be satisfied by discontinuities that are both tangential and normal to the line of discontinuity. In the case of normal discontinuity, disruption of the solidity of the material can be excluded due to the possibility of local thinning or thickening of the plate.

Thus, due to local carrier of the development of slip zones, the problem of elasto-plastic deformation in a plate can be reduced to the problem of the deformation of a linearly elastic plate, displacements of which along certain lines experience a discontinuity. The actual forces along these lines should satisfy the assumed plasticity condition. The shape and length of the lines of discontinuity (slip) should be defined during the process of the solution of the problem. In the general case this is a rather difficult problem, particularly as regards the determination of the shape of the lines of discontinuity. However, in several particular cases the form of the slip lines can be predicted. Then the solution of the problem is greatly simplified.

We will examine[2] the problem of the development of the first slip zones due to uniaxial tension of thin plates containing a slit or a round hole. The size of the slit and hole will be assumed small, and the plates will be made of isotropic, elastically ideally plastic material. We will use the Tresa-Saint Venant condition as the plasticity condition:

$$\tau_{max} = \frac{1}{2}\sigma_{\tau} = \text{const.} \tag{V.185}$$

The complex potentials in the presence of slip zones are represented in the form

$$\varphi(z) = \varphi_0(z) + \varphi_*(z);$$
$$\psi(z) = \psi_0(z) + \psi_*(z), \tag{V.186}$$

---

[1]See P. M. Vitvitskiy, M. Ya. Leonov [3], where the stated approach to the solution of the problems of the development of slip zones in thin plates is clearly formulated. Essentially, this method, although it is not clearly formulated, is used in the work of D. S. Dugdale [1]. The analogous method of the analysis of the simplest cases of plastic deformation of plates is used by A. R. Rzhanitsin [1].

[2]See I. M. Vitvitskiy, M. Ya. Leonov [3]. The dependence of the length of slip zones on the forces of tension for the case of a plate with a slit is also found by D. S. Dugdale [1].

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where the functions $\phi_0(z)$ and $\psi_0(z)$ define the stress and deformation fields under the assumption that the plate is always deformed elastically (basic field); by additional functions $\phi_*(z)$ and $\psi_*(z)$ we will consider the presence of slip bands (discontinuities of displacements). /445

Tension of Plate Containing Slit. Let a thin infinite plate containing a rectilinear slit of length $2l$ be subjected to tension perpendicular to the slit by forces $\sigma_y^{(\infty)} = p$ (Figure V.19).

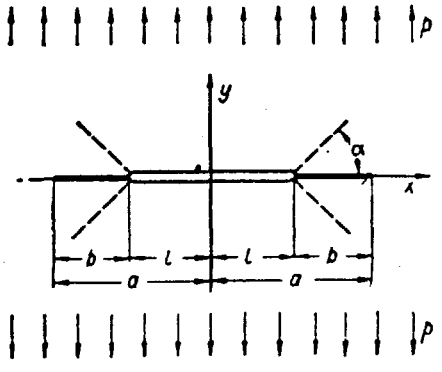


Figure V.19.

As follows from §2, Chapter II, the magnitude of forces p would not be small, the elasticity condition would not be satisfied on the ends of a narrow slit, and either plastic deformations or cracks due to the brittle destruction of the material may occur here.

The maximum stresses in the plate occur on the line of extension of the slit (on the line y = 0 when $|x| \geqslant l$), whereupon

$$\sigma_y(x, 0) > \sigma_x(x, 0) > \sigma_z(x, y) = 0, \qquad \tau_{xy}(x, 0) = 0. \tag{V.187}$$

Therefore plasticity condition (V.185) on the abscissa acquires the form

$$\sigma_y(x, 0) = \sigma_T. \tag{V.188}$$

The slip zones coincide with the Ox axis and are inclined to the plane of the plate at angle $\pm\pi/4$. The first slip zones thus extend along the abscissa[1]. We will assume that they occupy segments $l \leqslant |x| \leqslant a$ under some load. During displacements along the slip zones, local thinning of the plate occurs in these segments, which is the result of discontinuity of the component v(x, 0) of the displacement vector that is normal to the Ox axis.

Since a narrow slit can also be regarded as a line of discontinuity of displacements v(x, 0), we will assume that such discontinuity occurs in the entire segment $|x| \leqslant a$, whereupon, in this segment, we will have

$$\tau_{xy}(x, 0) = 0; \quad \sigma_y(x, 0) = \begin{cases} 0 & \text{for } |x| < l, \\ \sigma_T & \text{for } l < |x| < a. \end{cases} \tag{V.189}$$

[1]After the first elementary plastic displacements, the stress field, generally speaking, changes somewhat, although (as we will see later), relation (V.187) is satisfied. This insures the development of the first slip zones along the abscissa.

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Then, as the functions corresponding to tension of a heavy plate by forces /446
$\sigma_y^{(\infty)} = p$, we should use

$$\varphi_0(z) = \frac{1}{4} pz, \qquad \psi_0(z) = \frac{1}{2} pz. \tag{V.190}$$

The functions $\phi_*(z)$ and $\psi_*(z)$ which take into account the presence of discontinuity of displacements can be found by solving the following auxiliary problem. In an isotropic elastic plate, on segment $|x| \leqslant a$ of the abscissa, let displacements $v(t, 0)$ experience a discontinuity of the magnitude $\lambda(t)$, and let displacements $u(t, 0)$ and stresses σ_y and τ_{xy} be continuous, i.e.

$$\left.\begin{aligned} [u(t) + iv(t)]^+ - [u(t) + iv(t)]^- &= i\lambda(t), \\ [X_n(t) + iY_n(t)]^+ - [X_n(t) + iY_n(t)]^- &= 0 \end{aligned}\right\} \text{ for } |t| = |x| < a, \tag{V.191}$$

where + and - denote the left hand and right hand boundary values, respectively. We will assume that the function $\lambda(t)$ satisfies Gel'der's condition and $\lambda(a) = \lambda(-a) = 0$.

On the basis of formulas (I.9), (I.10) and relations (V.191) we have

$$\left.\begin{aligned} \varphi_*^+(t) - \varphi_*^-(t) &= \frac{2\mu}{\varkappa + 1} i\lambda(t), \\ \psi_*^+(t) - \psi_*^-(t) &= \frac{2\mu}{\varkappa + 1} i[\lambda(t) - t\lambda'(t)] \end{aligned}\right\} \text{ for } |t| < a. \tag{V.192}$$

Thus, we have arrived at the problem of linear conjugation (Hilbert's problem) for functions $\phi_*(z)$ and $\psi_*(z)$ that are piece-wise holomorphic on the entire plane. The solution of this problem is known[1]:

$$\begin{aligned} \varphi_*(z) &= D \int_{-a}^{a} \frac{\lambda(t)\, dt}{t - z}, \\ \psi_*(z) &= D \int_{-a}^{a} \frac{\lambda(t) - t\lambda'(t)}{t - z}\, dt, \end{aligned} \tag{V.193}$$

where

$$D = \frac{\mu}{\pi(\varkappa + 1)}.$$

[1]See N. I. Muskhelishvili [1].

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The derivatives of these functions, through which we determine the stresses, can be represented in the form

$$\varphi_*'(z) = \Phi_*(z) = D\int_{-a}^{a}\frac{\lambda'(t)\,dt}{t-z},$$
$$\psi_*(z) = \Psi_*(z) = -Dz\int_{-a}^{a}\frac{\lambda'(t)\,dt}{(t-z)^2}. \qquad (V.194)$$

Let us return now to the basic problem of slip zones. In order to solve this problem we must determine the function $\lambda'(t)$ and parameter a such that condition (V.189) be satisfied and the boundedness and continuity of stresses at points $z = \pm a$ be preserved. Recalling formulas (I.2), (I.9), (I.6) and (I.13), we find from condition (V.189) the equation for the determination of the function $\lambda'(t)$ and parameter a: /447

$$\frac{1}{\pi i}\int_{-a}^{a}\frac{\lambda'(t)\,dt}{t-x} = f(x) \equiv \begin{cases}\dfrac{ip}{2\pi D} & \text{for } |x| < l,\\ \dfrac{i}{2\pi D}(p-\sigma_\tau) & \text{for } l < |x| < a.\end{cases} \qquad (V.195)$$

It is easy to prove that only the solution of equation (V.195), which is equal to zero at the points $|x| = a$, insures continuity of stresses in the latter. Such a solution, as we know[1], exists under the condition

$$\int_{-a}^{a}\frac{f(x)}{\sqrt{a^2-x^2}} = 0 \qquad (V.196)$$

and is given by the formula

$$\lambda'(t) = \frac{\sqrt{a^2-t^2}}{\pi i}\int_{-a}^{a}\frac{f(x)}{\sqrt{a^2-x^2}}\frac{dx}{x-t}. \qquad (V.197)$$

By substituting into (V.197) the value f(x) and calculating the integrals, we find the function

$$\lambda'(t) = \frac{\sigma_\tau}{2\pi^2 D}\ln\left|\frac{l\sqrt{a^2-t^2}-t\sqrt{a^2-l^2}}{l\sqrt{a^2-t^2}+t\sqrt{a^2-l^2}}\right| \quad (|t| < a) \qquad (V.198)$$

and the parameter which defines the length of the slip zone:

$$a = l\sec\frac{\pi p}{2\sigma_\tau}. \qquad (V.199)$$

---

[1]See N. I. Muskhelishvili [2].

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From (V.198) and (V.194) we find the functions

$$\Phi_*(z) = -\frac{\sigma_T}{2\pi i}\ln\frac{l^2 - z^2 + li\sqrt{a^2 - l^2} - z\sqrt{z^2 - a^2}}{l^2 - z^2 - li\sqrt{a^2 - l^2} - z\sqrt{z^2 - a^2}},$$
$$\Psi_*(z) = \frac{\sigma_T}{\pi}\frac{zl\sqrt{a^2 - l^2}}{(z^2 - l^2)\sqrt{z^2 - a^2}}. \qquad (V.200)$$

Here, in the expression $\sqrt{z^2 - a^2}$, we take the branch that is holomorphic on plane z, with discontinuity along segment $|x| \leqslant a$, such that $\sqrt{z^2 - a^2} \to z$ when $|z| \to \infty$; in the logarithmic function we will take the branch that is holomorphic on the plane with the same segment, with the exception of the points $z = \pm l$ and which acquires a zero value at infinity.

The stresses in a plate with a slit are /448

$$\sigma_y + \sigma_x = p - \frac{2}{\pi}\sigma_T \arg\ln\frac{l^2 - z^2 + li\sqrt{a^2 - l^2} - z\sqrt{z^2 - a^2}}{l^2 - z^2 - li\sqrt{a^2 - l^2} - z\sqrt{z^2 - a^2}},$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = p + \frac{2}{\pi}\sigma_T\frac{(z - \bar{z})l\sqrt{a^2 - l^2}}{(z^2 - l^2)\sqrt{z^2 - a^2}}. \qquad (V.201)$$

Figure V.20 shows the graphs of stresses σ_x and σ_y through cross sections Ox and Oy for $p = 1/2\ \sigma_T$ (the length of the slip zone is $b = a - l = (\sqrt{2} - 1)l \approx \approx 0.41\ l$).

The solution thus found pertains to the first stage of elasto-plastic equilibrium of the plate, where the slip zones extend along the abscissa. However, new slip zones occur under some load, which are inclined at angle α to the first (shown in Figure V.19 by broken line). This is the second stage of elasto-plastic equilibrium of the plate. The new slip zones are generated when the maximum tangential stresses $\tau_{max} = \frac{1}{2}\left|\sigma_y - \sigma_x + 2i\tau_{xy}\right|$, which act in the surfaces perpendicular to the plane of the plate, achieve the yield points in the vicinity of the ends of the slit. These stresses are found from formulas (V.201), assuming $z = l + re^{i\vartheta}$ and after determining their maximum value as a function of ϑ when $r \to 0$:

$$\tau_{max} = \frac{1}{2}\left(\frac{\sigma_T}{\pi} + \sqrt{p^2 + \frac{\sigma_T^2}{\pi^2}}\right). \qquad (V.202)$$

On the basis of formulas (II.23) and (II.201), we can show that these stresses act in the surfaces whose slope α to the abscissa satisfies the relation

$$\tan 2\alpha = -\pi\frac{p}{\sigma_T} \qquad \left(\frac{\pi}{4} < \alpha < \frac{\pi}{2}\right). \qquad (V.203)$$

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On the basis of plasticity condition (V.185) we find from (V.202) that the new slip zones are generated under load

$$p = \sigma_T \sqrt{1 - \frac{2}{\pi}} \approx 0{,}60\sigma_T. \qquad \text{(V.204)}$$

From formulas (V.203) and (V.204) we know that the initial slope of the new slip zones is

$$\alpha = \frac{1}{2}[\pi - \operatorname{arctan} \sqrt{\pi(\pi - 2)}] \approx \frac{\pi}{3}. \qquad \text{(V.205)}$$

Thus, experiments with tension of plates made of soft steel with a slit verify the character of development of plastic deformations, which is the outcome of the analytical solution: at first, slip zones always appear on the lines of extension of the slit, and then, under a certain load new slip zones develop at some angle to the first. The relationship found between the length of the first slip zones and the load satisfactorily agrees with the theoretical relationship[1]. /449

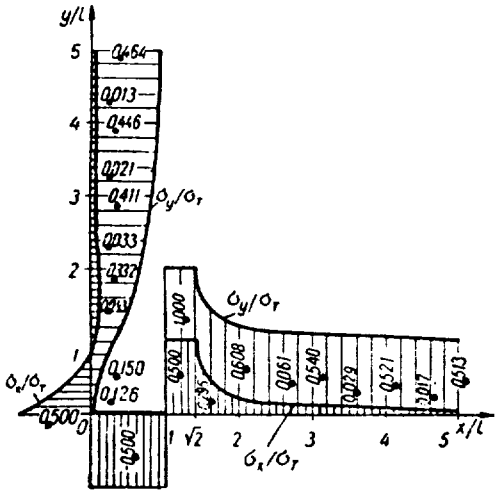


Figure V.20.

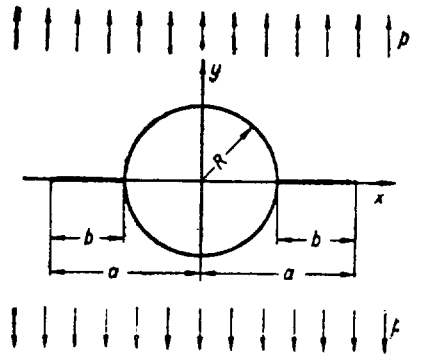


Figure V.21.

Tension of Plate with Round Hole. Let a thin infinite plate with a round hole of radius R be under tension by forces $\sigma_y^{(\infty)} = p$ (Figure V.21). Analysis of the field of elastic stresses (see §2, Chapter II) shows that in the vicinity of points $z = \pm R$, where the stresses are maximum, relation (V.187) is valid, and consequently, the elasticity condition has the form (V.188).

[1]See M. Ya. Leonov, P. M. Vitvitskiy, S. Ya. Yarema [1] and also D. S. Dugdale [1].

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Therefore the slip zones that occur when $p \geqslant 1/3\ \sigma_T$ are distributed on the abscissa and the displacements in them occur in the same manner as in the first slip zones in a plate with a slit.

We will determine the functions $\Phi_*(z)$ and $\Psi_*(z)$ for a plane containing a round hole in the presence, on segments $R \leqslant |x| \quad a$, of discontinuity of displacements $v^+(t, 0) - v^-(t, 0) = \lambda(t)$, whereupon $\lambda(\pm a) = 0$. For this purpose we will represent them in the form

$$\Phi_*(z) = \Phi_*^{(1)}(z) + \Phi_*^{(2)}(z), \quad \Psi_*(z) = \Psi_*^{(1)}(z) + \Psi_*^{(2)}(z), \tag{V.206}$$

where the functions $\Phi_*^{(1)}(z)$ and $\Psi_*^{(1)}(z)$ correspond to the above-stated discontinuity of displacements in a plane without a hole (they are given by formulas (V.194)), and the presence of the hole is taken into account by $\Phi_*^{(2)}(z)$ and $\Psi_*^{(2)}(z)$, i.e. the absence of external stresses on the contour of the hole is insured.

The functions $\Phi_*^{(2)}(x)$ and $\Psi_*^{(2)}(x)$ are found from the known formulas[1] by /450 solving the problem for an infinite plane with a hole of radius R, on the contour of which are applied normal and tangential stress $N + iT$ that are opposite in sign to stresses $\sigma_r + i\tau_{r\theta}$, which are caused on the contour of the hole $|z| = R$ by discontinuity of displacements on the above-mentioned segments in the plane without a hole. Stresses $\sigma_r + i\tau_{r\theta}$ are easy to find by substituting the functions $\phi_*(z)$ and $\psi_*(z)$ from (V.193) into formula (I.18) for $z = Re^{i\theta}$.

By combining, respectively, the functions $\Phi_*^{(1)}(x)$, $\Psi_*^{(1)}(z)$ and $\Phi_*^{(2)}(z)$, $\Psi_*^{(2)}(z)$, we obtain

$$\Phi_*(z) = -D[\lambda(R) - \lambda(-R)]\frac{1}{z} + D\int_L \left[\frac{1}{t-z} + \frac{R^4(z-t)}{zt(zt-R^2)^2}\right]\lambda'(t)\,dt,$$

$$\Psi_*(z) = -D[\lambda(R) - \lambda(-R)]\left(\frac{1}{z} + 2\frac{R^2}{z^3}\right) + \tag{V.207}$$

$$+ D\int_L \left\{-\frac{z}{(t-z)^2} + \frac{R^2}{z^2 t}\left[1 + \frac{zt}{zt-R^2} + \frac{R^4(z-2t)}{z(zt-R^2)^2} + \frac{2R^4 t(z-t)}{(zt-R^2)^3}\right]\right\}\lambda'(t)\,dt,$$

where the integrals are taken along the line $L = [-a, -R] + [R, a]$. If we know

---

[1] See N. I. Muskhelishvili [1].

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these functions, then it is easy to determine by formulas (I.13) the stresses $\sigma_y^* = (x, 0)$ caused by the discontinuity of displacements. After combining the latter stress with stress σ_y^0 (x, 0) of the basic field (it is given by the appropriate formulas of §2, Chapter II), then, on the basis of plasticity condition (V.188) we find the integral equation for the determination of the function $\lambda'(t)$ and parameter a:

$$4D\int_R^a\left\{\frac{t}{t^2-x^2}+\frac{R^4}{t}\left[\frac{1}{R^2x^2}+\frac{R^4(x^2-t^2)}{x^2(t^2x^2-R^4)^2}+\right.\right.$$
$$\left.\left.+4R^2t^2\frac{(t^2-R^2)(x^2-R^2)}{(t^2x^2-R^4)^3}\right]\right\}\lambda'(t)\,dt+\frac{1}{2}p\left(2+\frac{R^2}{x^2}+3\frac{R^4}{x^4}\right)=\sigma_\tau \qquad (V.208)$$
$$(R<|x|<a).$$

The conditions $\lambda(t) = \lambda(-t)$ and $\lambda'(t) = -\lambda'(-t)$, which obviously occur due to the symmetry of the problem, are taken into account in equation (V.208).

After substituting $x^2/R^2 = \xi$, $t^2/R^2 = \eta$, $\lambda'(t) = \lambda'(R\sqrt{\eta}) = \lambda_0'(\eta)$, $a^2/R^2 = \alpha$, equation (V.208) is converted to the form

$$\int_1^\alpha\left[\frac{1}{\eta-\xi}+\frac{1}{\xi\eta}+\frac{\xi-\eta}{\eta\xi(\eta\xi-1)^2}+\frac{4(\eta-1)(\xi-1)}{(\xi\eta-1)^3}\right]\lambda_0'(\eta)\,d\eta+$$
$$+\frac{p}{4D}\left(2+\frac{1}{\xi}+\frac{3}{\xi^2}\right)=\frac{\sigma_\tau}{2D}\quad(1<\xi<\alpha). \qquad (V.209)$$

The approximate solution of equation (V.209) will be found in the form of a polynomial with unknown coefficients a_n: /451

$$\lambda_0'(\eta)\approx\frac{\sigma_\tau}{D}\sum_{n=0}^m a_n\eta^n. \qquad (V.210)$$

To insure the boundedness and continuity of stresses at the points z = ±a, it is necessary to assume $\lambda_0'(\alpha) = 0$. By substituting (V.210) into (V.209), we obtain

$$\sum_{n=0}^m a_nJ_n(\alpha,\xi)+\frac{p}{4\sigma_\tau}\left(2+\frac{1}{\xi}+\frac{3}{\xi^2}\right)\approx\frac{1}{2}\quad(1<\xi<\alpha), \qquad (V.211)$$

where

$$J_n(\alpha,\xi)=\int_1^\alpha\left[\frac{1}{\eta-\xi}+\frac{1}{\xi\eta}+\right.$$
$$\left.+\frac{\xi-\eta}{\xi\eta(\xi\eta-1)^2}+\frac{4(\eta-1)(\xi-1)}{(\xi\eta-1)^3}\right]\eta^n d\eta\quad(n=0,1,\ldots,m).$$

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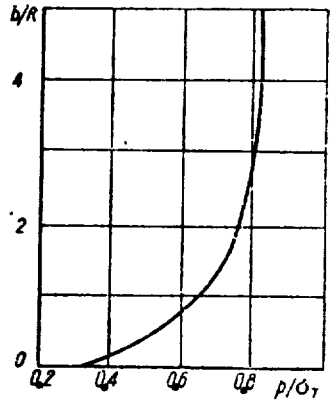


Figure V.22.

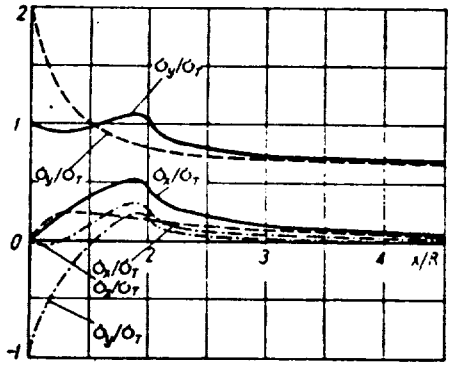


Figure V.23.

In order to determine the coefficients a_n, we will require that equation (V.211) be satisfied at the given points $1 \leqslant \xi_i \leqslant \alpha$ $(i = 0, 1, \ldots, m)$. Consequently, we obtain a system of $m + 1$ equations. From this system and from condition $\lambda_0'(\alpha) = 0$, if the various values $\alpha \geqslant 1$ are given, we find a_n and the ratio p/σ_T. In this manner we can find the approximate relationship between the length of the slip zones $b = a - R = R\sqrt{\alpha} - 1)$ and load p, and also the approximate value of the function $\lambda_0'(\eta)$. Then we can determine the stress field. /452

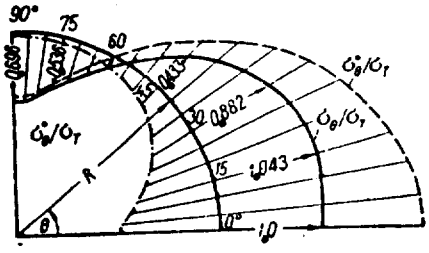


Figure V.24.

Figure V.22 illustrates the relationship between the length of slip zones b/R and load p/σ_T, and Figure V.23 shows the graphs[1] of stresses σ_y, σ_x on the abscissa; Figure V.24 illustrates the graphs of stresses σ_θ on the contour of the hole for the case where $b = R$, i.e. $p = 0.66\ \sigma_T$. The graphs are constructed for the three-place approximation of the value $\lambda_2'(\eta)$ (V.210) and for $\xi_0 = 1$, $\xi_1 = (1 + \alpha)/2$, $\xi_2 = \alpha = a^2/R^2$.

[1]The graphs of stresses for the basic field, under the assumption that the plate is deformed constantly (even when $p > \sigma_T/3$) elastically, are represented by the broken lines; the dot-dash lines represent stresses caused by the slip zones (discontinuity of displacements); the solid lines represent total stresses.

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§9. Critical Loads Caused by the Beginning of the Development of Cracks Around a Hole

We will assume that a body behaves elastically all the way up to destruction. In this case the destruction of the body will occur after the development of equilibrium cracks of brittle destruction in the zone of higher stresses[1].

We will analyze the stress state in an infinite elastic plate, weakened by a curvilinear hole or by two cracks (cross sections) of length l, when the contour L of the hole and of the cracks is free of external loads, and where, at sufficiently distant points from the hole and cracks, i.e. at infinity, are applied constant forces of tension $\sigma_y^{(\infty)} = p$ (uniaxial or multifold tension), as illustrated in Figure V.25.

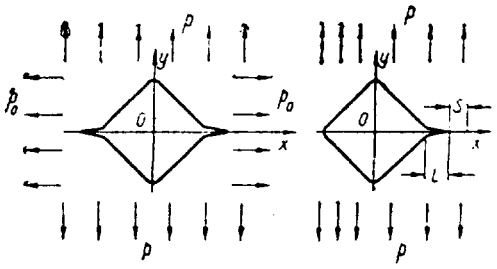


Figure V.25.

In the following we will analyze holes with smooth contours, for which the Ox axis will be the axis of symmetry and the equations of these contours will have the form

$$x = R\left[\cos\varphi + \sum_{n=1}^{N} a_n \cos n\varphi\right],$$
$$y = R\left[\sin\varphi - \sum_{n=1}^{N} a_n \sin n\varphi\right], \qquad (V.212)$$

where R, a_n are real parameters; N and n are whole positive numbers. /453

Through k we will denote the number of cracks. For the case of k = 2 we will analyze contour equations (V.212) with odd n.

Boundary condition (I.11) and the condition conjugate to it are, in the given case

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = 0,$$
$$\overline{\varphi(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\varphi'(\sigma) + \psi(\sigma) = 0. \qquad (V.213)$$

---

[1]Analysis of the stress state around variously oriented, both rectilinear and curvilinear, cracks is given in Chapter VIII.

[2]The solution of this problem was found by A. A. Kaminskiy [1]. For the particular case of a round hole, the solution was obtained earlier by O. L. Bowie [1]. See also §5, Chapter VIII, where the solutions of other authors are presented.

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The function $\omega(\zeta)$ which maps the exterior of contour (V.212) with one and two equal rectilinear cross sections of length l in plane z, on the exterior of unit circle γ in plane ζ will have the form

$$z = \omega(\zeta) = R\left[\zeta^* + \sum_{n=1}^{N} \frac{a_n}{\zeta^{*n}}\right], \tag{V.214}$$

where when k = 1

$$\zeta^* = \frac{L_0+1}{4}\left(\zeta + \frac{1}{\zeta}\right) + \frac{L_0-1}{2} + \\ + \sqrt{\left[\frac{L_0+1}{4}\left(\zeta + \frac{1}{\zeta}\right) + \frac{L_0-1}{2}\right]^2 - 1};$$

when k = 2

$$\zeta^* = \frac{L_0}{2}\left(\zeta + \frac{1}{\zeta}\right) + \sqrt{\frac{L_0^2}{4}\left(\zeta + \frac{1}{\zeta}\right)^2 - 1}.$$

Here $L_0 = \frac{1}{2}\left(1 + l_0 + \frac{1}{1+l_0}\right)$, l_0 is a real parameter.

In the case of an elliptical hole with semiaxes a, b, the mapping function (V.214) will have the following form:

when k = 1

$$\omega(\zeta) = R\left\{(1+m)\left[\frac{L_0+1}{4}\left(\zeta + \frac{1}{\zeta}\right) + \frac{L_0-1}{2}\right] + \right. \\ \left. + (1-m)\sqrt{\left[\frac{L_0+1}{4}\left(\zeta + \frac{1}{\zeta}\right) + \frac{L_0-1}{2}\right]^2 - 1}\right\}; \tag{V.215}$$

when k = 2

$$\omega(\zeta) = R\left\{\frac{L_0(1+m)}{2}\left(\zeta + \frac{1}{\zeta}\right) + (1-m)\sqrt{\frac{L_0^2}{4}\left(\zeta + \frac{1}{\zeta}\right)^2 - 1}\right\}. \tag{V.216}$$

Here /454

$$R = \frac{a+b}{2}, \qquad m = \frac{a-b}{a+b},$$

$$1 + l_0 = \frac{1}{a+b}\left[a + l + \sqrt{(a+l)^2 - (a^2-b^2)}\right].$$

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Mapping function (V.214) can be expanded into a series and represented in the form

$$\omega(\zeta) = R_1\left[\zeta + \sum_{n=1}^{\infty} \tilde{c}_n \zeta^{1-kn}\right], \tag{V.217}$$

where R_1, $\tilde{c}_n$ are real parameters.

Series (V.217) and expansion $\omega'(\zeta)$ converge everywhere to $|\zeta| \geqslant 1$, with the exception of points b_k of contour γ, which correspond to the points of intersection of the edges of the cracks with the contour of the hole. The edges of the equilibrium cracks should come together smoothly[1], and therefore we cannot approximate the mapping function (V.214) by the methods outlined in §2, Chapter I, since by such approximation the ends of the cracks will be rounded.

By maintaining in (V.217) N + 1 first terms of the series, we substitute coefficients τ_n by certain similar coefficients c_n such that for the new mapping function

$$\omega_N(\zeta) = R_1\left[\zeta + \sum_{n=1}^{N} c_n \zeta^{1-kn}\right] \tag{V.218}$$

the condition

$$\omega_N(\zeta) = R_1(1-\zeta^{-k})\,Q_N(\zeta), \tag{V.219}$$

will be satisfied, where Q_N is a polynomial with respect to negative powers of ζ, all roots of which lie within the unit circle in plane ζ. In constructing function (V.218) it is also required that $\omega_N(\zeta) \to \omega(\zeta)$, $\omega_N'(\zeta) \to \omega'(\zeta)$ for $N \to \infty$ everywhere in $|\zeta| \geqslant 1$, with the exception of points b_k of contour γ. Function (V.218) thus constructed will map the exterior of some new contour L' onto the exterior of contour γ, upon which are preserved the points of regression in the edges of the cracks, and only the angles at the intersection of the edges of the cracks with the contour of the hole are rounded off. Stress functions corresponding to $\omega_N(\zeta)$ will be denoted in the form $\phi_N(\zeta)$, $\psi_N(\zeta)$.

---

[1]See G. I. Barenblatt [1, 2].

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A singularity of the solution, occurring due to the presence of angular points of regression of the contour, can be explained by function $\psi_N(\zeta)$ and has no effect on the other function $\phi_N(\zeta)$, hence the function $\psi_N(\zeta)$, at the points γ corresponding to the points of regression of contour L', has a pole of the first power[1].

In the case of multifold tension "at infinity" by forces p, we will determine the function /455

$$\varphi_N(\zeta) = R_1 p\left[\frac{\zeta}{2} + \sum_{n=1}^{N} \alpha_n \zeta^{1-kn}\right]. \tag{V.220}$$

Here α_n are real coefficients. The second boundary condition of (V.213) for functions $\phi_N(\zeta)$ and $\psi_N(\zeta)$ are represented in the form

$$\omega_N'(\sigma)\,\psi_N(\sigma) = -\varphi_N\left(\frac{1}{\sigma}\right)\omega_N'(\sigma) - \omega_N\left(\frac{1}{\sigma}\right)\varphi_N'(\sigma). \tag{V.221}$$

By equating the coefficients for identical powers of σ in the expansion of the right and left hand sides of (V.221), we obtain a system of linear algebraic equations for the determination of α_n:

$$\alpha_p + \sum_{n=1}^{N-p} \alpha_{n+p}\, c_n\,(1 - kn) + \sum_{n=1}^{N-p} c_{p+n}\,\alpha_n\,(1 - kn) + \frac{c_p}{2} = 0. \tag{V.222}$$

By multiplying both sides of (V.221) by $1/2\pi i \cdot 1/\sigma - \zeta$, where ζ is a point outside of γ, and integrating with respect to γ, we obtain

$$\omega_N'(\zeta)\,\psi_N(\zeta) = -\varphi_N\left(\frac{1}{\zeta}\right)\omega_N'(\zeta) - \omega_N\left(\frac{1}{\zeta}\right)\varphi_N'(\zeta). \tag{V.223}$$

Since $\omega_N(\zeta) \to \omega(\zeta)$ and $\omega_N'(\zeta) \to \omega'(\zeta)$, we may assume[2] that when $N \to \infty$ $\phi_N(\zeta) \to \phi(\zeta)$; $\phi_N'(\zeta) \to \phi'(\zeta)$; $(1 - \zeta^{-k})\psi_N(\zeta) \to (1 - \zeta^{-k})\psi(\zeta)$ everywhere in $|\zeta| \geqslant 1$, with the exception of points b_k, which correspond to the angular points of contour L, which differ from the points of regression.

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[1]See S. M. Belonosov [1].
[2]See N. I. Muskhelishvili [1].

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Transitioning to the limit in the right hand and left hand sides of (V.223) for $N \to \infty$ for real z (on the extension of the cracks), we obtain

$$\psi(\zeta) = -\varphi\left(\frac{1}{\zeta}\right) - \frac{\omega(\zeta)}{\omega'(\zeta)}\varphi'(\zeta). \tag{V.224}$$

For the case when forces of tension p are applied "at infinity" in the direction perpendicular to the surfaces of the cracks, the stress function can be represented in the form

$$\varphi_N(\zeta) = pR_1\left[\frac{\zeta}{4} + \sum_{n=1}^{N}\alpha_n\zeta^{1-kn}\right], \tag{V.225}$$

where α_n are real coefficients.

Repeating all calculations and considerations given above, we find that $\psi(\zeta)$ for real z (on the extension of the cracks) also are expressed by formula (V.224), and coefficients α_n of function $\phi_N(\zeta)$ are defined by algebraic equation systems: /456

when k = 1

$$\alpha_p + \sum_{n=1}^{N-p}\alpha_{n+p}c_n(1-n) + \sum_{n=1}^{N-p}c_{n+p}\alpha_n(1-n) + \frac{c_p}{4} = \begin{cases}0, & p \neq 2,\\ -\frac{1}{2}, & p = 2;\end{cases} \tag{V.226}$$

when k = 2

$$\alpha_p + \sum_{n=1}^{N-p}\alpha_{n+p}c_n(1-2n) + \sum_{n=1}^{N-p}c_{n+p}\alpha_n(1-2n) + \frac{c_p}{4} = \begin{cases}0, & p > 1,\\ -\frac{1}{2}, & p = 1;\end{cases} \tag{V.227}$$

$$p = 1, 2, 3, \ldots, N.$$

After substituting the variable z by $\omega(\zeta)$, formulas (I.13) are transformed to the form

$$\begin{aligned}\sigma_x + \sigma_y &= 4\,\mathrm{Re}\{\Phi(\zeta)\},\\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2\left\{\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi'(\zeta) + \Psi(\zeta)\right\}.\end{aligned} \tag{V.228}$$

Due to the total symmetry of the problem we may confine our analysis to the right hand crack alone (if there are two of them).

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The stresses near the end of the crack are found from (V.228), recalling relations (V.214) and (V.224):

$$\sigma_y(x,0)=\frac{1}{\omega'(1+\varrho)}\left[\varphi'(1+\varrho)+\frac{1}{(1+\varrho)^2}\varphi'\left(\frac{1}{1+\varrho}\right)\right], \tag{V.229}$$

where

$$\varrho=f(l,r_k)\sqrt{s};$$

s is a small distance from the examined point on the x axis to the end of the crack (see Figure V.25) and $f(l, r_k)$ is a known function of the length of the crack l and characteristic parameters of hole r_k.

Elliptical Hole. Relation (V.229) is converted to the following form:

when k = 1

$$\sigma_y(x,0)=\frac{p}{2}\left[\tilde{\varphi}'(\zeta)+\frac{1}{\zeta^2}\tilde{\varphi}'\left(\frac{1}{\zeta}\right)\right]\sqrt{\frac{a+\frac{l}{2}\left[1-\frac{1}{(1+l_0)^2}\right]\left[1+\frac{1}{1+l_0}\right]}{2s\left[1-\frac{m}{(1+l_0)^2}\right]\left[1+\frac{m}{1+l_0}\right]}}; \tag{V.230}$$

when k = 2 /457

$$\sigma_y(x,0)=\frac{p}{2}\left[\tilde{\varphi}'(\zeta)+\frac{1}{\zeta^2}\tilde{\varphi}'\left(\frac{1}{\zeta}\right)\right]\sqrt{\frac{a+l}{2s}\cdot\frac{1-\frac{1}{(1+l_0)^4}}{1-\frac{m^2}{(1+l_0)^4}}}, \tag{V.231}$$

where

$$\tilde{\varphi}'(\zeta)=\frac{\varphi'(\zeta)}{R_1 p}.$$

The load will be critical if the following condition is satisfied[1]:

$$\sigma_y(x,0)=\frac{K}{\pi\sqrt{s}}+O(1), \tag{V.232}$$

where K is the coupling modulus.

[1]See G. I. Barenblatt [2].

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From (V.230)-(V.232) we find the expressions for the determination of the critical load required for the beginning of the development of cracks near an elliptical hole:

when k = 1

$$p_* = \frac{K}{\pi\tilde{\varphi}'(1)}\sqrt{\frac{2}{a+\frac{l}{2}}\cdot\frac{\left[1-\frac{m}{(1+l_0)^2}\right]\left[1+\frac{m}{1+l_0}\right]}{\left[1-\frac{1}{(1+l_0)^2}\right]\left[1+\frac{1}{1+l_0}\right]}}; \qquad \text{(V.233)}$$

when k = 2

$$p_* = \frac{K}{\pi\tilde{\varphi}'(1)}\sqrt{\frac{2}{a+l}\cdot\frac{1-\frac{m^2}{(1+l_0)^4}}{1-\frac{1}{(1+l_0)^4}}}. \qquad \text{(V.234)}$$

Since the functions $\phi'(\zeta)$ and $\phi_N'(\zeta)$ at the point $\zeta = 1$ do not have singularities and when $N \to \infty \phi_N'(1) \to \phi'(1)$, then by taking N sufficiently large, we can make the difference between $\phi_N'(1)$ and $\phi'(1)$ as small as we like. Therefore, for rather large N, by substituting in (V.233) and (V.234) $\tilde{\phi}'(1)$ by $\tilde{\phi}_N'(1) = \phi_N'(1)/R_1 p$, we obtain approximate formulas which enable us with a sufficient degree of accuracy to calculate the critical load.

Figure V.26 shows the curves $\tilde{p}_* = \pi p_*/K\sqrt{R}$ as a function of $\lambda = l/R$ for various m = (a - b)/(a + b) in the case of multifold tension when k = 2, constructed by formulas (V.233) and (V.234), where the function $\tilde{\phi}'(1)$ was approximated by function $\phi_N'(1)$. Hence, mapping function (V.218) retains 34 /458
terms. Figure V.27 represents the analogous curves for the case of uniaxial tension of a plane[1].

The data presented show that in the case of multifold tension for all $0 < m < 1$ even when $l > R/2$, the function $\phi_N'(1)$ differs from unity by the order of magnitude of 5%, and as λ increases this difference diminishes. Thus, in the given case, by substituting $\phi'(1)$ by one, we obtain simple approximate formulas:

[1]As shown by A. A. Kaminskiy [1] when k = 1 the curves of the dependence of $\tilde{p}_*$ on λ correspond qualitatively to the curves shown in Figures V.26 and V.27.

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when k = 1

$$p_* = \frac{K}{\pi}\sqrt{\frac{2}{a+\frac{l}{2}}\cdot\frac{\left[1-\frac{m}{(1+l_0)^2}\right]\left[1+\frac{m}{1+l_0}\right]}{\left[1-\frac{1}{(1+l_0)^2}\right]\left[1+\frac{1}{1+l_0}\right]}}; \qquad \text{(V.235)}$$

when k = 2

$$p_* = \frac{K}{\pi}\sqrt{\frac{2}{a+l}\cdot\frac{1-\frac{m^2}{(1+l_0)^4}}{1-\frac{1}{(1+l_0)^4}}}. \qquad \text{(V.236)}$$

From relations (V.233) and (V.234) when m = 1, i.e. in the case of a rectilinear insulated crack, we obtain the known Griffith's formula[1]:

$$p_* = \frac{K}{\pi}\sqrt{\frac{2}{c}}, \qquad \text{(V.237)}$$ /459

where

$$c = a + \frac{l}{2} \text{ for } k = 1 \text{ and } c = a + l \text{ for } k = 2.$$

Also examined[2] is the limiting case of an elliptical hole when b > a and a → 0, i.e. when m → -1. Here it is assumed that the edges of a vertical slit, which are parallel to forces of tension p, do not come into contact with each other.

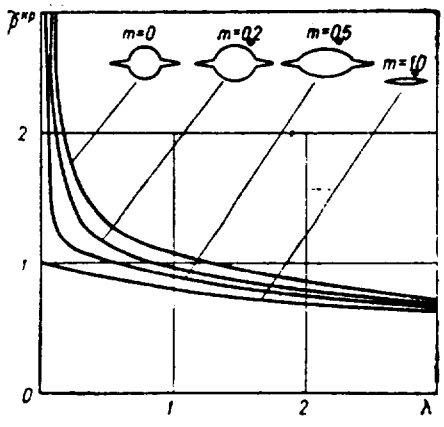


Figure V.26.

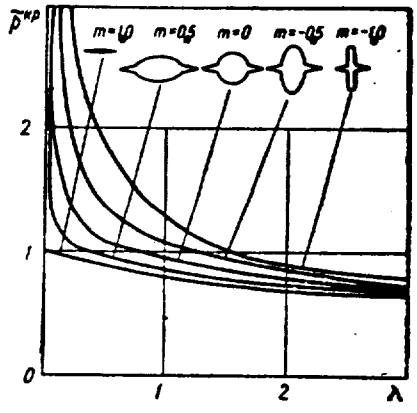


Figure V.27.

[1]See Griffith [1, 2].
[2]See A. A. Kaminskiy [1, 2], I. Yu. Babich, A. A. Kaminskiy [1].

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The critical load for a cross-shaped crack formed at the limit by an elliptical hole and two horizontal cracks, extending to the contour of the hole for $m \to -1$, differ from the critical load (V.237) for a rectilinear crack by the order of magnitude of 10%, which is evidence of the slight effect that the vertical slit has on the development of a horizontal crack.

Analyses conducted with the help of the computer have shown that for cracks of great length, it is necessary to retain a comparatively small number of terms in the mapping function (V.218) in order to obtain the required accuracy (of the order of 5%) of results; as the length of the crack decreases, it is necessary to retain in mapping function (V.218) a rather large number of terms, 24 and 34, in order to obtain the same degree of accuracy. As shown by calculations, an increase in the number of terms in mapping function (V.218) by 10 results in a comparitively small deviation in the magnitude of the critical load.

Round Hole[1]. If in relations (V.233) and (V.234) we assume that $m = 0$, we obtain the critical load for a round hole of radius R. In the case of multifold tension, from (V.233) and (V.234), for $m = 0$, we obtain for all values of λ, simple formulas[2]:

for $k = 1$

$$p_* = \frac{K}{\pi}\sqrt{\frac{2}{R+\frac{l}{2}}\cdot\frac{1}{\left[1-\left(\frac{1}{1+\lambda}\right)^2\right]\left[1+\frac{1}{1+\lambda}\right]}}; \qquad \text{(V.238)}$$

for $k = 2$

$$p_* = \frac{K}{\pi}\sqrt{\frac{2}{R+l}\cdot\frac{1}{1-\left(\frac{1}{1+\lambda}\right)^4}}. \qquad \text{(V.239)}$$

Formulas (V.238) and (V.239) and the curves of the dependence of $\tilde{p}$ on (see Figures V.26 and V.27), obtained by A. A. Kaminskiy [1], are in good agreement with the results of O. L. Bowie [1], obtained by a more difficult method with the aid of Griffith's method. /460

Effect of Local Stress Field near Hole on Development of Cracks. In the limiting case for $\lambda \to 0$,

---

[1]The problem of the development of cracks near a round hole, in a statement different from the one presented here is examined in the works of H. F. Bueckner [1], P. M. Vitvitskiy and M. Ya. Leonov [1] by different methods. The solution of P. M. Vitvitskiy and M. Ya. Leonov is given in §5, Chapter VIII.
[2]See A. A. Kaminskiy [1].

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$$\frac{(p_*)^{**}}{(p_*)^{*}} \to \frac{k^{**}}{k^{*}},$$

where the case of uniaxial tension is denoted by two asterisks, multifold tension, by one asterisk and k** and k* denote, respectively, stress concentration factors on the contour of the hole at the points of intersection of the contour of the hole with the Ox axis. This fact is evidence of the considerable effect that the local stress field has on the development of small cracks.

As follows from §2, Chapter II, a slit that coincides with the direction of the forces of tension p has no effect on the stress state of the plane. Hence[1]

$$\frac{(p_*)^{**}}{(p_*)^{*}} = 1.$$

Calculations show that in all examined cases, as the length of the crack increases

$$\frac{(p_*)^{**}}{(p_*)^{*}} \to 1;$$

in the case of a round hole with cracks, even when $l > R$, the stress field near the hole, for all practical purposes, has little effect (about 5%) on the magnitude of the critical load:

λ	0.1	0.3	0.5	0.7	0.9	1.0	2.0	3.0	4.0
$\frac{(p_*)^{**}}{(p_*)^{*}}$	1.3936	1.2543	1.1721	1.1214	1.0885	1.0766	1.0237	1.0100	1.0053

In the case of an elliptical hole, the zone of influence of the hole on the development of a crack becomes smaller, where $m \to 1$, i.e. when the ellipse is converted into a horizontal slit.

[1] See Griffith [1].

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CHAPTER VI. EFFECT OF ASYMMETRY OF STRESS TENSOR ON STRESS DISTRIBUTION NEAR HOLES

Abstract. This chapter discusses the influence of the couple-stresses of the theory of elasticity on the stress concentration near holes. Closed solutions are given of the plane boundary value problems of the couple stresses of the theory of elasticity for the domain with circular hole both free and reinforced. The "boundary form perturbation" method was applied to solve the problems on stress concentration near curvilinear holes such as elliptic, square and triangular.

§1. General Comments

Classical elasticity theory explains quite satisfactorily the behavior of /464
real solid bodies located under various loads, in all cases where the "grain" of the structure of the bodies under examination is not characteristic for these phenomena. However, classical elasticity theory cannot satisfactorily explain certain phenomena which can be observed in real elastic bodies. In particular, from the point of view of theoretical solutions of problems of classical elasticity theory, it is not possible to explain and predict the laws of propagation of short sound waves in crystalline solid bodies, polycrystalline metals, and high polymers.

Nor does classical elasticity theory explain the effect of the stress gradient on the fatigue characteristics of polycrystalline materials.

The cause of the above discrepancies obviously can be attributed primarily to the fact that the solid model of a rigid body upon which classicial elasticity theory is based cannot reproduce the elastic properties of real bodies, which are determined by their specific structure.

Obviously, in order to explain these phenomena, dense medium mechanics requires a new model of the solid body, in which the properties attributed to the specific structure of real bodies can be reproduced accurately.

We will examine one of the possible approaches to the construction of such a model.

Thus, we will assume that small displacements of material points in a dense elastic medium are defined by two vector fields:

$$\vec{u} = \vec{u}(x, y, z),$$
$$\vec{\omega} = \vec{\omega}(x, y, z). \qquad \text{(VI.1)}$$

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The first vector $\vec{u}$ characterizes the small displacements, and the second vector $\vec{\omega}$, the small rotations. The vector $\vec{\omega}$, generally speaking, can be regarded as kinematically independent of the vector $\vec{u}$.

The stress state at each point of this dense medium in any plane characterized by the perpendicular line $\vec{n}$, will be determined by the vectors of ordinary $\vec{\tau}^{(n)}$ and moment $\vec{\mu}^{(n)}$ stresses. The mass forces in this medium will also be of two forms: ordinary $\vec{F}$ and moment $\vec{M}$.

We know from the dense medium theory that vector $\vec{u}$ defines simultaneously both deformation /465

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i, j = 1, 2, 3) \tag{VI.2}$$

of an infinitely small vicinity of the examined point of the dense medium and rotation (as a rigid whole)

$$\vec{\omega} = \frac{1}{2}\operatorname{rot}\vec{u} \tag{VI.3}$$

of the examined vicinity of the point of the dense medium.

If to this model external forces are applied, or if the corresponding displacements are imparted to the surface points of the model, then, under the corresponding conditions, a stress state will occur within it, characterized by an asymmetric stress tensor, i.e. stresses will occur for which the law of conjugation of tangential stresses will not be satisifed.

W. Voigt [1] first examined in 1887 a model of a medium with rotational interaction of its particles for the analysis of the elastic properties of crystals.

The first attempt to construct elasticity theory with an asymmetric stress tensor was undertaken in 1909 by E. and G. Cosserat [1]. They based this branch of elasticity theory on the aforementioned model of a dense elastic medium with rotational interaction of the particles.

There are different approaches to the description of displacements of particles of the new dense medium model.

In the works of E. L. Aero and Ye. V. Kuvshinskiy [1, 2], R. D. Mindlin and G. F. Tirsten [1], Yu. N. Nemish [1-4], Khartrenft and Si [1], Khopmen and Shouena [1], Veytsman [1], V. T. Koyter [1] and M. Misicu [1-3], the conception of classical elasticity theory is retained, i.e. it is felt that displacements

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$\vec{u}$ of the points of this medium and their rigid small rotations $\vec{\omega}$ are completely defined by the vector $\vec{u}(x, y, z)$ for $\vec{\omega} = 1/2 \operatorname{rot} \vec{u}$. We will call this variant of elasticity theory with an asymmetric stress tensor variant one.

V. A. Pal'mov [1, 2], V. Novatskiy [1] and N. Neuber [1] for the description of displacements of particles of the examined medium introduced along with the ordinary stress field $\vec{u}(x, y, z)$, a kinematically independent field of vectors $\vec{\Phi}(x, y, z)$ which characterize small rotations of the particles of the medium. This variant of moment elasticity theory will be referred to as variant two.

It is easy to see that general rotation of the particles of the medium in variant two of this theory will consist of the sum of two independent terms: rotation $\vec{\omega} = 1/2 \operatorname{rot} \vec{u}$, produced by vector $\vec{u}$, and rotation produced by the vector $\vec{\Phi}$. Naturally, these variants, i.e. different assumptions concerning the nature of deformation of the new dense medium model, lead to different numbers of elastic constants. Thus, in variant one of moment elasticity theory, for an ideally isotropic medium, the elastic behavior of the new model is characterized by four elastic constants: E, ν, l and η, of which E is Young's modulus, ν is Poissons's ratio, l and η are new material constants (l is length and η is a dimensionless constant of the Poisson type). In variant two the elastic behavior of the new isotropic model is characterized by six elastic constants: E, ν, l, η_1, η_2, η_3, of which E and ν are, respectively, Young's modulus and Poisson's ratio, l is a new constant denoting length, η_1, η_2 and η_3 are new dimensionless constants. The elastic behavior of the new anisotropic (in the same general case of anisotropy) model of moment elasticity theory in variant two will be characterized[1] by 171 elastic constants[2]. /466

Below we will examine a plane problem of elasticity theory with an asymmetric stress tensor, assuming that deformation of the medium is completely characterized by displacement vector $\vec{u}$, while the vector $\vec{\omega}$, which defines "rigid" rotation of the particles of this medium, is expressed by formula (VI.3). We will assume the medium to be ideally elastic and isotropic.

The most significant corrections to the solutions of problems of classical elasticity theory should be anticipated for those cases where the stress state has the greatest stress gradient. This class of problems will include, in particular the problems of stress concentration near holes.

We know that by selecting the appropriate shape of the hole and orientation in relation to external forces, we can determine the large stress gradient

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[1] See H. Neuber [1].

[2] In classical elasticity theory the general case of anisotropy of a medium requires the knowledge of 21 elastic constants.

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near the hole. Therefore it is very important to analyze the effect of the asymmetry of the stress tensor on stress concentration around holes.

The results derived in this chapter for the plane problem, applicable to variant one of moment theory, will be valid[1] also for asymmetric elasticity theory in variant two, since the basic equations for the plane problem, in particular for plane deformation, in these variants, as indicated by V. A. Pal'mov [1, 2], coincide with an accuracy up to constant factors.

2. Plane Problem of Moment Elasticity Theory of Isotropic Medium

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Basic Equations and Relations. The plane problem of moment elasticity theory in the absence of three dimensional forces and moments reduces to the integration of a system of equilibrium equations[2] in stresses and moment stresses

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$
$$\frac{\partial \mu_x}{\partial x} + \frac{\partial \mu_y}{\partial y} + \tau_{xy} - \tau_{yx} = 0 \quad (\tau_{xy} \neq \tau_{yx}) \tag{VI.4}$$

and compatibility conditions

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu \nabla^2 (\sigma_x + \sigma_y) = \frac{\partial^2}{\partial x \partial y}(\tau_{xy} + \tau_{yx}),$$
$$\frac{\partial \mu_x}{\partial y} = \frac{\partial \mu_y}{\partial x},$$
$$\mu_x = l^2 \left\{ \frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2 \frac{\partial}{\partial y}[\sigma_y - \nu(\sigma_x + \sigma_y)] \right\}, \tag{VI.5}$$
$$\mu_y = l^2 \left\{ \frac{\partial}{\partial x}[\sigma_y - \nu(\sigma_x + \sigma_y)] - \frac{\partial}{\partial y}(\tau_{xy} + \tau_{yx}) \right\}$$

under the corresponding boundary conditions.

In the case of plane deformation for an isotropic elastic medium, the elasticity relations are of the form

---

[1] See G. N. Savin [2].
[2] See R. D. Mindlin [1].

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$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{1+\nu}{E}[\sigma_x - \nu(\sigma_x + \sigma_y)],$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{1+\nu}{E}[\sigma_y - \nu(\sigma_x + \sigma_y)],$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1+\nu}{2E}(\tau_{xy} + \tau_{yx}), \qquad \text{(VI.6)}$$
$$\sigma_z = \nu(\sigma_x + \sigma_y),$$

where E is Young's modulus; ν is Poisson's ratio.

Following the general trend of linear elasticity theory, we assume that the curvatures κ_x and κ_y of the fibers are directly proportional to moment stresses μ_x and μ_y, i.e.

$$\varkappa_x = \frac{\mu_x}{4\eta}, \quad \varkappa_y = \frac{\mu_y}{4\eta}, \qquad \text{(VI.7)}$$

where η is a new constant of the material (deflection-torsion modulus).

The connection between stresses σ_x, σ_y, τ_{xy}, τ_{yx} and moment stresses μ_x and μ_y is accomplished with the aid of the relations[1] /468

$$\mu_x = l^2\left\{\frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2\frac{\partial}{\partial y}[\sigma_y - \nu(\sigma_x + \sigma_y)]\right\},$$
$$\mu_y = l^2\left\{\frac{\partial}{\partial x}[\sigma_y - \nu(\sigma_x + \sigma_y)] - \frac{\partial}{\partial y}(\tau_{xy} + \tau_{yx})\right\}; \qquad \text{(VI.8)}$$

hence

$$l^2 = \frac{2(1+\nu)\eta}{E} = \frac{\eta}{G}. \qquad \text{(VI.9)}$$

It is easy to see from formulas (VI.7) and (VI.8) that the deflection-torsion modulus η has the dimension of force, and in view of the fact that displacement modulus G has the dimension of force divided by the square of the length, the constant l has the dimension of length, which can be used, instead of the constant η, as the new material constant.

Stress Functions. If stress functions U and F are given by relations[2]

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[1] See H. Schaefer [1], R. D. Mindlin [1].
[2] See H. Neuber [1], R. D. Mindlin [1].

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$$\sigma_x = \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial y},$$

$$\tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 F}{\partial y^2}, \quad \tau_{yx} = -\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}, \tag{VI.10}$$

$$\mu_x = \frac{\partial F}{\partial x}, \quad \mu_y = \frac{\partial F}{\partial y},$$

then, in the absence of forces and three dimensional moment, the plane problem of moment elasticity theory reduces to the solution of the equations

$$\nabla^2 \nabla^2 U = 0,$$
$$\nabla^2 (F - l^2 \nabla^2 F) = 0 \tag{VI.11}$$

under certain boundary conditions.

Stress functions U and F are not independent; they are related by conditions

$$\frac{\partial}{\partial x}(F - l^2 \nabla^2 F) = -2(1 - \nu) l^2 \frac{\partial}{\partial y} \nabla^2 U,$$
$$\frac{\partial}{\partial y}(F - l^2 \nabla^2 F) = 2(1 - \nu) l^2 \frac{\partial}{\partial x} \nabla^2 U, \tag{VI.12}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{VI.13}$$

Stresses and moment stresses σ_r, σ_θ, $\tau_{r\theta}$, $\tau_{\theta r}$, μ_r and μ_θ in polar coordinates (r, θ) are expressed through the values σ_x, σ_y, τ_{xy}, τ_{yx}, μ_x and μ_y corresponding to them in Cartesian coordinates: /469

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + (\tau_{xy} + \tau_{yx}) \sin\theta \cos\theta,$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - (\tau_{xy} + \tau_{yx}) \sin\theta \cos\theta,$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x) \sin\theta \cos\theta + \tau_{xy} \cos^2\theta - \tau_{yx} \sin^2\theta,$$
$$\tau_{\theta r} = (\sigma_y - \sigma_x) \sin\theta \cos\theta - \tau_{xy} \sin^2\theta + \tau_{yx} \cos^2\theta, \tag{VI.14}$$
$$\mu_r = \mu_x \cos\theta + \mu_y \sin\theta,$$
$$\mu_\theta = -\mu_x \sin\theta + \mu_y \cos\theta,$$

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where

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}. \tag{VI.15}$$

If relations (VI.10) are substituted into formulas (VI.14) considering the form of operators

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial\theta}, \end{aligned} \tag{VI.16}$$

then, by formulas (VI.14) we can prescribe the form

$$\begin{aligned} \sigma_r &= \frac{1}{r}\cdot\frac{\partial U}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 U}{\partial\theta^2} - \frac{1}{r}\cdot\frac{\partial^2 F}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial F}{\partial\theta}, \\ \sigma_\theta &= \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial^2 E}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial F}{\partial\theta}, \\ \tau_{r\theta} &= -\frac{1}{r}\cdot\frac{\partial^2 U}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial U}{\partial\theta} - \frac{1}{r}\cdot\frac{\partial E}{\partial r} - \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}, \\ \tau_{\theta r} &= -\frac{1}{r}\cdot\frac{\partial^2 U}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial U}{\partial\theta} + \frac{\partial^2 F}{\partial r^2}, \\ \mu_r &= \frac{\partial F}{\partial r}, \quad \mu_\theta = \frac{1}{r}\cdot\frac{\partial F}{\partial\theta}. \end{aligned} \tag{VI.17}$$

In polar coordinates conditions (VI.12) will acquire the form

$$\begin{aligned} \frac{\partial}{\partial r}(F - l^2\nabla^2 F) &= -2(1-\nu)\,l^2\,\frac{1}{r}\cdot\frac{\partial}{\partial\theta}\nabla^2 U, \\ \frac{\partial}{\partial\theta}(F - l^2\nabla^2 F) &= 2(1-\nu)\,l^2 r\,\frac{\partial}{\partial r}\nabla^2 U, \end{aligned} \tag{VI.18}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}. \tag{VI.19}$$

Basic Boundary Problems of Statically Elastic Body[1]. /470

First Basic Boundary Problem. The first basic boundary problem is defined elastic equilibrium of a body if the components of external forces (stresses

[1]See G. N. Savin [2], §3, p. 7.

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X_n, Y_n and moment stresses M_n) acting on the surface S of the examined body are known.

The boundary conditions in the case of the plane problem, i.e. on the contour L, have the form

$$\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{yx} \cos(n, y), \\ Y_n &= \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y), \\ M_n &= \mu_x \cos(n, x) + \mu_y \cos(n, y). \end{aligned} \tag{VI.20}$$

Here $\vec{n}$ is the unit perpendicular to contour L at the given point.

Boundary conditions (VI.20) are readily expressed through stress functions U and F:

$$\begin{aligned} X_n &= \frac{d}{ds}\left(\frac{\partial U}{\partial y} - \frac{\partial F}{\partial x}\right), \\ Y_n &= -\frac{d}{ds}\left(\frac{\partial U}{\partial x} + \frac{\partial F}{\partial y}\right), \\ M_n &= \frac{\partial F}{\partial n}, \end{aligned} \tag{VI.21}$$

where we recall that

$$\cos(n, x) = \frac{dy}{ds}, \quad \cos(n, y) = -\frac{dx}{ds}. \tag{VI.22}$$

Second Basic Boundary Problem. The second basic boundary problem is to find elastic equilibrium of a body if the components of displacement vector $\vec{u}$ and the component of rotation vector $\vec{\omega}$ = 1/2 rot $\vec{u}$, which lies in the plane tangent to surface S at the examined point, are known on its surface S.

The boundary conditions on contour L have the form

$$u = f_1(x, y), \quad v = f_2(x, y), \quad \omega_z = \omega = f_3(x, y), \tag{VI.23}$$

where $f_1(x, y)$, $f_2(x, y)$ and $f_3(x, y)$ are the given functions on the boundary of the examined range, i.e. on contour L.

Third (Mixed) Boundary Problem. The third boundary problem is to find the stress components as functions of the coordinates in accordance with the given external forces X_n, Y_n and M_n on part S_1 of the surface and according to the

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given displacements and rotations on the remaining part $S_2 = S - S_1$ of the surface. For the plane problem, the role of surface S will be played by contour L which bounds the given range.

Thus, in the case of the mixed plane boundary problem, the boundary conditions on part L_1 of the contour have the form (VI.20), and on the remainder $L_2 = L - L_1$ of the contour, the boundary conditions have the form (VI.23).

§3. Application of Theory of Complex Variable Functions to Solution of Boundary Problems of Moment Elasticity Theory

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Certain Relations in Complex Variables. Turning in formulas (VI.17) to complex variables $z = x + iy$ and $\bar{z} = x - iy$, we obtain

$$\begin{aligned}
\sigma_r &= 2\left[\frac{\partial^2 U}{\partial z \partial \bar{z}} - \operatorname{Re}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 U}{\partial z^2}\right) + \operatorname{Im}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 F}{\partial z^2}\right)\right],\\
\sigma_\theta &= 2\left[\frac{\partial^2 U}{\partial z \partial \bar{z}} + \operatorname{Re}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 U}{\partial z^2}\right) - \operatorname{Im}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 F}{\partial z^2}\right)\right],\\
\tau_{r\theta} &= 2\left[-\frac{\partial^2 F}{\partial z \partial \bar{z}} + \operatorname{Re}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 F}{\partial z^2}\right) + \operatorname{Im}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 U}{\partial z^2}\right)\right],\\
\tau_{\theta r} &= 2\left[\frac{\partial^2 F}{\partial z \partial \bar{z}} + \operatorname{Re}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 F}{\partial z^2}\right) + \operatorname{Im}\left(\frac{z}{\bar{z}} \cdot \frac{\partial^2 U}{\partial z^2}\right)\right],\\
\mu_r &= 2\operatorname{Re}\left(\sqrt{\frac{z}{\bar{z}}} \cdot \frac{\partial F}{\partial \bar{z}}\right); \quad \mu_\theta = -2\operatorname{Im}\left(\sqrt{\frac{z}{\bar{z}}} \cdot \frac{\partial F}{\partial z}\right).
\end{aligned} \tag{VI.24}$$

We will take the general solution of the second equation of system (VI.11) in the form

$$iF = f(z) - \overline{f(z)} + i\Omega, \tag{VI.25}$$

where f(z) is some harmonic function and Ω is the solution of equation

$$\Omega - l^2 \nabla^2 \Omega = 0. \tag{VI.26}$$

The biharmonic function can be represented by E. Gurs' formula

$$U = \operatorname{Re}[\bar{z}\varphi(z) + \chi(z)], \tag{VI.27}$$

where $\phi(z)$, $\chi(z)$ are functions that are holomorphic in a singly-connected range.

By substituting expressions (VI.25) and (VI.26) into conditions (VI.18), we obtain

$$f(z) = 4(1 - \nu) l^2 \varphi'(z). \tag{VI.28}$$

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If functions (VI.25) and (VI.27) are substituted into (VI.24), recalling condition (VI.28), then, after certain transformations, we obtain the formulas for stress components[1]:

$$\sigma_r = \operatorname{Re}[2\Phi(z) - H(z, \bar{z})],$$
$$\sigma_\theta = \operatorname{Re}[2\Phi(z) + H(z, \bar{z})],$$
$$\tau_{r\theta} = -\frac{1}{2}\nabla^2\Omega + \operatorname{Im} H(z, \bar{z}), \qquad \text{(VI.29)} \quad \underline{/472}$$
$$\tau_{\theta r} = \frac{1}{2}\nabla^2\Omega + \operatorname{Im} H(z, \bar{z}),$$
$$\sigma_r - i\tau_{r\theta} = \Phi(z) + \overline{\Phi(z)} + \frac{i}{2l^2}\Omega - H(z, \bar{z}),$$

where

$$\mu_r = \operatorname{Re} L(z, \bar{z}), \quad \mu_\theta = -\operatorname{Im} L(z, \bar{z}),$$

$$H(z, \bar{z}) = \frac{z}{\bar{z}}\left[\bar{z}\Phi'(z) + 8(1-\nu)l^2\Phi''(z) + \Psi(z) + 2i\frac{\partial^2\Omega}{\partial z^2}\right],$$
$$L(z, \bar{z}) = 2\sqrt{\frac{z}{\bar{z}}}\left[-4i(1-\nu)l^2\Phi'(z) + \frac{\partial\Omega}{\partial z}\right], \qquad \text{(VI.30)}$$
$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z), \quad \psi(z) = \chi'(z).$$

From (VI.29) it is easy to find

$$\sigma_\theta + \sigma_r = 4\operatorname{Re}[\Phi(z)], \quad \tau_{\theta r} - \tau_{r\theta} = \frac{1}{l^2}\Omega,$$
$$\sigma_\theta - \sigma_r + i(\tau_{r\theta} + \tau_{\theta r}) =$$
$$= 2e^{2i\theta}\left[\bar{z}\Phi'(z) + 8(1-\nu)l^2\Phi''(z) + \Psi(z) + 2i\frac{\partial^2\Omega}{\partial z^2}\right], \qquad \text{(VI.31)}$$
$$\mu_r - i\mu_\theta = 2e^{i\theta}\left[-4i(1-\nu)l^2\Phi'(z) + \frac{\partial\Omega}{\partial z}\right].$$

From elasticity equations (VI.6), recalling the representation of functions (VI.27) and (VI.28), we find

$$2\mu(u + iv) = (3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - 8(1-\nu)l^2\overline{\varphi''(z)} -$$
$$- \overline{\psi(z)} + 2i\frac{\partial\Omega}{\partial\bar{z}}. \qquad \text{(VI.32)}$$

By representing rotation in the form

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2i}\left[\frac{\partial(u + iv)}{\partial z} - \frac{\partial(u - iv)}{\partial\bar{z}}\right] \qquad \text{(VI.33)}$$

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[1] See Yu. N. Nemish [1].

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and substituting here the values u + iv and u - iv from (VI.32), we obtain

$$4\eta\omega = \Omega - 4i(1-\nu)l^2[\varphi'(z) - \overline{\varphi'(z)}]. \tag{VI.34}$$

Boundary conditions (VI.20), (VI.21) and (VI.23) of the basic problems of moment elasticity theory can be represented in the following form:

for the first basic problem

$$\begin{gathered}\Phi(z) + \overline{\Phi(z)} + \frac{i}{2l^2}\Omega - e^{2i\theta}\left[\bar{z}\Phi'(z) + 8(1-\nu)l^2\Phi''(z) + \Psi(z) + \right. \\ \left. + 2i\frac{\partial^2\Omega}{\partial z^2}\right] = N - iT, \\ \operatorname{Re}\left\{e^{i\theta}\left[-4i(1-\nu)l^2\Phi'(z) + \frac{\partial\Omega}{\partial z}\right]\right\} = M;\end{gathered} \tag{VI.35}$$

for the second basic problem /473

$$\begin{gathered}(3-4\nu)\varphi(z) - z\overline{\varphi'(z)} - 8(1-\nu)l^2\overline{\varphi''(z)} - \overline{\psi(z)} + 2i\frac{\partial\Omega}{\partial\bar{z}} = 2\mu(q_1 + iq_2), \\ \Omega + 8(1-\nu)l^2\operatorname{Im}[\varphi'(z)] = 4\eta q_3,\end{gathered} \tag{VI.36}$$

where M, N, T, q_1, q_2, q_3 are the given functions on boundary L of the examined range.

In solving specific problems, complex potentials $\phi(z)$ and $\psi(z)$ will be taken in the same general form[1] as for the corresponding problems of classical elasticity theory.

The total solution of Helmholtz' equation (VI.26) has the form[2]

$$\begin{gathered}\Omega = c_0K_0(r/l) + \sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta)K_n(r/l) + \\ + h_0I_0(r/l) + \sum_{n=1}^{\infty}(h_n\cos n\theta + t_n\sin n\theta)I_n(r/l).\end{gathered} \tag{VI.37}$$

Here $I_n(r/l)$ (n = 0, 1, ...) are modified Bessel functions of kind I; $K_n(r/l)$ are modified Bessel functions of kind II (Macdonald's functions).

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[1]See N. I. Muskhelishvili [1], G. N. Savin [1].
[2]See G. N. Vatson [1].

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In solving specific problems of stress concentration for infinite ranges with holes, it is necessary to retain in equation (VI.37) the first part only, since functions $I_n(r/l)$ when $r \to \infty$, increase without bounds. Analogously in examining finite ranges that contain the point $r = 0$, in solving the problem of stress concentration, it is necessary to retain in equation (VI.37) only its second part.

Statement of Problem of Stress Concentration near Holes. We will examine an elastic isotropic plate (plane deformation) or thin plate (plane stress state), where, under the corresponding conditions, a stress state that is characterized by an asymmetric stress tensor can occur[1].

We will assume that in an elastic plate that is in an unstressed state characterized by components

$$\sigma_x^0,\ \sigma_y^0,\ \tau_{xy}^0,\ \tau_{yx}^0,\ \mu_x^0,\ \mu_y^0, \tag{VI.38}$$

a hole of arbitrary shape is made, the contour of which has no angular points. The hole, generally speaking, causes certain redistributions of stresses in this plate, particularly in the range directly adjacent to the hole, and /474 instead of stress state (VI.38), we will have a new stress distribution:

$$\sigma_x^*,\ \sigma_y^*,\ \tau_{xy}^*,\ \tau_{yx}^*,\ \mu_x^*,\ \mu_y^*. \tag{VI.39}$$

Due to the linearity of the problem, stress state (VI.39) can be represented in the form

$$\begin{aligned} \sigma_x^* &= \sigma_x^0 + \sigma_x, & \sigma_y^* &= \sigma_y^0 + \sigma_y, \\ \tau_{xy}^* &= \tau_{xy}^0 + \tau_{xy}, & \tau_{yx}^* &= \tau_{yx}^0 + \tau_{yx}, \\ \mu_x^* &= \mu_x^0 + \mu_x, & \mu_y^* &= \mu_y^0 + \mu_y, \end{aligned} \tag{VI.40}$$

where

$$\sigma_x,\ \sigma_y,\ \tau_{xy},\ \tau_{yx},\ \mu_x,\ \mu_y - \tag{VI.41}$$

are moments of an additional stress state caused by the hole.

To the basic stress state (VI.38), and also to stress states (VI.39) and (VI.41), as follows from (VI.11), will correspond two stress functions each:

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[1]As seen from relations (VI.8), moment stresses μ_x and μ_y will not differ from zero in just any stress state.

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$U^0(x, y)$ and $F^0(x, y)$; $U^*(x, y)$ and $F^*(x, y)$; $U(x, y)$ and $F(x, y)$, which due to (VI.40) will be interelated by relations

$$U^*(x, y) = U^0(x, y) + U(x, y), \qquad F^*(x, y) = F^0(x, y) + F(x, y). \tag{VI.42}$$

We know that the effect of the hole on the stress state in the examined range has a local character, and therefore stress components (VI.41), by measure of distance from the hole, will vanish rapidly. Hence, and also from (VI.40), (VI.42) and (VI.10), we see that stress functions $U(x, y)$ and $F(x, y)$, at sufficiently distant points of the plane from the hole, should satisfy the conditions

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_{\substack{x\to\infty\\y\to\infty}} \to 0; \quad \left(\frac{\partial^2 U}{\partial x \partial y}\right)_{\substack{x\to\infty\\y\to\infty}} \to 0; \quad \left(\frac{\partial^2 U}{\partial y^2}\right)_{\substack{x\to\infty\\y\to\infty}} \to 0;$$
$$\left(\frac{\partial F}{\partial x}\right)_{\substack{x\to\infty\\y\to\infty}} \to 0; \quad \left(\frac{\partial F}{\partial y}\right)_{\substack{x\to\infty\\y\to\infty}} \to 0. \tag{VI.43}$$

Conditions (VI.43) will also be the conditions at "infinity" for the desired functions $U(x, y)$ and $F(x, y)$. From (VI.10) it is also clear that the function $U(x, y)$ is also defined by the given stress components with an accuracy up to an expression of the form $C_1x + C_2y + C_3$, and the function $F(x, y)$, with an accuracy up to the constant C_4, where C_1, C_2, C_3, C_4 are arbitrary (real) constants.

In the following discussion we will consider the effect of the asymmetry of the stress tensor on stress concentration near holes, i.e. the effect of moment stresses on the magnitudes of concentration coefficients /475

$$k_1 = \frac{\sigma_r^*}{\sigma_r^0}, \quad k_2 = \frac{\sigma_\theta^*}{\sigma_\theta^0}, \quad k_3 = \frac{\tau_{r\theta}^*}{\tau_{r\theta}^0}, \quad k_4 = \frac{\tau_{\theta r}^*}{\tau_{\theta r}^0}, \quad k_5 = \frac{\mu_r^*}{\mu_r^0}, \quad k_6 = \frac{\mu_\theta^*}{\mu_\theta^0}. \tag{VI.44}$$

On the contour of a hole that is free of external forces it is obvious that $\sigma_r^* = \tau_{r\theta}^* = \mu_r^* = 0$; consequently $k_1 = k_3 = k_5 = 0$ and only the coefficients k_2, k_4 and k_6, of which concentration coefficient k_2 will be of greatest interest to us, will be nonzero values.

It follows from formula (VI.8) that if the basic stress state is homogeneous, i.e. $\sigma_x^0 = \text{const}$ $\sigma_y^0 = \text{const}$, $\tau_{xy}^0 = \text{const}$ and $\tau_{yx}^0 = \text{const}$, there will be no moment stresses, i.e.

$$\mu_x^0 = \mu_y^0 = 0, \tag{VI.45}$$

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and

$$\tau^0_{xy} = \tau^0_{yx}.$$

If, however, the basic stress state depends linearly on the x and y coordinates, then components μ^0_x and μ^0_y will be constants, i.e.

$$\mu^0_x = \text{const}, \quad \mu^0_y = \text{const}. \tag{VI.46}$$

Since $\mu_x = \partial F/\partial x$ and $\mu_y = \partial F/\partial y$, then in the cases (VI.45) and (VI.46) we will have

$$\frac{\partial^2 F^0}{\partial x^2} = \frac{\partial^2 F^0}{\partial x \partial y} = \frac{\partial^2 F^0}{\partial y^2} = 0 \tag{VI.47}$$

and consequently, stress function $F^0(x, y)$ in these cases will have the form $F^0(x, y) = C_1 x + C_2 y + C_3$.

From (VI.10 and (VI.47) we find that for the case (VI.45) the basic stress state (VI.38) will be characterized by only one stress function $U^0(x, y)$. This enables us to assume that, in these cases, the moment stresses μ_x and μ_y that occur in the immediate vicinity of the hole do not alter the qualitative character of stress distribution near the examined hole in comparison with the analogous problem of classical elasticity theory, but affect only the dimensions of the zone of perturbations near the hole and the magnitude of stress concentration coefficients within it.

By integrating the second equation of system (VI.11), we find

$$F - l^2 \nabla^2 F = Q(x, y), \tag{VI.48}$$

where

$$Q(x, y) = 2(1 - \nu) l^2 \left\{ \int \frac{\partial}{\partial x} (\nabla^2 U)\, dy - \int \frac{\partial}{\partial y} (\nabla^2 U)\, dx + \right.$$
$$\left. + \frac{1}{2} \iint \left[\frac{\partial^2}{\partial y^2} (\nabla^2 U) - \frac{\partial^2}{\partial x^2} (\nabla^2 U) \right] dx dy \right\}. \tag{VI.49}$$

It follows from equation system (VI.11) that Q(x, y) is a harmonic function which represents the right hand side of differential equation (VI.48), and simultaneously represents the partial solution of this equation. /476

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§4. Stress Concentration near Round Hole

Solution of First Basic Problem. Uniaxial Tension of Plate with Round Hole[1]. We will examine the stress state near a round hole of radius a_0 (Figure VI.1). We will analyze the case of tension of a plane "at infinity" by forces p = const along the Ox axis, i.e. we will assume that

$$\sigma_x^0 = p, \quad \sigma_y^0 = \tau_{xy}^0 = \tau_{yx}^0 = \mu_x^0 = \mu_y^0 = 0. \tag{VI.50}$$

Complex Kolosov-Muskhelishvili potentials for this case, which are given by classical elasticity theory, have the form[2]

$$\varphi^*(z) = \frac{p}{4}\left(z + \frac{2a_0^2}{z}\right), \quad \psi^*(z) = -\frac{p}{2}\left(z + \frac{a_0^2}{z} - \frac{a_0^4}{z^3}\right). \tag{VI.51}$$

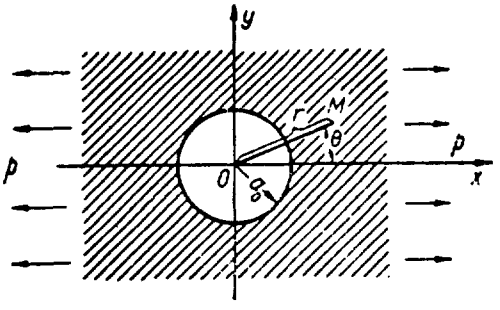


Figure VI.1.

Since the basic stress state (VI.50) satisfies condition (VI.46), then according to the above statements, the complex potentials for the stated problem in moment elasticity theory will be taken in the form

$$\varphi^*(z) = \frac{p}{4}z + \frac{C}{z},$$
$$\psi^*(z) = -\frac{p}{2}z + \frac{A}{z} - \frac{2B}{z^3}. \tag{VI.52}$$

The first terms in (VI.52) characterize the basic stress state (VI.50), and the following characterize the perturbed stress state occurring due to the presence of the hole.

From (VI.52) we obtain

$$\chi^*(z) = \int \psi^*(z)\,dz + \text{const} = -\frac{p}{4}z^2 + A\ln z + \frac{B}{z^2} + \text{const}. \tag{VI.53}$$

If the functions $\phi^*(z)$ and $\chi^*(z)$ are known, then, from E. Gurs' formula (VI.27), we obtain

$$U^*(r,\theta) = \frac{pr^2}{4}(1 - \cos 2\theta) + A\ln r + \left(\frac{B}{r^2} + C\right)\cos 2\theta. \tag{VI.54}$$

---

[1]See G. N. Savin [2]. This problem was first solved by R. D. Mindlin [1] and later by a somewhat different method by Yu. N. Nemish [1].
[2]See N. I. Muskhelishvili [1].

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In the case under examination the terms in formula (VI.42) will have the form /477

$$U^0(r, \theta) = \frac{pr^2}{4}(1 - \cos 2\theta),$$
$$U(r, \theta) = A \ln r + \left(\frac{B}{r^2} + C\right)\cos 2\theta. \tag{VI.55}$$

Since

$$\nabla^2 U^*(r, \theta) = p - \frac{4C}{r^2}\cos 2\theta, \tag{VI.56}$$

we find from conditions (VI.18)

$$F - l^2\nabla^2 F = 8C(1 - \nu)l^2\frac{1}{r^2}\sin 2\theta. \tag{VI.57}$$

According to previous considerations, the right hand side of equation (VI.57) is a harmonic function, and consequently

$$F_{part}(r, \theta) = 8C(1 - \nu)l^2\frac{1}{r^2}\sin 2\theta. \tag{VI.58}$$

The total solution of equation (VI.57) can be represented in the form

$$F(r, \theta) = \Omega(r, \theta) + F_{part}(r, \theta), \tag{VI.59}$$

where Ω is the solution of Helmholtz equation (VI.26).

The form of the partial solution of (VI.58) and the general considerations demonstrate that the solution of equation (VI.26) should be found in the form

$$\Omega(r, \theta) = R(r/l)\sin 2\theta. \tag{VI.60}$$

By substituting the function (VI.60) into equation (VI.26) considering that of the two modified functions $I_2(r/l)$ and $K_2(r/l)$, only the latter satisfies the conditions "at infinity," we obtain

$$R(r/l) = DK_2(r/l), \tag{VI.61}$$

where D is a real constant.

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Considering formulas (VI.59)-(VI.61), we obtain[1]

$$F^*(r, \theta) = F(r, \theta) = \left[DK_2(r/l) + \frac{8C(1-\nu)l^2}{r^2}\right]\sin 2\theta. \qquad \text{(VI.62)}$$

Thus functions (VI.54) and (VI.62) are determined with an accuracy up to constant coefficients A, B, C and D, which should be determined from boundary condition (VI.20). However, if the contour of the hole is free of external forces, then instead of condition (VI.20) the boundary conditions should be taken in the form /478

$$\sigma_r^* = \tau_{r\theta}^* = \mu_r^* = 0. \qquad \text{(VI.63)}$$

We will note that if in formulas (VI.14) and (VI.17) the stress components σ_x, σ_y, ..., μ_y, σ_r, σ_θ, ..., μ_θ and stress functions U and F are substituted respectively by the values σ_x^*, σ_y^*, ..., μ_y^*; σ_r^*, σ_θ^*, ..., μ_θ^*; U* and F*, we obtain the relations required for the general desired stress state (VI.39). If we use in the expressions obtained, instead of stress functions U*(r, θ) and F*(r, θ), their values from (VI.54) and (VI.62), recalling recurrent relations[2]

$$K_{m+1}(r/l) = \frac{2ml}{r}K_m(r/l) + K_{m-1}(r/l),$$
$$K'_m(r/l) = -\frac{m}{r}K_m(r/l) - \frac{1}{l}K_{m-1}(r/l) \qquad (m = 0, 1, 2, \ldots) \qquad \text{(VI.64)}$$

between the three functions $K_{m-1}(r/l)$, $K_m(r/l)$ and $K_{m+1}(r/l)$, we will find, finally, the formulas for stress components in polar coordinates for the given case:

$$\sigma_r^* = \frac{p}{2}(1 + \cos 2\theta) + \frac{A}{r^2} - \left(\frac{4C}{r^2} + 6\frac{B - \text{Æ}}{r^4}\right)\cos 2\theta +$$
$$+ \frac{2D}{lr}\left[3\frac{l}{r}K_0(r/l) + \left(1 + 6\frac{l^2}{r^2}\right)K_1(r/l)\right]\cos 2\theta;$$
$$\sigma_\theta^* = \frac{p}{2}(1 - \cos 2\theta) - \frac{A}{r^2} + 6\frac{B - \text{Æ}}{r^4}\cos 2\theta -$$
$$- \frac{2D}{lr}\left[3\frac{l}{r}K_0(r/l) + \left(1 + 6\frac{l^2}{r^2}\right)K_1(r/l)\right]\cos 2\theta;$$

---

[1]We will recall that in the examined case of the basic stress state (VI.50), the function $F^0(r, \theta) \equiv 0$.

[2]See G. N. Vatson [1].

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$$\tau^*_{r\theta} = -\left(\frac{p}{2} + 2\frac{C}{r^2} + 6\frac{B-\mathfrak{B}}{r^4}\right)\sin 2\theta +$$

$$+ \frac{D}{lr}\left[6\frac{l}{r}K_0(r/l) + \left(1 + 12\frac{l^2}{r^2}\right)K_1(r/l)\right]\sin 2\theta; \qquad \text{(VI.65)}$$

$$\tau^*_{\theta r} = -\left(\frac{p}{2} + 2\frac{C}{r^2} + 6\frac{B-\mathfrak{B}}{r^4}\right)\sin 2\theta +$$

$$+ \frac{D}{l^2}\left[\left(1 + 6\frac{l^2}{r^2}\right)K_0(r/l) + \left(3\frac{l}{r} + 12\frac{l^3}{r^3}\right)K_1(r/l)\right]\sin 2\theta;$$

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$$\mu^*_r = -\frac{2\mathfrak{B}}{r^3}\sin 2\theta - \frac{D}{l}\left[2\frac{l}{r}K_0(r/l) + \left(1 + 4\frac{l^2}{r^2}\right)K_1(r/l)\right]\sin 2\theta; \qquad \text{(VI.66)}$$

$$\mu^*_\theta = \frac{2\mathfrak{B}}{r^3}\cos 2\theta + \frac{2D}{r}\left[K_0(r/l) + 2\frac{l}{r}K_1(r/l)\right]\cos 2\theta,$$

where, for convenience, we introduce the definition

$$\mathfrak{B} = \beta_1 C = 8(1-\nu)l^2 C.$$

By substituting from (VI.65) the expressions for σ^*_r, $\tau^*_{r\theta}$ and μ^*_r when $r = a_0$ into boundary condition (VI.63), equating to zero the coefficients for identical sins and cosines, we find the following equation system

$$\frac{p}{2} + \frac{A}{a_0^2} = 0;$$

$$\frac{p}{2} - 4\frac{C}{a_0^2} - \frac{6}{a_0^4}(B-\mathfrak{B}) + \frac{2D}{la_0}\left[3\frac{l}{a_0}K_0(a_0/l) + \left(1 + 6\frac{l^2}{a_0^2}\right)K_1(a_0/l)\right] = 0;$$

$$-\frac{p}{2} - 2\frac{C}{a_0^2} - \frac{6}{a_0^4}(B-\mathfrak{B}) + \frac{D}{la_0}\left[6\frac{l}{a_0}K_0(a_0/l) + \left(1 + 12\frac{l^2}{a_0^2}\right)K_1(a_0/l)\right] = 0; \qquad \text{(VI.67)}$$

$$-2\frac{\mathfrak{B}}{a_0^3} - \frac{D}{l}\left[2\frac{l}{a_0}K_0(a_0/l) + \left(1 + 4\frac{l^2}{a_0^2}\right)K_1(a_0/l)\right] = 0.$$

By solving system (VI.67), we obtain

$$A = -\frac{pa_0^2}{2}; \qquad B = -\frac{pa_0^4(1-\mathcal{H}_1)}{4(1+\mathcal{H}_1)};$$

$$C = \frac{pa_0^2}{2(1+\mathcal{H}_1)}; \qquad D = -\frac{pa_0 l\mathcal{H}_1}{(1+\mathcal{H}_1)K_1(a_0/l)}; \qquad \text{(VI.68)}$$

$$\mathfrak{B} = \frac{4(1-\nu)a_0^2 l^2 p}{1+\mathcal{H}_1},$$

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where

$$\mathcal{H}_1 = \frac{8(1-\nu)}{4 + \frac{a_0^2}{l^2} + 2\frac{a_0}{l}\frac{K_0(a_0/l)}{K_1(a_0/l)}}. \tag{VI.69}$$

In view of the fact that

$$\lim_{\frac{a_0}{l}\to\infty} \frac{K_1(a_0/l)}{K_0(a_0/l)} = 1, \tag{VI.70}$$

it is easy to find from (VI.65) the known solution of G. Kirsch.

Stresses (VI.65) on the contour of the hole are

$$\sigma_\theta^* = p\left(1 - \frac{2\cos 2\theta}{1+\mathcal{H}_1}\right). \tag{VI.71}$$

The maximal σ_θ^* will occur (see Figure VI.1) at the points $\theta = \pm\pi/2$ /480

$$(\sigma_\theta^*)_{\max} = p\frac{3+\mathcal{H}_1}{1+\mathcal{H}_1}. \tag{VI.72}$$

Since for the given Poisson's ratio ν the value H_1 depends on the new elastic constant l of the material and the radius of the hole a_0, then

$$(k_{\max})_{\max} = \frac{(\sigma_{\theta.\max}^*)_{\max}}{p} = 3 \tag{VI.73}$$

when $l = 0$, i.e., when $H_1 = 0$.

However, $(H_1)_{\min}$ will be achieved when $a_0/l = 3$, i.e.,

$$(\mathcal{H}_1)_{\min} = 0{,}44(1-\nu), \tag{VI.74}$$

and consequently

$$(k_{\max})_{\min} = \frac{3{,}44 - 0{,}44\nu}{1{,}44 - 0{,}44\nu}. \tag{VI.75}$$

R. D. Mindlin [1] specifies the range of change of the ratio a_0/l that is equal to $3 \leqslant a_0/l < \infty$.

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Since $0 < \nu < 0.5$, then

$$2{,}39 < (k_{max})_{min} < 2{,}64, \tag{VI.76}$$

which is considerably less than k = 3 obtained by the classical elasticity theory.

In order to explain the effect of the asymmetry of the stress tensor on the magnitude of the perturbed zone near the hole, we have calculated the stresses σ_θ^* and σ_r^* (VI.65), respectively, through cross sections $\theta = \pi/2$ and $\theta = 0$

$$\begin{aligned}(\sigma_\theta^*)_{\theta=\pi/2} &= p - \frac{A}{r^2} - 6\frac{B-\mathscr{E}}{r^4} + \frac{2D}{lr}\left[3\frac{l}{r}K_0(r/l) + \left(1 + 6\frac{l^2}{r^2}\right)K_1(r/l)\right];\\ (\sigma_r^*)_{\theta=0} &= p + \frac{A}{r^2} - 4\frac{C}{r^2} - 6\frac{B-\mathscr{E}}{r^4} + \frac{2D}{lr}\left[3\frac{l}{r}K_0(r/l) + \left(1 + 6\frac{l^2}{r^2}\right)K_1(r/l)\right].\end{aligned} \tag{VI.77}$$

The solid curves in Figure VI.2 correspond to the values σ_θ^* and σ_r^* (VI.77) when $\nu = 0.25$, $a_0/l = 3$ and $a_0 = 30$ cm, and the broken curves correspond to the values σ_θ and σ_r through the same cross sections according to classical elasticity theory. From these graphs we see that the moment stresses when $a_0/l = 3$ decrease the perturbed zone near the hole by a factor of about 1.3.

Pure Deflection of Plate Weakened by Round Hole[1]. We will find the elastic equilibrium of an infinite plate weakened by a round hole of radius a_0, free of external forces, located in a field of pure deflection by forces py (Figure VI.3), i.e., "at infinity"

$$\sigma_x^0 = py, \quad \sigma_y^0 = \tau_{xy}^0 = \tau_{yx}^0 = 0. \tag{VI.78}$$

From relations (VI.8)

$$\mu_x^0 = -2p(1-\nu)l^2, \quad \mu_y^0 = 0. \tag{VI.79}$$

To the basic stress state (VI.78) and (VI.79) will correspond stress functions /481

$$U^0 = \frac{1}{6}py^3, \quad F^0 = -2p(1-\nu)l^2x. \tag{VI.80}$$

[1]The solution of this problem was first found by Yu. N. Nemish [1].

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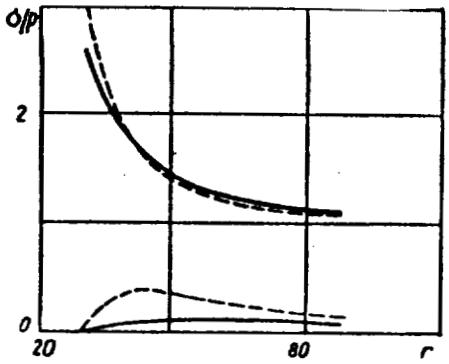


Figure VI.2.

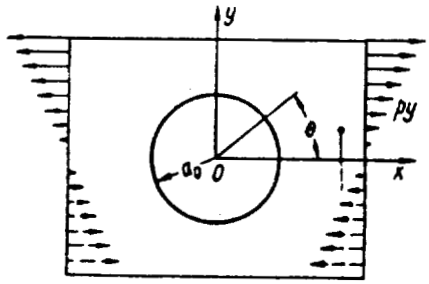


Figure VI.3.

The solution of this problem could be obtained by the method described above, but we will use a different approach to the solution of such problems; here we will follow the directions outlined in N. I. Muskhelishvili's monograph [1]. For this purpose we will use boundary condition (VI.35). We will assume that the functions M and N - iT given on the contour of a round hole are expanded into convergent complex Fourier series:

$$N - iT = \sum_{-\infty}^{\infty} A_n e^{in\theta}, \qquad M = \sum_{-\infty}^{\infty} B_n e^{in\theta}, \quad B_n = \overline{B}_{-n}. \tag{VI.81}$$

Since, by definition of the problem, the contour of the hole is free of external forces, then in (VI.81) we will assume that $A_n = B_n = 0$ for all n.

According to the foregoing (see §3) the functions are

$$\Phi(z) = -\frac{pi}{4}z + \sum_{n=1}^{\infty} a_n z^{-n}, \qquad \Psi(z) = \frac{pi}{4}z + \sum_{n=1}^{\infty} b_n z^{-n}, \tag{VI.82}$$

where the coefficients a_n and b_n (n = 1, 2, 3 ...), generally speaking, are complex constants:

$$a_n = a_n' + ia_n''; \qquad b_n = b_n' + ib_n''. \tag{VI.83}$$

The solution of Helmholtz equation (VI.26) will be found in the form /482

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$$\Omega = c_0 K_0(r/l) + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta) K_n(r/l), \tag{VI.84}$$

where c_n, d_n are real constants.

Now, by substituting the expansions of (VI.81), (VI.82) and (VI.84) into boundary condition (VI.35), considering recurrent relation (VI.64) for McDonald's functions, after equating to zero the coefficients for identical sins and cosines, we obtain an equation system, from which we find

$$a_3^* = -\frac{pa_0^4 \mathcal{H}_2}{4(\mathcal{H}_2 + \mathcal{H}_3)}, \qquad b_5^* = -\frac{pa_0^6\left(\mathcal{H}_2 - \frac{1}{3}\mathcal{H}_3\right)}{\mathcal{H}_2 + \mathcal{H}_3},$$

$$b_3^* = \frac{pa_0^4}{4} + 2p(1-\nu) l^2 a_0^2 \left(1 + \frac{1}{\mathcal{H}_4}\right), \tag{VI.85}$$

$$c_1 = -\frac{2p(1-\nu) l^2}{a_0 \mathcal{H}_4 K_1(a_0/l)}, \qquad c_3 = \frac{6p(1-\nu) l^3}{(\mathcal{H}_2 + \mathcal{H}_3) K_2(a_0/l)},$$

where

$$\mathcal{H}_2 = 1 + 3\frac{l^2}{a_0^2} \frac{8 + \frac{a_0^2}{l^2} + 4\frac{a_0}{l} \cdot \frac{K_0(a_0/l)}{K_1(a_0/l)}}{2 + \frac{a_0}{l} \cdot \frac{K_0(a_0/l)}{K_1(a_0/l)}};$$

$$\mathcal{H}_3 = 24(1-\nu)\frac{l^2}{a_0^2}; \qquad \mathcal{H}_4 = 1 + \frac{a_0}{l}\frac{K_0(a_0/l)}{K_1(a_0/l)}. \tag{VI.86}$$

In the case at hand

$$\Phi(z) = i\left(\frac{p}{4} z + \frac{a_3^*}{z^3}\right); \qquad \Psi(z) = i\left(\frac{p}{4} z + \frac{b_3^*}{z^3} + \frac{b_5^*}{z^5}\right);$$

$$\Omega = c_1 K_1(r/l)\cos\theta + c_3 K_3(r/l)\cos 3\theta. \tag{VI.87}$$

Functions (VI.87) give us the desired stress state. On the basis of formulas (VI.31)

$$\sigma_\theta + \sigma_r = pr\sin\theta + 4\frac{a_3^*}{r^3}\sin 3\theta;$$

$$\tau_{\theta r} - \tau_{r\theta} = \frac{1}{l^2}[c_1 K_1(r/l)\cos\theta + c_3 K_3(r/l)\cos 3\theta];$$

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$$\sigma_\theta - \sigma_r + i(\tau_{\theta r} + \tau_{r\theta}) = 2i\left\{\frac{pr}{4}(e^{3i\theta} - e^{i\theta}) + \frac{b_3''}{r^3}e^{-i\theta} + \right.$$
$$+\left[-3\frac{a_3''}{r^3} + \frac{96(1-\nu)l^2a_3''}{r^5} + \frac{b_5''}{r^5}\right]e^{-3i\theta} +$$
$$\left. + \frac{c_1}{4l^2}[K_1(r/l)e^{i\theta} + K_3(r/l)e^{-i\theta}] + \frac{c_3}{4l^2}K_1[(r/l)e^{3i\theta} + K_5(r/l)e^{-3i\theta}]\right\};$$
$$\mu_r - i\mu_\theta = -8(1-\nu)l^2\left(\frac{p}{4}e^{i\theta} + \frac{3a_3''}{r^4}e^{-3i\theta}\right) -$$
$$-\frac{1}{2l}\{c_1[K_0(r/l)e^{i\theta} + K_2(r/l)e^{-i\theta}] + c_3[K_2(r/l)e^{3i\theta} + K_4(r/l)e^{-3i\theta}]\}. \quad \text{(VI.88)}$$

Stresses σ_θ and μ_θ when $l \neq 0$, $r = a_0$ and $\theta = \pi/2$ will have the form

$$\sigma_\theta = pa_0\left(2 - \frac{\mathcal{H}_3}{\mathcal{H}_2 + \mathcal{H}_3}\right),$$
$$\mu_\theta = 2p(1-\nu)l^2\left(1 - \frac{3}{\mathcal{H}_2 + \mathcal{H}_3} + \frac{1}{\mathcal{H}_4}\right). \quad \text{(VI.89)}$$

Recalling (VI.85) and condition (VI.70) it easy to find from (VI.88) the solution of the corresponding problem of classical elasticity theory.

Solution of Second Basic Problem of Moment Elasticity Theory. Uniaxial Tension. Suppose we are required to determine the stress state of an infinite isotropic plate[1] weakened by a round hole of radius a_0, in which is fixed an absolutely rigid disc (see Figure VI.1). At infinity the plate is under the effect of forces $\sigma_x = p$, i.e., the basic stress state is characterized by formulas (VI.50). The problem amounts to the solution of differential equation system (VI.11) for boundary conditions (VI.36).

We will assume that the functions $2\mu(q_1 + iq_2)$ and $4\eta q_3$ given on contour L are expanded into convergent complex Fourier series:

$$2\mu(q_1 + iq_2) = \sum_{n=-\infty}^{\infty} Q_n e^{in\theta};$$
$$4\eta q_3 = \sum_{n=-\infty}^{\infty} P_n e^{in\theta} \qquad (P_n = \overline{P}_{-n}). \quad \text{(VI.90)}$$

[1]The basic equations for plane stress state during uniaxial tension are presented by G. N. Savin [2], §3, p. 13. In the present work are presented the analogous equations for the case of pure deflection (see below).

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We will note that in the case of an absolutely rigid disc the coefficients of the series are $Q_n = P_n = 0$ for all n. /484

The complex potentials, as in the classical elasticity theory, are

$$\varphi(z) = -\frac{X+iY}{2\pi(\varkappa+1)}\ln z + \Gamma z + \sum_{n=1}^{\infty} a_n z^{-n},$$
$$\psi(z) = \frac{X-iY}{2\pi(\varkappa+1)}\ln z + \Gamma' z + \sum_{n=1}^{\infty} b_n z^{-n}, \qquad \text{(VI.91)}$$

where a_n and b_n have the form (VI.83)

$$\Gamma = \frac{1}{4}(N_1+N_2), \qquad \Gamma' = -\frac{1}{2}(N_1-N_2)e^{-2i\alpha}, \qquad \text{(VI.92)}$$

whereupon X, Y are components of the main vector of all forces applied to contour L; N_1 and N_2 are the principal stresses at infinity; α is the angle between the direction of the force corresponding to N_1 and the Ox axis.

The function $\Omega(r, \theta)$, as before, will be taken in the form (VI.84), and the problem will be solved in the assumption that X = Y = 0.

From boundary conditions (VI.36) we find the coefficients of the expansions of functions (VI.91) and function (VI.84)

$$a_1' = -\frac{pa_0^2T_1}{2\varkappa}, \qquad b_1' = \frac{1}{4}(\varkappa-1)pa_0^2,$$
$$b_3' = -\frac{pa_0^4T_1}{2\varkappa}\left[1+(\varkappa+1)\frac{l}{a_0}\frac{K_1(a_0/l)}{K_2(a_0/l)}\right],$$
$$d_2 = \frac{(\varkappa+1)l^2pT_1}{\varkappa K_2(a_0/l)}, \qquad a_2' = a_3' = \ldots = a_n' = 0, \qquad \text{(VI.93)}$$
$$b_2' = b_4' = \ldots = b_n' = 0; \qquad d_1 = d_3 = \ldots = d_n = 0,$$

where

$$T_1 = \frac{2+\frac{a_0}{l}\cdot\frac{K_0(a_0/l)}{K_1(a_0/l)}}{1-\frac{1}{\varkappa}+\frac{a_0}{l}\cdot\frac{K_0(a_0/l)}{K_1(a_0/l)}}. \qquad \text{(VI.94)}$$

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Thus,

$$\varphi(z) = \Gamma z + \frac{a_1'}{z}, \qquad \psi(z) = \Gamma' z + \frac{b_1'}{z} + \frac{b_3'}{z^3},$$
$$\Omega = d_2 K_2(r/l)\sin 2\theta. \tag{VI.95}$$

The stress state in the plate is found by substituting the stress components: /485

$$\sigma_r = 2\Gamma + b_1' r^{-2} - \left\{\Gamma' + 4a_1' r^{-2} - 12(\varkappa+1) l^2 r^{-4} a_1' - 3b_3' r^{-4} - \right.$$
$$\left. - \frac{d_2}{4l^2}[K_4(r/l) - K_0(r/l)]\right\}\cos 2\theta,$$
$$\sigma_\theta = 2\Gamma - b_1' r^{-2} + \left\{\Gamma' - 12(\varkappa+1) l^2 r^{-4} a_1' - 3b_3' r^{-4} - \right.$$
$$\left. - \frac{d_2}{4l^2}[K_4(r/l) - K_0(r/l)]\right\}\cos 2\theta,$$
$$\tau_{r\theta} = \left\{\Gamma' - \frac{2a_1'}{r^2} + \frac{12(\varkappa+1) l^2 a_1'}{r^4} + \frac{3b_3'}{r^4} + \right.$$
$$\left. + \frac{d_2}{4l^2}[K_4(r/l) - 2K_2(r/l) + K_0(r/l)]\right\}\sin 2\theta, \tag{VI.96}$$
$$\tau_{\theta r} = \left\{\Gamma' - \frac{2a_1'}{r^2} + \frac{12(\varkappa+1) l^2 a_1'}{r^4} + \frac{3b_3'}{r^4} + \right.$$
$$\left. + \frac{d_2}{4l^2}[K_4(r/l) + 2K_2(r/l) + K_0(r/l)]\right\}\sin 2\theta$$

and

$$\mu_r = \left\{\frac{4(\varkappa+1) l^2 a_1'}{r^3} + \frac{d_2}{2l}[K_3(r/l) - K_1(r/l)]\right\}\cos 2\theta,$$
$$\mu_\theta = -\left\{\frac{4(\varkappa+1) l^2 a_1'}{r^3} + \frac{d_2}{2l}[K_3(r/l) + K_1(r/l)]\right\}\sin 2\theta. \tag{VI.97}$$

Hence in the case of the plane stress[1] state $\alpha = (3 - \nu)/1 + \nu$. On the contour of the hole, for all $r = a_0$

[1]For the derivation of the basic equations see G. N. Savin [2].

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$$\sigma_r = \frac{p}{2}\left\{\frac{\varkappa+1}{2} + \frac{1}{\varkappa}\left[\varkappa + T_1 + (\varkappa+1)T_1\frac{l}{a_0}\cdot\frac{K_1(a_0/l)}{K_2(a_0/l)}\right]\cos 2\theta\right\},$$

$$\sigma_\theta = \frac{p}{2}\left\{\frac{3-\varkappa}{2} + \frac{1}{\varkappa}\left[-\varkappa + 3T_1 - (\varkappa+1)T_1\frac{l}{a_0}\cdot\frac{K_1(a_0/l)}{K_2(a_0/l)}\right]\cos 2\theta\right\},$$

$$\tau_{r\theta} = \frac{p}{2\varkappa}\left[\varkappa + T_1 + (\varkappa+1)T_1\frac{l}{a_0}\cdot\frac{K_1(a_0/l)}{K_2(a_0/l)}\right]\sin 2\theta, \qquad \text{(VI.98)}$$

$$\tau_{\theta r} = \frac{p}{2\varkappa}\left[-\varkappa + (1+2\varkappa)T_1 - (\varkappa+1)T_1\frac{l}{a_0}\cdot\frac{K_1(a_0/l)}{K_2(a_0/l)}\right]\sin 2\theta;$$

$$\mu_r = -\frac{(\varkappa+1)lpT_1}{2\varkappa}\cdot\frac{K_1(a_0/l)}{K_2(a_0/l)}\sin 2\theta, \quad \mu_\theta = 0,$$

where T_1 is defined by formula (IV.94).

Pure Deflection. We will determine the elastic equilibrium of an infinite plate with an absolutely rigid disc fixed in a round hole of radius a_0, located in a field of pure deflection by forces $\sigma_x = py$ (see Figure VI.3). /486

The basic stress state is characterized by stresses (VI.78) and (VI.79), and the stress functions for the basic stress state have the form (VI.80).

From boundary conditions (VI.36), where, instead of ν it is assumed[1] that $\nu^* = \nu/1 + \nu$, the coefficients of the desired functions for the additional stress state caused by the presence of the hole are

$$a_2^* = -\frac{pa_0^4}{8\varkappa(1-T_2)}, \qquad b_0^* = \frac{1}{4}pa_0^2(1+T_3),$$

$$b_2^* = \frac{1}{8}pa_0^4(\varkappa+2T_3), \qquad b_4^* = -\frac{pa_0^6(2+\varkappa T_2)}{8\varkappa(1-T_2)}, \qquad \text{(VI.99)}$$

$$c_1 = \frac{(\varkappa+1)pa_0l^2}{2K_1(a_0/l)}, \qquad c_3 = -\frac{pa_0^2lT_2}{4(1-T_2)K_2(a_0/l)},$$

where

$$T_2 = \frac{2(\varkappa+1)}{\varkappa}\cdot\frac{2+\frac{a_0}{l}\frac{K_0(a_0/l)}{K_1(a_0/l)}}{8+\frac{a_0^2}{l^2}+4\frac{a_0}{l}\frac{K_0(a_0/l)}{K_1(a_0/l)}}; \qquad T_3 = (\varkappa+1)\frac{l^2}{a_0^2}\left[2+\frac{a_0}{l}\frac{K_0(a_0/l)}{K_1(a_0/l)}\right]. \qquad \text{(VI.100)}$$

In the given case

[1] See G. N. Savin [2].

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$$\varphi(z) = i\left(-\frac{pz^2}{8} + \frac{a_2^*}{z^2}\right), \qquad \psi(z) = i\left(\frac{pz^2}{8} + b_0^* + \frac{b_2^*}{z^2} + \frac{b_4^*}{z^4}\right), \tag{VI.101}$$

$$\Omega = \{c_1 K_1(r/l)\cos\theta + c_3 K_3(r/l)\cos 3\theta\}.$$

The stress components are

$$\sigma_\theta = \left[\frac{3}{4}pr - 2\frac{b_2^*}{r^3} + 2\frac{c_1}{lr}K_2(r/l)\right]\sin\theta + \left\{2\frac{a_2^*}{r^3} - 48(\varkappa+1)l^2\frac{a_2^*}{r^5} - \right.$$

$$\left. - \frac{pr}{4} - 4\frac{b_4^*}{r^5} + \frac{c_3}{2l^2}[K_5(r/l) - K_1(r/l)]\right\}\sin 3\theta;$$

$$\sigma_r = \left[\frac{1}{4}pr + 2\frac{b_2^*}{r^3} - 2\frac{c_1}{lr}K_2(r/l)\right]\sin\theta + \left\{-10\frac{a_2^*}{r^3} + 48(\varkappa+1)l^2\frac{a_2^*}{r^5} + \right.$$

$$\left. + \frac{1}{4}pr + 4\frac{b_4^*}{r^5} - \frac{c_3}{2l^2}[K_5(r/l) - K_1(r/l)]\right\}\sin 3\theta;$$

$$\tau_{\theta r} = \left\{-\frac{1}{4}pr - 2\frac{b_2^*}{r^3} + \frac{c_1}{2l^2}[K_3(r/l) + 3K_1(r/l)]\right\}\cos\theta + \qquad \text{/487}$$

$$+ \left\{\frac{1}{4}pr + 6\frac{a_2^*}{r^3} - 48(\varkappa+1)l^2\frac{a_2^*}{r^5} - 4\frac{b_4^*}{r^5} + \frac{c_3}{2l^2}[K_5(r/l) + \right.$$

$$\left. + 2K_3(r/l) + K_1(r/l)]\right\}\cos 3\theta; \tag{VI.102}$$

$$\tau_{r\theta} = \left\{-\frac{1}{4}pr - 2\frac{b_2^*}{r^3} + \frac{c_1}{2l^2}[K_3(r/l) - K_1(r/l)]\right\}\cos\theta + \left\{\frac{1}{4}pr + \right.$$

$$\left. + 6\frac{a_2^*}{r^3} - 48(\varkappa+1)l^2\frac{a_2^*}{r^5} - 4\frac{b_4^*}{r^5} + \frac{c_3}{2l^2}[K_5(r/l) - 2K_3(r/l) + K_1(r/l)]\right\}\cos 3\theta;$$

$$\mu_r = -\left\{\frac{1}{2}p(\varkappa+1)l^2 + \frac{c_1}{l}[K_2(r/l) + K_0(r/l)]\right\}\cos\theta + $$

$$+ \left\{12(\varkappa+1)l^2\frac{a_2^*}{r^4} - \frac{c_3}{l}[K_4(r/l) + K_2(r/l)]\right\}\cos 3\theta;$$

$$\mu_\theta = \left\{\frac{1}{2}p(\varkappa+1)l^2 - \frac{c_1}{l}[K_2(r/l) - K_0(r/l)]\right\}\sin\theta + $$

$$+ \left\{12(\varkappa+1)l^2\frac{a_2^*}{r^4} - \frac{c_3}{l}[K_4(r/l) - K_2(r/l)]\right\}\sin 3\theta.$$

The coefficients a_2'', b_0'', ..., c_3 in relation (VI.101) and (VI.102) are defined by formulas (VI.99).

On the contour of the hole for $r = a_0$

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$$\sigma_\theta = \frac{1}{4}(3-\varkappa)\,pa_0\left[\sin\theta + \frac{\sin 3\theta}{\varkappa(1-T_2)}\right],$$

$$\sigma_r = \frac{1}{4}(1+\varkappa)\,pa_0\left[\sin\theta + \frac{\sin 3\theta}{\varkappa(1-T_2)}\right],$$

$$\tau_{\theta r} = \frac{1}{4}(1+\varkappa)\,pa_0\left[\cos\theta - \frac{\cos 3\theta}{\varkappa(1-T_2)}\right], \qquad \text{(VI.103)}$$

$$\tau_{r\theta} = -\frac{1}{4}(1+\varkappa)\,pa_0\left[\cos\theta - \frac{\cos 3\theta}{\varkappa(1-T_2)}\right].$$

$$\mu_r = -\frac{1}{2}(\varkappa+1)\,pa_0 l\,\frac{K_2(a_0/l)}{K_1(a_0/l)}\cos\theta + \frac{pa_0^2 T_2}{4(1-T_2)}\cos 3\theta, \qquad \mu_\theta = 0.$$

From the solutions found for the second basic problem for $a_0/l \to \infty$, we obtain the values of stress components σ_r, σ_θ and $\tau_{r\theta}$ for the analogous problems of classical elasticity theory.

From (VI.98) and (VI.103) we have /488

$$\lim_{a_0/l\to\infty} \tau_{r\theta} = -\lim_{a_0/l\to\infty} \tau_{\theta r}, \qquad \text{(VI.104)}$$

i.e., when $a_0/l \to \infty$ we nevertheless do not obtain in its entirety the symmetrical stress tensor ($\tau_{r\theta} \neq \tau_{\theta r}$) upon which classical elasticity theory is based.

§5. Stress Concentration near Arbitrary Curvilinear Hole

Boundary Form Perturbation Method[1]. We will examine holes, the contours of which have no angular points and are given by equations

$$x^* = R\left(\cos\vartheta + \varepsilon\sum_{k=1}^{n} c_k\cos k\vartheta\right), \qquad \text{(VI.105)}$$

$$y^* = R\left(\sin\vartheta - \varepsilon\sum_{k=1}^{n} c_k\sin k\vartheta\right),$$

[1]This method in application to plane problems of moment elasticity theory for infinite singly-connected ranges is described in greater detail by G. N. Savin [2]. For infinite ranges weakened by a finite number of arbitrarily distributed holes, the contours of which are smooth curves, the method for the solution of the problem of plane moment elasticity theory is given in the work of G. N. Savin and A. N. Guz [2].

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where $c_1 = 1$, c_k are constant coefficients.

Equations (VI.105) represent real and imaginary parts of the function

$$z^* = \omega(\zeta) = R\left(\zeta + \varepsilon\sum_{k=1}^{n}\frac{c_k}{\zeta^k}\right), \qquad (VI.106)$$

$$z^* = x^* + iy^* = r^* e^{i\theta}, \quad \zeta = \rho e^{i\phi},$$

which conformally maps the exterior of the unit circle onto range Σ, representing the exterior of the examined hole (Figure VI.4).

For simplicity we will assume that the holes under examination have at least one axis of symmetry, for instance the Ox axis. Under this condition, coefficients c_k ($k = 2, 3, \ldots$) and functions (VI.105) will be real values if the Ox axis is used as the axis of symmetry.

The constant R in (VI.106), which characterizes absolute dimensions of the hole and its orientation in relation to the selected Oxy coordinate system, will be regarded as a real value. The parameter ε will be assumed as real and small, changing in the range $0 < |\varepsilon| \ll 1$, characterizing the degree of "deviation" of the examined hole from the round. /489

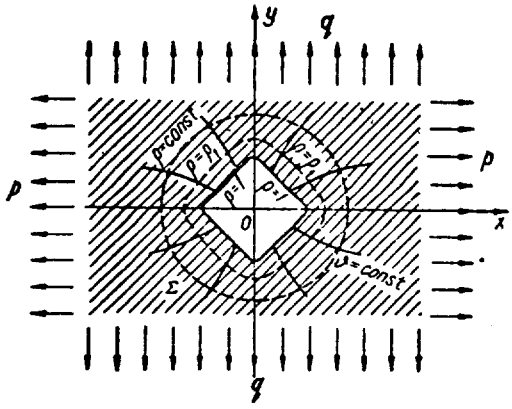


Figure VI.4.

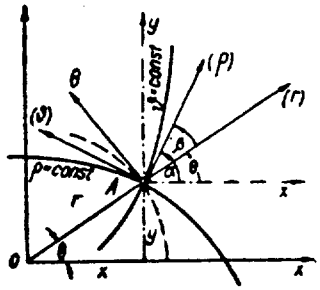


Figure VI.5.

In the following discussion it is convenient to change to dimensionless Cartesian x, y and polar r and θ coordinates:

$$z = \frac{z^*}{R} = x + iy = re^{i\theta}. \qquad (VI.107)$$

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Equations (VI.105) and (VI.106) in dimensionless coordinates will have the forms, respectively,

$$x = \cos\vartheta + \varepsilon \sum_{k=1}^{n} c_k \cos k\vartheta,$$

$$y = \sin\vartheta - \varepsilon \sum_{k=1}^{n} c_k \sin k\vartheta \qquad \text{(VI.108)}$$

and

$$z = \frac{1}{R}\omega(\zeta) = \zeta + \varepsilon f(\zeta), \qquad \text{(VI.109)}$$

where

$$f(\zeta) = \frac{1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots + \frac{c_n}{\zeta^n}; \qquad \text{(VI.110)}$$

$$\zeta = \rho e^{i\vartheta}.$$

Equation (VI.108) represents parametric equations of the "natural" curvilinear orthogonal coordinate system ρ = const and ϑ = const (see Figure VI.4). /490

In accordance with (VI.40) the total stresses near the hole are

$$\begin{aligned} \sigma_\rho^* &= \sigma_\rho^0 + \sigma_\rho, \quad \sigma_\vartheta^* = \sigma_\vartheta^0 + \sigma_\vartheta, \quad \mu_\vartheta^* = \mu_\vartheta, \\ \tau_{\rho\vartheta}^* &= \tau_{\rho\vartheta}^0 + \tau_{\rho\vartheta}, \quad \tau_{\vartheta\rho}^* = \tau_{\vartheta\rho}^0 + \tau_{\vartheta\rho}, \quad \mu_\rho^* = \mu_\rho, \end{aligned} \qquad \text{(VI.111)}$$

where σ_ρ^0, σ_ϑ^0, ..., $\tau_{\vartheta\rho}^0$ are components of the basic stress state, σ_ρ, σ_ϑ, ..., μ_ϑ are additional stress components occurring due to the presence of a hole. Stress components σ_ρ, σ_ϑ, $\tau_{\rho\vartheta}$, $\tau_{\vartheta\rho}$, μ_ρ and μ_ϑ in the curvilinear coordinate system (ρ, ϑ), expressed through σ_r, σ_θ, $\tau_{r\theta}$, $\tau_{\theta r}$, μ_r and μ_θ in polar coordinate system (r, θ), as follows from (VI.14), will have the form

$$\sigma_\rho = \sigma_r \cos^2\beta + \sigma_\theta \sin^2\beta + \frac{\tau_{r\theta} + \tau_{\theta r}}{2} \sin 2\beta,$$

$$\sigma_\vartheta = \sigma_r \sin^2\beta + \sigma_\theta \cos^2\beta - \frac{\tau_{r\theta} + \tau_{\theta r}}{2} \sin 2\beta,$$

$$\tau_{\rho\vartheta} = \frac{\sigma_\theta - \sigma_r}{2} \sin 2\beta + \tau_{r\theta} \cos^2\beta - \tau_{\theta r} \sin^2\beta,$$

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$$\tau_{\vartheta\varrho} = \frac{\sigma_\vartheta - \sigma_r}{2}\sin 2\beta - \tau_{r\vartheta}\sin^2\beta + \tau_{\vartheta r}\cos^2\beta,$$
$$\mu_\varrho = \mu_r\cos\beta + \mu_\vartheta\sin\beta,$$
$$\mu_\vartheta = -\mu_r\sin\beta + \mu_\vartheta\cos\beta, \qquad \text{(VI.112)}$$

where β is the angle of rotation of the axes (ρ) $A(\vartheta)$ in relation to the polar coordinate system $(r)A(\theta)$ (Figure VI.5),

The stress components in polar coordinates (r, θ) (see Figure VI.5), as follows from (VI.17), are defined by the formulas

$$\sigma_r = \frac{1}{R^2}\left[\frac{1}{r}\cdot\frac{\partial U}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 U}{\partial\theta^2} - \frac{1}{r}\cdot\frac{\partial^2 F}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial F}{\partial\theta}\right],$$
$$\sigma_\vartheta = \frac{1}{R^2}\left[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial^2 F}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial F}{\partial\theta}\right],$$
$$\tau_{r\vartheta} = -\frac{1}{R^2}\left[\frac{1}{r}\cdot\frac{\partial^2 U}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial U}{\partial\theta} + \frac{1}{r}\cdot\frac{\partial F}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial\theta^2}\right],$$
$$\tau_{\vartheta r} = -\frac{1}{R^2}\left[\frac{1}{r}\cdot\frac{\partial^2 U}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial U}{\partial\theta} - \frac{\partial^2 F}{\partial r^2}\right], \qquad \text{(VI.113)}$$
$$\mu_r = \frac{1}{R}\cdot\frac{\partial F}{\partial r},$$
$$\mu_\vartheta = \frac{1}{R}\cdot\frac{1}{r}\cdot\frac{\partial F}{\partial\theta}.$$

In equations (VI.11) the Laplace operator ∇^2 should be taken in "dimensionless" polar coordinates (r, θ), i.e., in the form /491

$$\nabla^2 = \frac{1}{R^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right). \qquad \text{(VI.114)}$$

and we will consider the functions U(r, θ) and F(r, θ) as additional stress functions corresponding to the additional stress state (VI.41).

However, stress functions $U^{(0)}(r, \theta)$ and $F^{(0)}(r, \theta)$ are determined in accordance with the given basic stress state (VI.38), since the arbitrariness of these functions has no effect on the final stress state near the hole.

From (IV.109) it is easy to find

$$x(\varrho,\vartheta) = \varrho\cos\vartheta + \varepsilon\frac{f(\zeta) + \overline{f(\zeta)}}{2},$$
$$y(\varrho,\vartheta) = \varrho\sin\vartheta + \varepsilon\frac{f(\zeta) - \overline{f(\zeta)}}{2i}, \qquad \text{(VI.115)}$$

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and also

$$r(\varrho,\vartheta)=\sqrt{x^2+y^2}=\sqrt{\varrho^2+\varepsilon[\bar{\zeta}f(\zeta)+\zeta\overline{f(\zeta)}]+\varepsilon^2 f(\zeta)\overline{f(\zeta)}}, \tag{VI.116}$$

$$\theta(\varrho,\vartheta)=\arctan\frac{y}{x}=\arctan\frac{\sin\vartheta+\varepsilon\dfrac{f(\zeta)-\overline{f(\zeta)}}{2i\varrho}}{\cos\vartheta+\varepsilon\dfrac{f(\zeta)+\overline{f(\zeta)}}{2\varrho}}. \tag{VI.117}$$

We see from Figure VI.15 that

$$e^{i\alpha}=e^{i(\theta+\beta)}=e^{i\theta}\cdot e^{i\beta}. \tag{VI.118}$$

We will represent the vector lying on the A(ρ) axis, i.e., tangent to the line ϑ = const, in the form

$$dz=|dz|\cdot e^{i\alpha}=\frac{1}{R}|\omega'(\zeta)|\cdot|d\zeta|\cdot e^{i\alpha}\ (\zeta=\varrho e^{i\vartheta}).$$

Consequently, $|d\zeta|=|d\varrho e^{i\vartheta}+i\varrho e^{i\vartheta}d\vartheta|$.

Along the line ϑ = const $d\vartheta = 0$, therefore $|d\zeta|=d\varrho$.

Consequently,

$$e^{i\alpha}=\frac{dz}{|dz|}=\frac{\omega'(\zeta)e^{i\vartheta}d\varrho}{|\omega'(\zeta)|d\varrho}=\frac{\zeta}{\varrho}\cdot\frac{\omega'(\zeta)}{|\omega'(\zeta)|}; \tag{VI.119}$$

$$e^{i\beta}=\frac{\zeta}{\varrho}\cdot\frac{\sqrt{1+\varepsilon f'(\zeta)}}{\sqrt{1+\varepsilon\overline{f'(\zeta)}}}\cdot\frac{\sqrt{\varrho^2+\varepsilon[\bar{\zeta}f(\zeta)+\zeta\overline{f(\zeta)}]+\varepsilon^2 f(\zeta)\overline{f(\zeta)}}}{\zeta+\varepsilon f(\zeta)}. \tag{VI.120}$$

In order to use the boundary form perturbation method it is necessary to have all values and functions in formulas (VI.112) and (VI.113) in the form of expansions by degrees of the small parameter ε. We will begin these expansions with the function (VI.116). After expanding r into a series by degrees of ε, we obtain /492

$$r=\varrho\left\{1+\varepsilon\frac{\bar{\zeta}f(\zeta)+\zeta\overline{f(\zeta)}}{2\varrho^2}-\varepsilon^2\frac{[\bar{\zeta}f(\zeta)-\zeta\overline{f(\zeta)}]^2}{8\varrho^4}+\ldots\right\}. \tag{VI.121}$$

Functions (VI.117) and (VI.120) are also expanded into series with respect to ε:

$$\theta(\varrho,\vartheta)=\vartheta+\varepsilon\left[\frac{f(\zeta)-\overline{f(\zeta)}}{2\varrho i}\cos\vartheta-\frac{f(\zeta)+\overline{f(\zeta)}}{2\varrho}\sin\vartheta\right]+$$
$$+\frac{\varepsilon^2}{4}\left[\frac{f^2(\zeta)+\overline{f^2(\zeta)}}{\zeta\bar{\zeta}}\sin 2\vartheta-\frac{f^2(\zeta)-\overline{f^2(\zeta)}}{\zeta\bar{\zeta}i}\cos\vartheta\right]+\ldots. \tag{VI.122}$$

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Since $z = re^{i\theta}$, then

$$e^{i\theta} = \frac{z}{r} = \frac{\zeta + ef(\zeta)}{r}. \tag{VI.123}$$

Recalling (VI.116), we obtain

$$e^{i\theta} = \frac{\zeta + ef(\zeta)}{\sqrt{\varrho^2 + e[\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}] + e^2 f(\zeta)\overline{f(\zeta)}}}. \tag{VI.124}$$

From (VI.118), (VI.119) and (VI.123) we readily find

$$e^{i\beta} = e^{i\alpha}\cdot e^{-i\theta} = \frac{\zeta}{\varrho}\frac{\omega'(\zeta)}{|\omega'(\zeta)|}\left[\frac{\omega(\zeta)}{Rr}\right]^{-1} \tag{VI.125}$$

or

$$e^{i\beta} = \frac{\zeta}{|\zeta|}\cdot\frac{\omega'(\zeta)}{|\omega'(\zeta)|}\cdot\frac{\overline{\omega(\zeta)}}{|\overline{\omega(\zeta)}|}. \tag{VI.126}$$

By substituting in (VI.126) the expressions from (VI.109) recalling that

$$|\omega'(\zeta)| = \sqrt{\omega'(\zeta)\,\overline{\omega'(\zeta)}},$$

we obtain

$$\begin{aligned} e^{i\beta} = 1 + e\frac{\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta) + \zeta\bar{\zeta}[f'(\zeta) - \overline{f'(\zeta)}]}{2\zeta\bar{\zeta}} - \\ - \frac{e^2}{8}\left[2\frac{f(\zeta)\overline{f(\zeta)}}{\zeta\bar{\zeta}} + \frac{\overline{f^2(\zeta)}}{\bar{\zeta}^2} + 2\frac{\overline{f(\zeta)}}{\bar{\zeta}}\overline{f'(\zeta)} + 2\frac{f(\zeta)}{\zeta}f'(\zeta) + f'^2(\zeta) + \right. \\ \left. + 2f'(\zeta)\overline{f'(\zeta)} - 3\frac{f^2(\zeta)}{\zeta^2} - 3\overline{f'^2(\zeta)} - 2\frac{\overline{f(\zeta)}}{\zeta}\overline{f'(\zeta)} - 2\frac{\overline{f(\zeta)}}{\bar{\zeta}}f'(\zeta)\right] + \ldots. \end{aligned} \tag{VI.127}$$

We now represent, in the form of an expansion with respect to ε, the arbitrary function $V\{r(\rho, \vartheta, \varepsilon)\ \theta(\rho, \vartheta, \varepsilon)\}$, which contains the required number of continuous derivatives:

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$$V(r, \theta) =$$

$$= V\left\{\sqrt{\varrho^2 + \varepsilon\left[\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}\right] + \varepsilon^2 f(\zeta)\overline{f(\zeta)}},\ \arctan\frac{\sin\vartheta + \varepsilon\dfrac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}}{\cos\vartheta + \varepsilon\dfrac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}}\right\} =$$

$$= V(\varrho, \vartheta) + \varepsilon\left\{J_1\frac{\partial}{\partial\varrho} + J_2\frac{\partial}{\partial\vartheta}\right\}V(\varrho, \vartheta) + \tag{VI.128}$$

$$+ \frac{\varepsilon}{2!}\left\{J_3\frac{\partial^2}{\partial\varrho^2} + J_4\frac{\partial^2}{\partial\varrho\partial\vartheta} + J_5\frac{\partial^2}{\partial\vartheta^2} + J_6\frac{\partial}{\partial\varrho} + J_7\frac{\partial}{\partial\vartheta}\right\}V(\varrho, \vartheta) + \ldots,$$

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where, for brevity, we introduce the definitions:

$$J_1 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{2\varrho},$$

$$J_2 = \frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}\sin\vartheta,$$

$$J_3 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{4\varrho^2},$$

$$J_4 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{\varrho^2}\left[\frac{f(\zeta) - \overline{f(\zeta)}}{2i}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{2}\sin\vartheta\right], \tag{VI.129}$$

$$J_5 = \frac{4if(\zeta)\overline{f(\zeta)} - 2i\left[f^2(\zeta) + \overline{f^2(\zeta)}\right]\cos 2\vartheta - 2\left[f^2(\zeta) - \overline{f^2(\zeta)}\right]\sin 2\vartheta}{8i\varrho^2},$$

$$J_6 = \frac{\left[\bar{\zeta}f(\zeta) - \zeta\overline{f(\zeta)}\right]^2}{4\varrho^3},$$

$$J_7 = \frac{\left[f^2(\zeta) + \overline{f^2(\zeta)}\right]i\sin 2\vartheta - \left[f^2(\zeta) - \overline{f^2(\zeta)}\right]\cos 2\vartheta}{2i\varrho}.$$

We will assume that the basic stress state (VI.38) is characterized by a symmetrical stress tensor, i.e., $\mu_x^0 = \mu_y^0 = 0$. In this case the function $\nabla^2 U = 2[\phi'(z) + \overline{\phi'(z)}]$ will be determined with an accuracy up to undefined coefficients, since the functions $\phi(z)$ and $\psi(z)$, which characterize the additional stress state occurring due to the presence of the hole, should be taken in the general form, which is given by classical elasticity theory[1]. The latter coefficient should be found from the boundary conditions of the problem.

We will represent the solution of biharmonic equation system (VI.11) in polar coordinates (r, θ) in the form

[1]See N. I. Muskhelishvili [1], §36.

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$$U(r, \theta, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k U_k(r, \theta). \qquad \text{(VI.130)}$$

If we substitute the right hand side of function (VI.130) into the first equation of system (VI.11) and equate the coefficients for identical degrees of ε to zero, we obtain a differential equation for any approximation in the form /494

$$\nabla^2\nabla^2 U_k(r, \theta) = 0 \qquad (k = 0, 1. 2, \ldots). \qquad \text{(VI.131)}$$

In our case it is not necessary to solve equation (VI.131), since by taking the functions $\phi(z)$ and $\psi(z)$ in the general form, which is given for the analogous case by classical elasticity theory, but only with undefined coefficients, we will have function (VI.130). By expanding the latter into the Maclaurin series by degrees of ε, we find the right hand side of (VI.130).

We will represent the solution of equation

$$\nabla^2 (F - l^2\nabla^2 F) = 0 \qquad \text{(VI.132)}$$

in the form of series by degrees of ε, i.e.,

$$F(r, \theta, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k F_k(r, \theta). \qquad \text{(VI.133)}$$

By substituting (VI.133) into equation (VI.132), we find differential equations for any approximation in the form

$$\nabla^2 [F_k(r, \theta) - l^2\nabla^2 F_k(r, \theta)] = 0, \qquad \text{(VI.134)}$$

where the operator ∇^2 has the form (VI.114).

The components of the additional stress state caused by the presence of the hole in the "natural" curvilinear orthogonal coordinate system (ρ, ϑ), given by mapping function $z = 1/R\ \omega(\zeta)$ (VI.109), will also be represented in the form of series by degrees of small parameter ε:

$$\sigma_\rho = \sum_{k=0}^{\infty} \varepsilon^k \sigma_\rho^{(k)}, \qquad \sigma_\vartheta = \sum_{k=0}^{\infty} \varepsilon^k \sigma_\vartheta^{(k)},$$

$$\tau_{\rho\vartheta} = \sum_{k=0}^{\infty} \varepsilon^k \tau_{\rho\vartheta}^{(k)}, \qquad \tau_{\vartheta\rho} = \sum_{k=0}^{\infty} \varepsilon^k \tau_{\vartheta\rho}^{(k)},$$

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$$\mu_\varrho = \sum_{k=0}^{\infty} \varepsilon^k \mu_\varrho^{(k)}, \qquad \mu_\vartheta = \sum_{k=0}^{\infty} \varepsilon^k \mu_\vartheta^{(k)}. \tag{VI.135}$$

By substituting expansions for σ_ρ, ..., μ_ϑ (VI.135) into the left hand sides of relations (VI.112) and the expressions for the components σ_r, ..., μ_θ from (VI.113) into the right hand sides, recalling the expansions for $e^{i\beta}$ (VI.127), (VI.121), (VI.122) and functions $V(r, \theta)$ (VI.128), and equating in the right hand and left hand parts, the coefficients for identical degrees of ε, we obtain for the k-th approximation the stress components in the curvilinear orthogonal coordinate system $(\rho\vartheta)$: /495

$$\begin{aligned}
\sigma_\varrho^{(k)} &= \tilde{\sigma}_\varrho^{(k)} + \sum_{m=0}^{k-1} \left[L_1^{(k-m)} \tilde{\sigma}_\varrho^{(m)} + L_2^{(k-m)} (\tilde{\sigma}_\vartheta^{(m)} - \tilde{\sigma}_\varrho^{(m)}) + \frac{1}{2} L_3^{(k-m)} (\tilde{\tau}_{\varrho\vartheta}^{(m)} + \tilde{\tau}_{\vartheta\varrho}^{(m)}) \right];\\
\sigma_\vartheta^{(k)} &= \tilde{\sigma}_\vartheta^{(k)} + \sum_{m=0}^{k-1} \left[L_1^{(k-m)} \tilde{\sigma}_\vartheta^{(m)} - L_2^{(k-m)} (\tilde{\sigma}_\vartheta^{(m)} - \tilde{\sigma}_\varrho^{(m)}) - \frac{1}{2} L_3^{(k-m)} (\tilde{\tau}_{\varrho\vartheta}^{(m)} + \tilde{\tau}_{\vartheta\varrho}^{(m)}) \right];\\
\tau_{\varrho\vartheta}^{(k)} &= \tilde{\tau}_{\varrho\vartheta}^{(k)} + \sum_{m=0}^{k-1} \left[L_1^{(k-m)} \tilde{\tau}_{\varrho\vartheta}^{(m)} - L_2^{(k-m)} (\tilde{\tau}_{\varrho\vartheta}^{(m)} + \tilde{\tau}_{\vartheta\varrho}^{(m)}) + \frac{1}{2} L_3^{(k-m)} (\tilde{\sigma}_\vartheta^{(m)} - \tilde{\sigma}_\varrho^{(m)}) \right];\\
\tau_{\vartheta\varrho}^{(k)} - \tau_{\varrho\vartheta}^{(k)} &= \tilde{\tau}_{\vartheta\varrho}^{(k)} - \tilde{\tau}_{\varrho\vartheta}^{(k)} + \sum_{m=0}^{k-1} L_1^{(k-m)} (\tilde{\tau}_{\vartheta\varrho}^{(m)} - \tilde{\tau}_{\varrho\vartheta}^{(m)});\\
\mu_\varrho^{(k)} &= \tilde{\mu}_\varrho^{(k)} + \sum_{m=0}^{k-1} [L_5^{(k-m)} \tilde{\mu}_\varrho^{(m)} + L_6^{(k-m)} \tilde{\mu}_\vartheta^{(m)}];\\
\mu_\vartheta^{(k)} &= \tilde{\mu}_\vartheta^{(k)} + \sum_{m=0}^{k-1} [L_5^{(k-m)} \tilde{\mu}_\vartheta^{(m)} - L_6^{(k-m)} \tilde{\mu}_\varrho^{(m)}],
\end{aligned} \tag{VI.136}$$

where, for brevity, we introduce the definitions:

$$\begin{aligned}
\tilde{\sigma}_\varrho^{(k)} &= \frac{1}{R^2} \left\{ \left[\frac{1}{\varrho} \cdot \frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2} \cdot \frac{\partial^2}{\partial \vartheta^2} \right] U_k(\varrho, \vartheta) - \frac{\partial^2}{\partial \vartheta \partial \varrho} \left[\frac{1}{\varrho} F_k(\varrho, \vartheta) \right] \right\};\\
\tilde{\sigma}_\vartheta^{(k)} &= \frac{1}{R^2} \left\{ \frac{\partial^2}{\partial \varrho^2} U_k(\varrho, \vartheta) + \frac{\partial^2}{\partial \varrho \partial \vartheta} \left[\frac{1}{\varrho} F_k(\varrho, \vartheta) \right] \right\};\\
\tilde{\tau}_{\varrho\vartheta}^{(k)} &= -\frac{1}{R^2} \left\{ \frac{\partial^2}{\partial \varrho \partial \vartheta} \left[\frac{1}{\varrho} U_k(\varrho, \vartheta) \right] + \left[\frac{1}{\varrho} \cdot \frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2} \cdot \frac{\partial^2}{\partial \vartheta^2} \right] F_k(\varrho, \vartheta) \right\};\\
\tilde{\tau}_{\vartheta\varrho}^{(k)} &= -\frac{1}{R^2} \left\{ \frac{\partial^2}{\partial \varrho \partial \vartheta} \left[\frac{1}{\varrho} U_k(\varrho, \vartheta) \right] - \frac{\partial^2}{\partial \varrho^2} F_k(\varrho, \vartheta) \right\};\\
\tilde{\mu}_\varrho^{(k)} &= \frac{1}{R} \cdot \frac{\partial}{\partial \varrho} F_k(\varrho, \vartheta); \qquad \tilde{\mu}_\vartheta^{(k)} = \frac{1}{R} \cdot \frac{1}{\varrho} \cdot \frac{\partial}{\partial \vartheta} F_k(\varrho, \vartheta).
\end{aligned} \tag{VI.137}$$

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By $U_k(\rho, \vartheta)$ and $F_k(\rho, \vartheta)$ in (VI.137) we refer, respectively, to the solution $U_k(r, \theta)$ of biharmonic equations (VI.131) and the solution $F_k(r, \theta)$ of equation (VI.134), in the k-th approximation in polar coordinates (r, θ), if in these solutions we simply substitute the letters r by ρ and θ by ϑ.

Formula (VI.136) still contains five operators $L_1^{(k-m)}$, $L_2^{(k-m)}$, $L_3^{(k-m)}$, $L_5^{(k-m)}$ and $L_6^{(k-m)}$. We will introduce these operators in expanded form[1] for mapping function $z = \frac{1}{R}\omega(\zeta)$, taken in the form (VI.109) in the first, second and third approximations[2]: /496

$$L_1^{(0)} = L_2^{(0)} = L_3^{(0)} = L_5^{(0)} = L_6^{(0)} \equiv 0;$$

$$L_1^{(1)} = P_1\frac{\partial}{\partial\varrho} + P_2\frac{\partial}{\partial\vartheta};$$

$$L_1^{(2)} = P_3\frac{\partial^2}{\partial\varrho^2} + P_4\frac{\partial^2}{\partial\varrho\partial\vartheta} + P_5\frac{\partial^2}{\partial\vartheta^2} + P_6\frac{\partial}{\partial\varrho} + P_7\frac{\partial}{\partial\vartheta};$$

$$L_2^{(1)} \equiv 0; \qquad L_3^{(1)} = \frac{\bar\zeta\zeta[f'(\zeta) - \overline{f'(\zeta)}] + \zeta\overline{f(\zeta)} - \bar\zeta f(\zeta)}{i\zeta\bar\zeta}; \tag{VI.138}$$

$$L_3^{(2)} = \frac{\bar\zeta^2 f^2(\zeta) - \zeta^2\overline{f^2(\zeta)} + \zeta^2\bar\zeta^2[\overline{f''(\zeta)} - f''(\zeta)]}{2i\zeta^2\bar\zeta^2} + P_8\frac{\partial}{\partial\varrho} + P_9\frac{\partial}{\partial\vartheta};$$

$$L_5^{(1)} = L_1^{(1)}; \qquad L_5^{(2)} = L_1^{(2)} - \frac{1}{2}L_2^{(2)}; \qquad L_6^{(1)} = \frac{1}{2}L_3^{(1)};$$

$$L_6^{(2)} = \frac{1}{2}L_3^{(2)},$$

where the following definitions are introduced:

$$P_1 = \frac{\bar\zeta f(\zeta) + \zeta\overline{f(\zeta)}}{2\varrho};$$

$$P_2 = \frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}\cos\vartheta - \frac{f(\zeta) - \overline{f(\zeta)}}{2\varrho}\sin\vartheta;$$

$$P_3 = \frac{[\bar\zeta f(\zeta) + \zeta\overline{f(\zeta)}]^2}{8\varrho^2};$$

$$P_4 = \frac{\bar\zeta f(\zeta) + \zeta\overline{f(\zeta)}}{\varrho^2}\left[\frac{f(\zeta) - \overline{f(\zeta)}}{4i}\cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{4}\sin\vartheta\right];$$

$$P_5 = \frac{2if(\zeta)\overline{f(\zeta)} - i[f^2(\zeta) + \overline{f^2(\zeta)}]\cos 2\vartheta - [f^2(\zeta) - \overline{f^2(\zeta)}]\sin 2\vartheta}{8i\varrho^2};$$

$$P_6 = -\frac{[\bar\zeta f(\zeta) - \zeta\overline{f(\zeta)}]^2}{8\varrho^3};$$

---

[1] See G. N. Savin, A. N. Guz' [1].

[2] By using as the first approximation, the solution of the problem for $\varepsilon = 0$, i.e., for a round hole.

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$$P_7 = \frac{[f^2(\zeta) + \overline{f^2(\zeta)}]i \sin 2\vartheta - [f^2(\zeta) - \overline{f^2(\zeta)}] \cos 2\vartheta}{4i\varrho^2};$$

$$P_8 = \frac{\bar{\zeta} f(\zeta) + \zeta \overline{f(\zeta)}}{2\varrho}\left[\frac{f'(\zeta) - \overline{f'(\zeta)}}{i} + \frac{\zeta \overline{f(\zeta)} - \bar{\zeta} f(\zeta)}{i\zeta\bar{\zeta}}\right]; \qquad \text{(VI.139)}$$

$$P_9 = \left[\frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho} \cos\vartheta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho} \sin\vartheta\right]\left[\frac{f'(\zeta) - \overline{f'(\zeta)}}{i} + \frac{\zeta \overline{f(\zeta)} - \bar{\zeta} f(\zeta)}{i\zeta\bar{\zeta}}\right].$$

And so, the problem of stress concentration near a round hole, the contour of which has no angular points and is given by equation (VI.115) for $\rho = 1$, by the boundary form perturbation method, reduces to the sequential solution of the corresponding boundary problems for a round hole. We will illustrate the application of this method for the simple example of an elliptical hole. /497

Uniaxial Tension of Range with Elliptical Hole. We will analyze the stress state near an elliptical hole with semiaxes a and b (Figure VI.6) for a plate under uniaxial tension "at infinity" (plane stress state or plane deformation[1]) by forces ρ = const along the Ox axis, i.e., in the basic stress state, characterized by formula (VI.50).

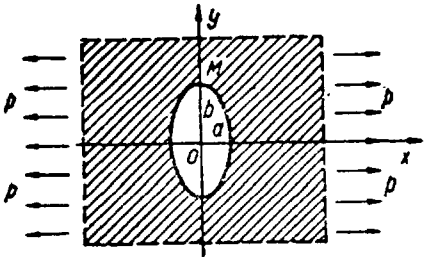


Figure VI.6.

The mapping function is found from (VI.109) by retaining in (VI.110) only one term:

$$z = \zeta + \frac{\varepsilon}{\zeta}, \qquad (\zeta = \varrho e^{i\vartheta}), \qquad \text{(VI.140)}$$

where the parameter

$$\varepsilon = \frac{a - b}{a + b}$$

characterizes the "degree" of deviation of the examined ellipse from the unit circle.

The Ox axis will always coincide with the a semiaxis of the ellipse and the Oy axis, with the b semiaxis (see Figure VI.6).

It is obvious that when $\varepsilon > 0$, semiaxis $a > b$, and the ellipse will be extended along the Ox axis. When $b > a$, the parameter $\varepsilon < 0$, and the ellipse will be extended along the Oy axis, i.e., arranged as indicated in Figure VI.6. When $b = a$ the parameter $\varepsilon = 0$, and the ellipse will be converted into a circle.

[1]Since the equations of plane deformation and plane stress state coincide with an accuracy up to constant coefficients (see G. N. Savin [2], §3, p. 13).

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In the basic stress state (VI.50), the case $\varepsilon < 0$ (Figure VI.6) will be of greatest interest. Stress functions U* and F*, corresponding to stress state (VI.40), will be found in form (VI.42). In the case of basic stress states (VI.50), the functions are $F^{(0)} \equiv 0$ and $F^* = F$.

We will take the biharmonic stress function $U^* = U^{(0)} + U$ in the form given by classical elasticity theory for the analogous problem, but with undefined coefficients. Thus, the function $U^*(r, \theta, \varepsilon) = U^{(0)}(r, \theta, \varepsilon) + U(r, \theta, \varepsilon)$ will be represented in the form

$$U^*(r, \theta, \varepsilon) = \mathrm{Re}\{\bar{z}[\varphi_1^{(0)}(z)+\varphi_1(z)]+\chi_1^{(0)}(z)+\chi_1(z)\}, \tag{VI.141}$$

where $z = re^{i\theta}$, $\phi_1^{(0)}(z)$, $\chi_1^{(0)}(z)$ and $\phi_1(z)$, $\chi_1(z)$ are complex Kolosov-Muskhelishvili potentials, respectively, for the basic and additional stress states.

For basic stress state (VI.50) classical theory[1] gives /498

$$\varphi_1^{(0)}(z) = \frac{p}{4}z, \qquad \psi_1^{(0)}(z) = -\frac{p}{2}z,$$
$$\chi_1^{(0)}(z) = \int \psi_1^{(0)}(z)\,dz = -\frac{p}{4}z^2 + \text{const}. \tag{VI.142}$$

From (VI.140)-(VI.142) we find the stress function for the basic stress state:

$$U^{(0)}(\varrho, \vartheta, \varepsilon) = \frac{p}{4}\left(\varrho^2 - 2\varepsilon + \frac{\varepsilon}{\varrho}\right)(1 - \cos 2\vartheta). \tag{VI.143}$$

Components σ_ϑ^0, σ_ρ^0 and $\tau_{\rho\vartheta}^0 = \tau_{\vartheta\rho}^0$ of basic stress state (VI.50) in curvilinear orthogonal coordinates ρ = const and ϑ = const (see Figure VI.4) are determined by formulas[2]

$$\sigma_\varrho^0 + \sigma_\vartheta^0 = 4\mathrm{Re}\left[\frac{\varphi^{(0)'}(\zeta)}{\omega'(\zeta)}\right],$$
$$\sigma_\vartheta^0 - \sigma_\varrho^0 + i(\tau_{\varrho\vartheta}^0 + \tau_{\vartheta\varrho}^0) = \frac{2\zeta^2}{\varrho^2\overline{\omega'(\zeta)}}\left\{\overline{\omega(\zeta)}\left[\frac{\varphi^{(0)'}(\zeta)}{\omega'(\zeta)}\right]' + \psi^{(0)'}(\zeta)\right\}, \tag{VI.144}$$

where $\tau_{\varrho\vartheta}^0 = \tau_{\vartheta\varrho}^0$; $\zeta = \varrho e^{i\vartheta}$;

$$\varphi^{(0)}(\zeta) = \frac{p}{4}\left(\zeta + \frac{\varepsilon}{\zeta}\right); \qquad \psi^{(0)}(\zeta) = -\frac{p}{2}\left(\zeta + \frac{\varepsilon}{\zeta}\right);$$
$$\omega(\zeta) = \zeta + \frac{\varepsilon}{\zeta}.$$

[1]See N. I. Muskhelishvili [1].
[2]See N. I. Muskhelishvili [1], §50.

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By separating in (VI.144) the real parts from the imaginary, we find the formulas for σ_ρ^0, σ_ϑ^0, $\tau_{\rho\vartheta}^0$ and $\tau_{\vartheta\rho}^0$. We will represent them in the form of an expansion with respect to the small parameter

$$\begin{aligned}
\sigma_\varrho^0 &= \frac{p}{2}\left[1+\cos 2\vartheta-\varepsilon\left(-\frac{1}{\varrho^2}+\frac{1}{\varrho^2}\cos 4\vartheta\right)+\right.\\
&\qquad\left.+\varepsilon^2\left(-\frac{1}{\varrho^4}\cos 2\vartheta+\frac{1}{\varrho^4}\cos 6\vartheta\right)+\ldots\right],\\
\sigma_\vartheta^0 &= \frac{p}{2}\left[1-\cos 2\vartheta-\varepsilon\left(\frac{1}{\varrho^2}-\frac{1}{\varrho^2}\cos 4\vartheta\right)+\varepsilon^2\left(\frac{1}{\varrho^4}\cos 2\vartheta-\frac{1}{\varrho^4}\cos 6\vartheta\right)+\ldots\right],\\
\tau_{\varrho\vartheta}^0 &= \tau_{\vartheta\varrho}^0 = \frac{p}{2}\left[-\sin 2\vartheta+\frac{\varepsilon}{\varrho^2}\sin 4\vartheta+\varepsilon^2\left(\frac{1}{\varrho^4}\sin 2\vartheta-\frac{1}{\varrho^4}\sin 6\vartheta\right)+\ldots\right].
\end{aligned} \tag{VI.145}$$

We will turn now to the determination of the function $U(r, \theta, \varepsilon)$ for the additional stress state caused by the presence of an elliptical hole. /499

The functions for the analogous problem of classical elasticity theory[1] have the form

$$\begin{aligned}
\varphi^{(cl)}(\zeta) &= \frac{p}{2}\frac{1+\varepsilon}{\zeta},\\
\psi^{(cl)}(\zeta) &= -\frac{p}{2}\left[\frac{\varepsilon^2-1+(\varepsilon+1)^2\zeta^2}{\zeta(\zeta^2+\varepsilon)}\right].
\end{aligned} \tag{VI.146}$$

Functions (VI.146) are conveniently represented as functions of the variable $z = \zeta + \varepsilon/\zeta$ (VI.140).

By substituting in (VI.146) instead of ζ its value[2] from (VI.140),

$$\zeta = \frac{1}{2}(z+\sqrt{z^2+4\varepsilon}), \tag{VI.147}$$

we obtain

$$\begin{aligned}
\varphi_1^{(cl)}(z,\varepsilon) &= p\frac{1+\varepsilon}{z+\sqrt{z^2+4\varepsilon}},\\
\psi_1^{(cl)}(z,\varepsilon) &= -p\frac{4(\varepsilon^2-1)+(\varepsilon+1)^2(z+\sqrt{z^2+4\varepsilon})^2}{(z+\sqrt{z^2+4\varepsilon})[(z+\sqrt{z^2+4\varepsilon})^2+4\varepsilon]}.
\end{aligned} \tag{VI.148}$$

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[1] For distinction from other functions, these functions are denoted by the superscript (cl).

[2] In accordance with the examined case, represented in Figure VI.6, as in formula (VI.145), the parameter ε is replaced by $\varepsilon\left(\varepsilon=\left|\frac{a-b}{a+b}\right|\right)$.

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By expanding functions (VI.148) into Maclaurin series with respect to ε, we will have

$$\varphi_1^{(cl)}(z, \varepsilon) = \varphi_1^{(cl)}(z, 0) - \varepsilon\varphi_1^{(cl)'}(z, 0) + \frac{\varepsilon^2}{2!}\varphi_1^{(cl)''}(z, 0) + \ldots, \tag{VI.149}$$

$$\psi_1^{(cl)}(z, \varepsilon) = \psi_1^{(cl)}(z, 0) - \varepsilon\psi_1^{(cl)'}(z, 0) + \frac{\varepsilon^2}{2!}\psi_1^{(cl)''}(z, 0) + \ldots, \tag{VI.150}$$

where

$$\begin{aligned} \varphi_1^{(cl)}(z, 0) &= \frac{p}{2}\cdot\frac{1}{z}, \\ \varphi_1^{(cl)'}(z, 0) &= -\frac{p}{2}\left(\frac{1}{z} - \frac{1}{z^3}\right), \\ \varphi_1^{(cl)''}(z, 0) &= -p\left(\frac{1}{z^3} - \frac{2}{z^5}\right); \end{aligned} \tag{VI.151}$$

$$\begin{aligned} \psi_1^{(cl)}(z, 0) &= -\frac{p}{2}\cdot\frac{1}{z} + \frac{p}{2}\cdot\frac{1}{z^3}, \\ \psi_1^{(cl)'}(z, 0) &= p\frac{1}{z} + p\frac{1}{z^3} - 2p\frac{1}{z^5}, \\ \psi_1^{(cl)''}(z, 0) &= -p\frac{1}{z} + 3p\frac{1}{z^3} - 6p\frac{1}{z^5} + 15p\frac{1}{z^7}. \end{aligned} \tag{VI.152}$$

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We will also make use of the function $\chi_1^{(cl)}(z, \varepsilon) = \int\psi_1^{(cl)}(z, \varepsilon)dz$. By integrating with respect to z the function (VI.150), we obtain

$$\begin{aligned} \chi_1^{(cl)}(z, \varepsilon) &= -\frac{p}{2}\ln z - \frac{p}{4}\cdot\frac{1}{z^2} - \varepsilon\left[p\ln z + \frac{p}{2}\cdot\frac{1}{z^2} - \frac{p}{2}\cdot\frac{1}{z^4}\right] + \\ &+ \varepsilon^2\left[-\frac{p}{2}\ln z - \frac{3p}{4}\cdot\frac{1}{z^2} + \frac{3p}{4}\cdot\frac{1}{z^4} - \frac{5p}{4}\cdot\frac{1}{z^6}\right] + \ldots; \\ \varphi_1^{(cl)}(z, \varepsilon) &= \frac{p}{2}\cdot\frac{1}{z} - \varepsilon\left[-\frac{p}{2z} + \frac{p}{2z^3}\right] + \varepsilon^2\left[\frac{p}{z^5} - \frac{p}{2z^3}\right] + \ldots. \end{aligned} \tag{VI.153}$$

After determining function (VI.153) we may proceed to the construction of functions $\phi(z, \varepsilon)$ and $\chi(z, \varepsilon)$ for our problem. These functions will be taken in the form of expansions with respect to small parameter ε:

$$\begin{aligned} \chi(z, \varepsilon) &= A\ln z + \frac{B}{z^2} - \varepsilon\left(a_1\ln z + \frac{a_2}{z^2} + \frac{a_3}{z^4}\right) + \\ &+ \varepsilon^2\left(a_4\ln z + \frac{a_5}{z^2} + \frac{a_6}{z^4} + \frac{a_7}{z^6}\right) - \ldots, \\ \varphi(z, \varepsilon) &= \frac{C}{z} - \varepsilon\left(\frac{a_8}{z} + \frac{a_9}{z^3}\right) + \varepsilon^2\left(\frac{a_{10}}{z} + \frac{a_{11}}{z^3} + \frac{a_{12}}{z^5}\right) - \ldots. \end{aligned} \tag{VI.154}$$

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By knowing function (VI.154) we can determine stress functions $U(r^*, \theta, \varepsilon)$ according to E. Gurs' formula (VI.27), conveniently represented in polar coordinates $r^* = Rr$ and θ in the form of an expansion with respect to small parameter ε:

$$U(r^*, \theta, \varepsilon) = A \ln r^* + \left(\frac{B}{r^{*2}} + C\right)\cos 2\theta - \varepsilon\left[a_1 \ln r^* + \left(\frac{a_3}{r^{*2}} + a_8\right)\cos 2\theta + \right.$$
$$\left. + \left(\frac{a_3}{r^{*4}} + \frac{a_9}{r^{*2}}\right)\cos 4\theta\right] + \varepsilon^2\left[a_4 \ln r^* + \left(\frac{a_5}{r^{*2}} + a_{10}\right)\cos 2\theta + \left(\frac{a_6}{r^{*4}} + \frac{a_{11}}{r^{*2}}\right)\cos 4\theta + \right. \quad \text{(VI.155)}$$
$$\left. + \left(\frac{a_7}{r^{*6}} + \frac{a_{12}}{r^{*4}}\right)\cos 6\theta\right] + \ldots .$$

Assuming that $\varepsilon = 0$ in function (VI.155), we obtain the function $U(r^*, \theta)$ (VI.55), which we quite naturally expected. The undefined coefficients A, B, C, a_1, a_2, ..., a_{12} in function (VI.155) should be found from boundary conditions (VI.63).

It follows from (VI.155) that in dimensionless coordinates $r = r^*/R$ and θ, the functions are /501

$$U_0(r, \theta) = A \ln Rr + \left(\frac{B}{R^2 r^2} + C\right)\cos 2\theta,$$
$$U_1(r, \theta) = a_1 \ln Rr + \left(\frac{a_3}{R^2 r^2} + a_8\right)\cos 2\theta + \left(\frac{a_9}{R^4 r^4} + \frac{a_9}{R^2 r^2}\right)\cos 4\theta, \quad \text{(VI.156)}$$
$$U_2(r, \theta) = a_4 \ln Rr + \left(\frac{a_5}{R^2 r^2} + a_{10}\right)\cos 2\theta + \left(\frac{a_6}{R^4 r^4} + \frac{a_{11}}{R^2 r^2}\right)\cos 4\theta +$$
$$+ \left(\frac{a_7}{R^6 r^6} + \frac{a_{12}}{R^4 r^4}\right)\cos 6\theta.$$

From (VI.149) and (VI.154) we readily find

$$\nabla^2 U = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right] = -4\left[\frac{C}{r^{*2}}\cos 2\theta - \varepsilon\left(\frac{a_8}{r^{*2}}\cos 2\theta + \frac{3a_9}{r^{*4}}\cos 4\theta\right) + \right.$$
$$\left. + \varepsilon^2\left(\frac{a_{10}}{r^{*2}}\cos 2\theta + \frac{3a_{11}}{r^{*4}}\cos 4\theta + \frac{5a_{12}}{r^{*6}}\cos 6\theta\right) + \ldots\right]. \quad \text{(VI.157)}$$

The function $F(r, \theta)$ for the selected function $\nabla^2 U$ (VI.157) will be found in dimensionless polar coordinates r, θ from conditions (VI.18), where the operator ∇^2 should be used in the form (VI.114) in the same coordinates.

By integrating (VI.34) for $\nabla^2 U$ (VI.157) and including the integration constant in the function $F(r^*, \theta)$, we obtain for the latter the following differential equation:

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$$F - l^2\nabla^2 F = 8(1-\nu)l^2\left[\frac{C}{r^{*2}}\sin 2\theta - \varepsilon\left(\frac{a_8}{r^{*2}}\sin 2\theta + \frac{3a_9}{r^{*4}}\sin 4\theta\right) + \right.$$
$$\left. + \varepsilon^2\left(\frac{a_{10}}{r^{*2}}\sin 2\theta + \frac{3a_{11}}{r^{*4}}\sin 4\theta + \frac{5a_{12}}{r^{*6}}\sin 6\theta\right) + \ldots\right], \quad \text{(VI.158)}$$

the partial solution of which in polar coordinates (r, θ) will be found in the form

$$F_{\text{part}}(r,\theta) = 8(1-\nu)l^2\left[\frac{C}{R^2r^2}\sin 2\theta - \varepsilon\left(\frac{a_8}{R^2r^2}\sin 2\theta + \frac{3a_9}{R^4r^4}\sin 4\theta\right) + \right.$$
$$\left. + \varepsilon^2\left(\frac{a_{10}}{R^2r^2}\sin 2\theta + \frac{3a_{11}}{R^4r^4}\sin 4\theta + \frac{5a_{12}}{R^6r^6}\sin 6\theta\right) + \ldots\right]. \quad \text{(VI.159)}$$

The complete solution of equation (VI.158) will be found in the form (VI.59).

We will look for the solution $\Omega(r, \theta, \varepsilon)$ of equation (VI.26) in the form of an expansion by degrees of small parameter ε:

$$\Omega\left(\frac{Rr}{l}, \theta, \varepsilon\right) = R_0\left(\frac{Rr}{l}\right)\sin 2\theta - \varepsilon\left[R_1\left(\frac{Rr}{l}\right)\sin 2\theta + R_2\left(\frac{Rr}{l}\right)\sin 4\theta\right] +$$
$$+ \varepsilon^2\left[R_3\left(\frac{Rr}{l}\right)\sin 2\theta + R_4\left(\frac{Rr}{l}\right)\sin 4\theta + R_5\left(\frac{Rr}{l}\right)\sin 6\theta\right] + \ldots . \quad \text{(VI.160)}$$

By substituting the function (VI.160) into equation (VI.26) and equating the coefficients for identical degrees of ε for each function R_0, R_1, R_2, R_3, R_4 and R_5, we obtain the corresponding equations of the Bessel type, from which we find[1] /502

$$R_0 = DK_2(r/l^*), \quad R_1 = a_{13}K_2(r/l^*),$$
$$R_2 = a_{14}K_4(r/l^*), \quad R_3 = a_{15}K_2(r/l^*),$$
$$R_4 = a_{16}K_4(r/l^*), \quad R_5 = a_{17}K_6(r/l^*), \quad \text{(VI.161)}$$

where K_2, K_4 and K_6 are modified Bessel functions of kind II (McDonald's functions or modified Hankel functions) and, for brevity, we introduce the definition:

$$l^* = \frac{l}{R}. \quad \text{(VI.162)}$$

[1] In the solution of equation (VI.26) we retain only those functions which satisfy condition (VI.43), i.e., conditions at infinity.

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Thus,

$$\Omega\left(\frac{r}{l^*},\ \theta,\ \varepsilon\right) = DK_2(r/l^*)\sin 2\theta - \varepsilon\,[a_{13}K_2(r/l^*)\sin 2\theta + \\ + a_{14}K_4(r/l^*)\sin 4\theta] + \varepsilon^2\,[a_{15}K_2(r/l^*)\sin 2\theta + a_{16}K_4(r/l^*)\sin 4\theta + \\ + a_{17}K_6(r/l^*)\sin 6\theta] + \ldots . \tag{VI.163}$$

Consequently, the function is

$$F(r,\theta,\varepsilon) = \left[DK_2(r/l^*) + \frac{8(1-\nu)\,l^2C}{R^2r^2}\right]\sin 2\theta - \\ - \varepsilon\left\{\left[a_{13}K_2(r/l^*) + \frac{8(1-\nu)\,l^2a_8}{R^2r^2}\right]\sin 2\theta + \right. \\ \left. + \left[a_{14}K_4(r/l^*) + \frac{24(1-\nu)\,l^2a_9}{R^4r^4}\right]\sin 4\theta\right\} + \\ + \varepsilon^2\left\{\left[a_{15}K_2(r/l^*) + \frac{8(1-\nu)\,l^2a_{10}}{R^2r^2}\right]\sin 2\theta + \left[a_{16}K_4(r/l^*) + \right.\right. \\ \left.\left. + \frac{24(1-\nu)\,l^2a_{11}}{R^4r^4}\right]\sin 4\theta + \left[a_{17}K_6(r/l^*) + \frac{40(1-\nu)\,l^2a_{12}}{R^6r^6}\right]\sin 6\theta\right\} + \ldots , \tag{VI.164}$$

where

$$D, C, a_8, a_9, \ldots, a_{17} \tag{VI.165}$$

are integration constants requiring further determination. From (VI.164) it follows that

$$F_0(r,\theta) = \left[DK_2 + \frac{\beta_1^* C}{r^2}\right]\sin 2\theta; \\ F_1(n,\theta) = \left[a_{13}K_2 + \frac{\beta_1^* a_8}{r^2}\right]\sin 2\theta + \left[a_{14}K_4 + \frac{3\beta_1^* a_9}{R^2r^4}\right]\sin 4\theta; \tag{VI.166}$$

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$$F_2(r,\theta) = \left[a_{15}K_2 + \frac{\beta_1^* a_{10}}{r^2}\right]\sin 2\theta + \left[a_{16}K_4 + \frac{3\beta_1^* a_{11}}{R^2r^4}\right]\sin 4\theta + \\ + \left[a_{17}K_6 + \frac{5\beta_1^* a_{12}}{R^4r^6}\right]\sin 6\theta,$$

where

$$\beta_1^* = 8(1-\nu)\,l^2/R^2.$$

Integration constants (VI.165) in functions (VI.155) and (VI.164) should be determined from boundary condition (VI.63) on the contour of the elliptical

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hole (see Figure VI.6). The operators $L_1^{(0)}, \ldots, L_6^{(2)}$ (VI.138) for the mapping function $z = \omega(\zeta)$ (VI.140), taken with an accuracy up to ε^2, inclusively, will have the form

$$
\begin{aligned}
&L_1^{(0)} = L_2^{(0)} = L_3^{(0)} = L_5^{(0)} = L_6^{(0)} = 0;\\
&L_1^{(1)} = \frac{\cos 2\vartheta}{\varrho}\cdot\frac{\partial}{\partial\varrho} - \frac{\sin 2\vartheta}{\varrho^2}\cdot\frac{\partial}{\partial\vartheta};\\
&L_1^{(2)} = \frac{1+\cos 4\vartheta}{4\varrho^2}\cdot\frac{\partial^2}{\partial\varrho^2} - \frac{\sin 4\vartheta}{2\varrho^2}\cdot\frac{\partial^2}{\partial\varrho\partial\vartheta}\cdot\frac{1}{\varrho} + \frac{1-\cos 4\vartheta}{4\varrho^4}\left(\frac{\partial^2}{\partial\vartheta^2} + \varrho\frac{\partial}{\partial\varrho}\right);\\
&L_2^{(1)} \equiv 0; \quad L_3^{(1)} = 4\frac{\sin 2\vartheta}{\varrho^2}; \quad L_2^{(2)} = 2\frac{1-\cos 4\vartheta}{\varrho^4};\\
&L_3^{(2)} = 2\frac{\sin 4\vartheta}{\varrho^3}\cdot\frac{\partial}{\partial\varrho} - 2\frac{1-\cos 4\vartheta}{\varrho^4}\cdot\frac{\partial}{\partial\vartheta}.
\end{aligned}
\tag{VI.167}
$$

Now, by substituting into (VI.111) the values found for stress components σ_ρ^0, σ_ϑ^0, $\tau_{\rho\vartheta}^0 = \tau_{\vartheta\rho}^0$ in (VI.145) and the values of the components σ_ρ, σ_ϑ, ..., μ_ϑ of (VI.135), retaining in each of them only three terms and recalling: a) the values of $\sigma_\rho^{(k)}$, $\sigma_\vartheta^{(k)}$, $\tau_{\rho\vartheta}^{(k)}$, $\tau_{\vartheta\rho}^{(k)}$, $\mu_\rho^{(k)}$, $\mu_\vartheta^{(k)}$ (VI.136) (k = 0, 1, 2); b) the form of operator (VI.167); c) the values of $\tilde{\sigma}_\rho^{(k)}$, $\tilde{\sigma}_\vartheta^{(k)}$, ..., $\tilde{\mu}_\vartheta^{(k)}$ (VI.137), we find[1] the stress components around the elliptical hole:

$$
\begin{aligned}
&\sigma_\varrho^* = \sigma_\varrho^0 + \sigma_\varrho^{(0)} - \varepsilon\sigma_\varrho^{(1)} + \varepsilon^2\sigma_\varrho^{(2)},\\
&\sigma_\vartheta^* = \sigma_\vartheta^0 + \sigma_\vartheta^{(0)} - \varepsilon\sigma_\vartheta^{(1)} + \varepsilon^2\sigma_\vartheta^{(2)};\\
&\tau_{\varrho\vartheta}^* = \tau_{\varrho\vartheta}^0 + \tau_{\varrho\vartheta}^{(0)} - \varepsilon\tau_{\varrho\vartheta}^{(1)} + \varepsilon^2\tau_{\varrho\vartheta}^{(2)},\\
&\tau_{\vartheta\varrho}^* = \tau_{\vartheta\varrho}^0 + \tau_{\vartheta\varrho}^{(0)} - \varepsilon\tau_{\vartheta\varrho}^{(1)} + \varepsilon^2\tau_{\vartheta\varrho}^{(2)},\\
&\mu_\varrho^* = \mu_\varrho^0 + \mu_\varrho^{(0)} - \varepsilon\mu_\varrho^{(1)} + \varepsilon^2\mu_\varrho^{(2)},\\
&\mu_\vartheta^* = \mu_\vartheta^0 + \mu_\vartheta^{(0)} - \varepsilon\mu_\vartheta^{(1)} + \varepsilon^2\mu_\vartheta^{(2)}.
\end{aligned}
\tag{VI.168}
$$

By introducing from (VI.167) the expressions found for components σ_ρ^*, $\tau_{\rho\vartheta}^*$ /504
and μ_ρ^* into boundary conditions (VI.63), we obtain a linear equation system, from which we determine the desired coefficients:

[1] In the functions $U_k(r, \theta)$ (VI.156) and $F_k(r, \theta)$ (VI.166) (k = 0, 1, 2) the letters r and θ should be substituted simply by ρ and ϑ, respectively.

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$$A = -\frac{pR^2}{2}; \qquad B = -\frac{pR^4(1-\mathcal{H})}{4(1+\mathcal{H})};$$

$$C = \frac{pR^2}{2(1+\mathcal{H})}; \qquad D = -\frac{pRl\mathcal{H}}{(1+\mathcal{H})K_1(R/l)};$$

$$a_1 = \frac{pR^2}{1+\mathcal{H}}; \qquad a_2 = \frac{pR^4}{2(1+\mathcal{H})}; \qquad \text{(VI.169)}$$

$$a_8 = -\frac{pR^2}{2(1+\mathcal{H})}; \quad a_{13} = \frac{pRl\mathcal{H}}{(1+\mathcal{H})K_1(R/l)};$$

$$a_9 = -\frac{R^2(T_2 - T_3 + 3T_1T_4)}{6(1+6\beta_1^*T_4)}; \quad a_{14} = \frac{T_1 - 2\beta_1^*(T_3 - T_2)}{K_4'(R/l)(1+6\beta_1^*T_4)};$$

$$a_3 = -\frac{R^2}{40}\{R^2(T_2 + T_3 + 5T_1T_5) + 30a_9[1 + 2\beta_1^*(T_5 - 2)]\},$$

where

$$\beta_1^* = 8(1-\nu)\frac{l^2}{R^2}; \quad \mathcal{H} = \frac{8(1-\nu)}{4 + \frac{R^2}{l^2} + 2\frac{R}{l}\frac{K_0(R/l)}{K_1(R/l)}};$$

$$T_1 = \frac{pR^2}{2(1+\mathcal{H})}\left\{-7\beta_1^* + \mathcal{H}\left[5 + 28\frac{l^2}{R^2} + \left(\frac{R}{l} + 14\frac{l}{R}\right)\frac{K_0(R/l)}{K_1(R/l)}\right]\right\};$$

$$T_2 = \frac{pR^2}{1+\mathcal{H}}\left\{1 + 15\beta_1^* - \mathcal{H}\left[19 + 60\frac{l^2}{R^2} + \left(2\frac{R}{l} + 30\frac{l}{R}\right)\frac{K_0(R/l)}{K_1(R/l)}\right]\right\};$$

$$T_3 = \frac{pR^2}{2(1+\mathcal{H})}\left\{8 + 30\beta_1^* - \mathcal{H}\left[32 + 120\frac{l^2}{R^2} + \left(\frac{R}{l} + 60\frac{l}{R}\right)\frac{K_0(R/l)}{K_1(R/l)}\right]\right\}; \qquad \text{(VI.170)}$$

$$T_4 = \frac{1 + 8\frac{l^2}{R^2} + 4\frac{l}{R}\cdot\frac{K_0(R/l)}{K_1(R/l)}}{1 + 40\frac{l^2}{R^2} + 192\frac{l^4}{R^4} + 8\frac{l}{R}\left(1 + 12\frac{l^2}{R^2}\right)\frac{K_0(R/l)}{K_1(R/l)}};$$

$$T_5 = \frac{1 + 72\frac{l^2}{R^2} + 384\frac{l^4}{R^4} + \left(12\frac{l}{R} + 192\frac{l^3}{R^3}\right)\frac{K_0(R/l)}{K_1(R/l)}}{1 + 40\frac{l^2}{R^2} + 192\frac{l^4}{R^4} + 8\frac{l}{R}\left(1 + 12\frac{l^2}{R^2}\right)\frac{K_0(R/l)}{K_1(R/l)}}$$

From recurrent relations (VI.64) we obtain

$$K_4(R/l) = \left(1 + \frac{24l^2}{R^2}\right)K_0(R/l) + \frac{8l}{R}\left(1 + \frac{6l^2}{R^2}\right)K_1(R/l);$$

$$K_4'(R/l) = -8\left(1 + 12\frac{l^2}{R^2}\right)K_0(R/l) - \frac{l}{R}\left(40 + 192\frac{l^2}{R^2} + \frac{R^2}{l^2}\right)K_1(R/l). \qquad \text{(VI.171)}$$

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The coefficients a_4, a_5, a_6, a_7, a_{10}, a_{11}, a_{12}, a_{15}, a_{16}, a_{17} in the function $U_k(r, \theta)$ of (VI.156) and the function $F_k(r, \theta)$ of (VI.166) are found from boundary conditions for the third approximation. /505

From (VI.170) under conditions (VI.70), we obtain

$$\lim_{\frac{R}{l}\to\infty} T_1 = 0; \quad \lim_{\frac{R}{l}\to\infty} T_2 = pR^2; \quad \lim_{\frac{R}{l}\to\infty} T_3 = 4pR^2;$$
$$\lim_{\frac{R}{l}\to\infty} T_4 = \lim_{\frac{R}{l}\to\infty} T_5 = 1. \tag{VI.172}$$

Thus, from (VI.169) under condition (VI.172), we will have

$$A = -\frac{pR^2}{2}; \quad B = -\frac{pR^4}{4}; \quad C = \frac{pR^2}{2}; \quad a_1 = pR^2;$$
$$a_2 = \frac{pR^4}{2}; \quad a_3 = -\frac{pR^4}{2}; \quad a_8 = -\frac{pR^2}{2}; \quad a_9 = \frac{pR^4}{2}. \tag{VI.173}$$

The values of coefficients (VI.173) coincide[1] with the corresponding values of the coefficients of the functions $\phi_1^{(c1)}(z, \varepsilon)$ and $\chi_1^{(c1)}(z, \varepsilon)$.

From (VI.169) we have formulas (VI.168) for a round hole. It is of great importance to analyze the effect of the asymmetry of the stress tensor on the magnitude of stress concentration coefficients near an elliptical hole at the point of the contour with the least radius of curvature (see Figure VI.6) (in our case this point of the contour will be the point M) and on the size of the zone of perturbation near the examined hole.

In order to answer the first question, it is necessary to calculate the values of σ_ϑ^* for $\rho = 1$ and $\vartheta = \pi/2$:

$$(\sigma_\vartheta^*)_{\substack{\rho=1\\ \vartheta=\frac{\pi}{2}}} = p + \frac{1}{R^2}\left[-A - \frac{6B}{R^2} + \frac{6\beta_1 C}{R^2} - 2D(K_2'(R/l) - K_2(R/l))\right] -$$
$$- \frac{\varepsilon}{R^2}\left[-a_1 - \frac{24B}{R^2} + \frac{24\beta_1 C}{R^2} + 2D(K_2''(R/l) - 2K_2'(R/l) + 2K_2(R/l)) + \frac{20a_2}{R^4} + \right. \tag{VI.174}$$
$$+ \frac{6a_9}{R^2}\left(1 - \frac{10\beta_1}{R^2}\right) + 4a_{14}(K_4'(R/l) - K_4(R/l)) - 2A - \frac{6a_2}{R^2} + \frac{6\beta_1 a_8}{R^2} -$$
$$\left. - 2a_{13}(K_2'(R/l) - K_2(R/l))\right],$$

[1]Recalling the form of the Laplace operator in polar coordinates (VI.19) and in dimensionless polar coordinate (VI.114).

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and to the second question, it is sufficient to calculate the values of σ_ϑ^* and σ_ρ^*, respectively, by degrees $\vartheta = \pi/2$ and $\vartheta = 0$, and to compare the stress-strain diagrams for these stresses obtained with the same stress-strain diagrams yielded by classical elasticity theory. Such a comparison of the stress-strain diagrams of σ_ρ^* and σ_ϑ^* is presented in Figures VI.7 and VI.8 for $\varepsilon = 0.2$, where the solid curves correspond to moment elasticity theory, and the broken curves, to classical elasticity theory. Stresses σ_ρ^* through cross section $\vartheta = 0$ (see Figure VI.6) are shown on Figure VI.7 and σ_ϑ^* through cross section $\vartheta = \pi/2$ (see Figure VI.6), on Figure VI.8. The numerical values of the stress components presented on these figures were obtained for $\nu = 0.25$ and $(R/l)_{min} = 3$. /506

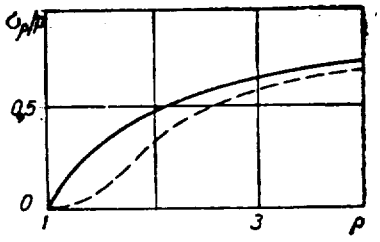


Figure VI.7.

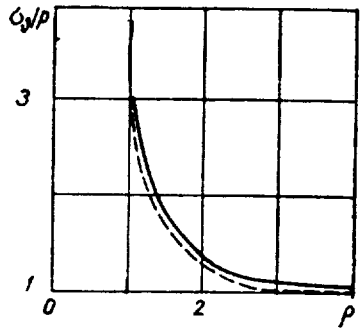


Figure VI.8.

Multifold Tension of Range Weakened by an Elliptical Hole[1]. We will analyze stress concentration near an elliptical hole with semiaxes a and b (Figure VI.9) under uniform multifold tension at infinity by forces p = const, i.e., in the basic stress state

$$\sigma_x^0 = \sigma_y^0 = p, \quad \tau_{xy}^0 = \tau_{yx}^0 = \mu_x^0 = \mu_y^0 = 0. \tag{VI.175}$$

Proceeding as in the case of uniaxial tension, we obtain

$$\varphi_1^{(cl)}(z, \varepsilon) = -\varepsilon \frac{p}{z} - \varepsilon^3 \frac{p}{z^3} + \ldots,$$
$$\chi_1^{(cl)}(z, \varepsilon) = -p \ln z + \varepsilon \frac{p}{z^2} + \varepsilon^2 p \left(-\ln z + \frac{3}{2z^4}\right) + \ldots. \tag{VI.176}$$

After finding functions (VI.176) we may, on the basis of the above considerations, select the functions for the examined case of moment elasticity theory:

[1]A somewhat different solution is given by Yu. N. Nemish [2].

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$$\varphi(z, \varepsilon) = \varepsilon \frac{b_1}{z} + \varepsilon^2 \frac{b_3}{z^3} + \ldots;$$

$$\chi(z, \varepsilon) = b_3 \ln z + \varepsilon \frac{b_4}{z^2} + \varepsilon^2 \left(b_5 \ln z + \frac{b_6}{z^4} \right) + \ldots . \qquad \text{(VI.177)}$$

If we know the functions $\phi(z, \varepsilon)$ and $\chi(z, \varepsilon)$, then by E. Gurs' formula we can determine the function $U(r^*, \theta, \varepsilon)$ which, as earlier, we represent in the form of an expansion with respect to small parameter ε: /507

$$U(r^*, \theta, \varepsilon) = \operatorname{Re}[\bar{z}\varphi(z) + \chi(z)] = b_3 \ln r^* + \varepsilon \left(b_1 + \frac{b_4}{r^{*2}} \right) \cos 2\theta +$$

$$+ \varepsilon^2 \left[b_5 \ln r^* + \left(\frac{b_2}{r^{*2}} + \frac{b_6}{r^{*4}} \right) \cos 4\theta \right] + \ldots . \qquad \text{(VI.178)}$$

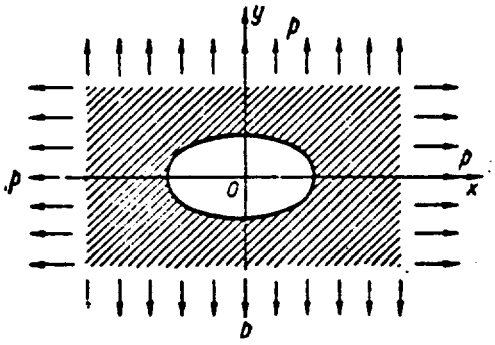


Figure VI.9.

It follows from (VI.178) that in dimensionless polar coordinates r and θ, the functions are

$$U_0(r, \theta) = b_3 \ln Rr,$$

$$U_1(r, \theta) = \left(b_1 + \frac{b_4}{R^2 r^2} \right) \cos 2\theta, \qquad \text{(VI.179)}$$

$$U_2(r, \theta) = b_5 \ln Rr + \left(\frac{b_5}{R^2 r^2} + \frac{b_6}{R^4 r^4} \right) \cos 4\theta.$$

After determining, on the basis of (VI.178), the function $\nabla^2 U$, we obtain the equation for the function $F(r, \theta)$

$$F - l^2 \nabla^2 F = 8(1 - \nu) l^2 \times$$

$$\times \left[\varepsilon \frac{b_1}{R^2 r^2} \sin 2\theta + \varepsilon^2 \frac{3 b_3}{R^4 r^4} \sin 4\theta + \ldots \right] \qquad \text{(VI.180)}$$

Using analogous considerations, as we did in the case of uniaxial tension, we find the solution of equation (VI.180) in the form

$$F(r, \theta) = \varepsilon \left[\beta_1^* \frac{b_1}{r^2} \sin 2\theta + b_7 K_2(r/l^*) \sin 2\theta \right] +$$

$$+ \varepsilon^2 \left[3 \frac{\beta_1^* b_3}{R^4 r^4} \sin 4\theta + b_8 K_4(r/l^*) \sin 4\theta \right] + \ldots , \qquad \text{(VI.181)}$$

where, as before,

$$\beta_1^* = 8(1 - \nu) l^2 / R^2.$$

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Consequently,

$$\begin{aligned} F_0(r,\theta) &= 0, \\ F_1(r,\theta) &= \left[\frac{\beta_1^* b_1}{r^2} + b_7 K_2(r/l^*)\right]\sin 2\theta, \\ F_2(r,\theta) &= \left[\frac{3\beta_1^* b_2}{R^2 r^4} + b_8 K_4(r/l^*)\right]\sin 4\theta. \end{aligned} \tag{VI.182}$$

The constant coefficients b_n (n = 1, 2, ..., 8) in (VI.179) and (VI.182) are determined from boundary condition (VI.63) on the contour of an elliptical hole, i.e., when $\rho = 1$.

For the case under consideration /508

$$\begin{aligned} \sigma_Q^* &= \sigma_Q^0 + \sigma_Q^{(0)} + \varepsilon\sigma_Q^{(1)} + \varepsilon^2\sigma_Q^{(2)}, \\ \tau_{Q\Theta}^* &= \tau_{Q\Theta}^0 + \tau_{Q\Theta}^{(0)} + \varepsilon\tau_{Q\Theta}^{(1)} + \varepsilon^2\tau_{Q\Theta}^{(2)}, \\ \mu_Q^* &= \mu_Q^0 + \mu_Q^{(0)} + \varepsilon\mu_Q^{(1)} + \varepsilon^2\mu_Q^{(2)}, \end{aligned} \tag{VI.183}$$

where, as it is easy to check from (VI.175), $\sigma_\rho^0 = p$; $\tau_{\rho\vartheta}^0 = \mu_\rho^0 = 0$, and stress components $\sigma_\rho^{(0)}$, $\sigma_\rho^{(1)}$, ..., $\mu_\rho^{(2)}$ are determined by the given functions of stresses (VI.179) and (VI.182) by formulas (VI.136) and (VI.137).

Hence, by satisfying boundary conditions (VI.63), we find

$$\begin{aligned} b_1 &= -\frac{pR^2}{1+Q_1}; \quad b_2 = -\frac{pR^4}{6(1+Q_1)(1+Q_2)}\left\{2(3+Q_1) + Q_2\left(\frac{7}{2}+Q_4\right) - Q_2\right\}; \\ b_3 &= -pR^2; \quad b_4 = \frac{pR^4}{1+Q_1}; \quad b_5 = -pR^2\frac{1-Q_1}{1+Q_1}; \\ b_6 &= -\frac{pR^6}{20}\left\{6p - \frac{2p}{1+Q_1}[9+\beta_1^*(15+Q_5)] + \frac{6}{R^4}(3-10\beta_1^*)b_2 + \right. \\ &\quad \left. + \frac{4}{R^2}\left[K_4'\left(\frac{R}{l}\right) - K_4\left(\frac{R}{l}\right)\right]b_8\right\}, \\ b_7 &= -\frac{2\beta_1^* pR^2}{(1+Q_1)K_2'\left(\frac{R}{l}\right)}; \quad b_8 = -\frac{2\beta_1^* pR^2}{(1+Q_1)(1+Q_2)K_4'\left(\frac{R}{l}\right)}\left\{\frac{1}{2}(5+4Q_1) - \right. \\ &\quad \left. - Q_2 - Q_4\right\}. \end{aligned} \tag{VI.184}$$

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where

$$Q_1 = \beta_1^* \left[1 + 2 \frac{K_2\left(\frac{R}{l}\right)}{K_2'\left(\frac{R}{l}\right)} \right];$$

$$Q_2 = 6\beta_1^* \left[1 + 4 \frac{K_4\left(\frac{R}{l}\right)}{K_4'\left(\frac{R}{l}\right)} \right];$$

$$Q_3 = \beta_1^* \frac{3K_2^*\left(\frac{R}{l}\right) + 5K_2'\left(\frac{R}{l}\right) - 8K_2\left(\frac{R}{l}\right)}{K_2'\left(\frac{R}{l}\right)}; \qquad \text{(VI.185)}$$

$$Q_4 = \frac{K_2^*\left(\frac{R}{l}\right) - 2K_2'\left(\frac{R}{l}\right) + 4K_2\left(\frac{R}{l}\right)}{2K_2'\left(\frac{R}{l}\right)};$$

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$$Q_5 = \frac{2K_2^*\left(\frac{R}{l}\right) - 5K_2'\left(\frac{R}{l}\right) + 8K_2\left(\frac{R}{l}\right)}{K_2'\left(\frac{R}{l}\right)}.$$

From (VI.185) we obtain

$$\lim_{\frac{R}{l}\to\infty} \beta_1^* = \lim_{\frac{R}{l}\to\infty} Q_1 = \lim_{\frac{R}{l}\to\infty} Q_2 = \lim_{\frac{R}{l}\to\infty} Q_3 = 0;$$

$$\lim_{\frac{R}{l}\to\infty} Q_4 = \lim_{\frac{R}{l}\to\infty} Q_5 = \infty. \qquad \text{(VI.186)}$$

From (VI.184) under conditions (VI.186) we obtain

$$b_1 = -pR^2; \quad b_2 = -pR^4; \quad b_3 = -pR^2; \quad b_4 = pR^4;$$

$$b_5 = -pR^2; b_6 = \frac{3}{2} pR^6; \quad b_7 = b_8 = \infty. \qquad \text{(VI.187)}$$

Recalling the form of the Laplace operator (VI.114) in dimensionless polar coordinates (r, θ), we see that the values of coefficients (VI.187) coincide with the corresponding values of (VI.176).

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The stresses are

$$\sigma_\varrho^* = p + \frac{b_3}{R^2\varrho^2} + \frac{\varepsilon}{R^2}\left\{b_1\left(-\frac{4}{\varrho^2} + \frac{6\beta_1^*}{\varrho^4}\right) - \frac{2b_3}{\varrho^4} - \frac{6b_4}{R^2\varrho^4} - 2b_7\frac{K_2'(\varrho/l^*)\varrho - K_2(\varrho/l^*)}{\varrho^2}\right\}\cos 2\vartheta + \frac{\varepsilon^2}{R^2}\left\{b_1\left(-\frac{4}{\varrho^4} + \frac{6\beta_1^*}{\varrho^6}\right) - \frac{3b_3}{\varrho^6} - \frac{6b_4}{R^2\varrho^6} + \frac{b_5}{\varrho^2} - b_7\frac{K_2'(\varrho/l^*)\varrho - 4K_2(\varrho/l^*)}{\varrho^4} + \left[b_1\left(\frac{12}{\varrho^4} - \frac{30\beta_1^*}{\varrho^6}\right) + b_2\left(-\frac{18}{R^2\varrho^4} + \frac{60\beta_1^*}{R^2\varrho^6}\right) + \frac{6b_3}{\varrho^6} + \frac{30b_4}{R^2\varrho^6} - \frac{20b_6}{R^4\varrho^6} - b_7\frac{2K_2''(\varrho/l^*)\varrho^2 - 5K_2'(\varrho/l^*)\varrho + 8K_2(\varrho/l^*)}{\varrho^4} - 4b_8\frac{K_4'(\varrho/l^*)\varrho - K_4(\varrho/l^*)}{\varrho^2}\right]\cos 4\vartheta\right\}; \quad \text{(VI.188)}$$

$$\sigma_\vartheta^* = p - \frac{b_3}{R^2\varrho^2} + \frac{\varepsilon}{R^2}\left\{-\frac{6\beta_1^* b_1}{\varrho^4} + \frac{2b_3}{\varrho^4} + \frac{6b_4}{R^2\varrho^4} + 2b_7\frac{K_2'\varrho - K_2}{\varrho^2}\right\}\cos 2\vartheta + \frac{\varepsilon^2}{R^2}\left\{b_1\left(\frac{4}{\varrho^4} - \frac{6\beta_1^*}{\varrho^6}\right) + \frac{3b_3}{\varrho^6} + \frac{6b_4}{R^2\varrho^6} - \frac{b_5}{\varrho^2} + b_7\frac{K_2'\varrho - 4K_2}{\varrho^4} + \left[b_1\left(-\frac{4}{\varrho^4} + \frac{30\beta_1^*}{\varrho^6}\right) + b_2\left(\frac{6}{R^2\varrho^4} - \frac{60\beta_1^*}{R^2\varrho^6}\right) - \frac{6b_3}{\varrho^6} - \frac{30b_4}{R^2\varrho^6} + 20\frac{b_6}{R^4\varrho^6} + b_7\frac{2K_2''\varrho^2 - 5K_2'\varrho + 8K_2}{\varrho^4} + 4b_8\frac{K_4'\varrho - K_4}{\varrho^2}\right]\cos 4\vartheta\right\}, \quad \text{(VI.189)}$$

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where the coefficients b_1, b_2, ..., b_8 are determined by formulas (VI.184).

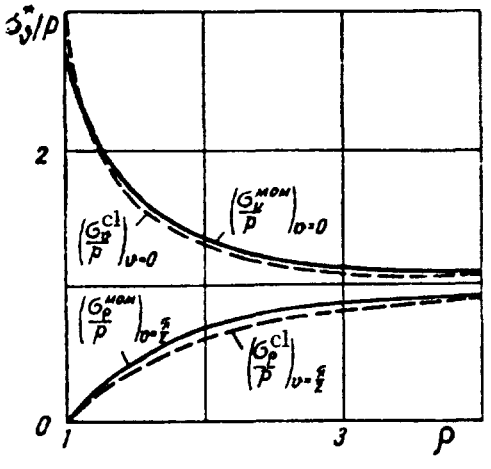


Figure VI.10.

Figure VI.10 represents the graphs characterizing the change of the coefficient of concentration $k_2 = \sigma_\vartheta/p$ through cross section $\vartheta = 0$ when $\varepsilon = 0.2$, $\nu = 0.25$ and $(R/l)_{min} = 3$ (solid curve). The broken curve corresponds to the value of the stress concentration coefficient k_2 in the same cross section, found on the basis of formulas of classical elasticity theory. Here, too, is represented change of concentration coefficient $k_1 = \sigma_\rho/p$ through cross section $\vartheta = \pi/2$ (the symbols are the same as those used for coefficient k_2). The curves showing the change of the concentration

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coefficient by classical elasticity theory were constructed on the basis of formulas

$$(\sigma_\varrho^*)_{\vartheta=\frac{\pi}{2}} = p\left\{1 - \frac{1}{\varrho^2} - 4\varepsilon\left(\frac{1}{\varrho^3} - \frac{1}{\varrho^4}\right) + \varepsilon^2\left(-\frac{1}{\varrho^2} + \frac{10}{\varrho^4} - \frac{9}{\varrho^6}\right)\right\}; \tag{VI.190}$$

$$(\sigma_\vartheta^*)_{\vartheta=0} = p\left\{1 + \frac{1}{\varrho^2} + \frac{4\varepsilon}{\varrho^4} + \varepsilon^2\left(\frac{1}{\varrho^2} - \frac{6}{\varrho^4} + \frac{9}{\varrho^6}\right)\right\}. \tag{VI.191}$$

Triangular and Square Holes in Field of Uniform Multifold Tension[1]. We will assume that the function (VI.109) that accomplishes conformal mapping of an infinite elastic isotropic plane z, weakened by a curvilinear hole with contour Γ, onto an infinite plane ζ with a round hole of unit radius, has the form

$$z = \omega(\zeta) = R\left(\zeta + \frac{\varepsilon}{\zeta^N}\right) \quad (z = re^{i\alpha},\ \zeta = \varrho e^{i\vartheta}), \tag{VI.192}$$

i.e., the function (VI.110) is $f(\zeta) = 1/\zeta^N$. The parameters R, ε and N (a whole positive number) characterize the dimensions and shapes of the holes.

We will notice that by selecting respectively ε and N, we obtain the mapping functions for elliptical, square and triangular holes (the latter having rounded corners).

We will determine the concentration of stresses near holes corresponding to mapping function (VI.192) in the case of uniform multifold tension "at infinity" by constant forces p = const. Here the basic stress state has the form (VI.175). /511

According to (VI.25) and (VI.27) for the k-th approximation of stress functions (VI.130) and (VI.134), we have

$$U_k = \mathrm{Re}[\bar{z}\varphi_k(z) + \chi_k(z)],$$
$$F_k = 8(1-\nu)l^2\,\mathrm{Im}\,\varphi_k'(z) + \Omega_k. \tag{VI.193}$$

The functions ϕ_k, χ_k and Ω_k are conveniently taken in the form

[1] See Yu. N. Nemish [2].

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$$\varphi_k = \sum_{n=1}^{\infty} A_n^{(k)} R^{n+1} z^{-n};$$

$$\psi_k = \chi_k' = \sum_{n=1}^{\infty} B_n^{(k)} R^{n+1} z^{-n}; \qquad \text{(VI.194)}$$

$$\Omega_k = R^2 \sum_{-\infty}^{\infty} C_n^{(k)} K_n\left(\frac{R}{l} r\right) e^{in\alpha}.$$

By determining differential operators L_i^j (VI.138) for mapping function (VI.192), considering the complex representation of the k-th approximation of stress functions U_k and F_k (VI.193), according to formulas (VI.136) we find the components of the stress state in the k-th approximation.

By considering the expressions for the components of ordinary and moment stresses in the zero, first and second approximations, we find from boundary conditions (VI.183) algebraic equation systems, the solutions of which yield

$$B_1^{(0)} = -p; \quad A_N^{(1)} = -\frac{p}{1+Q_1}; \quad B_{N+2}^{(1)} = -\frac{(N+1)p}{1+Q_1};$$

$$C_{N+1}^{(1)} = -\frac{N(N+1)tp}{(1+Q_1)K_{N+1}'\left(\frac{R}{l}\right)}; \quad B_1^{(2)} = -Np\frac{1-Q_1}{1+Q_1};$$

$$A_{2N+1}^{(2)} = -\frac{Np}{2(2N+1)(1+Q_1)(1+Q_2)}\left\{2[1+N(2+Q_1)]+\right.$$

$$\left.+Q_2\left[\frac{3N+4}{2}+\frac{T_{N+1}^{+(4)}\left(\frac{R}{l}\right)}{K_{N+1}'\left(\frac{R}{l}\right)}\right]-Q_3\right\}; \qquad \text{(VI.195)}$$

$$C_{2N+2}^{(2)} = -\frac{N(N+1)tp}{(1+Q_1)(1+Q_2)K_{2N+2}'\left(\frac{R}{l}\right)}\left\{\frac{N}{2}(5+4Q_1)-\frac{T_{N+1}^{+(4)}\left(\frac{R}{l}\right)}{K_{N+1}'\left(\frac{R}{l}\right)}-Q_2\right\}; \qquad \text{/512}$$

$$(2N+3)B_{2N+3}^{(2)} = (N^2+2N+3)p-\frac{(N+1)p}{1+Q_1}\left\{3(N+2)+\right.$$

$$\left.+Nt\left[(N+2)(2N+3)+\frac{T_{N+1}^{+(1)}\left(\frac{R}{l}\right)}{K_{N+1}'\left(\frac{R}{l}\right)}\right]\right\}+2(2N+1)[N+2-$$

$$-(N+1)(2N+3)t]A_{2N+1}^{(2)}+2(N+1)M_{2N+2}^{(1)}\left(\frac{R}{l}\right)C_{2N+2}^{(2)}.$$

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Here

$$t = 8(1-\nu)\frac{l^2}{R^2};$$

$$Q_j = \frac{1}{2} tj(N+1)[j(N+1)-1]\left[1 + j(N+1)\frac{K_{j(N+1)}\left(\frac{R}{l}\right)}{K'_{j(N+1)}\left(\frac{R}{l}\right)}\right] \quad (j = 1; 2); \qquad \text{(VI.196)}$$

$$Q_3 = (N+1)\,t\,\frac{T^{+(1)}_{N+1}\left(\frac{R}{l}\right) - T^{+(2)}_{N+1}\left(\frac{R}{l}\right)}{K'_{N+1}\left(\frac{R}{l}\right)};$$

$$T_n^{\pm(1)}\left(\frac{R}{l}\varrho\right) = \frac{1}{2\varrho^{N+1}}\left[n\varrho M_n^{(1)'}\left(\frac{R}{l}\varrho\right) \mp n^2 M_n^{(1)}\left(\frac{R}{l}\varrho\right) \pm (N+1)\,M_n^{(3)}\left(\frac{R}{l}\varrho\right)\right];$$

$$T_n^{\pm(2)}\left(\frac{R}{l}\varrho\right) = \frac{1}{2\varrho^{N+1}}\left[\varrho M_n^{(2)'}\left(\frac{R}{l}\varrho\right) \mp nM_n^{(2)}\left(\frac{R}{l}\varrho\right) \mp 2(N+1)\,nM_n^{(1)}\left(\frac{R}{l}\varrho\right)\right];$$

$$T_n^{\pm(4)}\left(\frac{R}{l}\varrho\right) = \frac{1}{2\varrho^{N+1}}\left[\varrho K_n^{*}\left(\frac{R}{l}\varrho\right) \mp nK'_n\left(\frac{R}{l}\varrho\right) \pm (N+1)\,nK_n\left(\frac{R}{l}\varrho\right)\right];$$

$$M_n^{(1)}\left(\frac{R}{l}\varrho\right) = \frac{1}{\varrho^2}\left[\varrho K'_n\left(\frac{R}{l}\varrho\right) - K_n\left(\frac{R}{l}\varrho\right)\right];$$

$$M_n^{(2)}\left(\frac{R}{l}\varrho\right) = \frac{1}{\varrho^2}\left[\varrho K'_n\left(\frac{R}{l}\varrho\right) - n^2 K_n\left(\frac{R}{l}\varrho\right)\right];$$

$$M_n^{(3)}\left(\frac{R}{l}\varrho\right) = \frac{1}{\varrho^2}\left[\varrho^2 K_n^{*}\left(\frac{R}{l}\varrho\right) - \varrho K'_n\left(\frac{R}{l}\varrho\right) + n^2 K_n\left(\frac{R}{l}\varrho\right)\right].$$

The formulas for stresses σ^*_ϑ and σ^*_ρ, with an accuracy up to ε^2, will be: /513

$$\begin{aligned}
\sigma^*_\vartheta = {} & p - \frac{1}{\varrho^2}B_1^{(0)} + \varepsilon\left\{\frac{2}{\varrho^{N+3}}B_1^{(0)} + \frac{N}{\varrho^{N+1}}\left[N - 1 - \frac{(N+1)(N+2)t}{\varrho^2}\right]A_N^{(1)} - \right. \\
& \left. - \frac{N+2}{\varrho^{N+3}}B_{N+2}^{(1)} + (N+1)\,M_{N+1}^{(1)}\left(\frac{R}{l}\varrho\right)C_{N+1}^{(1)}\right\}\cos(N+1)\vartheta + \\
& + \varepsilon^2\left\{\frac{N(N+2)}{\varrho^{2N+4}}B_1^{(0)} + \frac{N(N+1)}{\varrho^{2N+2}}\left[N + 1 - \frac{N(N+2)t}{\varrho^2}\right]A_N^{(1)} - \frac{N(N+2)}{\varrho^{2N+2}}B_{N+2}^{(1)} + \right. \\
& + T^{-(1)}_{N+1}\left(\frac{R}{l}\varrho\right)C_{N+1}^{(1)} - \frac{1}{\varrho^2}B_1^{(2)} + \left[-\frac{N^2+2N+3}{\varrho^{2N+4}}B_1^{(0)} - \frac{N(N+1)}{\varrho^{2N+2}}\times\right. \\
& \times\left(2N - \frac{(N+2)(2N+3)t}{\varrho^2}\right)A_N^{(1)} + \frac{2(2N+1)}{\varrho^{2N+2}}\left(N - \frac{(N+1)(2N+3)t}{\varrho^2}\right)A_{2N+1}^{(2)} + \\
& + \frac{(N+2)(2N+3)}{\varrho^{2N+4}}B_{N+2}^{(1)} - \frac{2N+3}{\varrho^{2N+4}}B_{2N+3}^{(2)} + T^{+(1)}_{N+1}\left(\frac{R}{l}\varrho\right)C_{N+1}^{(1)} + \\
& \left.\left. + 2(N+1)\,M_{2N+2}^{(1)}\left(\frac{R}{l}\varrho\right)C_{2N+2}^{(2)}\right]\cos 2(N+1)\vartheta\right\};
\end{aligned} \qquad \text{(VI.197)}$$

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$$\sigma_\varrho^* = -\sigma_\vartheta^* + 2p - \frac{4N\varepsilon}{\varrho^{N+1}} A_N^{(1)} \cos(N+1)\vartheta +$$

$$+ \frac{4\varepsilon^2}{\varrho^{2N+2}} [N(N+1) A_N^{(1)} - (2N+1) A_{2N+1}^{(2)}] \cos 2(N+1)\vartheta.$$

We will notice that by limit transition for $R/l \to \infty$ we obtain from (VI.195)-(VI.197) the approximate formulas of classical elasticity theory. However, in the case of a plate with the hole of type (VI.192), which is under uniform tension by forces p in the directions of the x and y axes, according to N. I. Muskhelishvili's method [1] for stresses σ_ϑ^* on contour Γ in classical elasticity theory, we obtain the precise formula

$$\sigma_\vartheta^*|_\Gamma = 2p \frac{1 - N^2\varepsilon^2}{1 - 2N\varepsilon\cos(N+1)\vartheta + N^2\varepsilon^2}. \qquad \text{(VI.198)}$$

By selecting the corresponding values of ε and N from the above formulas we can solve the problems under examination for the following holes:

elliptical $\varepsilon = \frac{a-b}{a+b}$; $N = 1$; $R = \frac{a+b}{2}$ (a, b are the semiaxes of the ellipse);

equilateral triangle N = 2, $\varepsilon = \pm 1/3$ or $\varepsilon = \pm 1/4$;

square N = 3, $\varepsilon = \pm 1/6$ or $\varepsilon = \pm 1/9$.

Figure VI.11 illustrates the distribution of $\sigma_\vartheta^{*(cl)}/p$ and $\sigma_\vartheta^{*(mom)}/p$ along half of the contour of a triangular hole when $\varepsilon = -1/4$, and Figure VI.12 illustrates the same along one-fourth of the contour of a square hole for $\varepsilon = -1/9$. The graphs were constructed on the basis of formulas (VI.197), i.e., with consideration of the second approximation. The numerical values of the stresses were found for Poisson's ratio $\nu = 0.25$ and $R/l = 3$. /514

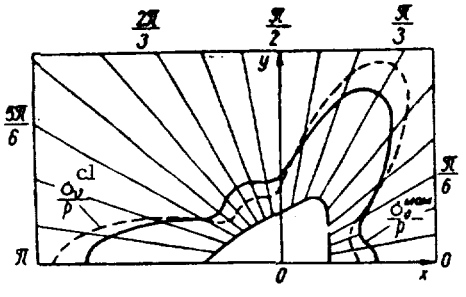


Figure VI.11.

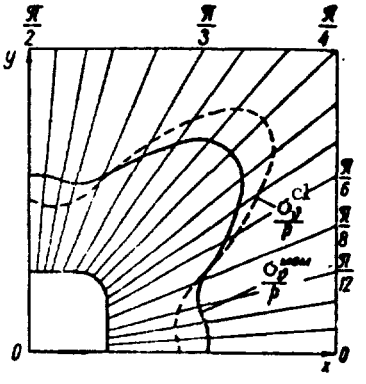


Figure VI.12.

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By analyzing the results obtained we see that for free holes (circular, elliptical, triangular and square) there is a general principle: the maximum value of the concentration coefficient for moment theory decreases as a function of the magnitude of Poisson's ratio ν and ratios R/l.

Plane Problem for Multiply-Connected (Infinite) Ranges[1]. We will examine an infinite range weakened by a finite number of arbitrarily arranged holes (Figure VI.13), the contours L_k ($k = 1, 2, \ldots, m$) of which are smooth curves and are given by the functions

$$z_k = \omega_k(\zeta_k) = R_k[\zeta_k + \varepsilon\eta_k f_k(\zeta_k)];$$
$$\varepsilon < 1;\ \eta_k < 1;\ \varepsilon\eta_k = \varepsilon_k;\ \varepsilon = \max\{\varepsilon_k\} \qquad (VI.199)$$
$$k = 1, 2, \ldots, m.$$

Here $f_k(\zeta_k)$ are functions of the form $f_k(\zeta) = \sum_{p=1}^{n} \frac{C_p^{(k)}}{\zeta^p}$, where $C_p^{(k)}$ are known constant coefficients.

The function $z_k(\zeta)$ accomplishes conformal mapping of an infinite plane with a round hole of unit radius onto an infinite plane with a hole L_k.

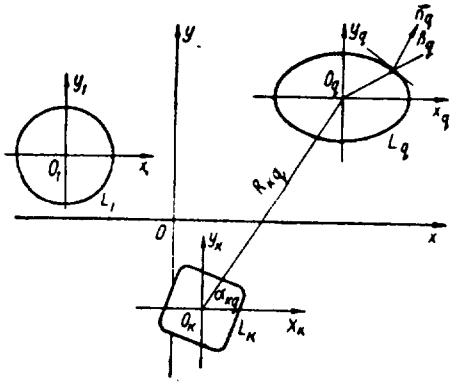


Figure VI.13.

To each contour L_k we will relate the coordinate system (x_k, y_k), and (x, y) to an arbitrary point on the plane (Figure VI.13): /515

$$z = x + iy = re^{i\theta};\ z_k = x_k + iy_k = r_k e^{i\theta_k},$$
$$\zeta_k = \rho_k e^{i\theta_k};\ z = z_k + l_k;\ z_k = z_q + R_{kq}e^{i\alpha_{kq}}. \qquad (VI.200)$$

The coordinate line $\rho_k = 1$ coincides with the contour L_k of the k-th hole; the parameter $r_0 R_k$ characterizes absolute dimensions of the k-th hole; β_q is the angle between r_k and ρ_q (or $\vec{n}_q$) lines.

The use of the function z_k (VI.199) makes it possible to reduce the stated problem for stress functions to a series m of boundary problems for a range corresponding to connectedness with round holes of unit radius in the plane ζ_k.

---

[1] See G. N. Savin and A. N. Guz' [2] and also A. N. Guz' and G. N. Savin [1].
[2] All coordinates in linear parameters are dimensionless and related to r_0.

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The solution of the problem for a multiply-connected range reduces to the determination of stress functions U and F that characterize the additional stress state (VI.41) near the holes, from equations (VI.11) and (VI.12) under boundary conditions (VI.20) or (VI.21), which are equivalent to them. These functions should satisfy "at infinity" both conditions (VI.43) and the conditions of uniqueness of displacements.

Due to the linearity of the problem, the solution of equation system (VI.11) for an m-connected range is represented in the form of the sum of complete solutions for the corresponding singly-connected ranges bounded by contours L_k (k = 1, 2, ..., m), i.e.,

$$U(r, \theta, \varepsilon) = \sum_{k=1}^{m} U^{(k)}(r_k, \theta_k, \varepsilon);$$
$$F(r, \theta, \varepsilon) = \sum_{k=1}^{m} F^{(k)}(r_k, \theta_k, \varepsilon), \qquad \text{(VI.201)}$$

where for each k the functions $U^{(k)}$ and $F^{(k)}$ acquire the forms, respectively, (VI.130) and (VI.133). Each pair of functions $U^{(k)}$ and $F^{(k)}$ should satisfy individually the conditions "at infinity," and furthermore these functions are such that the main vector and main moment of forces applied to the L_k (k = = 1, 2, ..., m) contour should be equal to zero.

Additional difficulties arising in the solution of the problem for multiply-/516 connected ranges, are encountered in the transformation of 2(m - 1) functions $U^{(k)}$ and $F^{(k)}$ to the k-th coordinate system (r_k, θ_k) (Figure VI.13) and the representation of these functions in the form of series with separated variables.

The conversion to the k-th coordinate system is essential for the satisfaction of the boundary conditions of the problem on each contour of the hole and consequently for the determination of the total system of linear algebraic equations from which the unknown coefficients of the desired functions $U^{(k)}$ and $F^{(k)}$ are found.

In order that the problem be homogeneous, it is necessary in the desired functions $U^{(k)}$ and $F^{(k)}$ to convert from polar coordinates to a new curvilinear orthogonal coordinate system (ρ_k, ϑ_k), given by mapping function $\omega_k(\zeta_k)$ (VI.199).

The problem is greatly simplified in the case of *round* holes, i.e., for an infinite plane weakened by a finite number of arbitrarily arranged, nonintersecting round holes[1].

[1]See A. N. Guz' and G. N. Savin [1].

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In this case stress functions U and F for an m-connected range is written in the form

$$U = \sum_{q=1}^{m} A_0^{(q)} \ln r_q + \sum_{q=1}^{m} \sum_{p=1}^{\infty} \left[\begin{matrix} A_p^{(q)} \\ B_p^{(q)} \end{matrix} r_q^{-p} + \begin{matrix} C_p^{(q)} \\ D_p^{(q)} \end{matrix} r^{-p+2} \right] \begin{matrix} \cos \\ \sin \end{matrix} p\vartheta_q;$$

$$F = \sum_{q=1}^{m} F_0^{(q)} K_0\left(r_q \frac{r_0}{l}\right) + \sum_{q=1}^{m} \sum_{p=1}^{\infty} \begin{matrix} F_p^{(q)} \\ E_p^{(q)} \end{matrix} K_p\left(r_q \frac{r_0}{l}\right) \begin{matrix} \cos \\ \sin \end{matrix} p\theta_q \mp \qquad \text{(VI.202)}$$

$$\mp 8(1-\nu) l^2 \sum_{q=1}^{m} \sum_{p=1}^{\infty} (p-1) \begin{matrix} D_p^{(q)} \\ C_p^{(q)} \end{matrix} r_q^{-p} \begin{matrix} \cos \\ \sin \end{matrix} p\theta_q,$$

i.e., the desired solution of (VI.202) is constructed as the sum of the complete solutions for the corresponding infinite ranges with a round hole of radius $r_0 R_k$.

If the main vector and main moment of forces applied to each contour L_k is equal to zero, then the conditions of uniqueness of displacements and the conditions "at infinity" will be satisfied if

$$C_1^{(q)} \equiv D_1^{(q)} = 0 \quad (q = 1, 2, \ldots, m). \qquad \text{(VI.203)}$$

In representing the solutions of (VI.202) in the k-th coordinate system it is necessary to recalculate all functions, including the cylindrical functions. In order to recalculate the latter it is necessary to use the theorem of addition of these functions.

It is convenient to introduce into (VI.102) the following new constants: /517

$$A_p^{(q)} = x_{p,1}^{(q)}; \quad C_p^{(q)} = x_{p,2}^{(q)}; \quad E_p^{(q)} = I_p\left(R_q \frac{r_0}{l}\right) x_{p,3}^{(q)};$$

$$B_p^{(q)} = x_{p,4}^{(q)}; \quad D_p^{(q)} = x_{p,5}^{(q)}; \quad F_p^{(q)} = I_p\left(R_q \frac{r_0}{l}\right) x_{p,6}^{(q)}. \qquad \text{(VI.204)}$$

From the boundary conditions on the k-th contour we obtain an infinite equation system in the form:

$$\dot{B}_n^{(k)} X_n^{(k)} + \sum_{q=1}^{m}{}' \sum_{p=0}^{\infty} B_{n,p}^{(k,q)} X_p^{(q)} = \ddot{B}_n^{(k)} \qquad \text{(VI.205)}$$

$$\left(\begin{matrix} k = 1, 2, \ldots, m \\ n = 0, 1, 2, \ldots, \infty \end{matrix} \right).$$

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Here and in the following the "prime mark" near the sum denotes that in the corresponding sum the term for q = k is omitted; $X_n^{(k)}$; $B^{**(k)}$ are six-variate vector columns; $B_n^{*(k)}$ and $B_{n,p}^{(k,q)}$ are six-variate matrices;

$$X_n^{(k)} = \{x_n^{(k)}\}; \qquad \overset{\bullet\bullet}{B}{}_n^{(k)} = \{b_j^{\bullet\bullet}(n, k)\};$$

$$\overset{\bullet}{B}{}_n^{(k)} = \| b_{ij}^{\bullet}(n, k) \|; \qquad B_{n,p}^{(k,q)} = \| b_{ij}(n, p, k, q) \|.$$

This system (VI.205) can be transformed to canonic form:

$$X_n^{(k)} + \sum_{q=1}^{m} \sum_{p=0}^{\infty} A_{n,p}^{(k,q)} X_p^{(q)} = b_n^{(k)}. \qquad \text{(VI.206)}$$

where $A_{n,p}^{(k,q)} = ||a_{ij}(n, p, k, q)||$ is a six-variate matrix; $b_n = \{b_j(n, k)\}$; $X_n^{(k)} = \{x_{n,j}^{(k)}\}$ (i, j = 1, 2, ..., 6) are six-variate vector columns.

The asymptotic representations of the cylindrical functions for large indices and the summation of certain series makes it possible to show that infinite system (VI.206) is quasiregular[1] for any vicinity of nonintersecting contours and for a certain smoothness[2] of the right hand sides of the boundary problems.

It so happens that the uniqueness of the solution of the stated boundary problem is ensured by the satisfaction of the condition of applicability of Hulbert's alternative.

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[1]This means that the solutions obtained from truncated system (VI.206) for an increased number of terms in these truncated systems will approach the precise solution of infinite system (VI.206), i.e., will approach the desired (precise) solution of the problem.

[2]In particular, system (VI.206) will be quasiregular under the condition that the functions σr_k and $\tau r_k \theta_k$ on the L_k-th contour are continuous, and that their first derivatives satisfy Dirichlet's condition, and further, if μr_k are continuous functions on L_k with their first derivatives; the second derivative of μr_k should satisfy Dirichlet's condition.

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CHAPTER VII. TEMPERATURE STRESS DISTRIBUTION AROUND FREE AND REINFORCED HOLES

Abstract. This chapter deals with the thermoelastic problems of the theory of elasticity. Basic equations are derived for the plane thermoelastic problem as well as for the problem of bending of the thin plates. The stress concentration is studied near various curvilinear holes with uniform heat flow. A case is also considered of a plate weakened by a biperiodical system of circular holes having the same form and size.

In the preceding chapters we examined the effect of various types of holes /520
on stress distribution caused by the effect of a force load on a body at constant temperature. If the temperature of the body is changed, then, in addition to the stresses caused by the force load, other stresses, the so-called temperature stresses, can occur within the body. The determination of these stresses is the object of the temperature problem of elasticity theory. Obviously the magnitude and law of distribution of temperature stresses depends on the temperature change of the body, which is found by solving the problem of thermoconductivity. It is quite natural, therefore, that the temperature problem of elasticity theory be analyzed in conjunction with the corresponding problem of thermoconductivity theory.

In the present chapter we will present the basic equations of the plane temperature problem of elasticity theory and the problems of deflection of thin plates as well as the equations of the two-dimensional problem of the theory of thermoconductivity. The examples given here do not represent all existing solutions of the temperature problem of elasticity theory, but illustrate only the characteristic features of the problem of the effect of holes on stress distribution in elastic bodies possessing constant thermoelastic properties. A more complete treatment of this problem, and also the solutions of many specific problems, can be found in the monographs of N. N. Lebedev [1], E. Melan and G. Parkus [1], G. Parkus [1], V. Novatskiy [1], B. Boli and J. Weiner [1], A. D. Kovalenko [1], A. V. Lykov [1], G. Karslou and D. Yeger [1], which pertain to temperature problems of elasticity theory and problems of the theory of thermoconductivity.

§1. Basic Equations of Plane Temperature Problem of Elasticity Theory and of Problem of the Theory of Deflection of Thin Plates

Initial Equations of the Temperature Problem of Elasticity Theory. If the temperature of a deformed solid body changes by the magnitude t, which is, generally speaking, a function of the coordinates and time, then stresses can occur within it which are caused by the incompatibility of purely thermal deformation. To prove this we will deliberately break the body down into small elements in such a way that the temperature within each one of them can be assumed to be uniformly distributed. Then each of these elements, since it

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does not encounter resistance from the others, undergoes pure thermal expansion, characterized in the case of an isotropic body by tensor components of temperature deformation. /521

$$\varepsilon_x^{(t)} = \varepsilon_y^{(t)} = \varepsilon_Z^{(t)} = \alpha_t t, \quad \gamma_{xy}^{(t)} = \gamma_{yZ}^{(t)} = \gamma_{Zx}^{(t)} = 0, \tag{VII.1}$$

where α_t is the temperature coefficient of linear expansion.

If purely thermal deformation is incompatible, i.e., if the elements thus deformed do not constitute a solid body, then internal (temperature) stresses that restore its continuity should develop within it.

The tensor components of forced deformation

$$\begin{aligned} \varepsilon_x^{(s)} &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_Z)], & \gamma_{xy}^{(s)} &= \frac{1+\nu}{E}\tau_{xy}, \\ \varepsilon_y^{(s)} &= \frac{1}{E}[\sigma_y - \nu(\sigma_Z + \sigma_x)], & \gamma_{yZ}^{(s)} &= \frac{1+\nu}{E}\tau_{yZ}, \\ \varepsilon_Z^{(s)} &= \frac{1}{E}[\sigma_Z - \nu(\sigma_x + \sigma_y)], & \gamma_{Zx}^{(s)} &= \frac{1+\nu}{E}\tau_{Zx} \end{aligned} \tag{VII.2}$$

caused in an elastic isotropic body by these stresses should be such that the tensor components of total deformation

$$\begin{aligned} &\varepsilon_x = \varepsilon_x^{(s)} + \alpha_t t, \ \varepsilon_y = \varepsilon_y^{(s)} + \alpha_t t, \ \varepsilon_Z = \varepsilon_Z^{(s)} + \alpha_t t, \\ &\gamma_{xy} = \gamma_{xy}^{(s)}, \qquad \gamma_{yZ} = \gamma_{yZ}^{(s)}, \qquad \gamma_{Zx} = \gamma_{Zx}^{(s)} \end{aligned} \tag{VII.3}$$

will satisfy the conditions of continuity. In this case the tensor of total deformation is potential and Cauchy's relations are applicable for it.

By excluding from formulas (VII.2) and (VII.3) the tensor components of force deformation, we obtain the relations

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_Z)] + \alpha_t t, & \gamma_{xy} &= \frac{1+\nu}{E}\tau_{xy}, \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_Z + \sigma_x)] + \alpha_t t, & \gamma_{yZ} &= \frac{1+\nu}{E}\tau_{yZ}, \\ \varepsilon_Z &= \frac{1}{E}[\sigma_Z - \nu(\sigma_x + \sigma_y)] + \alpha_t t, & \gamma_{Zx} &= \frac{1+\nu}{E}\tau_{Zx}, \end{aligned} \tag{VII.4}$$

which generalize Hooke's law for the case where the temperature of the body is changed by the magnitude t.

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Plane Problem. If we proceed from the assumptions used in the formulation of the plane problem of elasticity theory, then, instead of relations (VII.4), we may write

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha_t t,\ \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha_t t,\ \gamma_{xy} = \frac{1+\nu}{E}\tau_{xy}, \qquad \text{(VII.5)}$$

where in the case of plane deformation, t = t(x, y, τ) and instead of α_t, ν, E /522 we will use

$$\alpha_t^* = \alpha_t(1+\nu),\ \nu^* = \frac{\nu}{1-\nu},\ E^* = \frac{E}{1-\nu^2}, \qquad \text{(VII.6)}$$

and also

$$\sigma_Z = \nu(\sigma_x + \sigma_y) - \alpha_t E t. \qquad \text{(VII.7)}$$

In the case of the generalized plane stress state (thin plates), by stress components σ_x, σ_y, $t_{x,y}$ and temperature t in equation (VII.5), we mean their mean values through thickness:

$$\sigma_x^* = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x dZ,\quad \sigma_y^* = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y dZ,\quad \tau_{x,y}^* = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} dZ,$$
$$T_1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} t dZ. \qquad \text{(VII.8)}$$

To prevent deflection of the plate from the center of its plane, the function t(x, y, Z, τ) should be symmetrical with respect to the middle of the plate.

The stress tensor components should, as usual, satisfy equilibrium equations (I.1) and compatibility equation

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y},$$

which, after the substitution of (VII.5) in it, acquires the form

$$\Delta(\sigma_x + \sigma_y) = -\alpha_t E \Delta t \qquad \text{(VII.9)}$$

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and is, as we see, heterogeneous. Therefore if we introduce the stress function through relations (I.6), then we obtain, for its determination, the heterogeneous biharmonic equation

$$\Delta\Delta U = -\alpha_t E \Delta t, \tag{VII.10}$$

the general solution of which can be represented in the form

$$U = \operatorname{Re}[\bar{z}\varphi_1(z) + \chi_1(z)] - \frac{\alpha_t E}{4}\iint t dz d\bar{z}. \tag{VII.11}$$

Hence, for the determination of the two analytical functions $\phi_1(z)$ and $\psi_1(z) = \chi_1'(z)$, instead of (I.9) and (I.10), we obtain the contour conditions

$$\varphi_1(z) + \overline{z\varphi_1'(z)} + \overline{\psi_1(z)} = f_1 + if_2 + \frac{\alpha E}{2}\int t dz + C \text{ on } L, \tag{VII.12}$$

$$\varkappa\varphi_1(z) - \overline{z\varphi_1'(z)} - \overline{\psi_1(z)} = 2\mu\left[g_1 + ig_2 - \frac{\alpha_t(1+\nu)}{2}\int t dz\right] \text{ on } L. \tag{VII.13}$$

In the case of the first basic problem, it is sometimes more convenient to use instead of (VII.12) the contour condition /523

$$\Phi_1(z) + \overline{\Phi_1(z)} - [\bar{z}\Phi_1'(z) + \Psi_1(z)]e^{2i\theta} = N - iT + \frac{\alpha_t E}{2}\left(t - e^{2i\theta}\int \frac{\partial t}{\partial z} dz\right) \text{ on } L, \tag{VII.14}$$

where N and T are normal and tangential, respectively, to L, component stresses, and $\Phi_1(z) = \phi_1'(z)$, $\Psi(z) = \psi_1'(z)$.

If the functions $\phi_1(z)$ and $\psi_1(z)$ are known, then the stress components and displacement components are found from formulas

$$2\mu(u + iv) = \varkappa\varphi_1(z) - \overline{z\varphi_1'(z)} - \overline{\psi_1(z)} + \frac{\alpha_t E}{2}\int t dz; \tag{VII.15}$$

$$\sigma_x + \sigma_y = 2[\varphi_1'(z) + \overline{\varphi_1'(z)}] - \alpha_t E t,$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[z\varphi_1'(z) + \overline{\psi_1'(z)}] - \alpha_t E \int \frac{\partial t}{\partial z} d\bar{z}. \tag{VII.16}$$

Relations (VII.12)-(VII.16), as far as we know, were first established by N. N. Lebedev [1]. In the case of thin plates they are valid, as we pointed out earlier, for the symmetrical distribution of temperature with respect to the middle of the plate.

Deflection of Thin Plates. A temperature field that is asymmetrical with respect to the middle of the plane can be reduced to its deflection. If we

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proceed from the assumptions that are ordinarily used in the theory of deflection of thin plates and make use[1] of relations (VII.4) for the determination of the stresses in the plate caused by its sagging w, we obtain

$$\sigma_x = -\frac{EZ}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - \frac{\alpha_t E t}{1-\nu},$$
$$\sigma_y = -\frac{EZ}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - \frac{\alpha_t E t}{1-\nu}, \qquad \text{(VII.17)}$$
$$\tau_{xy} = -\frac{EZ}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}.$$

Analogously, we obtain for the moments and shear forces[2]

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \frac{2\alpha_t(1+\nu)}{h}T^*\right],$$
$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} + \frac{2\alpha_t(1+\nu)}{h}T^*\right], \qquad \text{(VII.18)}$$
$$H_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y};$$

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$$N_x = -D\frac{\partial}{\partial x}\left(\Delta w + \frac{2\alpha_t(1+\nu)}{h}T^*\right), \qquad \text{(VII.19)}$$
$$N_y = -D\frac{\partial}{\partial y}\left(\Delta w + \frac{2\alpha_t(1+\nu)}{h}T^*\right),$$

where

$$T^* = \frac{6}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} tZdZ. \qquad \text{(VII.20)}$$

By substituting the values of N_x and N_y from (VII.19) into the third equilibrium equation[3], we will obtain for the determination of the sag of a thin isotropic plate, the heterogeneous biharmonic equation

$$\Delta\Delta w = -\frac{2\alpha_t(1+\nu)}{h}\Delta T^*, \qquad \text{(VII.21)}$$

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[1]See G. N. Savin [1], formula (6.3).
[2]See G. N. Savin [1], formulas (6.7), (6.8).
[3]See G. N. Savin [1].

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the general solution of which has the form

$$w = \mathrm{Re}\left[\bar{z}\varphi(z) + \overline{\chi(z)}\right] - \frac{\alpha_t(1+\nu)}{2h}\iint T^* dz d\bar{z}. \qquad \text{(VII.22)}$$

Hence, for the determination of analytical functions[1] $\phi(z)$ and $\psi(\ ) = \chi'(z)$, we will have the following contour conditions

$$n\varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} = f_1 + if_2 + \frac{\alpha_t(1+\nu)}{h}\int T^* dz + iCz + C_1 \text{ on } L', \qquad \text{(VII.23)}$$

$$\varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} = g_1 + ig_2 + \frac{\alpha_t(1+\nu)}{h}\int T^* dz \text{ on } L. \qquad \text{(VII.24)}$$

If the functions $\phi(z)$ and $\psi(z)$ are known, then the moments and shear forces, as demonstrated by R. N. Shvets, are found from the formulas

$$\begin{aligned} M_x + M_y &= -2(1+\nu)D\left[\varphi'(z) + \overline{\varphi'(z)} + \frac{\alpha_t(1-\nu)}{h}T^*\right], \\ M_y - M_x + 2iH_{xy} &= 2(1-\nu)D\left[\bar{z}\varphi''(z) + \psi'(z) - \frac{\alpha_t(1+\nu)}{h}\int \frac{\partial T^*}{\partial z}\,d\bar{z}\right], \\ N_x - iN_y &= -4D\varphi''(z). \end{aligned} \qquad \text{(VII.25)}$$

In the general case of heating within a plate, both deflection stresses and stresses that correspond to the generalized plane stress state can occur within it. Then the general stress state, due to the linearity of the problem, /525 is determined by the formulas

$$\begin{aligned} \sigma_x &= \sigma_x^{\cdot} + 12M_x\frac{Z}{h^3} + \frac{\alpha_t E}{1-\nu}\left(T + \frac{2Z}{h}T^* - t\right), \\ \sigma_y &= \sigma_y^{\cdot} + 12M_y\frac{Z}{h^3} + \frac{\alpha_t E}{1-\nu}\left(T + \frac{2Z}{h}T^* - t\right), \\ \tau_{xy} &= \tau_{xy}^{\cdot} + 12H_{xy}\frac{Z}{h^3}. \end{aligned} \qquad \text{(VII.26)}$$

Thus, for the determination of stresses in the case of plane deformation, it is essential to know the plane temperature field $t(x, y, \tau)$. To determine the mean stresses and deflection moments of thin plates through thickness, it is necessary to find the mean temperatures T and T* through the thickness, and also the temperature $t(x, y, Z, \tau)$ for the determination of the stresses.

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[1] See G. N. Savin [1].

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§2. Basic Equations of the Theory of Thermoconductivity

Three-Dimensional Problem. To determine the temperature field in a solid body we will use the equations of thermoconductivity

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial Z^2} = \frac{1}{a} \cdot \frac{\partial t}{\partial \tau} \qquad \text{(VII.27)}$$

where τ is time; $a = \lambda/c$ is temperature conductivity (λ is thermoconductivity, c is heat capacity of unit volume).

At the initial moment of time $\tau = 0$, the temperature of the body should also satisfy the initial condition

$$t(x, y, Z, 0) = f(x, y, Z) \qquad \text{(VII.28)}$$

and the conditions on the boundary of body S.

Usually, four types of boundary conditions are distinguished: boundary conditions of the first type

$$t = f_1(M, \tau), \quad M \in S, \quad \tau > 0; \qquad \text{(VII.29)}$$

boundary conditions of the second type

$$\frac{\partial t}{\partial n} = f_2(M, \tau), \quad M \in S, \quad \tau > 0; \qquad \text{(VII.30)}$$

boundary conditions of the third type

$$\frac{\partial t}{\partial n} + k(t - t_m) = 0 \text{ on } S \text{ for } \tau > 0. \qquad \text{(VII.31)}$$

In the case of ideal thermal contact of two bodies, the boundary conditions will have the form: /526

$$t_1 = t_2, \ \lambda_1 \frac{\partial t_1}{\partial n} = \lambda_2 \frac{\partial t_2}{\partial n} \text{ on } S \text{ for } \tau > 0, \qquad \text{(VII.32)}$$

where n is the external line normal to the surface of the body, $k = \alpha_n/\lambda$ is a relative coefficient (α_n is the absolute coefficient of thermoconductivity into the medium with temperature t_m).

Plane Temperature Field. If the temperature of the body does not depend on the Z coordinate, then the temperature field is plane. The latter is

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possible in cylindrical bodies of arbitrary length, including thin plates, the ends of which are thermally insulated, and the initial condition (VII.28) and boundary conditions on the cylindrical surfaces are identical in any cross section. Hence the temperature $t(x, y, \tau)$ will satisfy the two-dimensional equation of thermoconductivity

$$\Delta t = \frac{1}{a} \cdot \frac{\partial t}{\partial \tau}, \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{VII.33}$$

initial condition

$$t(x, y, \tau) = f_1(x, y) \qquad \tau = 0 \tag{VII.34}$$

and one of boundary conditions (VII.29)-(VII.32) on contour L, which bounds the cross section of the body.

Thus, the determination of the plane field reduces to a two-dimensional boundary problem of thermoconductivity.

Equations of Thermoconductivity of Thin Plates. In the general case when the temperature in the plate changes through the thickness, the temperature field has to be found from equation (VII.27) and the corresponding boundary problems. However, as mentioned earlier, the generalized stress state of the plate and the deflecting moments are characterized by the values

$$T = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} t dZ, \qquad T^* = \frac{6}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} Zt dZ, \tag{VII.35}$$

the determination of which, as in the case of the plane field, can be reduced to the solution of some two-dimensional boundary problem of thermoconductivity.

We will formulate this problem for the case where heat conducted from the ends of the plate $Z = \pm h/2$ into the medium in accordance with boundary condition (VII.31).

By multiplying equation (VII.27) by $\frac{1}{h}dZ$ and $\frac{6}{h^2}ZdZ$ and integrating within the range -h/2 to +h/2, we obtain

$$\begin{aligned} \Delta T - \frac{1}{h}(k^+t^+ + k^-t^-) &= \frac{1}{a} \cdot \frac{\partial T}{\partial \tau} - \frac{1}{h}(k^+t_m^+ + k^-t_m^-), \\ \Delta T^* - \frac{3}{h}(k^+t^+ - k^-t^-) - \frac{6}{h^2}(t^+ - t^-) &= \frac{1}{a} \cdot \frac{\partial T^*}{\partial \tau} - \frac{3}{h}(k^+t_m^+ - k^-t_m^-), \end{aligned} \tag{VII.36}$$

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where t^+, t^- is temperature and k^+, k^- are relative coefficients of thermo-conductivity on surfaces $Z = \pm h/2$, respectively; t_m^+, t_m^- is the temperature of the medium flowing over each of these surfaces.

In order to exclude from (VII.36) the known temperatures t^+ and t^- on planes $Z = \pm h/2$, we will express these values through the desired values T and T* (VII.35). By representing for this purpose equations (VII.27) in the form

$$\frac{\partial^2 t}{\partial Z^2} + p^2 t = 0, \tag{VII.37}$$

where $p^2 = \Delta - \frac{1}{a}\cdot\frac{\partial}{\partial \tau}$, and employing the symbolic method, we write

$$t = \cos pZ\, t_0 + \frac{\sin pZ}{p} t_0^*, \tag{VII.38}$$

where t_0 is the temperature, and t_0^* is its derivative with respect to Z on the center of the plate.

By expressing t_0 and t_0^* with the aid of (VII.35) through T and T*, we obtain the relation

$$t = \frac{p\frac{h}{2}\cos pZ}{\sin p\frac{h}{2}} T + \frac{\left(p\frac{h}{2}\right)^2}{3} \frac{\sin pZ}{\sin p\frac{h}{2} - p\frac{h}{2}\cos p\frac{h}{2}} T^*, \tag{VII.39}$$

by which we can determine through simple differentiation the temperature through its average characteristics T and T*. By introducing (VII.39) into (VII.36), we obtain[1] for the determination of these characteristics, a system of equations of infinitely high order

$$\left(p\frac{h}{2}\right)^2 T - kp\frac{h}{2}\cot p\frac{h}{2}\, T - \frac{k^*}{3}\frac{\left(p\frac{h}{2}\right)^2}{1 - p\frac{h}{2}\cot p\frac{h}{2}} T^* = -(kt_m + k^* t_m^*),$$

$$\left(p\frac{h}{2}\right)^2 T^* - (1+k)\frac{\left(p\frac{h}{2}\right)^2}{1 - p\frac{h}{2}\cot p\frac{h}{2}} T^* - 3k^* p\frac{h}{2}\cot p\frac{h}{2}\, T = -3(kt_m^* + k^* t_m) \tag{VII.40}$$

[1] See Ya. S. Podstrigach [4].

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where

$$k = \frac{h}{4}(k^+ + k^-), \qquad k^* = \frac{h}{4}(k^+ - k^-), \qquad t_m = \frac{1}{2}(t_m^+ + t_m^-),$$

$$t_m^* = \frac{1}{2}(t_m^+ - t_m^-).$$

If, for the description of the thermophysical properties of a thin plate, we introduce the adduced thermoconductivity $\lambda^* = \lambda h$, adduced heat capacity $c^* = ch$ and thermoresistance $r^* = h/\lambda$, then, by proceeding in (VII.40) to the limit for $h \to 0$, preserving the constants λ^*, c^*, r^* we obtain[1], for the determination of T and T*, the following system of two equations of the second order:

$$\lambda^* \Delta T - (\varepsilon T + \varepsilon^* T^*) = c^* \frac{\partial T}{\partial \tau} - (\varepsilon t_m + \varepsilon^* t_m^*),$$

$$\lambda^* \Delta T^* - 3(\varepsilon^* T + \varepsilon_0 T^*) = c^* \frac{\partial T^*}{\partial \tau} - 3(\varepsilon t_m^* + \varepsilon^* t_m), \qquad \text{(VII.41)}$$

where $\varepsilon = \frac{1}{r^+} + \frac{1}{r^-}$, $\varepsilon^* = \frac{1}{r^+} - \frac{1}{r^-}$, $\varepsilon_0 = \frac{1}{r^+} + \frac{1}{r^-} + \frac{4}{r^*}$; r^+, r^- is resistance to heat exchange on surfaces $Z = \pm h/2$, respectively.

Equation systems (VII.40) and (VII.41) were derived by Ya. S. Podstrigach [4]. Equation system (VII.41), with consideration of heat sources, was also derived by V. V. Bolotin [1] on the basis of the variation principle in the assumption of linear temperature distribution through the thickness of the plate.

Equation system (VII.41) for thermoconductivity of thin plates, as pointed out by I. A. Motovilovets [1] and V. M. Gembara [1], reduces to a single resolving solution

$$\lambda^* \Delta \theta_i - (\varepsilon + 3\mu_i \varepsilon^*)\theta_i = c^* \frac{\partial \theta_i}{\partial \tau} - [(\varepsilon + 3\mu_i \varepsilon^*) t_m + (\varepsilon^* + 3\mu_i \varepsilon) t_m^*], \qquad \text{(VII.42)}$$

where

$$\theta_i = T + \mu_i T^*, \quad i = 1, 2, \quad \mu_{1,2} = \frac{1}{6\varepsilon^*}[3\varepsilon_0 - \varepsilon \pm \sqrt{(3\varepsilon_0 - \varepsilon)^2 + 12\varepsilon^{*2}}].$$

Proceeding to the limit for $h \to 0$ in relation (VII.39), retaining here the constant Z/h along with Z, we obtain

[1] See Ya. S. Podstrigach [4]; V. V. Bolotin [1].

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$$t = T + \frac{2Z}{h} T^*. \qquad \text{(VII.43)}$$

If the problem of thermoconductivity is symmetrical with respect to the middle surface of the plate, i.e., $t_m^+ = t_m^- = t_m$, $\alpha^+ = \alpha^- = \alpha$, $T^* = 0$, then, considering that here, due to (VII.43) t = T, we obtain, instead of (VII.41), the equation /529

$$\Delta t - \varkappa^2 (t - t_m) = \frac{1}{a} \cdot \frac{\partial t}{\partial \tau}, \qquad \text{(VII.44)}$$

where

$$\varkappa^2 = \frac{2\alpha}{\lambda h}.$$

The boundary conditions which must be satisfied by the values T and T* are derived from (VII.28)-(VII.32):

the initial condition is

$$T = F_0(x, y, 0), \qquad T^* = F_0^*(x, y, 0); \qquad \text{(VII.45)}$$

the conditions of the first type are

$$T = F_1(M, \tau), \qquad T^* = F_1^*(M, \tau), \qquad M \in L, \quad \tau > 0; \qquad \text{(VII.46)}$$

the conditions of the second type are

$$\frac{\partial T}{\partial n} = F_2(M, \tau), \qquad \frac{\partial T^*}{\partial n} = F_2^*(M, \tau), \quad M \in L, \quad \tau > 0; \qquad \text{(VII.47)}$$

the conditions of the third type are

$$\frac{\partial T}{\partial n} + k(T - T_m) = 0, \qquad \frac{\partial T^*}{\partial n} + k(T^* - T_m^*) = 0, \quad M \in L, \quad \tau > 0; \qquad \text{(VII.48)}$$

the conditions of ideal thermal contact are

$$T_1 = T_2, \quad T_1^* = T_2^*, \quad \lambda_1 \frac{\partial T_1}{\partial n} = \lambda_2 \frac{\partial T_2}{\partial n}, \quad \lambda_1 \frac{\partial T_1^*}{\partial n} = \lambda_2 \frac{\partial T_2^*}{\partial n}, \quad M \in L, \quad \tau > 0, \qquad \text{(VII.49)}$$

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where n is normal to contour L, which bounds the middle plane of the plate.

Conditions of Thermal Exchange on Reinforced Edge of Plate with Hole. We will assume that the edge of a hole in a thin plate of thickness h is reinforced by a ring of some other material of the same thickness and of width h_r (Figure VII.1). Assuming that the heat exchange of the system with the surrounding medium obeys Newton's law, and that there is ideal thermal contact between the ring and the plate on surface S, then for the determination of the temperature field in accordance with (VII.44), we have the following equations:

for the plate

$$\Delta t - \varkappa^2 (t - t_c) = \frac{c}{\lambda} \cdot \frac{\partial t}{\partial \tau}; \tag{VII.50}$$

for the ring

$$\Delta t_r - \varkappa_r^2 (t_r - t_m) = \frac{c_r}{\lambda_r} \cdot \frac{\partial t_r}{\partial \tau}; \tag{VII.51}$$

the boundary conditions are /530

$$\lambda_r = \frac{\partial t_r}{\partial n} = \lambda \frac{\partial t}{\partial n}, \quad t_r = t \quad \text{on } S, \quad \lambda_r \frac{\partial t_r}{\partial n} = \alpha_S (t_r - t_m) \text{ on } S_c; \tag{VII.52}$$

$$\lambda \frac{\partial t}{\partial n} = -\alpha'_S (t - t_m) \text{ on } S'_c; \tag{VII.53}$$

the initial conditions are

$$t = t^{(0)}, \quad t_r = t_r^{(0)} \text{ for } \tau = 0, \tag{VII.54}$$

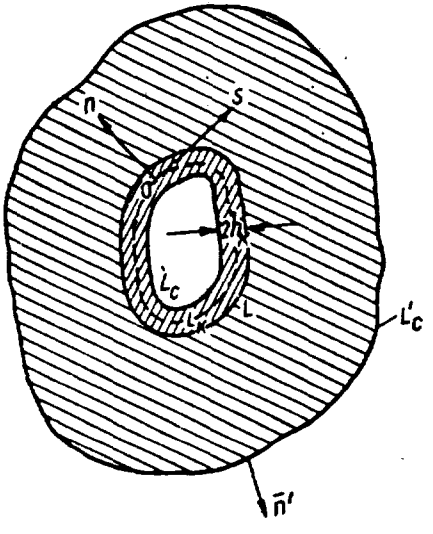


Figure VII.1.

where t, t_r, t_m are the temperatures, respectively, of the plate, ring and medium flowing over the surfaces of the system; $\varkappa_r^2 = 2\alpha_Z/\lambda_r h$, λ_r is the thermoconductivity of the material of the ring, c_r is its heat capacity, α_Z is the coefficient of heat emission from the lateral surfaces Z = ±h/2 of the rod, α_S is the coefficient of heat emission from the internal cylindrical surface S_c of the rod, α'_S is the coefficient of heat emission from the external cylindrical surface S'_c of the plate.

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Assuming that the width of the ring h_r is of the same order as the thickness h, we will regard it as a thin rod. We will formulate the condition that must be satisfied by the temperature of the plate on the reinforced edge, considering that the axis of the rod coincides with the contour of the plate. For this purpose we will relate the rod to coordinates (s, n) (Figure VII.1) and, by writing equation (VII.51) in these coordinates, we will disregard the values k, n (k is the curvature of the axis of the rod L_r) in comparison with unity. To determine the temperature of the ring we obtain the equation

$$\frac{\partial^2 t_r}{\partial n^2} + p^2 t_r = -\varkappa_r^2 t_m \qquad \text{(VII.55)}$$

the solution of which, with consideration of the two latter conditions (VII.52) has the following form:

$$t_m = \left\{ t^+ \left[\lambda_r p \cos p\left(\frac{h_r}{2} + n\right) + \alpha_s \sin p\left(\frac{h_r}{2} + n\right) \right] + \bar{t}_m \alpha_s \sin p\left(\frac{h_r}{2} - n\right) + \right.$$

$$\left. + \frac{\varkappa_r^2}{p} \sin p\left(\frac{h_r}{2} - n\right) \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} t_m \left[\lambda_r p \cos p\left(\frac{h_r}{2} + n\right) + \alpha_s \sin p\left(\frac{h_r}{2} + n\right) \right] dn \right\} \times \qquad \text{(VII.56)} \quad \underline{/531}$$

$$\times \left(\lambda_r p \cos p h_r + \alpha_s \sin p h_r\right)^{-1} + \frac{\varkappa_r^2}{p} \int_{\frac{h_r}{2}}^{n} \sin p (n_0 - n)\, t_m\, dn_0,$$

where the superscripts "plus" and "minus" indicate the value of the functions for $n = \pm h_r/2$, respectively; $p^2 = \partial^2/\partial s^2 - \kappa_r^2 - c_r/\lambda_r \cdot \partial/\partial\tau$ (ds is an element of arc L_r).

By substituting the expression for the temperature of the rod (VII.56) into the first boundary condition of (VII.52), we find that the temperature of the plate on the contour should satisfy the equation of infinitely high order

$$\left(\cos p h_r \frac{\sin p h_r}{p \lambda_r r_s}\right) \lambda \left(\frac{\partial t}{\partial n}\right)^+ - \left(\frac{\cos p h_r}{r_s} - \lambda_r p \sin p h_r\right) t^+ +$$

$$+ \frac{\bar{t}_m}{r_s} + \frac{\varkappa_r^2}{p} \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} \left[\lambda_r p \cos p\left(\frac{h_r}{2} + n\right) + \frac{1}{r_s} \sin p\left(\frac{h_r}{2} + n\right) \right] t_m\, dn = 0, \qquad \text{(VII.57)}$$

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where $r_S = 1/\alpha_S$ is resistance to heat exchange on the cylindrical surface S_c of the rod.

As in the derivation of the conditions of imperfect thermal contact between solid bodies[1], we will introduce the following thermophysical parameters: $\lambda_r^* = \lambda_r F$ is adduced thermoconductivity of the rod; $c_r^* = c_r F$ is its adduced heat capacity; $r_r^* = h_r/\lambda_r h$ is the adduced internal heat resistance; $r_Z^* = 1/\alpha_Z h_r$ is the adduced resistance to heat exchange on surfaces $Z = \pm h/2$ of the rod; $r_S^* = 1/\alpha_S h$ is the adduced resistance to heat exchange on cylindrical surface S_c of the rod; $F = hh_r$ is the area of cross section of the rod.

We will proceed in equation (VII.57) to the limit for $h_r \to 0$, using the theorem of the mean and preserving λ_r^*, c_r^*, r_r^*, r_Z^* constants, but disregarding the derivatives $\lambda_r^* \cdot r_r^*$, $c_r^* \cdot r_r^*$, $\alpha_Z^* \cdot r_r^*$. We obtain the condition of heat exchange on reinforced thin rod edge L of the plate[2]

$$\left[\lambda_r^{\bullet}\frac{\partial^2}{\partial s^2} + \left(1 + \frac{r_r^{\bullet}}{r_S^{\bullet}}\right)\lambda^{\bullet}\frac{\partial}{\partial n} - c_r^{\bullet}\frac{\partial}{\partial \tau}\right]t = \left(\frac{1}{r_S^{\bullet}} + \frac{2}{r_Z^{\bullet}}\right)(t - t_m). \qquad \text{(VII.58)}$$

Assuming here that the adduced thermoconductivity λ_r^* and heat capacity c_r^* of the rod are equal to zero, we arrive the condition of heat exchange on reinforced edge L of the plate

$$\frac{\partial t}{\partial n} = H_r^{\bullet}(t - t_m). \qquad \text{(VII.59)}$$

which coincides in form with Newton's condition (VII.52) on cylindrical surface S_c of a nonreinforced plate, in which the role of the relative coefficient of heat emission with the surface is filled by /532

$$H_r^{\bullet} = \frac{r_Z^{\bullet} + 2r_S^{\bullet}}{\lambda^{\bullet} r_Z^{\bullet}(r_r^{\bullet} + r_S^{\bullet})}.$$

In the case where $r_r^* = 0$, the value H_r^* in (VII.59) acquires the form

$$H_r^{\bullet} = \frac{1}{\lambda^{\bullet}}\left(\frac{1}{r_S^{\bullet}} + \frac{2}{r_Z^{\bullet}}\right). \qquad \text{(VII.60)}$$

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[1]See Ya. S. Podstrigach [2, 3].
[2]See Yu. M. Kolyano [1, 2].

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To determine the mean temperature through width h_r of the rod

$$T_r = \frac{1}{h_r} \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} t_r \, dn \qquad \text{(VII.61)}$$

we integrate (VII.56) in accordance with (VII.61) and in the expression obtained proceed to the limit for $h_r \to 0$, just as in the derivation of the condition of heat exchange (VII.58). We obtain an expression for the temperature in the rod

$$t_m = t^+ - \frac{t^+ - t_m^-}{2\left(1 + \frac{r_s^+}{r_r^*}\right)}, \qquad \text{(VII.62)}$$

which, when $r_r^* = 0$, will be equal to the temperature t^+ of the plate on its reinforced edge.

§3. Temperature Stresses Caused by Perturbation of Homogeneous Heat Flow near Holes[1]

Suppose we have an infinite plane with an arbitrary hole, bounded by smooth contour L. We will assume that at infinity there is homogeneous heat flow q, directed at angle α to the Ox axis, to which corresponds linear temperature distribution

$$t_\infty = q(x\cos\alpha + y\sin\alpha) + t_0. \qquad \text{(VII.63)}$$

A linear temperature field in a dense plate, as we know, has no effect on stresses if the plate is free of pinch. Because of the hole, the temperature field, remaining linear at infinity, experiences some perturbation in the vicinity of the hole, with the result that temperature stresses develop around it. This disturbance will be determined, obviously, by the form of contour L and boundary conditions on it. The temperature field in a plate with a hole is represented in the form /533

$$t = t_1 + t_\infty, \qquad \text{(VII.64)}$$

where t_1 is perturbation of the temperature field caused by the hole.

[1] In this section we will examine problems for the plane temperature field.

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In the following discussion we will disregard constant temperature t_0, since it will cause no stresses in a nonreinforced body.

Considering the contour of the hole to be thermally insulated, for the determination of the perturbation of a stationary plane temperature field by the hole on the basis of formulas (VII.31) and (VII.33), we obtain the equation of thermoconductivity

$$\Delta t_1 = 0 \tag{VII.65}$$

and boundary conditions

$$\frac{\partial t_1}{\partial n} = 0 \quad \text{on } L; \quad t_1 = 0 \quad \text{for } x = y = \infty. \tag{VII.66}$$

Let the range outside of the given contour of hole L be mapped conformally by the function $Z = \omega(\zeta)$ on the exterior of unit circle γ. Then, from (VII.63)-(VII.66), for the distribution of temperature in the range of variable $\zeta = \rho e^{i\vartheta}$, we obtain

$$t(\zeta, \bar{\zeta}) = \frac{qR}{2}\left[e^{i\alpha}\left(\bar{\zeta} + \frac{1}{\zeta}\right) + e^{-i\alpha}\left(\zeta + \frac{1}{\bar{\zeta}}\right)\right], \tag{VII.67}$$

where R is some real constant that depends on the dimension and shape of the hole.

The problem of temperature stress distribution in this case is analyzed in the works of A. L. Florence and J. N. Goodier [1-3], H. Deresiewicz [1], Atsumi Akira [1], Muramatsu Masamitsu, Atsumi Akira [1], I. V. Gayvas' [1].

By substituting variable z by ( ), formulas (VII.15) and (VII.16) are transformed to[1]

$$\sigma_\rho + \sigma_\phi = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}] - \alpha_t E t(\zeta, \bar{\zeta}), \tag{VII.68}$$

$$\sigma_\phi - \sigma_\rho + 2i\tau_{\rho\phi} = \frac{2\zeta^2}{\rho^2 \overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta) - \frac{\alpha_t E}{2}\int \overline{\omega'(\zeta)}\frac{\partial t}{\partial \zeta}\, d\bar{\zeta}\right]; \tag{VII.69}$$

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} + \frac{\alpha_t E}{2}\int \omega'(\zeta)\, t(\zeta, \bar{\zeta})\, d\zeta,$$

[1] See I. V. Gayvas' [1].

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and contour conditions (VII.14) in the case of the first basic problem in the absence of external load, acquire the form /534

$$\Phi(\sigma)+\overline{\Phi(\sigma)}-\frac{\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi'(\sigma)+\omega'(\sigma)\,\Psi(\sigma)\right]=$$
$$=\frac{\alpha_t E}{2}\left[t(\zeta,\bar{\zeta})-\frac{\zeta^2}{\overline{\omega'(\zeta)}}\int\overline{\omega'(\zeta)}\,\frac{\partial t}{\partial\zeta}\,d\bar{\zeta}\right]\Big|_{\zeta=\sigma}. \tag{VII.70}$$

Square Hole. We will take the mapping function in the form[1]

$$\omega(\zeta)=R\left(\zeta-\frac{1}{6\zeta^3}\right).$$

By substituting (VII.67) into (VII.70), we obtain a functional equation relative the functions $\Phi(\zeta)$ and $\Psi(\zeta)$:

$$\Phi(\sigma)+\overline{\Phi(\sigma)}-\frac{\sigma^2\omega'(\sigma)}{\overline{\omega'(\sigma)}}\left[\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\Phi'(\sigma)+\Psi(\sigma)\right]=$$
$$=\frac{\alpha_t EqR}{12(2+\sigma^4)}\left[\frac{e^{i\alpha}(18+5\sigma^4)}{\sigma}+e^{-i\alpha}\sigma(6+7\sigma^4)\right] \tag{VII.71}$$

or, if we convert to the conjugate values,

$$\Phi(\sigma)+\overline{\Phi(\sigma)}-\frac{1}{\sigma^2}\,\frac{\overline{\omega'(\sigma)}}{\omega'(\sigma)}\left[\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\Phi'(\sigma)}+\overline{\Psi(\sigma)}\right]=$$
$$=\frac{\alpha_t EqR}{12(2\sigma^4+1)}\left[e^{-i\alpha}\sigma(18\sigma^4+5)+e^{i\alpha}\,\frac{6\sigma^4+7}{\sigma}\right]. \tag{VII.72}$$

We will represent the functions in the form

$$\Phi(\zeta)=a_1\zeta+\sum_{n=1}^{\infty}a_{-n}\zeta^{-n},$$
$$\Psi(\zeta)=\sum_{n=1}^{\infty}b_{-n}\zeta^{-n}. \tag{VII.73}$$

By multiplying equation (VII.72) by $\frac{1}{2\pi i}\cdot\frac{1+2\sigma^4}{\sigma-\zeta}\,d\sigma$ and integrating with respect to γ, considering $|\zeta|>1$, we obtain

[1]See formula (I.43).

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$$(1+2\zeta^4)[a_1\zeta-\Phi(\zeta)]+2a_{-1}\zeta^3+2a_{-2}\zeta^2+2a_{-3}\zeta+2a_{-4}-\frac{4}{3}\cdot\frac{\bar{a}_1}{\zeta}=$$
$$=-\frac{7}{12}\alpha_t EqRe^{i\alpha}\frac{1}{\zeta}. \qquad \text{(VII.74)}$$

Analogously, from equation (VII.72), multiplying it by $\frac{1}{2\pi i}\cdot\frac{1+2\sigma^4}{\sigma^5}\cdot\frac{d\sigma}{\sigma-\zeta}$ /535 and integrating with respect to γ, recalling $|\zeta|>1$, we obtain

$$-\frac{1+2\zeta^4}{\zeta^5}\Phi(\zeta)+2a_1-\frac{4}{3}\frac{\bar{a}_1}{\zeta^6}-\frac{2}{3}\cdot\frac{\bar{a}_{-1}}{\zeta^4}-\frac{1}{3}\cdot\frac{\bar{a}_{-2}}{\zeta^3}+\frac{1}{3}\cdot\frac{\bar{a}_{-4}}{\zeta}+$$
$$+\frac{2\bar{b}_{-1}}{\zeta^2}+\frac{2\bar{b}_{-2}}{\zeta}=-\frac{\alpha_t EqR}{4}\left[\frac{5}{3}e^{-i\alpha}\frac{1}{\zeta^4}+e^{i\alpha}\frac{1}{\zeta^2}\left(2+\frac{7}{3}\cdot\frac{1}{\zeta^4}\right)\right]. \qquad \text{(VII.75)}$$

By comparing the expressions for $\Phi(\zeta)$ from (VII.74) and (VII.75), we obtain, for the determination of the coefficients, the following system of equations:

$$a_1+2a_{-3}=-\frac{2}{3}\bar{a}_{-1}+\frac{5}{12}\alpha_t EqRe^{-i\alpha},$$
$$2a_{-1}=2\bar{b}_{-1}+\frac{\alpha_t EqR}{2}e^{i\alpha}, \qquad \text{(VII.76)}$$
$$a_{-2}=a_{-4}=0.$$

The condition of uniqueness of the displacements requires that

$$\varkappa a_{-1}+\bar{b}_{-1}=-\frac{1}{4}\alpha_t EqRe^{i\alpha}. \qquad \text{(VII.77)}$$

The stresses at infinity will be equal to zero if

$$a_1=\frac{1}{4}\alpha_t EqRe^{-i\alpha}. \qquad \text{(VII.78)}$$

Solving jointly (VII.76)-(VII.78), we find

$$a_{-1}=0; \qquad a_{-3}=\frac{1}{12}\alpha_t EqRe^{-i\alpha}. \qquad \text{(VII.79)}$$

By introducing the values found for the coefficients into (VII.74), we will have

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$$\Phi(\zeta) = \frac{\alpha_t EqR}{4} \cdot \frac{1}{1 + 2\zeta^4} \left[\frac{e^{i\alpha}}{\zeta} + e^{-i\alpha} \frac{\zeta (5 + 6\zeta^4)}{3} \right]. \qquad \text{(VII.80)}$$

The function $\Psi(\zeta)$ is found from equation (VII.71) by multiplying it by $\frac{1}{2\pi i} \cdot \frac{2 + \sigma^4}{\sigma^2} \times \frac{d\sigma}{\sigma - \zeta}$ and integrating with respect to γ, considering $|\zeta| > 1$. Recalling here (VII.80), we obtain

$$\Psi(\zeta) = \frac{\alpha_t EqR}{6} \frac{\zeta^3}{(1 + 2\zeta^4)^3} \left[13e^{i\alpha}\zeta^2 (1 - 2\zeta^4) + \frac{e^{-i\alpha}}{3} (-36\zeta^8 + 16\zeta^4 - 9) \right]. \qquad \text{(VII.81)}$$

By substituting (VII.67), (VII.80) into (VII.68) and assuming $\zeta = \rho e^{i\vartheta}$ when $\rho = 1$, we obtain the formula for stresses on the contour of a square hole:

$$\sigma_\vartheta = -\frac{2}{3} \cdot \frac{\alpha_t EqR}{5 + 4\cos 4\vartheta} [5\cos(\vartheta - \alpha) + \cos(3\vartheta + \alpha)]. \qquad \text{(VII.82)}$$

The values of stresses σ_ϑ in fractions of α_tEqR along the contour of the hole, corresponding to functions (VII.80) and (VII.81) when $\alpha = \pi/2$, are presented below: /536

0	0,000	50	— 1,789
15	— 0,056	60	— 0,962
30	— 0,333	75	— 0,527
40	— 1,261	90	— 0,444
45	— 1,886		

Triangular Hole. The exterior of a triangular hole is mapped onto the exterior of unit circle γ by the function[1] $\omega(\zeta) = R\left(\zeta + \frac{1}{3\zeta^2}\right)$.

Functions $\Phi(\zeta)$ and $\Psi(\zeta)$ for this case are found in the form

$$\Phi(\zeta) = \frac{\alpha_t EqR}{4(3\zeta^3 - 2)} [-2e^{i\alpha}\zeta + 3e^{-i\alpha}\zeta(\zeta^3 - 1)],$$
$$\Psi(\zeta) = -\frac{\alpha_t EqR}{4} \cdot \frac{\zeta^3}{(3\zeta^3 - 2)^3} [22e^{i\alpha}\zeta(1 + 3\zeta^3) + 3e^{-i\alpha}(9\zeta^6 - \zeta^3 + 4)]. \qquad \text{(VII.83)}$$

[1] See formula (I.44).

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The stresses on the contour of the hole are

$$\sigma_\vartheta = \frac{\alpha_t EqR}{13 - 12\cos 3\vartheta}\left[-7\cos(\vartheta - \alpha) + 3\cos(2\vartheta + \alpha)\right]. \qquad \text{(VII.84,}$$

The values of σ_ϑ in fractions of $\alpha_t EqR$ on the contour of the hole are presented below for $\alpha = 0$ and $\alpha = \pi/2$, respectively:

$\alpha=0$		$\alpha=\pi/2$	
0	— 4,000	0	0,000
10	— 1,563	10	— 0,860
15	— 0,922	30	— 0,469
30	— 0,351	90	— 0,539
90	— 0,231	100	— 0,838
115	0,731	115	— 2,872
120	2,000	120	— 3,464
125	2,122	125	— 2,069
135	1,096	135	— 0,432
180	0,400	180	— 0,000

Elliptical Hole. The function that conformally maps the exterior of an elliptical hole with semiaxes a and b onto the exterior of unit circle γ has the form[1]

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right).$$

Omitting the intermediate calculations, we present the final expressions /537 for the functions

$$\Phi(\zeta) = \frac{\alpha_t EqR}{4}\cdot\frac{1}{\zeta^2 - m}\left(e^{-i\alpha}\zeta^3 - me^{i\alpha}\frac{1}{\zeta}\right),$$

$$\Psi(\zeta) = \frac{\alpha_t EqR}{4}\cdot\frac{\zeta}{(\zeta^2 - m)^3}\left\{\left[m\zeta^4 - 2(1 + 2m^2)\zeta^2 + m^3\right]e^{i\alpha} + \right. \qquad \text{(VII.85)}$$

$$\left. + \left[-\zeta^4 + 2m(m^2 + 2)\zeta^2 - m^2\right]e^{-i\alpha}\right\}.$$

The stress components are

[1]See formula (I.42).

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$$\sigma_\varrho = \frac{\alpha_t E \varrho R}{2(\varrho^4 - 2m\varrho^2\cos 2\vartheta + m^2)^2}[m\varrho(\varrho^4 - 1)(\varrho^2 - m^2)\cos(\vartheta + \alpha) +$$
$$+ \varrho(1 - \varrho^2)(\varrho^4 - m^4)\cos(\vartheta - \alpha)],$$
$$\sigma_\vartheta = \frac{\alpha_t EqR}{2(\varrho^4 - 2m\varrho^2\cos 2\vartheta + m^2)^2}[m\varrho(\varrho^2 + m^2)(1 + 4\varrho^2 + \varrho^4)\cos(\vartheta + \alpha) -$$
$$- \varrho(1 + \varrho^2)(\varrho^4 + 4m^2\varrho^2 + m^4)\cos(\vartheta - \alpha) - 2m^2\varrho^3(1 + \varrho^2)\cos(3\vartheta + \alpha) +$$
$$+ 2m\varrho^3(\varrho^2 + m^2)\cos(3\vartheta - \alpha)], \qquad \text{(VII.86)}$$
$$\tau_{\varrho\vartheta} = \frac{\alpha_t EqR}{2(\varrho^4 - 2m\varrho^2\cos 2\vartheta + m^2)^2}[m\varrho(\varrho^2 - m^2)(1 - \varrho^2)^2\sin(\vartheta + \alpha) +$$
$$+ \varrho(1 - \varrho^2)(\varrho^2 - m^2)^2\sin(\vartheta - \alpha)].$$

The stresses on the contour of the hole are

$$\sigma_\vartheta = \frac{\alpha_t EqR}{1 - \cos 2\vartheta + k^2(1 + \cos 2\vartheta)}[(1 - k^2)\cos(\vartheta + \alpha) - (1 + k^2)\cos(\vartheta - \alpha)], \qquad \text{(VII.87)}$$

where k = b/a.

Below are the values of σ_ϑ in fractions of $\alpha_t EqR$ on the contour of the hole when $\alpha = 0$:

ϑ°	b/a=2/3	b/a=3/2	ϑ°	b/a=2/3	b/a=3/2
0	— 1,000	— 1,000	50	— 0,370	— 0,954
10	— 0,949	— 1,002	60	— 0,258	— 0,857
20	— 0,820	— 1,005	70	— 0,163	— 0,671
30	— 0,660	— 1,006	80	— 0,078	— 0,377
40	— 0,505	— 0,994	90	0,000	0,000

Round Hole. Assuming in formulas (VII.85) m = 0, α = 0, we obtain stress functions $\Phi(\zeta)$ and $\Psi(\zeta)$ for a range with a round thermally isolated hole of radius R, subjected to the effect of a homogeneous heat flow.

The stress components are found from (VII.86) for m = 0 and α = 0: /538

$$\sigma_\varrho = \frac{\alpha_t EqR}{2}\left(\frac{1}{\varrho^3} - \frac{1}{\varrho}\right)\cos\vartheta,$$
$$\sigma_\vartheta = -\frac{\alpha_t EqR}{2}\left(\frac{1}{\varrho^3} + \frac{1}{\varrho}\right)\cos\vartheta, \qquad \text{(VII.88)}$$
$$\tau_{\varrho\vartheta} = \frac{\alpha_t EqR}{2}\left(\frac{1}{\varrho^3} - \frac{1}{\varrho}\right)\sin\vartheta.$$

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From (VII.88) it follows that the maximum absolute values of σ_ϑ on the contour of a round hole are found for $\vartheta = 0$ and $\vartheta = \pi$.

Below are the values of σ_ϑ in fractions of $\alpha_t EqR$ near the contour of the hole

0	— 1,000	105	0,258
15	— 0,965	120	0,500
30	— 0,866	135	0,707
45	— 0,707	150	0,866
60	— 0,500	165	0,965
75	— 0,258	180	1,000
90	0,000		

Narrow Slit. Under the effect of a homogeneous heat flow in a medium with an insulated slit, the stress state will be determined by functions (VII.85) when $m = \pm 1$. When $m = 1$, $\rho = 1$, we obtain from (VII.86) the expression for stress along the contour of a slit arranged along the Ox axis:

$$\sigma_\vartheta = -\frac{\alpha_t EqR}{2} \cdot \frac{\sin \alpha}{\sin \vartheta}. \qquad \text{(VII.89)}$$

If the heat flux is directed parallel to the slit, then the stress components are equal to zero. In the case where the direction of the heat flow is perpendicular to the slit, the stress on the contour will be

$$\sigma_\vartheta = -\frac{\alpha_t EqR}{2} \cdot \frac{1}{\sin \vartheta}. \qquad \text{(VII.90)}$$

It follows from (VII.90) that at the points ($\vartheta = 0$ and $\vartheta = \pi$) the stresses will increase to infinity.

§4. Axisymmetrical Temperature Field and Temperature Stresses in an Infinite Plate with a Round Reinforced Edge

We will determine the unsteady temperature field and temperature stresses in an infinite plate, free of external load, with a circular reinforced edge or a thin rod, considering the temperature of the external medium as a function of time only.

Temperature Field. Because of symmetry, instead of (VII.50) and (VII.58) /539 for the determination of the unsteady temperature field, we will have in the given plate:

equation of thermoconductivity

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} - \mathrm{Bi}^* (t - t_m) = \frac{\partial t}{\partial \mathrm{Fo}^*}; \qquad \text{(VII.91)}$$

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boundary condition

$$\frac{\partial t}{\partial r} - \mathrm{Bi}_r^* (t - t_m) = \beta_k^* \frac{\partial t}{\partial \mathrm{Fo}^*} \quad \text{for } r = 1, \tag{VII.92}$$

where $\mathrm{Bi}^* = 2\frac{\alpha R^2}{\lambda h}$; $\mathrm{Fo}^* = \frac{a\tau}{R^2}$ is Fourier's criterion; r relates the polar radius to the radius of the hole of the plate;

$$\mathrm{Bi}_r^* = \frac{(r_Z^* + 2r_S^*) R}{\lambda^* (r_r^* + r_S^*) r_Z^*} = H_r^* R, \qquad \beta_r^* = \frac{c_m m r_s^*}{c (r_S^* + r_r^*)}, \qquad m = \frac{h_r}{R}.$$

We will assume that the temperature of the system at the initial moment of time is equal to zero. Assuming, for instance, that the temperature of the medium is a harmonic function of time, i.e.,

$$t_m = t_0 e^{i\omega \mathrm{Fo}^*}, \tag{VII.93}$$

we find from (VII.91)-(VII.93), using the Laplace transformation, the unstationary temperature field of the plate[1]

$$t = t_0 e^{-\mathrm{Mi}} Q_0 (r, \mathrm{Fo}^*, \eta) + t_m \{i\omega [\varepsilon_0 (r, \eta) - \varepsilon_0 (r, \mathrm{Fo}^*, \eta)]\}, \tag{VII.94}$$

where

$$Q_n = \frac{2A^*}{\pi} \int_0^\infty e^{-\eta^2 \mathrm{Fo}^*} f_n (r, \eta)\, d\eta, \qquad \varepsilon_n = \frac{2A^*}{\pi} \int_0^\infty \frac{f_n (r, n)}{\mathrm{Bi}^* + i\omega + \eta^2} e^{-(\mathrm{Bi}^* + i\omega + \eta^2)\mathrm{Fo}^*} d\eta,$$

$$f_n (r, \eta) = \frac{J_n(r,\eta) [\eta Y_1 (\eta) + B (\eta) Y_0 (\eta)] - Y_n(r,\eta) [\eta J_1 (\eta) + B (\eta) J_0 (\eta)]}{\eta^{n+1} \{[\eta J_1 (\eta) + B (\eta) J_0 (\eta)]^2 + [\eta Y_1 (\eta) + B (\eta) Y_0 (\eta)]^2\}},$$

$n = 0, 1$; $A^* = \mathrm{Bi}_r^* - \beta_k^* \mathrm{Bi}^*$, $B (\eta) = A^* - \beta_k^* \eta^2$; $\mathrm{Mi} = \mathrm{Bi}^* \mathrm{Fo}^*$ is Mikheyev's criterion; $J_n(\eta)$, $Y_n(\eta)$ are Bessel's functions of kind I and II of the real argument.

For the asymptotic condition[2], we have instead of (VII.94)

$$t^{(as)} = \frac{t_m}{\omega^*} [C K_0 (r \sqrt{\omega^*}) + \mathrm{Bi}^*], \tag{VII.95}$$

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[1] See Yu. M. Kolyano [1, 2].
[2] See Ya. S. Podstrigach [1].

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where

$$C = \frac{A^* i\omega}{\sqrt{\omega^*} K_1(\sqrt{\omega^*}) + (\mathrm{Bi}_I^* + i\omega\beta_k^*) K_0(\sqrt{\omega^*})},$$

$K_n(\eta)$ is McDonald's function, $\omega^* = \mathrm{Bi}^* + i\omega$.

Assuming in (VII.94) $\omega = \upsilon$, we find the solution of the problem of thermo- /540
conductivity

$$t = t_0 [1 + e^{-\mathrm{Mi}} Q_0(r, \mathrm{Fo}^*, \eta)], \tag{VII.96}$$

that corresponds to the case where the temperature of the surrounding medium changes at the initial moment of time by some magnitude, remaining constant thereafter.

Determination of Stresses. To determine the stress-deformation state caused in the plate by temperature fields (VII.94) and (VII.96), we will use relations (VII.15) and (VII.16) of the temperature problem of elasticity theory for the generalized plane stress state, which in the case of radial symmetry become

$$\begin{aligned} \sigma_r + \sigma_\theta &= 2[\Phi(z) + \overline{\Phi(z)}] - \alpha_t E t, \\ \sigma_\theta - \sigma_r &= \left\{2[\bar{z}\Phi'(z) + \Psi(z)] - \alpha_t E \int \frac{\partial t}{\partial z} d\bar{z}\right\} e^{2i\theta}, \\ 2\mu u &= \left\{\varkappa\varphi(z) - z\overline{\varphi'(z)} - \psi(z) + \frac{\alpha_t E}{2} \int t dz\right\} e^{-i\theta}, \end{aligned} \tag{VII.97}$$

the condition of compatibility of deformations of the plate and rod is

$$u = \frac{\sigma_r}{mE_k} + \alpha_t^{(k)} t \text{ for } r = 1 \tag{VII.98}$$

and the conditions of the absence of stresses at infinity are

$$\sigma_r^{(\infty)} = \sigma_\theta^{(\infty)} = 0, \tag{VII.99}$$

where $\phi(z)$, $\psi(z)$ are functions of complex variable $z = ze^{i\theta}$. In the given case

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$$\varphi(z) = \Gamma z + \varphi_0(z),$$
$$\psi(z) = \Gamma' z + \psi_0(z), \tag{VII.100}$$

where Γ, Γ' are complex constants that determine the stress distribution at infinity and rotation of the body as a rigid whole; $\phi_0(z)$, $\psi_0(z)$ are holomorphous functions which, for a sufficiently large value of z, have an expansion of the form

$$\varphi_0(z) = \sum_{n=0}^{\infty} a_{-n} z^{-n}; \qquad \psi_0(z) = \sum_{n=0}^{\infty} b_{-n} z^{-n}; \tag{VII.101}$$

a_n, b_n are complex constants.

In the case where the temperature field has the form (VII.95), we find /541 from (VII.97)-(VII.99) that all coefficients of the functions $\phi(z)$ and $\psi(z)$ are equal to zero, with the exception of

$$\operatorname{Re}\Gamma = \frac{t_m}{4\omega^*}\mathrm{Bi}^*\alpha_t E, \qquad a_0 = b_0,$$
$$\operatorname{Re} b_{-1} = D\left[\alpha_t t_m \frac{\mathrm{Bi}^*}{m\omega^*} - \alpha_t^{(k)} t_\kappa\right] - \frac{\alpha_t t_m EC}{(\omega^*)^{3/2}} K_1(\sqrt{\omega^*}), \tag{VII.102}$$

where

$$D = \frac{1}{\frac{1+\nu}{E} + \frac{1}{mE_\kappa}}.$$

By substituting (VII.100)-(VII.102) into formulas (VII.97), we find the distribution of temperature stresses in the plate

$$\sigma_r^{(as)} = \left\{D\left[\alpha_t t_m \frac{\mathrm{Bi}^*}{m\omega^*} - \alpha_t^{(k)} t_r\right] + \alpha_t t_m EC(\omega^*)^{-3/2}[rK_1(r_1) - K_1(\sqrt{\omega^*})]\right\} r^{-2},$$
$$\sigma_\theta^{(as)} = -\sigma_r^{(as)} - \alpha_t E t_m \frac{C}{\omega^*} K_0(r_1), \tag{VII.103}$$

where

$$r_1 = r\sqrt{\omega^*}.$$

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In the case of temperature field (VII.96),

$$\sigma_r = \left\{D\left[\alpha_t t_0^* - \alpha_t^{(k)} t_r\right] + \alpha_t E t_0 e^{-\mathrm{Mi}}\left[Q_1(\mathrm{Fo}^*, \eta) - rQ_1(r, \mathrm{Fo}^*, \eta) + \frac{1-r^2}{2}\right]\right\} r^{-2},$$
$$\sigma_\theta = -\sigma_r + \alpha_t E (t_0^* - t), \qquad \text{(VII.104)}$$

where

$$t_0^* = t(1 - e^{-\mathrm{Mi}}).$$

On the edge of the plate (r = 1) we find from (VII.104)

$$\sigma_r = D\left[\alpha_t t_0^* - \alpha_t^{(k)} t_r\right], \qquad \sigma_\theta = -\sigma_r - \alpha_t E e^{-\mathrm{Mi}}(1 - Q_0^+) \qquad \text{(VII.105)}$$

where

$$Q_0^+ = \frac{4A^*}{\pi^2}\int_0^\infty \frac{e^{-\eta^2}\mathrm{Fo}^* d\eta}{\eta\left\{[\eta J_1(\eta) + B(\eta) J_0(\eta)]^2 + [\eta Y_1(\eta) + B(\eta) Y_0(\eta)]^2\right\}}.$$

For a nonreinforced plate, we should assume in (VII.105) that

$$A^* = B = \mathrm{Bi}_S = \frac{\alpha_S R}{\lambda}, \qquad D = 0.$$

Assuming the adduced heat capacity c_r and internal heat resistance r_r^* of the rod to be equal to zero, we will have, instead of (VII.105),

$$\sigma_r^* = \frac{1}{m_E + 1 + \nu}\{1 - e^{-\mathrm{Mi}} - \alpha_t^*[1 - e^{-\mathrm{Mi}} Q_0^+]\}, \qquad \text{(VII.106)}$$

/542

$$\sigma_\theta^* = -\sigma_r^* - e^{-\mathrm{Mi}}[1 - Q_0^+] \quad \text{for} \quad A^* = B = \mathrm{Bi}_r^*,$$

where

$$\sigma_t^* = \frac{\sigma_t}{\alpha_t E t_0}; \qquad m_E = \frac{E}{mE_r}; \qquad \alpha_t^* = \frac{\alpha_t^{(k)}}{\alpha_t}; \qquad \mathrm{Bi}_r^* = \frac{h_r H_r^*}{m}.$$

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The graphs of distribution of stresses (VII.106) for $h_r H_r^* = 93.1$ j/m·sec·deg as functions of Fo*, Bi*, m for a steel ($E = 2061 \times 10^7$ n/m^2, $\alpha_t = 12 \cdot 10^{-6}$ 1/deg, $\lambda = 46.556$ j/m·sec·deg) plate with a round hole, the edge of which is reinforced by a thin bronze ($E_r = 11772 \cdot 10^7$ n/m^2, $\alpha_t^{(k)} = 18 \cdot 10^{-6}$ 1/deg) rod, are presented in Figures VII.2 and VII.3. The graph of the function Q_0^+, given in the book of G. Karslou and D. Yeger [1] was used for the construction of these graphs.

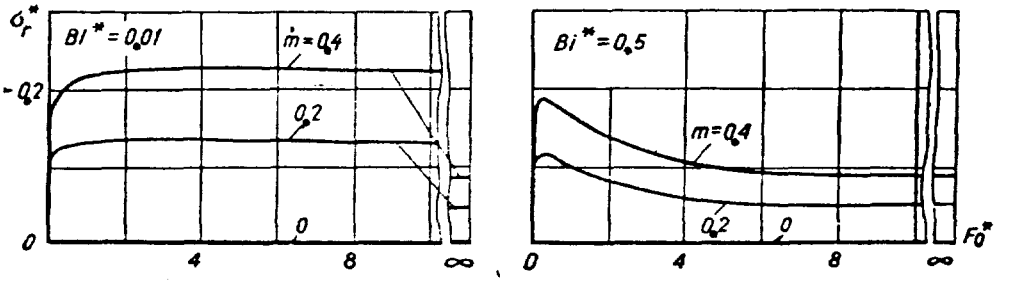


Figure VII.2.

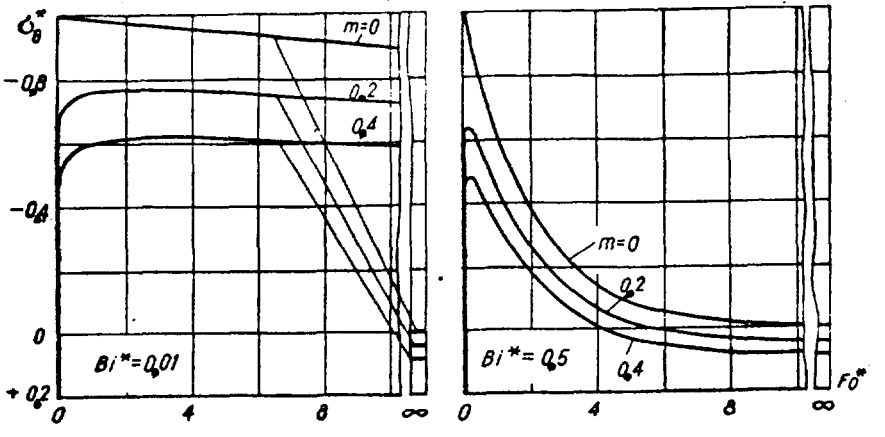


Figure VII.3.

It is clear from the graphs that the maximum stresses on the edge of the plate reinforced by the thin rod will be achieved sooner for the given case as heat emission from its lateral surfaces increases.

Radial stresses σ_r^* (Figure VII.2) are always positive. They decrease as the parameter of thinness of the wall of the reinforcing element $m = h_r R$ decreases.

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The ring stresses σ_θ^* (Figure VII.3) can change signs as time passes, changing from compressive stresses to tension. Hence as the parameter m in the range of compression decreases, they will increase in absolute value, and in the range of tension, they will decrease.

In the stationary thermal condition, i.e., when $Fo^* \to \infty$, temperature stresses (VII.106) will become

$$\sigma_r^* = -\sigma_\theta^* = \frac{1}{m_E + 1 + \nu}\left(1 - \frac{\alpha_t^{(k)}}{\alpha_t}\right).$$

As we see, in this case they do not depend on the thermophysical characteristics of the rod and plate.

If in (VII.104) we change to Cartesian coordinates, moving the origin of the coordinates to the contour of the hole, assuming $R \to \infty$, then we obtain stresses /543

$$\sigma_y = -\alpha_t E t_0\left\{\frac{A}{\sqrt{1-4\beta_k A}}\left[F_2^-(x) - F_1^-(x) - f_2(x) + f_1(x)\right] + e^{-Mi}\operatorname{erfc}\left(\frac{x}{2\sqrt{Fo}}\right)\right., \quad \text{(VII.107)}$$

which occur in a semiinfinite plate whose edge is reinforced by a thin elastic rod.

Here

$$F_i^{\pm}(x) = \frac{e^{Pi\,Fo}}{2\sqrt{Bi + Pi}}\left[e^{-x\sqrt{Bi+Pi}}\operatorname{erfc}\left(\frac{x}{2\sqrt{Fo}} - \sqrt{(Bi + Pi)\,Fo}\right) \pm\right.$$

$$\left. \pm e^{x\sqrt{Bi+Pi}}\operatorname{erfc}\left(\frac{x}{2\sqrt{Fo}} + \sqrt{(Bi + Pi)\,Fo}\right)\right]; \qquad f_i = \frac{(Bi_r + \beta_k Pi)\,F_i^+}{\sqrt{Bi + Pi}};$$

$$\rho_{1,2} = \frac{1 - 2\beta_k Bi \pm \sqrt{1 - 4\beta_k A}}{2\beta_k^2}; \qquad \operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-\xi^2} d\xi; \qquad x = \frac{x_0}{h};$$

$$Bi = 2\frac{\alpha h}{\lambda}; \quad Bi_r = H_r^* h; \quad A = Bi_r - \beta_k Bi; \quad \beta_k = \frac{\beta_k^*}{m^*}; \quad m^* = \frac{h}{R}; \quad Fo = \frac{a\tau}{h^2}.$$

§5. Temperature Stresses during Deflection of Thin Plate with Round Hole /544

Let an annular plate be heated by convective heat exchange with a medium bounded by concentric circles of radii $r = R_1$ and $r = R_2 (R_2 > R_1)$. The temperature of the medium flowing over the surfaces $Z = \pm h/2$ of the plate is

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constant and equal, respectively to t_m^+ and t_m^-. Conditions (VII.48) are satisfied on the contours of the plate. We will determine the temperature field established in the plate and the stresses caused by this field under the corresponding boundary conditions.

Determination of Temperature Field. The stationary temperature field in the plate for identical coefficients of heat emission from surfaces Z = ±h/2 is described by the equations

$$\lambda^*\Delta T - \varepsilon T = -\varepsilon\frac{t_m^+ + t_m^-}{2},$$

$$\lambda^*\Delta T^* - 3\varepsilon_0 T^* = -3\varepsilon\frac{t_m^+ - t_m^-}{2} \qquad \text{(VII.108)}$$

By solving equations (VII.108) under boundary conditions

$$\frac{\partial T}{\partial r} - k_1(T - T_m) = 0, \qquad \frac{\partial T^*}{\partial r} - k_1(T^* - T_m^*) = 0 \text{ for } r = R_1,$$

$$\frac{\partial T}{\partial r} + k_2(T - \bar{T}_m) = 0, \qquad \frac{\partial T^*}{\partial r} + k_2(T^* - \bar{T}_m^*) = 0 \text{ for } r = R_2, \qquad \text{(VII.109)}$$

we obtain

$$T = A_1 I_0(\beta_1 r) + B_1 K_0(\beta_1 r) + t_m$$

$$T^* = A_2 I_0(\beta_2 r) + B_2 K_0(\beta_2 r) + \frac{\varepsilon}{\varepsilon_0} t_m^* \qquad \text{(VII.110)}$$

where

$$A_1 = -\frac{1}{\Delta_1}[k_1(T_m - t_m) K(\beta_1 R_2) + k_2(\bar{T}_m - t_m) K(\beta_1 R_1)],$$

$$B_1 = -\frac{1}{\Delta_1}[k_1(T_m - t_m) I(\beta_1 R_2) + k_2(\bar{T}_m - t_m) I(\beta_1 R_1)],$$

$$A_2 = -\frac{1}{\Delta_2}\left[k_1\left(T_m^* - \frac{\varepsilon}{\varepsilon_0} t_m^*\right) K(\beta_2 R_2) + k_2\left(\bar{T}_m^* - \frac{\varepsilon}{\varepsilon_0} t_m^*\right) K(\beta_2 R_1)\right],$$

$$B_2 = -\frac{1}{\Delta_2}\left[k_1\left(T_m^* - \frac{\varepsilon}{\varepsilon_0} t_m^*\right) I(\beta_2 R_2) + k_2\left(\bar{T}_m^* - \frac{\varepsilon}{\varepsilon_0} t_m^*\right) I(\beta_2 R_1)\right],$$

$$\Delta_i = I(\beta_i R_1) K(\beta_i R_2) - I(\beta_i R_2) K(\beta_i R_1) \qquad (i = 1, 2),$$

$$I(\beta_1 R_i) = \beta_1 I_1(\beta_1 R_i) + (-1)^i k_i I_0(\beta_1 R_i),$$

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$$K(\beta_1 R_i) = \beta_1 K_1(\beta_1 R_i) - (-1)^i k_i K_0(\beta_1 R_i),$$

$$I(\beta_2 R_i) = \beta_2 I_1(\beta_2 R_i) + (-1)^i k_i I_0(\beta_2 R_i), \qquad \beta_1^2 = \frac{\varepsilon}{\lambda^*},$$

$$K(\beta_2 R_i) = \beta_2 K_1(\beta_2 R_i) - (-1)^i k_i K_0(\beta_2 R_i), \qquad \beta_2^2 = 3\frac{\varepsilon_0}{\lambda^*}, \tag{VII.111}$$

$$t_m = \frac{t_m^+ + t_m^-}{2}, \qquad t_m^* = \frac{t_m^+ - t_m^-}{2};$$

T_m, $\overline{T}_m$, T_m^*, $\overline{T}_m^*$, t_m^+, t_m^- are known constants, $I_0(\rho)$, $I_1(\rho)$ are Bessel functions of the first kind of the zero and first orders from the imaginary argument, $K_0(\rho)$, $K_1(\rho)$ are McDonald's functions of the zero and first orders.

If the edges of the plate are thermally insulated ($k_1 = k_2 = 0$), then $A_1 = A_2 = B_1 = B_2 = 0$, and the temperature in the mean plane of the plate will be constant:

$$T = t_m \qquad T^* = \frac{\varepsilon}{\varepsilon_0} t_m^*. \tag{VII.112}$$

Temperature Stresses and Moments. Let the edges of an annular plate be free of stresses. By substituting the values of temperature T and T* from (VII.110) into boundary conditions (VII.12) and (VII.23) assuming $f_1 = f_2 = 0$, we obtain

$$\varphi_1(z) = \frac{\alpha_t E}{2} a_0 z, \qquad \psi_1(z) = \alpha_t E b_0 \frac{1}{z},$$

$$\varphi(z) = \frac{\alpha_0}{2(1+\nu)} az, \qquad \psi(z) = \frac{\alpha_0}{1-\nu} b \frac{1}{z},$$

where

$$a_0 = \frac{2\alpha_t(1-\nu^2)}{h\beta_2}, \qquad \Delta_0 = R_2^2 - R_1^2,$$

$$a_0 = \frac{R_2}{\beta_1 \Delta_0}\left\{A_1 I_1(\beta_1 R_2) - B_1 K_1(\beta_1 R_2) - \frac{R_1}{R_2}[A_1 I_1(\beta_1 R_1) - B_1 K_1(\beta_1 R_1)]\right\} + \frac{1}{2} t_c,$$

$$b_0 = \frac{R_1 R_2^2}{\beta_1 \Delta_0}\left\{A_1 I_1(\beta_1 R_1) - B_1 K_1(\beta_1 R_1) - \frac{R_1}{R_2}[A_1 I_1(\beta_1 R_2) - B_1 K_1(\beta_1 R_2)]\right\},$$

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$$a = \frac{R_2}{\Delta}\left\{\frac{R_1}{R_2}[A_2 I_1(\beta_2 R_1) - B_2 K_1(\beta_2 R_1)] - A_2 I_1(\beta_2 R_2) + B_2 K_1(\beta_2 R_2)\right\} - \frac{3\beta_1^2}{2\beta_2} t_m^* \qquad \text{(VII.113)}$$

$$b = \frac{R_1 R_2^2}{\Delta_0}\left\{A_2 I_1(\beta_2 R_1) - B_2 K_1(\beta_2 R_1) - \frac{R_1}{R_2}[A_2 I_1(\beta_2 R_2) - B_2 K_1(\beta_2 R_2)]\right\}.$$

The stress and moment are /546

$$\sigma_{\theta\theta}^* = \alpha_t E\left(a_0 + b_0 \frac{1}{r^2}\right) - \sigma_{\theta\theta}^0,$$
$$M_{\theta\theta} = -\alpha_0 D\left(a_0 + b\frac{1}{r^2}\right) - M_{\theta\theta}^0, \qquad \text{(VII.114)}$$

where

$$\sigma_{\theta\theta}^0 = \alpha_t E\left\{A_1\left[I_0(\beta_1 r) - \frac{1}{\beta_1 r} I_1(\beta_1 r)\right] + B_1\left[K_0(\beta_1 r) + \frac{1}{\beta_1 r} K_1(\beta_1 r)\right] + \frac{1}{2} t_m\right\},$$
$$M_{\theta\theta}^0 = \alpha_0 \beta_2 D\left\{A_2\left\{I_0(\beta_2 r) - \frac{1}{\beta_2 r} I_1(\beta_2 r)\right] + B_2\left[K_0(\beta_2 r) + \frac{1}{\beta_2 r} K_1(\beta_2 r)\right] + \frac{e}{2\varepsilon_0} t_m^*\right\}. \qquad \text{(VII.115)}$$

If the edges of the plate are thermally insulated ($k_1 = k_2 = 0$), then from (VII.114) and (VII.115) it follows that the temperature stresses and moments are equal to zero. The plate, under the effect of temperature field (VII.112), is deflected without stresses into a spherical surface.

If R_2 in formulas (VII.111) and (VII.113) approaches infinity, we obtain the stress state in an infinite plate with a round hole of radius R_1. In this case $A_1 = A_2 = 0$

$$B_1 = \frac{k_1(T_m - t_m)}{K(\beta_1 R_1)}, \qquad a_0 = \frac{1}{2} t_m \qquad b_0 = -\frac{R_1 B_1}{\beta_1} K_1(\beta_1 R_1); \qquad \text{(VII.116)}$$

$$B_2 = \frac{k_1\left(T_m^* - \frac{e}{\varepsilon_m} t_m^*\right)}{K(\beta_2 R_1)}; \qquad a = -\frac{3}{2}\frac{\beta_1^2}{\beta_2} t_m^*; \qquad b = -R_1 B_2 K_1(\beta_2 R_1);$$

$$\sigma_{\theta\theta} = \frac{\alpha_t E B_1}{\beta_1 r}\left[\frac{R_1}{r} K_1(\beta_1 R_1) - K_1(\beta_1 r) - \beta_1 r K_0(\beta_1 r)\right],$$
$$M_{\theta\theta} = \frac{\alpha_0}{r} D B_2\left[\frac{R_1}{r} K_1(\beta_2 R_1) - K_1(\beta_2 r) - \beta_2 r K_0(\beta_2 r)\right]. \qquad \text{(VII.117)}$$

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There are no stresses in a dense plate.

Let the outer edge of the plate be pinched, and let the inner edge be free of stresses. Then from boundary conditions (VII.12), (VII.13), (VII.23) and (VII.24), with consideration of (VII.3), we obtain

$$\varphi_1(z) = \frac{\alpha_t E(1+\nu)}{2(1-\nu)} a_0 z, \qquad \psi_1(z) = \alpha_t E b_0 \frac{1}{z},$$
$$\varphi(z) = \frac{\alpha_0}{1-\nu} a_1 z, \qquad \psi(z) = \frac{\alpha_0}{1-\nu} b_1 \frac{1}{z}, \tag{VII.118}$$

where /547

$$a_0 = \frac{R_2}{\beta_1 \Delta_R}\left\{\frac{R_1}{R_2}[A_1 I_1(\beta_1 R_2) - B_1 K_1(\beta_1 R_1)] - A_1 I_1(\beta_1 R_2) + B_2 K_1(\beta_1 R_2)\right\} - \frac{1}{2} t_m$$

$$b_0 = \frac{R_1 R_2^2}{\beta_1 \Delta_R}\left\{\frac{1+\nu}{1-\nu}\cdot\frac{R_1}{R_2}[A_1 I_1(\beta_1 R_2) - B_1 K_1(\beta_1 R_2)] + A_1 I_1(\beta_1 R_1) - \right.$$
$$\left. - B_1 K_1(\beta_1 R_1) + \frac{\beta_1 R_1}{1-\nu} t_m\right\},$$

$$a_1 = \frac{R_2}{2\Delta_R}\left\{A_2 I_1(\beta_2 R_2) - B_2 K_1(\beta_2 R_2) - \frac{R_1}{R_2}[A_2 I_1(\beta_2 R_1) - B_2 K_1(\beta_2 R_1)] + \right.$$
$$\left. + \frac{3}{2}\cdot\frac{\beta_1^2 \Delta_0}{\beta_2} t_m^*\right\}, \tag{VII.119}$$

$$b_1 = \frac{R_1 R_2^2}{\Delta_R}\left\{\frac{1+\nu}{1-\nu}\cdot\frac{R_1}{R_2}[A_2 I_1(\beta_2 R_2) - B_2 K_1(\beta_2 R_2)] + A_2 I_1(\beta_2 R_1) - \right.$$
$$\left. - B_2 K_1(\beta_2 R_1) + \frac{3R_1\beta_1^2}{\beta_2(1-\nu)} t_m^*\right\}$$

$$\Delta_R = R_2^2 + \frac{1+\nu}{1-\nu} R_1^2.$$

The stress and deflecting moment are

$$\sigma_{\theta\theta}^* = \alpha_t E\left(\frac{1+\nu}{1-\nu} a_0 - \frac{b_0}{r^2}\right) - \sigma_{\theta\theta}^0,$$
$$M_{\theta\theta} = -\alpha_0 D\left(2\frac{1+\nu}{1-\nu} a_1 + \frac{b_1}{r_2}\right) - M_{\theta\theta}^0, \tag{VII.120}$$

where $\sigma_{\theta\theta}^0$, $M_{\theta\theta}^0$ are determined by formulas (VII.115).

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The stress state in a plate that is pinched at infinity and weakened by a round hole is

$$\sigma_{\theta\theta} = \alpha_t E\left\{B_1\left[K_0(\beta_1 r) + \frac{R_1}{\beta_1 r^2}K_1(\beta_1 R_1) - \frac{1}{\beta_1 r}K_1(\beta_1 r)\right] - \frac{t_m}{1-\nu}\left(1 + \frac{R_1^2}{r^2}\right)\right\},$$

$$M_{\theta\theta} = \alpha_0 D\left\{B_2\left[\frac{R_1}{\beta_2 r^2}K_1(\beta_2 R_1) - K_0(\beta_2 r) - \frac{1}{\beta_2 r}K_1(\beta_2 r)\right] - \frac{3\beta_1^2 t_m^*}{\beta_2(1-\nu)}\left(1 + \frac{R_1^2}{r^2}\right)\right\}, \quad \text{(VII.121)}$$

where B_1 and B_2 are defined by formulas (VII.116).

If the contour of a round hole is thermally insulated ($k_1 = 0$), then /548
$B_1 = B_2 = 0$ and from (VII.121) we find

$$\sigma_{\theta\theta}^* = -\alpha_t E\frac{t_m}{1-\nu}\left(1 + \frac{R_1^2}{r^2}\right),$$

$$M_{\theta\theta} = -\frac{\alpha_t e(1+\nu)}{he_0}Dt_m^*\left(1 + \frac{R_1^2}{r^2}\right). \quad \text{(VII.122)}$$

It is clear from (VII.122) that on the contour of the plate $r = R_1$, pinched at infinity, the values $\sigma_{\theta\theta}^*$ and $M_{\theta\theta}$ are twice as great as in a continuous plate.

§6. Temperature Stresses in Thin Plate Weakened by a Biperiodic System of Identical Round Holes

This problem was examined by L. A. Fil'shtinskiy [1] under the following conditions:

1) on the contours of round holes free of external forces, a constant (identical for all holes) temperature T_0 is given;

2) the centers of round holes of radius R form a biperiodic system with basic periods $\omega_1 = 2$; $\omega_2 = 2le^{i\alpha}$; $l > 0$; $\mathrm{Im}\omega_2 > 0$, and the congruent system of /549
points (centers of round holes) $D = m\omega_1 + n\omega_2$ ($m, n = 0; \pm 1; \pm 2; \ldots$) is symmetrical with respect to the coordinate axis Oxy, the origin of which is placed at one of the centers L_{00} of the holes (Figure VII.4);

3) the surface of the plate is in contact with a constant heat flow of intensity q.

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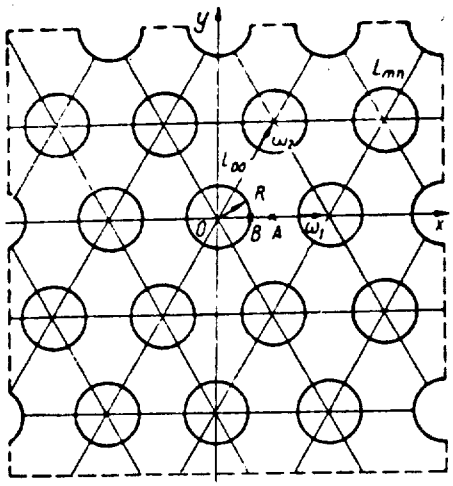


Figure VII.4.

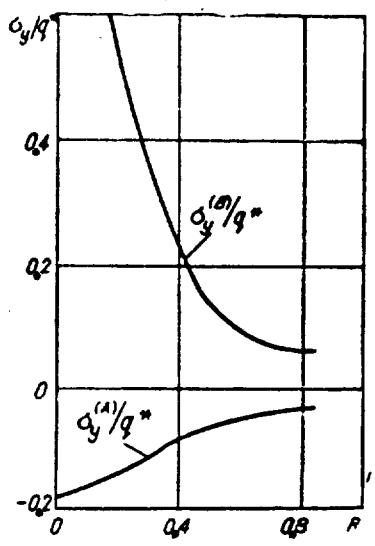


Figure VII.5.

The range under examination, in which we seek the distribution of temperature stresses, represents the exterior of equal circles L_{mn}, the centers of which are located at the points $\vec{P} = m\vec{\omega}_1 + n\vec{\omega}_2$ (see Figure VII.4) (m, n = 0; ±1, ±2;...).

Using the elliptical Weierstrass functions and derivatives, and also certain special functions, we can construct the solution for a broad class of biperiodic problems of the plane theories of elasticity, thermoelasticity, thermoconductivity and deflection.

Figure VII.5 illustrates the curves[1] σ_y/q^* at the points A and B (see Figure VII.4) for a right triangular lattice ($\omega_1 = 2$; $\omega_2 = 2e^{i\pi/3}$) where $q^* = E\alpha q/h\lambda^*$; λ^* is the coefficient of thermoconductivity of the material from which the plate is made; h is the thickness of the plate; q is the intensity of the heat flow; $\alpha = \pi/3$; E is Young's modulus of the material of the plate.

[1]See L. A. Fil'shtinskiy [1].

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CHAPTER VIII. STRESS DISTRIBUTION NEAR CRACKS. LIMIT LOADS DURING BRITTLE RUPTURE OF MATERIAL

Abstract. In this chapter are given the principal statements of the theory of equilibrium crack in the case of a simplified brittle body model. A large class of problems is considered for the determination of the critical load. The critical loads are determined for one and two colinear rectilinear cracks emerging at the circular hole contour due to the prescribed loads on infinity. The diagram plot of the ultimate stresses for a body subjected to the plane stress state is presented. A comparison of the diagrams with the experimental results is given.

§1. Introductory Comments

The results of analyses of stress concentration near an elliptical hole, /551
presented in §2, Chapter II, show that the greatest stresses σ_ϑ on the contour of the hole depend greatly on the curvature of the contour at the particular point. In the case of uniaxial tension along the Ox axis, these stresses achieve their maximum values at the points of intersection of the contour of the hole with the 2b axis of the ellipse, i.e., when $\vartheta = \pm\pi/2$ and are determined by the formula

$$\sigma_\vartheta = kp, \tag{VIII.1}$$

where the stress concentration coefficient is

$$k = 1 + 2\frac{b}{a} = 1 + 2\sqrt{\frac{b}{\rho}}; \tag{VIII.2}$$

p is the intensity of tension stresses at infinity, directed perpendicular to the 2b axis of the elliptical hole; ρ is the radius of curvature of the contour of the hole at points $\vartheta = \pm\pi/2$.

If the radius of curvature of the hole at this point is not too small and the material of the elastic plane is sufficiently brittle, then the limit value of the load is $p = p_*$, i.e., the value of the external forces of tension[1] at which the body will rupture is determined from the following conditions:

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[1] In this chapter the principal stresses N_1 and N_2 applied at infinitely remote points of the elastic plate are denoted through p and q, such that $N_1 \equiv p$, $N_2 \equiv q$.

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$$\sigma^{(\max)} = \sigma_{\text{rup}} \quad \text{or} \quad p_* = \frac{\sigma_{\text{rup}}}{k}, \tag{VIII.3}$$

where σ_{rup} is the resistance of the material to rupture.

If the radius of curvature ρ is infinitely decreased ($\rho \to 0$), then at the limit we obtain a plate that is weakened by a hole in the form a rectilinear slit with sharp ends. Such a sharp-ended slit in an elastic brittle body, when the maximum distance 2a between its opposite edges (walls) is small in comparison with the distances between its ends (between the ends of the hole), will represent a real crack.

It follows from formula (VIII.2) that concentration coefficient k found from the solution of the corresponding problem of classical elasticity theory increases when the radius of curvature ρ decreases, whereupon $k \to \infty$, when $\rho \to 0$. Hence, and from (VIII.3), we arrive at the conclusion that a plate (Figure VIII.1), weakened by a crack with a sharp notch, should rupture immediately if subjected to tension by forces $p > 0$, however small. In reality, however, this does not occur. A simple test (see §9) indicates that /552 a plate a sharp-ended crack will rupture under a finite (and, generally speaking, not small) value of tension forces p (Figure VIII.1). In other words: a plate made of a brittle material, weakened by a rectilinear isolated crack, is capable of sustaining a completely defined finite load of tension, the value of which depends on the dimension (length) of the crack and the physico-mechanical properties of the material.

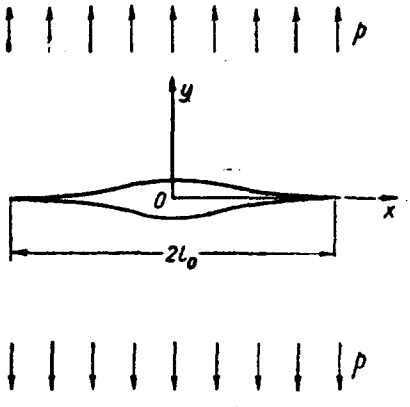


Figure VIII.1.

However, the solution of (VIII.1) and (VIII.2) obtained in classical elasticity theory, and which gives infinite large values of tension in the vicinity of great stress concentrations, is obviously not valid in the physical sense. This disagreement between theory and experiment, to which classical elasticity theory leads, can probably be attributed to the fact that the assumptions upon which the basic relations and equations of classical elasticity theory are based, are not applicable to the given class of problems. This discrepancy between theory and test can obviously be attributed to two factors: first, the equations of classical elasticity theory[1] which were derived in the assumption of negligible smallness (incomparison with unity) of elongations, displacements, and angles of rotation, and also that the products of the angles of rotation can be disregarded in comparison with elongations and displacements

[1]See V. V. Novozhilov [1], p. 179.

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cannot be applied to the given class of problems of stress concentration near cracks and near acute angles of holes, in general, and second, Hooke's law is assumed to be valid for all deformations. As shown in §8, Chapter V, slip zones, from which micro and macrocracks may originate under the appropriate conditions, occur near acute stress concentrations under the effect of comparatively small values of tension forces p.

The pattern of deformation of the material near these cracks will be different from that of the stress-deformation state predicted by classical elasticity theory, based on the continuous model of an ideally elastic body. Therefore, to determine the magnitude of the limit load of a brittle body with sharp-ended stress concentrators (cracks) it is necessary as the first step[1] to retain the method of classical elasticity theory, but to supplement the model /553
of classical theory by certain new properties, which will make it possible to describe more completely (in the physical sense) the stress-deformation state in the vicinity of such stress concentrators and to establish a better criterion for the evaluation of the strength of a solid body.

The analysis of equilibrium of a deformed solid body, weakened by defects in the form of sharply pointed cavities or cracks, also is the object of mathematical theory of equilibrium cracks. This theory (after the known works of A. A. Griffith) was developed only in recent years in the works of the Soviet researchers A. S. Khristianovich, G. I. Barenblatt, M. Ya. Leonov, V. V. Panasyuk, among others, and also in the works of certain foreign researchers (G. R. Irwin, E. O. Orowan and others).

An outline of the development of research on mathematical theory of equilibrium cracks, a bibliography of works in this area, and the formulation of certain general positions of the theory, are contained in the articles[2] of G. I. Barenblatt [1, 2], G. R. Irwin [1], J. I. Bluhm [1]; here we will present the solutions of only a few two-dimensional problems of this theory.

§2. Griffith's Problem

We will examine an infinite elastic plate of unit thickness, weakened by a rectilinear isolated crack of length $2l$ (see Figure VIII.1). We will assume that the material of which the plate is made is ideally brittle, i.e., retains the property of linear elasticity all the way to rupture and is capable of sustaining finite forces of tension. Let such a plate be subjected to tension by monotonically increasing stresses $\sigma_y^\infty = p$, applied at infinitely distant

---

[1]Generally speaking, in analyzing the stress state in the zone of concentration of stresses near acute angles in holes or at the ends of cracks, it is necessary to use the basic equations of nonlinear elasticity theory. This greatly complicates the problem, since it results in the solution of a system of nonlinear equations under rather complex boundary conditions.

[2]See also N. I. Muskhelishvili [2], §166.

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points of the plate and directed perpendicular to the plane of the crack. We will determine the magnitude of load $p = p_*$ at which the crack begins to increase in length, and under which the plate is ruptured[1].

As was mentioned earlier, the solution of this problem is impossible within the frameworks of classical elasticity theory. In order to solve this problem it is necessary to introduce into the analysis, the forces of the weakened interparticle coupling in this part of the deformed body, where the magnitude of deformations exceeds the limit of elasticity. In this case, such regions will be the vicinities of the ends of a real crack (sharply pointed stress concentrators). /554

The layers of a solid body, where the material is deformed beyond the limit of elasticity, can be separated from the body, forming certain microscopic slits (cracks), to the surfaces of which are applied forces corresponding to the action of the material removed. Consequently, the problem of the stress-deformation state of a solid body, where the body contains initial cracks as well as layers of the material, deformed beyond the limit of elasticity, reduces to the problem of the stress-deformation state in an elastic body weakened by the original cracks (slits) and by microscopic cracks, the surfaces of which attract each other due to certain forces. If the forces of interaction between the edges of the microcracks are defined, the problem reduces to some mixed problem of elasticity theory.

The determination of the forces of interaction between the edges of microcracks in the general case represents an extremely difficult problem. However, the forces of attraction $q(\lambda)$ between the edges of a microcrack can be represented approximately[2], for brittle materials, in the following[3] form (Figure VIII.2):

$$q(\lambda) = \begin{cases} \sigma_0 \text{ for } \lambda_m < \lambda < \delta_k, \\ 0 \text{ for } \lambda > \delta_k, \end{cases} \qquad \text{(VIII.4)}$$

where λ is the distance between the edges of the microcrack; σ_0 is the limit of brittle strength (resistance to rupture); λ_m is the limit value of the parameter

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[1]The solution of this problem was first found by A. A. Griffith [1, 2] with the help of the so-called energy method, which is based on the balance of work accomplished by external forces in increasing the length of the crack, and of the work expended thereby on the formation of new surfaces of the body. The solution given in this section is found in the works of V. V. Panasyuk [1, 2], M. Ya. Leonov and V. V. Panasyuk [1], P. M. Vitvitskiy and M. Ya. Leonov [1].

[2]It will be shown below that such an approximate representation of the function $q(\lambda)$ is sufficient for the solution of problems of cracks.

[3]See M. Ya. Leonov and V. V. Panasyuk [2], V. V. Panasyuk [1], P. M. Vitvitskiy, M. Ya. Leonov [1].

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λ at which the forces of interaction no longer obey Hooke's law; δ_k is the limit value of the parameter λ at which there is still interaction between the edges of the microcracks. The value of δ_k is defined such that the values of the effective energy surfaces of the brittle body and elastic model coincide, i.e., the value of δ_k from the equation (see Figure VIII.2)

$$\sigma_0 \delta_k = 2\gamma. \qquad \text{(VIII.5)}$$

Here γ is the density of the effective energy surface of the brittle or quasi-brittle material.

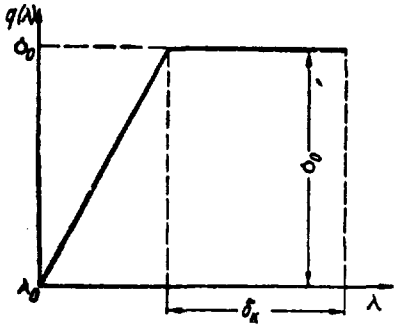


Figure VIII.2.

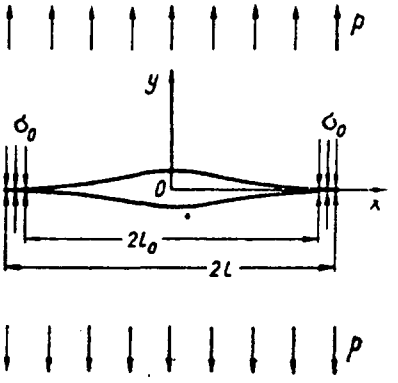


Figure VIII.3.

If the forces of interaction $q(\lambda)$ between the edges of the microcracks in a brittle material are determined by equation (VIII.4), then it is easy to introduce the following calculation model for the determination of equilibrium of a brittle body with cracks. The brittle body is represented as an elastic continuum, for which the following conditions are satisfied:

a) the normal tension stresses acting in it do not exceed the values of σ_0, i.e.,

$$\sigma^{(\max)} < \sigma_0; \qquad \text{(VIII.6)}$$

b) the relationship between stresses and deformations obeys Hooke's law if condition (VIII.6) is satisfied; /555

c) if the elastic body contains ranges in which the stress-deformation state that satisfies the conditions of linear elasticity theory and condition (VIII.6) (i.e., there are layers of the body in which the material is deformed beyond the limit of elasticity) is impossible, then such ranges of the body are regarded as microcracks (slits in an elastic body);

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d) the opposite sides of the microcracks are attracted to each other by stresses σ_0 if the distance between them does no exceed δ_k, and do not interact with each other in the opposite case; the distance between the sides of the cracks during deformation of the body is defined as a sum of elastic displacements of the opposing points of the sides of a slit in the elastic model.

For the examined problem the layers of the body in which the material is deformed beyond the limit of elasticity under stresses $p > 0$, however small, are located in the vicinity of the ends of the crack. Therefore, in the frameworks of the formulated model, the problem of the distribution of the initial crack of length $2l_0$ in a brittle plate (Figure VIII.3), where the plate is under tension by monotonically increasing stresses $\sigma_y^\infty = p$, reduces to the following plane problem of mathematical elasticity theory. In an infinite elastic plate there is a slit of length $2l$ ($-l \leqslant x \leqslant l$). The following stresses act on the surfaces of this slit:

$$\sigma_y(x,0) = \begin{cases} 0 & \text{for } -l_0 < x < l_0, \\ \sigma_0 & \text{for } l_0 < |x| < l; \end{cases} \qquad \text{(VIII.7)}$$

$$\tau_{xy}(x,0) = 0 \quad \text{for } -\infty < x < \infty,$$

and at infinitely distant points of the plate: /556

$$\sigma_y^\infty = p \quad \text{and} \quad \tau_{xy}^\infty = 0, \qquad \text{(VIII.8)}$$

where l is the size of the abscissas of the points that bound the range of the microcrack and the range of the elastically deformed material of the plate. The magnitude of l is not known beforehand; it must be determined during the solution of the problem. Outside of the slit $(-l, l)$, the material of the plate is deformed elastically and within this range of the body, classical elasticity theory is applicable.

If from the stress state in an elastic plate with slit $2l$ (Figure VIII.3) defined by boundary conditions (VIII.7) and (VIII.8), we calculate the homogeneous stress state $\sigma_y = p$, we obtain some complementary (auxiliary) stress state in this plate, which vanishes at infinity, and on the surface of the slit is defined by the following boundary conditions:

$$p_n(x) = \begin{cases} p & \text{for } -l_0 < x < l_0, \\ p - \sigma_0 & \text{for } l_0 < |x| < l; \end{cases} \qquad \text{(VIII.9)}$$

$$\tau_{xy}(x,0) = 0 \text{ for } -\infty < x < \infty,$$

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where $p_n(x) = -\sigma_y(x, 0)$ is normal pressure on the surfaces of the slit for the auxiliary problem.

We will determine for the auxiliary problem the value of vertical displacements of the sides of the slit, i.e., we will find[1] $v(x, 0)$ for $-l \leqslant x \leqslant l$ and $\sigma_y^{(1)}(x, 0)$ when $|x| \geqslant l$:

$$v(x,0) = c\int_{-l}^{l} p_n(\xi)\,\Gamma(l,x,\xi)\,d\xi \qquad (-l < x < l); \tag{VIII.10}$$

$$\sigma_y^{(1)}(x,0) = \frac{1}{\pi\sqrt{x^2-l^2}}\int_{-l}^{l}\frac{p_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\,d\xi \quad (x > l). \tag{VIII.11}$$

Here c is a constant which, for the case of plane deformation, is equal to $(1 - \nu^3)/(\pi E)$, and for the case of the generalized plane stress state $1/\pi E$ (E is Young's modulus, ν is Poisson's ratio) the function is

$$\Gamma(l,x,\xi) = \ln\frac{l^2 - x\xi + \sqrt{(l^2-x^2)(l^2-\xi^2)}}{l^2 - x\xi - \sqrt{(l^2-x^2)(l^2-\xi^2)}} \tag{VIII.12}$$

It is easy to see that displacements of the sides of the slit for the original (see Figure VIII.3) and auxiliary problem coincide, and stresses $\sigma_y(x, 0)$ for $|x| > l$ differ for the original problem from stresses $\sigma_y^{(1)}(x, 0)$ when $|x| > l$ by the magnitude p, i.e.,

$$\sigma_y(x,0) = \sigma_y^{(1)}(x,0) + p \quad \text{for } x > l. \tag{VIII.13}$$

Stresses $\sigma_y(x, 0)$ for $x \geqslant l$ (in the range where the material is deformed elastically) cannot exceed the limit of brittle strength σ_0, i.e., they should be limited for all $x \geqslant l$, as required by condition (VIII.6). From formulas (VIII.12) and (VIII.13), however, we see that condition (VIII.6) is violated when $x \to l$. Consequently, the parameter l should be defined such that condition (VIII.6) will be satisfied even when $x = l$. In accordance with formulas (VIII.12) and (VIII.13), the necessary (and sufficient) condition for this is the equation

/557

$$\lim_{x\to l+0}\int_{-l}^{l}\frac{p_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\,d\xi = 0, \tag{VIII.14}$$

[1] See M. Ya. Leonov and V. V. Panasyuk [2].

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which also serves as a condition for the determination of the (previously unknown) parameter l.

By substituting into equation (VIII.14) the value $p_n(\xi)$ in accordance with formula (VIII.9) and by completing the required calculations, we obtain

$$l = l_0 \sec \frac{\pi p}{2\sigma_0}. \qquad \text{(VIII.15)}$$

With such a value of l, on the basis of formulas (VIII.9), (VIII.11) and (VIII.13), stresses $\sigma_y(x, 0)$ will satisfy condition (VIII.6) for all $x \geqslant l$ whereupon, on the boundary of the microcrack and elastic range ($x = l$), continuity of stresses $\sigma_y(l, 0)$ is observed, i.e.,

$$\sigma_y(l + 0,0) = \sigma_y(l - 0,0) = \sigma_0.$$

Further, by using formulas (VIII.9), (VIII.10) and (VIII.15), we obtain

$$v(x,0) = c\sigma_0 \{(x - l_0)\Gamma(l,x,l_0) - (x + l_0)\Gamma(l,x,-l_0)\}, \qquad \text{(VIII.16)}$$

where l is expressed by formula (VIII.15) and $\Gamma(l, x, \pm l_0)$ is determined in accordance with equation (VIII.12) when $\xi = \pm l_0$.

By differentiating (VIII.16) with respect to x and assuming $x \to l$, we obtain

$$\lim_{x \to l-0} v'(x,0) = 0. \qquad \text{(VIII.17)}$$

Equation (VIII.17) shows that during the process of the opening of the crack (under the effect of external force), the opposite walls of the crack, in the vicinity of its ends, are smoothly interlocked (with zero angle of aperture). This result is also obtained in the general case of load $p_n(x)$, if we differentiate expression (VIII.10) with respect to x and find the value $v'(x, 0)$ when $x \to l$, and when condition (VIII.14) is satisfied.

Assuming in (VIII.16) $x = \pm l_0$, it is easy to find the distance

$$2v(\pm l_0,0) = -8cl_0\sigma_0 \ln \cos \frac{\pi p}{2\sigma_0}, \qquad \text{(VIII.18)}$$

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to which, during the process of deformation of the plate, the opposite sides of the crack separate in the vicinity of its ends, i.e., in the vicinity of the points $(\pm l_0, +0)$ and $(\pm l_0, -0)$.

If stresses $p = p_*$ are such that the distance is /558

$$2v(\pm l_0, 0) = \delta_k, \tag{VIII.19}$$

then, according to the properties of the assumed model in the vicinity of points $(\pm l_0, +0)$, the opposite sides of the microcrack cease to interact with each other and consequently, it becomes possible for the length of the original crack $2l_0$ to increase (see Figure VIII.3). Thus, equation (VIII.19) is the condition of spreading of the initial crack of length $2l_0$, i.e., is a condition of development of the process of rupture of a plate with a crack when the plate is under tension by stresses p.

Stresses $p = p_*$ which cause the spreading of the initial crack, i.e., when condition (VIII.19) is satisfied, are called the limit or critical stresses.

Thus, the magnitude of the limit stresses for the problem at hand, as follows from (VIII.18) and (VIII.19), is

$$p_* = \frac{2}{\pi}\sigma_0 \arccos\exp\left(-\frac{\delta_k}{8cl_0\sigma_0}\right) \tag{VIII.20}$$

It follows from (VIII.20) that when $l_0 \to 0$, the rupturing stresses are $p_* \to \sigma_0$, i.e., the strength of a plate with a "crack of zero length" is equal to the strength of the material containing no defects.

If the length of the original crack $2l_0$ is so great that it is possible to consider $\delta_k/l_0 \ll 1$, then, by retaining in formula (VIII.20) the small values (δ_k/l_0) only of the first order of smallness, we obtain

$$p_* = \frac{2}{\pi}\sigma_0\sqrt{\frac{2\delta_k}{8cl_0\sigma_0}} = \sqrt{\frac{\sigma_0\delta_k}{\pi^2 cl_0}}.$$

Hence, and on the basis of equation (VIII.5), we obtain the known formulas of A. A. Griffith [1, 2], respectively, for plane deformation and generalized plane stress state:

$$p_* = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l_0}} \text{ and } p_* = \sqrt{\frac{2E\gamma}{\pi l_0}}. \tag{VIII.21}$$

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§3. Condition of Limit State for Macroscopic Cracks

Macroscopic cracks will be defined as those cracks for which the following condition[1] is satisfied: the width of the terminal region of the crack, i.e., at the point where the forces of conjugation between the opposite sides of the crack are at work, is small in comparison with the dimensions of the entire crack[2].

For macroscopic cracks it is possible to obtain certain simple relations that will enable us to determine the magnitude of the limit load for a body (weakened by such cracks) during its brittle rupture, by knowing simply the principal part of the elastic forces of tension in the vicinity of the contour of the given crack. /559

Suppose that there is a rectilinear isolated macrocrack of length $2l_0$ in an elastic plate consisting of a brittle material (Figure VIII.4). We will introduce rectilinear Cartesian coordinate system xOy, as shown in Figure VIII.4 and assume that some normal pressure[3] $q_n(x)$ is applied to the edges of the crack under the condition $|x| \leqslant a$, where $a \leqslant l_0$, and that there are no stresses at the infinitely distant points of the plate. Within the frameworks of the model of an ideally brittle body, and analogously as in the preceding section, we will represent the vertical displacements of the sides of the crack in the form

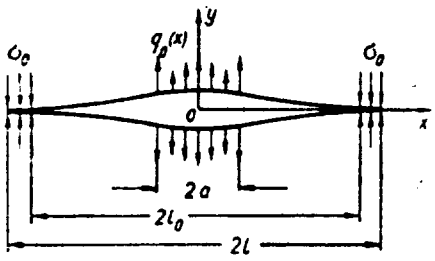


Figure VIII.4.

$$v(x,0) = c\int_{-l}^{l} q_n(\xi)\,\Gamma(l,x,\xi)\,d\xi \qquad (|x| < L), \tag{VIII.22}$$

where l is the abscissa of the boundary between the microcrack and the elastically deformed material;

---

[1] See Yu. P. Zheltkov and S. A. Khristianovich [1], G. I. Barenblatt [1].

[2] It can be shown that if, for instance, the characteristic linear dimension l_0 of the initial crack is greater than 10 μ, then such cracks can be regarded as macroscopic cracks, i.e., they satisfy the formulated condition.

[3] It can be shown, analogously as in 2 (see H. F. Bueckner [1]) that in the general case of tension of a plate at infinity by a system of external forces p = const; q_0 = const, the problem can be reduced to that of normal pressure $q_0(x)$ applied to the edges of the crack.

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$$q_n(\xi) = \begin{cases} q_0(\xi) & \text{for } |\xi| < a, \\ 0 & \text{for } a < |\xi| < l_0, \\ -\sigma_0 & \text{for } l_0 < |x| < l. \end{cases} \qquad \text{(VIII.23)}$$

For the case under consideration

$$\sigma_y(x,0) = \frac{1}{\pi\sqrt{x^2 - l^2}} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2 - \xi^2}}{x - \xi}\, d\xi \qquad (x > l). \qquad \text{(VIII.24)}$$

From condition of boundedness of stresses $\sigma_y(x, 0)$ when $x \to l + 0$ it follows that

$$\lim_{x \to l+0} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2 - \xi^2}}{x - \xi}\, d\xi = 0. \qquad \text{(VIII.25)}$$

By substituting the corresponding values of $q_n(\xi)$ from (VIII.23) into (VIII.25) /560 we obtain

$$\lim_{x \to l+0} \sigma_0 \left\{ \int_{-l}^{-l_0} \frac{\sqrt{l^2 - \xi^2}}{x - \xi}\, d\xi + \int_{l_0}^{l} \frac{\sqrt{l^2 - \xi^2}}{x - \xi}\, d\xi \right\} = \lim_{x \to l+0} \int_{-a}^{a} \frac{q_0(\xi)\sqrt{l^2 - \xi^2}}{x - \xi}\, d\xi. \qquad \text{(VIII.26)}$$

For convenience we will introduce the parameter

$$\varepsilon = \frac{l - l_0}{l_0}.$$

Since the crack under consideration is regarded as macroscopic, the inequality $l - l_0 \ll l_0$ should be satisfied, and therefore $\varepsilon \ll 1$. Considering this, equation (VIII.26), with an accuracy up to small values of the first order of smallness with respect to parameter ε, may be represented as follows:

$$2l_0\sigma_0\sqrt{2\varepsilon} = \int_{-a}^{a} \frac{q_0(\xi)\sqrt{l^2 - \xi^2}}{l - \xi}\, d\xi. \qquad \text{(VIII.27)}$$

Analogous to the preceding, from (VIII.22) we obtain, with the same accuracy with respect to ε,

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$$v(l_0,0) = -4cl_0\sigma_0\varepsilon + 2c\sqrt{2\varepsilon}\int_{-a}^{a}\frac{q_0(\xi)\sqrt{l^2-\xi^2}}{l-\xi}\,d\xi. \qquad \text{(VIII.28)}$$

Here it is necessary to bear in mind that according to the definition of macrocracks ($\varepsilon \ll 1$) we may for such cracks assume $l_0 + l \approx 2l_0 \approx 2l$.

If the parameters that define the load $q_0(x)$ are such that condition (VIII.19) is satisfied, then the parameter ε achieves some limit value ε_*. On the basis of (VIII.19) and also on the basis of equations (VIII.27) and (VIII.28) we find

$$\varepsilon_* = \frac{\delta_k}{8cl_0\sigma_0}. \qquad \text{(VIII.29)}$$

Whence

$$l_* - l_0 = \frac{\delta_k}{8c\sigma_0}, \qquad \text{(VIII.30)}$$

where l_* is the maximal magnitude of parameter l which is achieved as soon as the external load reaches its maximum value.

Since, in the right hand side of equation (VIII.30), values appear that do not depend on the character of the load and initial size of the crack, it follows from this equation that for the given material, under the given conditions, the terminal part of the macrocrack, in the case of its limit equilibrium, is constant.

The limit load $q_0^{\max}(x) = q_*$ for macrocracks should satisfy an equation first found by G. I. Barenblatt [2]:

$$\lim_{s\to 0}\{\sqrt{s}\,\sigma_y(s, q_*)\} = \frac{K}{\pi}, \qquad \text{(VIII.31)}$$

where $\sigma_y(s, q_*)$ are rupturing elastic stresses, calculated on the basis of classical elasticity theory for load q_*; s is a small distance of points of the body located in the plane of the crack, from the contour of the crack; K is the coupling modulus, equal for plane deformation and for the generalized plane stress state, respectively, to /561

$$K_1 = \sqrt{\frac{\pi E\gamma}{1-\nu^2}} \text{ и } K_2 = \sqrt{\pi E\gamma}.$$

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We will examine equation (VIII.26), which, with an accuracy up to small values of order ε (inclusively) can be represented in the form

$$\frac{2l_0\sigma_0\sqrt{2\varepsilon}}{\pi} = \lim_{x\to l_0+0}\{\sigma_y(x,q)\sqrt{x^2-l_0^2}\}, \tag{VIII.32}$$

where q is a parameter that characterizes the external load.

The function is

$$\sigma_y(x,q) = \frac{1}{\pi\sqrt{x^2-l_0^2}}\int_{-a}^{a}\frac{q_0(\xi)\sqrt{l_0^2-\xi^2}}{x-\xi}d\xi \quad (x \geqslant l_0). \tag{VIII.33}$$

Formula (VIII.33) represents exactly the same thing as elastic rupture stresses, calculated for a macroscopic crack of length $2l_0$ (Figure VIII.4), where, to the edges of the crack, are applied external forces $q_0(x)$ for $|x| \leqslant a$.

It is readily seen that the expression

$$\sqrt{x^2-l_0^2} = \sqrt{s}\sqrt{2l_0+s}, \tag{VIII.34}$$

where $s = x - l_0$, represents the distance between the vertex of the contour of the crack and the points of the body lying on the Ox axis (Figure VIII.4) when $x \geqslant l_0$.

Moreover, we will notice that relation (VIII.32) is valid for any load q(x) that does not exceed its limit value. If, however, the parameters that characterize the external load are such that the external achieves the magnitude of the limit value (in the given case $q = q_*$), then the parameter ε is determined by formula (VIII.29). Considering this and (VIII.32) we obtain for the determination of the magnitude of the limit load, the formula

$$\lim_{s\to 0}\{\sqrt{s}\,\sigma_y(s,q_*)\} = \frac{2\sigma_0}{\pi}\sqrt{l_0\varepsilon_*}. \tag{VIII.35}$$

The right hand side of formula (VIII.35) will be transformed, with the aid of formulas (VIII.5) and (VIII.29), to the form

$$\frac{2\sigma_0}{\pi}\sqrt{l_0\varepsilon_*} = \frac{1}{\pi}\sqrt{\frac{\sigma_0\delta_k}{2c}} = \frac{K}{\pi}. \tag{VIII.36}$$

We see from (VIII.36) that relations (VIII.35) and (VIII.31) coincide, as we were required to prove.

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In the following discussion we will use, for the determination of the magnitude of the limit load for a brittle body weakened by macroscopic cracks, relation (VIII.31), which was first established by G. I. Barenblatt [2] as an outcome of other considerations. /562

§4. Tension of Elastic Plate Weakened by Two Colinear Cracks[1]

Two Unlike Cracks. We will examine an infinite elastic plane with two microscopic cracks of different length, arranged on one straight line, which we will call the x axis (Figure VIII.5). Let monotonically increasing stresses $\sigma_y^\infty = p$, directed perpendicular to the line of the cracks, be applied at infinitely distant points of the plate. We will determine the magnitude of stresses

$$p_* = \min p, \tag{VIII.37}$$

due to which the cracks will spread.

Through a, b, -c, -d we will denote the abscissas of the points of the ends of a crack, as illustrated in Figure VIII.5, and through $p_*^{(a)}$, $p_*^{(b)}$, $p_*^{(c)}$, $p_*^{(d)}$, the values of stresses p at which the cracks will begin to spread in the direction of abscissas a, b, -c, -d, respectively.

The stresses in the examined plate[2] are

$$\sigma_y(x,0) = \frac{p(x^2 + c_1 x + c_2)}{\sqrt{(x-a)(x-b)(x+c)(x+d)}}, \tag{VIII.38}$$

where x acquires values corresponding to the points outside of the cracks; the coefficients c_1 and c_2 are determined from the conditions of uniqueness of displacements:

$$c_1 = \frac{(d-b)F(k) - 2d\Pi(n,k) + 2b\Pi(m,k) + (d+b)[J_2(n,k) - J_2(m,k)]}{F(k) - \Pi(n,k) - \Pi(m,k)}, \tag{VIII.39}$$

$$c_2 = c_1\left[d - (b+d)\frac{\Pi(n,k)}{F(k)}\right] + \left[-d^2 + 2d(d+b)\frac{\Pi(n,k)}{F(k)} - (b+d)^2\frac{J_2(n,k)}{F(k)}\right],$$

[1]The solution of the problem of the limit load for a plate with two colinear cracks of unequal length, with its refinement to calculation formulas, is given in the work of V. V. Panasyuk, B. L. Lozovyy [5]. The general approach to the solution of such a problem for the case of an arbitrary number of cracks is given in the work of G. I. Barenblatt, G. P. Cherepanov [1].

[2]See N. I. Muskhelishvili [1].

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where F_k, $\Pi(n, k)$, $\Pi(m, k)$ are total elliptical integrals of kinds I and III with the modulus k and parameters n and m; the integrals are /563

$$J_2(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1+n\sin^2\varphi)^2\sqrt{1-k^2\sin^2\varphi}},$$
$$J_2(m,k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi_1}{(1+m\sin^2\varphi_1)^2\sqrt{1-k^2\sin^2\varphi_1}}. \qquad \text{(VIII.40)}$$

The modulus k and parameters n and m are expressed through the values of the abscissas of the ends of a crack by the equalities

$$k^2 = \frac{(b-a)(d-c)}{(b+c)(d+a)}; \qquad n = \frac{b-a}{d+a}; \qquad m = \frac{d-c}{b+c}. \qquad \text{(VIII.41)}$$

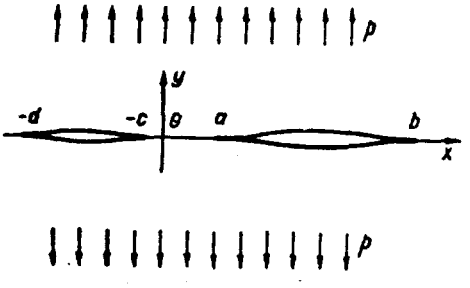


Figure VIII.5.

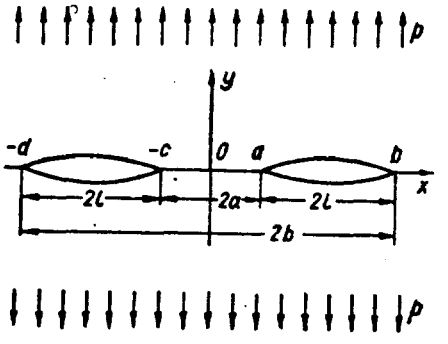


Figure VIII.6.

By substituting (VIII.38) into equation (VIII.31) and accomplishing limit transition for $s \to 0$ for each of the ends of a crack, we obtain the formulas for $p_*^{(a)}$, $p_*^{(b)}$, $p_*^{(c)}$ and $p_*^{(d)}$. Thus, for instance, for the ends (a, 0) and (b, 0) of a crack, $s = a - x$ for $x \to a - 0$ and $s = x - b$ for $x \to b + 0$, resepctively. On the basis of these equations and formula (VIII.38) and also equation (VIII.31), we obtain

$$p_*^{(a)} = \frac{-K}{\pi}\frac{\sqrt{(b-a)(a+c)(a+d)}}{a^2+c_1a+c_2},$$
$$p_*^{(b)} = \frac{K}{\pi}\frac{\sqrt{(b-a)(b+c)(b+d)}}{b^2+c_1b+c_2}, \qquad \text{(VIII.42)}$$

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where the coefficients c_1 and c_2 are determined by formulas (VIII.39)-(VIII.41). The analogous formulas can also be found for $p_*^{(c)}$ and $p_*^{(d)}$

Two Equal Cracks (Figure VIII.6). If, in formulas (VIII.42), we assume c = a and d = b, we obtain the formulas for the determination of the limit load in the case of plate weakened by two colinear cracks of equal length. In such a case (c = a and d = b), from formulas (VIII.42) we obtain /564

$$p_*^{(a)} = \frac{F(k)\sqrt{a(b^2-a^2)}}{(b^2-a^2)F(k) - 4b^2[\Pi(k,k) - J_2(k,k)]} \cdot \frac{\sqrt{2K}}{\pi},$$
$$p_*^{(b)} = \frac{F(k)\sqrt{b(b^2-a^2)}}{4b^2[\Pi(k,k) - J_2(k,k)]} \cdot \frac{\sqrt{2K}}{\pi}. \qquad \text{(VIII.43)}$$

If, further, we represent the integrals $J_2(k, k)$ in formulas (VIII.43) through a combination of integrals $\Pi(k, k)$, $E(k)$, $F(k)$ and then transform the elliptical integrals into a new modulus

$$e = \frac{\sqrt{b^2-a^2}}{b}; \qquad k = \frac{e^2}{(1+\sqrt{1-e^2})^2} = \frac{1-e_1}{1-e_1} \quad (e_1^2 = 1-e^2), \qquad \text{(VIII.44)}$$

as shown in the work of V. V. Panasyuk, B. L. Lozovyy [1], we then obtain[1]

$$p_*^{(a)} = \frac{be\sqrt{a}F(e)}{b^2E(e) - a^2F(e)}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)}},$$
$$p_*^{(b)} = \frac{eF(e)}{[F(e) - E(e)]\sqrt{b}}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)}}, \qquad \text{(VIII.45)}$$

where $F(e)$, $E(e)$ are the total elliptical integrals of kinds I and II with the modulus $e = \sqrt{\frac{b^2-a^2}{b^2}}$.

We will examine certain limit cases that derive from formulas (VIII.45). Thus, assuming that $a \to 0$, we obtain $e \to 1$ and

$$\lim_{e\to 1}\frac{eF(e)}{F(e) - E(e)} = 1, \qquad \lim_{e\to 1}\frac{be\sqrt{a}F(e)}{b^2E(e) - a^2F(e)} = 0.$$

On the basis of the latter for the case a = 0 (e = 1), i.e., for the case of a plate with one crack of length 2b, we find from formulas (VIII.45)

[1]Formulas (VIII.45) were derived in the works of T. J. Willmore [1], V. V. Panasyuk and B. L. Lozovyy [2] on the basis of the energy method.

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$$p_*^{(a)} = 0, \quad p_*^{(b)} = p_*^{(\Gamma)} = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)b}}. \qquad \text{(VIII.46)}$$

Formula (VIII.46) coincides, as should be expected, with the known formula (VIII.21) of Griffith for one isolated crack of length 2b.

We will examine another limit case where colinear cracks of equal length are located at a sufficient distance from each other, i.e., when it can be assumed that $a \to \infty$ and $b \to \infty$ but such that $b - a = 2l = \text{const}$, where $2l$ is the length of each of the above-mentioned cracks. In this case $e \to 0$, and the coefficients to the left of the radicals in (VIII.45) rapidly approach the expression $\sqrt{\frac{2}{b-a}}$ for $e = \sqrt{\frac{b^2-a^2}{b^2}} \to 0$. At the limit we will have /565

$$p_*^{(a)} = p_*^{(b)} = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}, \qquad \text{(VIII.47)}$$

i.e., formula (VIII.46). Hence each such crack will behave as an independent isolated (single) crack of length $2l$.

The graphs of change of conditions $p_*^{(a)}$ (curve 1) and $p_*^{(b)}$ (curve 2), calculated on the basis of formulas (VIII.45), as functions of a/l, are shown in Figure VIII.7. As we see, the values of limit load $p_*^{(a)}$ are always smaller than $p_*^{(b)}$, i.e., the development of two colinear cracks (of equal length) first proceeds in the direction of each other by way of the rupture of the plate; after they have joined, yet another (joined) crack of length 2b comes into being. It should be pointed out here that if the crosspiece 2a between the cracks is sufficiently small in comparison with the length $2l$ of each crack, such that $a/l \leqslant 0.1$, then the rupture of the crosspiece (when external load p achieves the value $p_*^{(a)}$) will not yet entail the rupture of the entire plate. In this case the rupture load for the plate is determined by formula (VIII.46). If, however, $a/l > 0.1$ then the destruction of the crosspiece will involve the rupture of the entire plate.

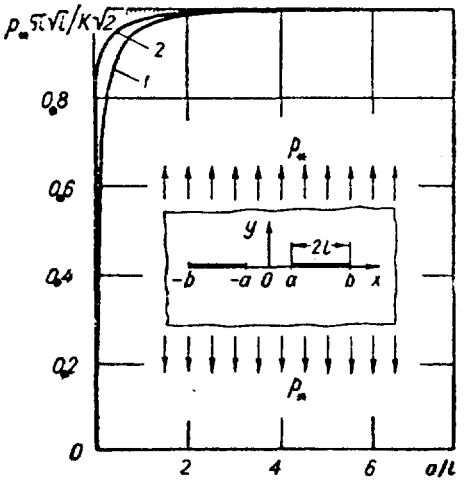


Figure VIII.7.

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Thus, the rupture load ($p = p_{rup}$) for a plate with two colinear cracks of equal length ($2l$), in the case where the plate is under tension by monotonically increasing forces $\sigma_y^\infty = p$, is

$$p_{\text{rup}} = \begin{cases} p_*^{(a)} & \text{for } p_*^{(a)} > p_*^{(\Gamma)}, \\ p_*^{(\Gamma)} & \text{for } p_*^{(a)} < p_*^{(\Gamma)}, \end{cases} \qquad \text{(VIII.48)}$$

where $p_*^{(a)}$ and $p_*^{(\Gamma)}$ are calculated on the basis of formulas (VIII.45) and (VIII.46), respectively.

From the graphs shown in Figure VIII.7, we may also conclude that if the crosspiece between the cracks is such that $a/l > 3$, then the cracks, for all practical purposes, can be regarded as isolated (having no effect on each other) and the rupture load can be found by formula (VIII.47).

Mutual Effect of Small and Large Cracks. Consider the case where the length of one of the cracks (Figure VIII.5) is small in comparison with that of the other. We will analyze, in this case, the effect of the small crack on the magnitude of the limit load for the large crack. For this purpose we will use formulas (VIII.42), assuming that $d - c \ll b - a$. Here, for brevity, we will assume in the following that $c = a$ and $(d - a)/d \ll 1$. In this case the parameter /566

$$m = \frac{d-a}{b+a} = \varepsilon \qquad (\varepsilon \ll 1)$$

is a small value in comparison with one. Hence, and from (VIII.41), we find

$$k^2 = n\varepsilon.$$

The total elliptical integrals of kinds I and III, and also integrals $J_2(n, k)$, $J(m, k)$ in formulas (VIII.43), can be expanded into series by degrees of small parameter ε. By retaining in these expansions only the terms containing ε in the first power, we obtain

$$\begin{aligned} F(k) &= \frac{\pi}{2}\left(1 + \frac{n\varepsilon}{4}\right) + 0(\varepsilon^2), \\ \Pi(n, k) &= \frac{\pi}{2}\frac{1}{\sqrt{1+n}}\left[1 + \frac{\varepsilon}{2}(\sqrt{1+n} - 1)\right] + 0(\varepsilon^2), \\ J_2(n, k) &= \frac{\pi}{2}\cdot\frac{1}{2(1+n)^{3/2}}\left[n + 2 + \frac{n\varepsilon}{2}\right] + 0(\varepsilon^2), \\ \Pi(m, k) &= \frac{\pi}{2}\left[1 + \frac{\varepsilon}{4}(n - 2)\right] + 0(\varepsilon^2), \\ J_2(m, k) &= \frac{\pi}{2}\left[1 + \frac{\varepsilon}{4}(n - 4)\right] + 0(\varepsilon^2). \end{aligned} \qquad \text{(VIII.49)}$$

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Recalling expansions (VIII.49), we may represent formulas (VIII.43) with an accuracy up to magnitudes of the order $0(\varepsilon^2)$ in the following form:

$$p_*^{(a)} \approx \left(1 + \frac{\varepsilon}{2} \cdot \frac{b+d}{a+d}\right) \sqrt{\frac{2a}{d+a}} \sqrt{\frac{4E\gamma}{\pi(1-\nu^2)(b-a)}}; \qquad \text{(VIII.50)}$$

$$p_*^{(b)} \approx \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{b+a}{b+d}} \sqrt{\frac{4E\gamma}{\pi(1-\nu^2)(b-a)}}, \qquad \text{(VIII.51)}$$

where

$$\varepsilon = \frac{d-a}{b+a}.$$

Formulas (VIII.50) and (VIII.51) afford, in each specific case, the possibility to evaluate the effect of the small crack on the magnitude of the limit load p_* for the large crack, located on the same straight line with the small one.

§5. The Effect of the Limit Load for an Elastic Plane Weakened by a Round Hole with Radial Cracks

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The problem of the rupture of an elastic plane weakened by a round hole with radial cracks of equal length was first analyzed by O. L. Bowie [1]. In this work is constructed the rational function $z = \omega(\zeta)$ which accomplishes, approximately, the conformal mapping of the exterior of the circle $|\zeta| \geqslant 1$ in plane ζ on the exterior of a circle with radial cracks in plane Z. With the aid of function $\omega(\zeta)$ and N. I. Muskhelishvili's method, the approximate method for the determination of stresses in the examined plate is given in the above-mentioned work, and then, on the basis of Griffith's method, the graphs for the determination of the magnitude of limit forces are constructed as functions of length l of the radial cracks. O. L. Bowie's calculations are extremely cumbersome. Another approach to the solution of such problems is outlined in the works of A. A. Kaminskiy [1, 2] (see also §9, Chapter V). Here we will present the approximate method for the solution of the problem of the limit load for an elastic plate weakened by a round hole with radial cracks. This method was presented in the work of V. V. Panasyuk [3].

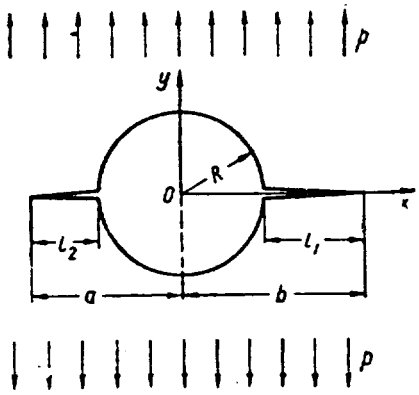


Figure VIII.8.

Round Hole with Two Radial Cracks. We will assume that an unbounded elastic plane xOy (Figure VIII.8) contains a round hole of radius R with two radial macrocracks of length l_1 and l_2 ($l_2 \leqslant l_1$), which are arranged on the extension of one of the diameters of

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the hole. We will assume further that the cracks are located on segments $-a \leqslant x \leqslant -R$ and $R \leqslant x \leqslant b$, where $a = R + l_2$, $b = R + l_1$, and that external forces that are symmetrical with respect to the plane of distribution of the cracks are applied at infinitely distant points of the plane (we will assume, for instance, that monotonically increasing stresses $\sigma_y^\infty = p$ are applied at infinitely distant points of the plate). We are required to determine the limit magnitude of external load for the problem under consideration.

To determine the limit load $p = p_*$ in the case of macroscopic cracks, it is necessary to determine, as was pointed out in §3, the intensity of rupturing elastic stresses in the vicinity of the ends of a crack. For the problem at hand, the precise determination of elastic rupturing stresses $\sigma_y(x, 0)$ in the vicinity of the ends of a crack is very difficult. In this connection we will represent the elastic stresses $\sigma_y(x, 0)$ for $x \leqslant -a$ and $x \geqslant b$ approximately, in the form of the sum

$$\sigma_y(x, 0) \approx \sigma_y^0(x, 0) + \sigma_y^{(1)}(x, 0), \tag{VIII.52}$$

where $x \leqslant -a$ or $x \geqslant b$; $\sigma_y^0(x, 0)$ are elastic tension stresses, which occur in the elastic plane containing the round hole, where a given system of external forces is applied to the plate, for instance stresses $\sigma_y^\infty = p$; $\sigma_y^{(1)}(x, 0)$ are elastic tension forces which occur in an elastic plane containing a rectilinear slit along the Ox axis when $-a \leqslant x \leqslant b$, when, to the edges of this slit on segments $-a \leqslant x \leqslant -R$ and $R \leqslant x \leqslant b$, is applied normal force /568

$$p_n(x) = -\sigma_y^0(x, 0). \tag{VIII.53}$$

The representation of stresses $\sigma_y(x, 0)$ in the form of sum (VIII.52) is valid if $R \ll l_2 \leqslant l_1$. In the case where the radius R of the hole is commensurate with the length of the cracks, sum (VIII.52) gives only a certain approximate value of stresses $\sigma_y(x, 0)$ for the problem at hand[1].

Stresses $\sigma_y^0(x, 0)$ in (VIII.52) can be calculated readily[2] for an arbitrary load, and in the following analysis we will regard them as known values. Stresses $\sigma_y^{(1)}(x, 0)$ can be determined[3] with the aid of the formula

[1]Such an approach to the determination of stresses $\sigma_y(x, y)$ can be regarded as somewhat of an analog of the method of series approximations proposed in the works of D. I. Sherman [1] and S. G. Mikhlin [1].
[2]See §1, Chapter II.
[3]See M. Ya. Leonov and V. V. Panasyuk [2].

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$$\sigma_y^{(1)}(x, 0) = \frac{1}{\pi \sqrt{(x-b)(x+a)}} \int_{-a}^{b} \frac{p_n(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi, \qquad \text{(VIII.54)}$$

where $x \leqslant -a$, $x \geqslant b$, $a \leqslant b$;

$$p_n(\xi) = \begin{cases} \sigma_y^0(\xi, 0) & \text{при} \quad -a \leqslant \xi \leqslant -R, \\ 0 & \text{при} \quad -R < \xi < R, \\ \sigma_y^0(\xi, 0) & \text{при} \quad R \leqslant \xi \leqslant b. \end{cases} \qquad \text{(VIII.55)}$$

Consequently, on the basis of relation (VIII.31) and formulas (VIII.52)-(VIII.55), we can determine the approximate value of the limit load $p = p_*$:

$$\lim_{x \to x_i} \left\{ \sqrt{|x - x_i|} \left[\sigma_y^0(x, 0) + \frac{1}{\pi \sqrt{(x-b)(x+a)}} \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi \right] \right\} = \frac{K}{\pi}, \qquad \text{(VIII.56)}$$

where $p_n^*(\xi)$ is defined by formula (VIII.55) for the limit value of the parameters characterizing the external load ($p = p_*$); x_i is the abscissa of one of the ends of the cracks (a; b).

Since stresses $\sigma_y^0(x, 0)$ do not depend on the parameters that characterize the dimensions of the crack, i.e., on abscissas a and b, then equation (VIII.56) /569 can be transformed readily to

$$\lim_{x \to x_i} \left\{ \frac{\sqrt{|x - x_i|}}{\sqrt{(x-b)(x+a)}} \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi \right\} = K \qquad \text{(VIII.57)}$$

Hence, to determine the limit value ($p = p_*^{(b)}$) of external forces applied to the plate (Figure VIII.8), due to which the cracks begin to spread in the direction of abscissa b, we have

$$\frac{1}{\sqrt{b+a}} \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{b-\xi} d\xi = K, \qquad \text{(VIII.58)}$$

where $p_n^*(\xi)$ is determined by formula (VIII.55) for $p = p_*^{(b)}$.

Using the analogous method, we can determine from equation (VIII.57) the equation for the determination of the magnitude of external forces $p = p_*^{(a)}$,

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due to which the cracks begin to spread in the direction of abscissa a. But since $a \leqslant b$ ($l_2 \leqslant l_1$) for the problem under examination, then obviously, $p_*^{(b)} \leqslant p_*^{(a)}$. Consequently, the limit forces for an elastic plate weakened by a round hole with radial cracks l_1 and l_2 where $l_2 \leqslant l_1$, will be the forces $p_* = p_*^{(b)}$.

Two Radial Cracks of Equal Length. Let us examine two cases in greater detail: a) when tension stresses $\sigma_y^\infty = p$ and $\sigma_x^\infty = 0$ act at infinitely distant points of an elastic plate weakened by a round hole with radial cracks l_1 and l_2 (see Figure VIII.8); b) when stresses $\sigma_y^\infty = q$ and $\sigma_x^\infty = q$ act at infinitely distant points of such a plate (multifold tension), and the contour of the round hole in the first and second problems is free of external forces.

For the above-stated examples of stress,

$$\sigma_y^0(x, 0) = p\left(1 + \frac{1}{2}\cdot\frac{R^2}{x^2} + \frac{3}{2}\cdot\frac{R^4}{x^4}\right); \tag{VIII.59}$$

$$\sigma_y^0(x, 0) = q\left(1 + \frac{R^2}{x^2}\right). \tag{VIII.60}$$

We find from (VIII.55) and (VIII.58)-(VIII.60) the limit value of forces $p = p_*$ and $q = q_*$:

$$p_* = \frac{K\sqrt{b+a}}{f_1(a, b)} \text{ и } q_* = \frac{K\sqrt{b+a}}{f_2(a, b)}; \tag{VIII.61}$$

where

$$f_1(a, b) = \int_{-a}^{-R}\left(1 + \frac{R^2}{2\xi^2} + \frac{3}{2}\cdot\frac{R^4}{\xi^4}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi +$$

$$+\int_R^b\left(1 + \frac{R^2}{2\xi^2} + \frac{3}{2}\frac{R^4}{\xi^4}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi; \tag{VIII.62}$$

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$$f_2(a, b) = \int_{-a}^{-R}\left(1 + \frac{R^2}{\xi^2}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi + \int_R^b\left(1 + \frac{R^2}{\xi^2}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi. \tag{VIII.63}$$

By calculating integrals (VIII.62) and (VIII.63), we obtain the expressions for the functions

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$$f_1(a,b) = A(a,b,R)\sqrt{(a+R)(b-R)} - A(a,b,-R)\sqrt{(a-R)(b+R)} -$$
$$- B(a,b,R)\ln\frac{(\sqrt{ab}+\sqrt{(a-R)(b+R)})^2+R^2}{(\sqrt{ab}+\sqrt{(a+R)(b-R)})^2+R^2} + \quad \text{(VIII.64)}$$
$$+ \frac{a+b}{2}\left(\pi + \arcsin\frac{a-b-2R}{a+b} - \arcsin\frac{a-b+2R}{a+b}\right),$$

where

$$A(a,b\pm R) = 1 \pm \frac{R}{b} + \frac{R^2}{b^2}\left(\frac{5}{8}+\frac{1}{8}\cdot\frac{b}{a}\right) \pm \frac{R^3}{b^3}\left(\frac{15}{16}+\frac{1}{4}\cdot\frac{b}{a}-\frac{3}{16}\cdot\frac{b^2}{a^2}\right);$$

$$B(a,b,R) = \frac{R^2\sqrt{ab}}{32a^3b^4}\{8a^2b^2(a+b)+3R^2(7a^3-3a^2b^2+5ab^2-b^3\};$$

$$f_2(a,b) = \left(1+\frac{R}{b}\right)\sqrt{(a+R)(b-R)} - \left(1-\frac{R}{b}\right)\sqrt{(a-R)(b+R)} -$$
$$- \frac{R^2\sqrt{ab}}{2ab^2}(a+b)\ln\frac{(\sqrt{ab}+\sqrt{(a-R)(b+R)})^2+R^2}{(\sqrt{ab}+\sqrt{(a+R)(b-R)})^2+R^2} + \quad \text{(VIII.65)}$$
$$+ \frac{a+b}{2}\left(\pi + \arcsin\frac{a-b-2R}{a+b} - \arcsin\frac{a-b+2R}{a+b}\right);\ a = R+l_2,\ b = R+l_1.$$

Then, with the aid of formulas (VIII.61), (VIII.64) and (VIII.65), it is easy to find the solution of the problems examined in the work of O. L. Bowie [1]. Thus, in the case where radial cracks are of the same length, i.e., $l_2 = l_1 = l$ and consequently a = b, we obtain from the above formulas

$$p_* = \frac{K}{\sqrt{2R(1+\varepsilon)}}\cdot\frac{1}{f_1(\varepsilon)};\quad q_* = \frac{K}{\sqrt{2R(1+\varepsilon)}}\cdot\frac{1}{f_2(\varepsilon)}, \quad \text{(VIII.66)}$$

where

$$f_1(\varepsilon) = \frac{\pi}{2} - \arctan\frac{1}{\sqrt{2\varepsilon+\varepsilon^2}} + \frac{\sqrt{2\varepsilon+\varepsilon^2}}{(1+\varepsilon)^4}(2+2\varepsilon+\varepsilon^2), \quad \text{(VIII.67)}$$

$$f_2(\varepsilon) = \frac{\pi}{2} - \arctan\frac{1}{\sqrt{2\varepsilon+\varepsilon^2}} + \frac{\sqrt{2\varepsilon+\varepsilon^2}}{(1+\varepsilon)^2},\quad \varepsilon = \frac{l}{R}. \quad \text{(VIII.68)}$$

In the case where an elastic plate weakened by a round hole with one radial crack (see Figure VIII.9) is under tension at infinitely distant points by stresses $\sigma_y^\infty = p$ and $\sigma_x^\infty = 0$ for $\sigma_y^\infty = q$ and $\sigma_x^\infty = q$, then the limit values of /571
these stresses are easily computed by formulas (VIII.61), (VIII.64) and (VIII.65), if we assume in these formulas that $l_2 = 0$ and $l_1 \neq 0$. In such a case we have (a = R)

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$$p_* = \frac{K\sqrt{1+\eta}}{\sqrt{R}(1+\varepsilon_1)} \cdot \frac{1}{f_1(\eta)}; \quad q_* = \frac{K\sqrt{1+\eta}}{\sqrt{R}(1+\varepsilon_1)} \cdot \frac{1}{f_2(\eta)}, \tag{VIII.69}$$

where

$$f_1(\eta) = A(\eta)\sqrt{2\eta(1-\eta)} - B(\eta)\ln\frac{1+\eta}{(1+\sqrt{2(1-\eta)})^2+\eta} + \frac{1+\eta}{2}\left(\frac{\pi}{2} + \arcsin\frac{1-3\eta}{1+\eta}\right),$$

$$A(\eta) = \frac{1}{16}(16 + 15\eta + 14\eta^2 + 15\eta^3),$$

$$B(\eta) = \frac{\eta\sqrt{\eta}}{32}(5 + 23\eta - 9\eta^2 + 21\eta^3), \tag{VIII.70}$$

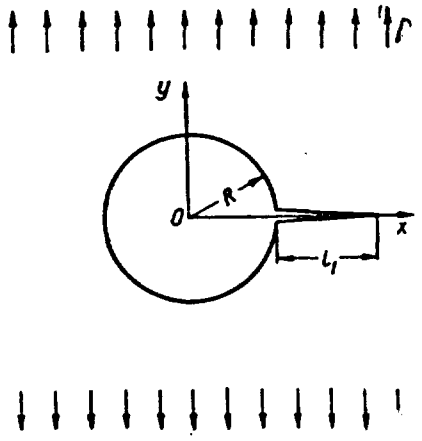


Figure VIII.9.

$$f_2(\eta) = \frac{1+\eta}{2}\left(2\sqrt{2\eta(1-\eta)} - \eta\sqrt{\eta}\ln\frac{1+\eta}{(1+\sqrt{2(1-\eta)})^2+\eta} + \frac{\pi}{2} + \arcsin\frac{1-3\eta}{1+\eta}\right);$$

$$\eta = \frac{R}{b} = \frac{1}{1+\varepsilon_1}, \quad \varepsilon_1 = \frac{l_1}{R}. \tag{VIII.71}$$

The graphs (solid lines) of change of limit loads $\pi p_* \sqrt{R}/K$ and $\pi q_* \sqrt{R}/K$ are constructed in Figures VIII.10 and VIII.11 on the basis of formulas (VIII.66)-(VIII.71), as functions of the relations $\varepsilon = l/R$ and $\varepsilon_1 = l_1/R$, where curves 1 correspond to the case of one radial crack (l_1), and curves 2, to the case of two radial cracks of identical length (l). For comparison, the functions of the above stated limit loads, as established in the work of O. L. Bowie [1], are shown in these figures by the broken lines. Comparison of the graphs shows that the constructed approximate solution, given by formulas (VIII.66)-(VIII.71) for $l/R > 0.5$, coincide satisfactorily with the calculations of O. L. Bowie. It should be noted here, however, that the solution of O. L. Bowie was found by the method of cumbersome calculations, whereas here this solution was found quite simply.

If the length of the radial cracks is small in comparison with the radius of the round hole, then the distribution of such cracks will obviously be determined by the magnitude of rupture stresses acting directly on the contour of the hole. It is easy to see from formulas (VIII.59) and (VIII.60) that in the case of uniaxial tension, the greatest rupture stresses on the contour of the hole will be equal to 3p, where as in the case of multifold tension, 2q. /572
Therefore, for the examined problems, the ratio of the limit equilibrium loads p_*/q_* should approach 2/3 when $l \to 0$.

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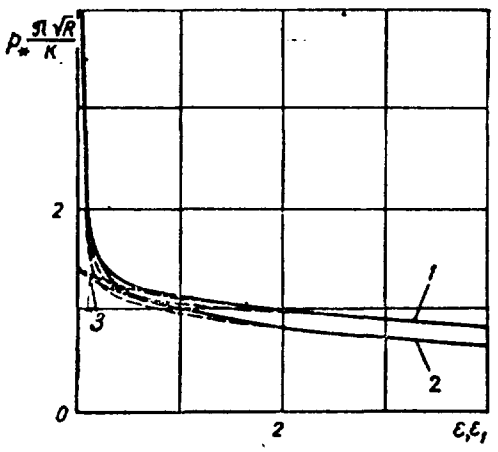


Figure VIII.10.

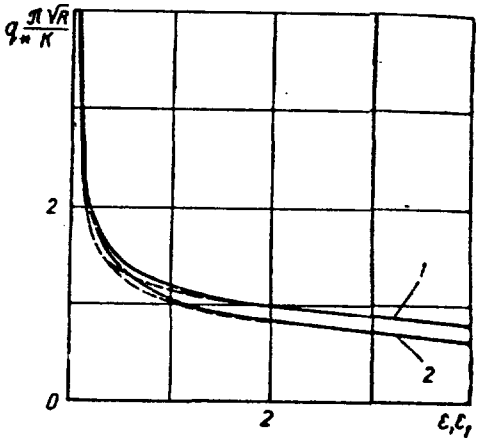


Figure VIII.11.

From (VIII.66)-(VIII.68) we obtain

$$\lim_{l\to 0} \frac{p_*}{q_*} = \lim_{\varepsilon\to 0} \frac{f_2(\varepsilon)}{f_1(\varepsilon)} = \frac{2}{3},$$

i.e., the formulas (VIII.66)-(VIII.71) at the limit for $l \to 0$ ($\varepsilon \to 0$) yield the expected result. We arrive at the very same conclusion also on the basis of formulas (VIII.69)-(VIII.71), where $\varepsilon_1 \to 0$. If, on the other hand, the length of the radial cracks is sufficiently great, for instance, $l \gg R$ (Figure VIII.8) such that we may assume that $\varepsilon \to \infty$, $R \to 0$ ($\varepsilon R = l = \text{const}$), then we find from formulas (VIII.66)-(VIII.68) or (VIII.69)-(VIII.71), the known Griffith's formulas for an isolated crack.

For comparison, the graphs (curves 3) of change of the critical load $\pi p_* \sqrt{R}/K$ are also constructed in Figure VIII.10 for the case where the plate contains no isolated rectilinear crack of length $2(R + l)$ or $2(R + 1/2\, l)$, but where forces of tension $\sigma_y^\infty = p$ act at infinitely distant points of the plate. Comparison of the curves in Figure VIII.10 shows that when $l/R > 1$ the limit values of stresses ($p = p_*$) for a plate weakened by a round hole with radial cracks, and for a plate weakened by a rectilinear crack of length $2(R + l)$, are so similar that the hole now has hardly any effect.

In the recently published works of O. L. Bowie [2, 3], the case of limit equilibrium of a beam with outer surface cracks is examined.

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§6. Tension of Elastic Plate with Arc-Shaped or Arbitrarily Oriented Rectilinear Crack[1]

/5

Arc-Shaped Crack. In the preceding sections we examined problems of the limit equilibrium of a plate with a rectilinear crack oriented perpendicular to the line of tension of a plate. Now we will determine the limit equilibrium state of plate with an arc-shaped crack. Thus, we will examine an infinite elastic plate with an arc-shaped macroscopic crack, which represents a slit along an arc of a circle of radius R. We will introduce a system of rectangular Cartesian coordinates xOy (Figure VIII.12). We will assume that the edges of the crack are free of external stresses and that monotonically increasing forces of tension p and q are uniformly applied in mutually perpendicular directions at infinitely distant points of the plate, where p are directed at angle α to the Ox axis (Figure VIII.12). We will determine the limit (critical) values of stresses $p = p_*$ and $q = q_*$, due to which the crack goes into the state of mobile equilibrium (begins to spread from one of its ends).

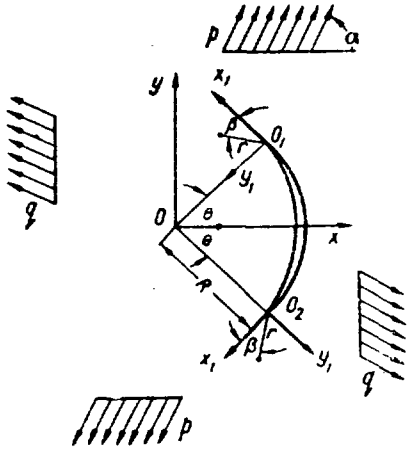


Figure VIII.12.

The determination of the magnitude of the limit load for the problem under examination is, generally speaking, impossible simply on the basis of equation (VIII.31), since, for the above-stated problem, the direction of the initial spreading of the crack is not known beforehand. However, it is natural to assume[2] that the initial direction of propagation of a curvilinear (or rectilinear arbitrarily oriented) macrocrack will coincide with the plane in which the normal forces of tension achieve the maximum possible intensity. As an outcome of this assumption, and on the basis of equation (VIII.31), we will obtain for the determination of the limit values of external forces for the problem under examination,

$$\lim_{r\to 0} \sqrt{r}\,\sigma_\beta^*(r, \beta_*) = \frac{K}{\sqrt{\pi}}, \qquad \text{(VIII.72)}$$

where r, β are polar coordinates with the origin at the vertex of the crack, and with the polar axis directed along the tangent to the contour of the crack (see Figure VIII.12); $\sigma_\beta(r, \beta)$ are normal tension stresses perpendicular to lines $\beta = \text{const}$; $\sigma_\beta^*(r, \beta)$ are stresses $\sigma_\beta(r, \beta)$ for $p = p_*$ and $q = q_*$.

/574

[1]The solutions of these problems were found by V. V. Panasyuk, L. T. Berezhnitskiy [1].
[2]See V. V. Panasyuk, L. T. Berezhnitskiy [1],Erdogan, Si [1].

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The angle $\beta = \beta_*$, which determines the initial direction of propagation of a crack, satisfies the relation

$$\lim_{r\to 0}\left\{\sqrt{r}\,\frac{\partial\sigma_\beta(r,\beta)}{\partial\beta}\right\}_{\beta=\beta_*} = 0. \qquad \text{(VIII.73)}$$

Thus, if the magnitude of elastic stresses in the vicinity of the vertices of a curvilinear crack is defined, then by using relations (VIII.72) and (VIII.73), it is easy to determine the magnitude of the limit load as well.

For the problem under examination (Figure VIII.12) the tensor components of stresses σ_r σ_β and $\tau_{r,\beta}$ in polar coordinate system (r, β), in the vicinity of the ends O_1 and O_2 of the crack, can be represented in the form[1]

$$\begin{aligned}
\sigma_r(r,\beta) &= \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left(5\cos\frac{\beta}{2}-\cos\frac{3\beta}{2}\right)+k_{2,i}\left(-5\sin\frac{\beta}{2}+3\sin\frac{3\beta}{2}\right)\right\}+\\
&\quad+4A_i\cos^2\beta+O(r^{\frac{1}{2}}),\\
\sigma_\beta(r,\beta) &= \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left(3\cos\frac{\beta}{2}+\cos\frac{3\beta}{2}\right)-3k_{2,i}\left(\sin\frac{\beta}{2}+\sin\frac{3\beta}{2}\right)\right\}+\\
&\quad+4A_i\sin^2\beta+O(r^{\frac{1}{2}}),\\
\tau_{r\beta}(r,\beta) &= \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left(\sin\frac{\beta}{2}+\sin\frac{3\beta}{2}\right)+k_{2,i}\left(\cos\frac{\beta}{2}+3\cos\frac{3\beta}{2}\right)\right\}-\\
&\quad-2A_i\sin 2\beta+O(r^{\frac{1}{2}}) \qquad (i=1,2),
\end{aligned} \qquad \text{(VIII.74)}$$

where the coefficients $k_{1,i}$ and $k_{2,i}$ (i = 1, 2) of the intensity of the stresses at the vertex of the crack O_i (i = 1, 2) are determined by formulas

$$\begin{aligned}
k_{1,1} &= \sqrt{R\sin\theta}\,\varphi_1(\alpha,\theta), \qquad k_{2,1} = \sqrt{R\sin\theta}\,\varphi_2(\alpha,\theta),\\
k_{1,2} &= \sqrt{R\sin\theta}\,\varphi_1(\alpha,-\theta), \quad k_{2,2} = \sqrt{R\sin\theta}\,\varphi_2(\alpha,-\theta),\\
A_1(\alpha,\theta) &= \frac{1}{4}(p+q)\,\frac{\sin^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}+\frac{p-q}{4}\left[\frac{\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}-\cos 2(\alpha-\theta)\right],
\end{aligned} \qquad \text{(VIII.75)}$$

$$A_2(\alpha,\theta) = A_1(\alpha,-\theta);$$

---

[1]See Si, Paris, Erdogan [1], V. V. Panasyuk, L. T. Berezhnitskiy [1].

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here

$$\varphi_1(\alpha,\theta)=\frac{1}{2}\left[\frac{p+q-(p-q)\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}\cos\frac{\theta}{2}+\right.$$
$$\left.+(p-q)\sin 2\alpha\sin^3\frac{\theta}{2}+(p-q)\cos\left(2\alpha-\frac{3\theta}{2}\right)\right];$$
$$\varphi_2(\alpha,\theta)=\frac{1}{2}\left[\frac{p+q-(p-q)\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}\sin\frac{\theta}{2}-\right. \tag{VIII.76}$$
$$\left.-(p-q)\sin 2\alpha\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}-(p-q)\sin\left(2\alpha-\frac{3\theta}{2}\right)\right].$$

From (VIII.75) and (VIII.76), for the case q = 0, we will have[1] the expressions

$$k_{1,1}=\frac{p\sqrt{R\sin\theta}}{2}\left[\frac{1-\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}\cos\frac{\theta}{2}+\right.$$
$$\left.+\sin 2\alpha\sin^3\frac{\theta}{2}+\cos\left(2\alpha-\frac{3\theta}{2}\right)\right],$$
$$k_{2,1}=\frac{p\sqrt{R\sin\theta}}{2}\left[\frac{1-\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+\sin^2\frac{\theta}{2}}\sin\frac{\theta}{2}-\right. \tag{VIII.77}$$
$$\left.-\sin 2\alpha\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}-\sin\left(2\alpha-\frac{3\theta}{2}\right)\right].$$

By using formulas (VIII.74), we can represent relations (VIII.72) and (VIII.73) in the form

$$k_{1,i}^{*}\left(3\cos\frac{\beta_*}{2}+\cos\frac{3\beta_*}{2}\right)-3k_{2,i}^{*}\left(\sin\frac{\beta_*}{2}+\sin\frac{3\beta_*}{2}\right)=\frac{4\sqrt{2K}}{\pi}, \tag{VIII.78}$$

$$k_{1,i}\left(\sin\frac{\beta_*}{2}+\sin\frac{3\beta_*}{2}\right)+k_{2,i}\left(\cos\frac{\beta_*}{2}+3\cos\frac{3\beta_*}{2}\right)=0, \tag{VIII.79}$$

[1]The coefficients $k_{1,i}$ and $k_{2,i}$ for an arc-shaped crack in an infinite elastic plate were calculated by a different method by Si, Paris, Erdogan [1]. However, in the expressions proposed by them for the coefficients $k_{1,1}$ and $k_{2,2}$ (only at vertex O_1 of an arc-shaped crack), when the elastic plate is under tension "at infinity" by forces p at angle α to the Ox axis, i.e., when q = 0 (Figure VIII.12), there are errors.

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where the intensity coefficients $k^*_{1,i}$ and $k^*_{2,i}$ are determined by formulas (VIII.75) and (VIII.76) for $p = p_*$ and $q = q_*$.

From equation (VIII.79) we find the values of angle β_* at which the stress $\sigma_\beta(r, \beta)$ achieves its maximum intensity: /576

$$\beta_* = \pm 2 \arcsin \sqrt{\frac{6n_i^2 + 1 - \sqrt{8n_i^2 + 1}}{2(9n_i^2 + 1)}} \quad \text{for } k_{1,i} > 0, \tag{VIII.80}$$

$$\beta_* = \pm 2 \arcsin \sqrt{\frac{6n_i^2 + 1 + \sqrt{8n_i^2 + 1}}{2(9n_i^2 + 1)}} \quad \text{for } k_{1,i} < 0, \tag{VIII.81}$$

where "plus" corresponds to the values $k_{2,i} < 0$ and "minus," to the values $k_{2,i} > 0$; $n_i = k_{2i}/k_{1,i}$ $(i = 1, 2)$.

Thus, on the basis of formulas (VIII.78)-(VIII.81) we can calculate the value of the limit load for a plate with a curvilinear isolated crack, if the coefficients $k_{1,i}$ and $k_{2,i}$ $(i = 1, 2)$ of intensity of the stresses are defined.

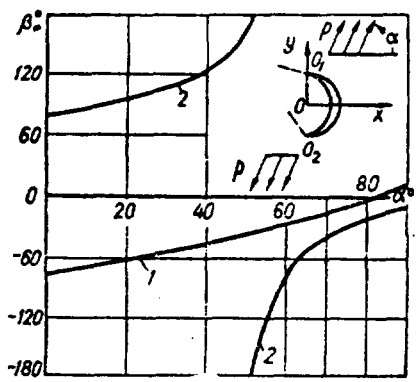


Figure VIII.13.

Let us examine in greater detail the case where an elastic plate, weakened by an arc-shaped crack in the form of a semicircle, is under tension at infinitely distant points by uniformly distributed stresses p, directed at angle α to the Ox axis (Figure VIII.13).

Assuming in formulas (VIII.75) and (VIII.76) $q = 0$, $p = p_*$ and $\theta = \pi/2$, we obtain for the coefficients $k^*_{1,i}$ and $k^*_{2,i}$, the expressions

$$k^*_{1,1} = Af_+(\alpha), \quad k^*_{2,1} = Ag_+(\alpha), \quad k^*_{1,2} = Af_-(\alpha), \quad k^*_{2,2} = -Ag_-(\alpha), \tag{VIII.82}$$

where

$$A = \frac{p_* \sqrt{R}}{12\sqrt{2}}; \quad f_\pm = 4 - 7\cos 2\alpha \pm 9 \sin 2\alpha; \quad g_\pm(\alpha) = 4 + 5\cos 2\alpha \pm 3 \sin 2\alpha.$$

Hence, on the basis of equation (VIII.78) it is easy to find

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$$p_{1*} = \frac{24K}{\pi\sqrt{R}} \cdot \frac{1}{\cos^2\frac{\beta_*}{2}\left[f_+(\alpha)\cos\frac{\beta_*}{2} - 3g_+(\alpha)\sin\frac{\beta_*}{2}\right]}, \qquad \text{(VIII.83)}$$

$$p_{2*} = \frac{24K}{\pi\sqrt{R}} \cdot \frac{1}{\cos^2\frac{\beta_*}{2}\left[f_-(\alpha)\cos\frac{\beta_*}{2} + 3g_-(\alpha)\sin\frac{\beta_*}{2}\right]}, \qquad \text{(VIII.84)}$$

where β_* is defined by formulas (VIII.80)-(VIII.82).

By using formulas (VIII.80), (VIII.81) and (VIII.82)-(VIII.84), it is easy to calculate the change of angle β_* and limit stresses p_{1*} and p_{2*} as functions of angle α ($0 \leqslant \alpha \leqslant \pi/2$). The results of such calculations are presented in Figures VIII.13 and VIII.14, where curves 1 pertain to the point O_1, and curves 2 to the point O_2. From the graphs presented in these figures it is easy to see that when $0 < \alpha < \pi/2$, the limit load $p_{1*} < p_{2*}$, i.e., a crack in the form a semicircle, begins to spread near the vertex, for which the angle β_* is smallest. /577

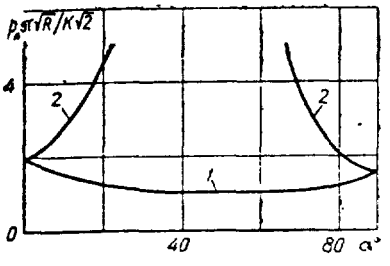


Figure VIII.14.

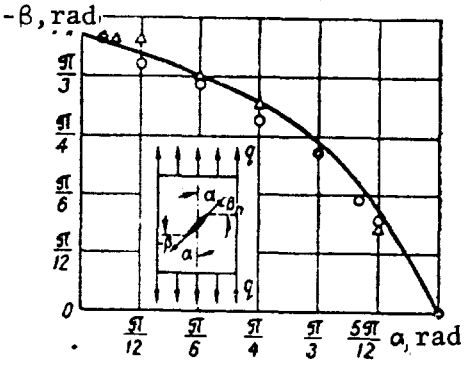


Figure VIII.15.

Arbitrarily Oriented Rectilinear Crack. By using the results obtained above, it is easy to determine the magnitude of limit stress for the case where an infinite plate with a rectilinear crack is subjected to tension by monotonically increasing stresses of intensity q, applied at the infinitely distant points of the plate and directed at angle α to the plane of propagation of the crack (Figure VIII.15).

Assuming in formulas (VIII.75) and (VIII.76) $\theta \to 0$, $R \to \infty$, but such that $R\theta = l = \text{const}$, and assuming $p = 0$ and $q = q_*$, we obtain

$$k^*_{1,l} = q_*\sqrt{l}\sin^2\alpha, \quad k_{2,l} = q_*\sqrt{l}\sin\alpha\cos\alpha \quad (i = 1,2),$$

where $0 \leqslant \alpha \leqslant \pi/2$, l is the half length of the crack.

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From this and in accordance with formulas (VIII.80) and (VIII.81), we have

$$\beta_* = -2\arcsin\sqrt{\frac{6\cot^2\alpha + 1 - \sqrt{8\cot^2\alpha + 1}}{2(9\cot^2\alpha + 1)}}. \qquad \text{(VIII.85)}$$

Further, by using equation (VIII.78) and coefficients $k^*_{1,i}$ and $k^*_{2,i}$, we obtain for the problem under examination

$$q_* = \frac{K\sqrt{2}}{\pi\sqrt{l}} \cdot \frac{1}{\cos^2\frac{\beta_1}{2}\left[\sin^2\alpha\cos\frac{\beta_*}{2} - 3\sin\alpha\cos\alpha\sin\frac{\beta_*}{2}\right]}, \qquad \text{(VIII.86)}$$

where angle β_* is defined by formula (VIII.85).

Assuming in formulas (VIII.85) and (VIII.86) $\alpha = \pi/2$ (see Figure VIII.15), we readily find, as a partial case, the known Griffith's relations [2]: /578

$$\beta_* = 0, \quad q_* = \frac{K\sqrt{2}}{\pi\sqrt{l}}.$$

Experimental Check[1] of Functions (VIII.85) and (VIII.86). The dependence of angle β_* and of limit load q_* on angle α in the case of uniaxial tension of a plate is represented in accordance with formulas (VIII.85) and (VIII.86) in Figure VIII.15 and VIII.16 in the form of solid lines. These functions were checked experimentally on plates of organic and silicate glass. For this purpose, plates (see Figure VIII.15) of the corresponding dimensions[2] were cut from sheet material. In the center of each plate was cut a hole with a diameter of about 6 mm, and with the aid of a glass cutter, a crack (slit) was made at some angle α to the longitudinal axis of the plate. Then, with the aid of a special device, by way of application of forces of tension to the edges of the hole in the direction perpendicular to the line of the slit, the initial crack was opened up through the entire thickness of the plate and was extended to a given length $2l$. After this the length ($2l$) of the original crack and angle α were measured. In this manner groups of plates were made of silicate and organic glass with cracks with different directions (different value of angle α for $0 < \alpha \leqslant \pi/2$) and different length $2l$. The test plates were then subjected to tension in the direction of the longitudinal axis of the plate on an MR-0.5 rupture machine (at a load rate of $6.6 \cdot 10^{-5}$ m/sec) to rupture. In this manner the magnitude of the limit (rupture) force $p = p_*$ at which the

[1]A more detailed description of the experiment is found in the work of V. V. Panasyuk, L. T. Berezhnitskiy and S. Ye. Kovchik [1].
[2]See V. V. Panasyuk, L. T. Berezhnitskiy and S. Ye. Kovchik [1], where detailed tables are available.

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crack would begin to spread and at which the plate would rupture, was measured. On the basis of the data of the experiment, the rupture stress was calculated:

$$q_{*\alpha} = \frac{P_{*\alpha}}{bh} \text{ [кг/мм}^2\text{]},$$

where b is width and h is thickness of plate; the index α denotes that the above-stated values pertain to the cracks directed at angle α to the longitudinal axis of the plate.

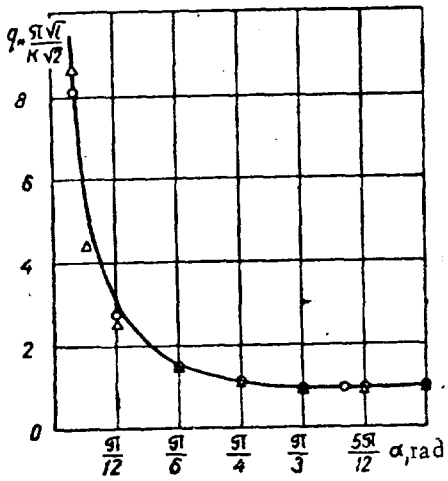


Figure VIII.16.

By determining the value $q_{*\alpha}$, it is easy to calculate the ratio

$$\eta(\alpha, l_\alpha) = \frac{q_{*\alpha}\sqrt{l_\alpha}}{q_*\left(\frac{\pi}{2}\right)\sqrt{l\left(\frac{\pi}{2}\right)}} = \frac{\pi q_{*\alpha}\sqrt{l_\alpha}}{K\sqrt{2}}.$$

The results of the experimental data are presented in Figure VIII.16. The average value of $\eta(\alpha, l_\alpha)$ for each group of plates, having identical angle α, are represented by the circular and triangular symbols on Figure VIII.16, where the triangles pertain to plates made of organic glass, and the circles, to plates made of silicate glass. The solid curve on this figure represents the values of the function /579

$$\eta(\alpha, l_\alpha) = \frac{\pi q_* \sqrt{l_\alpha}}{K\sqrt{2}},$$

derived from formula (VIII.86).

Furthermore angles (β_*) of the initial direction of propagation of the crack for both ends of the crack, i.e., $\beta_*^{(l)}$ (for the left end of the crack) and $\beta_*^{(r)}$ (for the right end of the crack), were measured for each plate. The average angles $\beta_*^{(a)} = 1/2(\beta_*^{(l)} + \beta_*^{(r)})$ for each group of plates are shown in Figure VIII.15 in the form of triangles (for organic glass) and circles (for silicate glass) as functions of angle α. The solid curve on this figure, which represents the dependence of angle β_* on α, was constructed on the basis of equation (VIII.85).

Thus, as seen from the graphs presented herein, the experimental data agree satisfactorily with theory.

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§7. Deflection of Rods (Beams) Weakened by Rectilinear Cracks[1]

We will analyze the limit equilibrium state of rods (beams) weakened by rectilinear cracks and deflected by external monotonically increasing forces acting at the middle of the plane of the rod. First we will find the distribution of elastic stresses in the rod (beam) in the vicinity of the crack, where known external loads are applied to the middle plane of the rod, far from the crack.

Deflection of Rod with Central Rectilinear Crack. We will analyze an elastic isotropic rod (beam) with a rectilinear crack located in the center of the rod, perpendicular to its longitudinal axis (Figure VIII.17). We will denote through 2h and 2δ, respectively, the width and thickness of the rod, and through $2l$, the length of the crack. We will introduce rectangular Cartesian coordinate system xOy, as shown in Figure VIII.17, and we will assume that the middle plane of the rod coincides with the xOy plane, and that the crack is located on the segment $-l \leqslant x \leqslant l$.

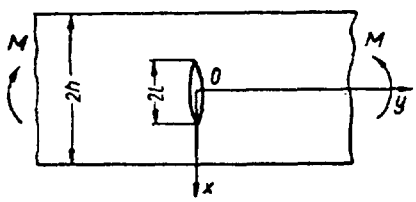


Figure VIII.17.

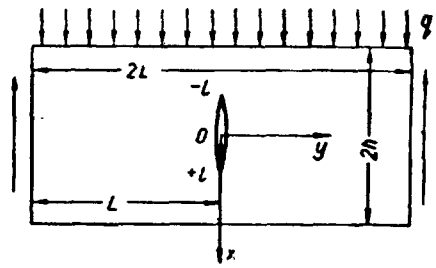


Figure VIII.18.

Let external loads (deflecting moments or uniformly distributed pressure) located in the middle plane of the rod, i.e., in the xOy plane, act on such a rod. Under the effect of external forces (in the zone of compressive stresses), the edges of the crack come into contact in some segment $\lambda_1 \leqslant x \leqslant \lambda_2$, which results in the development of contact stresses on this segment of the edges of the crack. Beyond this segment the edges of the crack will be free of contact stresses. The parameters λ_1 and λ_2, which define the boundary of the range of contact between the edges of the crack, must be determined during the solution of the problem. We will determine the stress-deformation state in the vicinity of the crack.

For the given problem, we will have, on the contour of the crack, the following boundary conditions: /580

[1]See B. L. Lozovoy, V. V. Panasyuk [1, 2], V. V. Panasyuk, B. L. Lozovoy [3, 4], where, apparently, the solution of this problem is first encountered.

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on the segment of contact, i.e., when $y = 0$, $\lambda_1 \quad x \quad \lambda_2$,

$$\sigma_y^+(x, 0) = \sigma_y^-(x, 0), \quad v^+(x, 0) = v^-(x, 0) = 0; \tag{VIII.87}$$

on the edges of the crack that are free of contact stresses,

$$\sigma_y^+(x, 0) = \sigma_y^-(x, 0) = 0. \tag{VIII.88}$$

where

$$\sigma_y^+(x, 0) = \sigma_y(x, +0), \quad \sigma_y^-(x, 0) = \sigma_y(x, -0).$$

Furthermore, we will assume that there are no tangential stresses on the contour of the crack, i.e.,

$$\tau_{xy}(x, 0) = 0 \text{ for } \quad -l \leqslant x \leqslant l. \tag{VIII.89}$$

Stress tensor components σ_x, σ_y, τ_{xy} and displacement vectors u, v are determined in the plane problem through two analytical functions $\Phi(z)$ and $\Omega(z)$ (see §1, Chapter I) by the following relations:

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}], \quad z = x + iy, \tag{VIII.90}$$

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{VIII.91}$$

$$2G(u' + iv') = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}$$

$$\left(u' \equiv \frac{\partial u}{\partial x}, \quad v' \equiv \frac{\partial v}{\partial x}\right). \tag{VIII.92}$$

We will assume that

$$\Phi_0(z) = A_0z^3 + A_1z^2 + A_2z + A_3, \tag{VIII.93}$$

$$\Omega_0(z) = B_0z^3 + B_1z^2 + B_2z + B_3, \tag{VIII.94}$$

where the coefficients A_n and B_n (n = 0, 1, 2, 3) are constants that determine the stress state in a rod without a crack for

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = \frac{M}{4J}, \quad A_3 = 0;$$

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = \frac{3M}{4J}, \quad B_3 = 0, \tag{VIII.95}$$

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where $J = 4\delta h^3/3$ is the moment of inertia of the rod; the functions (VIII.93) and (VIII.94) give the solution of the problem of pure deflection by moments M of an infinite rod (beam) without a crack (see Figure VIII.17). When /581

$$A_0 = \frac{q}{24J}, \quad A_1 = 0, \quad A_2 = \frac{q}{8J}\left(L^2 + \frac{3h^2}{5}\right), \quad A_3 = \frac{-qh^3}{12J};$$
$$B_0 = \frac{7q}{24J}, \quad B_1 = 0, \quad B_2 = \frac{q}{8J}\left(3L^2 - \frac{11h^2}{5}\right), \quad B_3 = \frac{qh^3}{12J}. \tag{VIII.96}$$

(symbols are given in Figure VIII.18), the functions (VIII.93) and (VIII.94) give the solution of the problem of deflection of a beam (rod) of length 2L without a crack, when the beam is loaded by uniform pressure of intensity q. It is assumed here that the beam is resting freely on two supports, and that the support reactions are defined as tangential forces applied to the ends of the beam.

In a rod (beam) that is under the effect of an external load, i.e., deflecting moments (see Figure VIII.17), or uniformuly distributed pressure (Figure VIII.18), let a crack of length $2l$ extend along the Ox axis. The presence of the crack in the rod results in the redistribution of stresses in the vicinity of this crack. The stress-deformation state in the rod at points distant from the crack will be characterized[1] for the above mentioned types of load by functions $\Phi_0(z)$ and $\Omega_0(z)$ for the corresponding values of coefficients A_n and B_n from (VIII.95) and (VIII.96). Boundary conditions (VIII.87)-(VIII.89) on the contour of the crack should be satisfied.

If in formulas (VIII.90) and (VIII.91) we proceed to the boundary values on the contour of the crack, i.e., assume $y \to \pm 0$, recalling boundary conditions (VIII.87) and (VIII.89), we will arrive at the problem of linear conjugation for the desired functions $\Phi(z)$ and $\Omega(z)$. The method for solving these problems is known[2]. The functions that satisfy condition (VIII.90) will have the form

$$\Phi(z) = \frac{1}{2\pi i\sqrt{z^2 - l^2}} \int_{\lambda_1}^{\lambda_2} \frac{\sqrt{t^2 - l^2}\,p(t)}{t - z}\,dt + \frac{p_n(z)}{\sqrt{z^2 - l^2}} + \frac{1}{2}\left[\Phi_0(z) - \Omega_0(z)\right], \tag{VIII.97}$$

$$\Omega(z) = \frac{1}{2\pi i\sqrt{z^2 - l^2}} \int_{\lambda_1}^{\lambda_2} \frac{\sqrt{t^2 - l^2}\,p(t)}{t - z}\,dt + \frac{p_n(z)}{\sqrt{z^2 - l^2}} - \frac{1}{2}\left[\Phi_0(z) - \Omega_0(z)\right], \tag{VIII.98}$$

where $\Phi_0(z)$ and $\Omega_0(z)$ are known functions (VIII.93) and (VIII.94);

$$p(t) = \sigma_y(t, 0) \quad (\lambda_1 \leqslant t \leqslant \lambda_2);$$
$$p_n(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_n. \tag{VIII.99}$$

---

[1] This assumption, as we will see below, will be satisfied with a very high degree of accuracy.

[2] See N. I. Muskhelishvili [1].

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The power of the polynomial $p_n(z)$ and its coefficients $(c_0, c_1, \ldots, c_n)$ are determined from the conditions of the behavior of functions $\Phi(z)$ and $\Omega(z)$ in the vicinity of infinitely distant points, i.e., when $|z| \to \infty$. These functions, for $|z| \to \infty$, should satisfy conditions /582

$$\Phi(z) \to \Phi_0(z), \qquad \Omega(z) \to \Omega_0(z). \tag{VIII.100}$$

To determine the magnitude of contact stresses $\sigma_y^+(x, 0)$, we will use formula (VIII.92). Proceeding in this formula and also in the formula conjugate to it, to the values on the contour of the crack, using for functions $\Phi(z)$ and $\Omega(z)$ their expressions from (VIII.97) and (VIII.98), we find

$$2Gi(v^{1+} - v^{1-}) = \frac{\varkappa+1}{1-i\varrho}\cdot\frac{1}{\sqrt{t_0^2-t^2}}\left\{\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2}\frac{\sqrt{t^2-t^2}\,p(t)}{t-t_0}\,dt + \right.$$
$$\left. + R_0t_0^4 + R_1t_0^3 + R_2t_0^2 + R_3t_0 + R_4\right\}, \tag{VIII.101}$$

where $\lambda_1 \leqslant t_0 \leqslant \lambda_2$; R_0, R_1, R_2, R_3, R_4 are constants which are expressed through the coefficients A_n, B_n $(n = 0, 1, 2, 3)$ of functions (VIII.93) and (VIII.94); t_0 is an arbitrary point on the contour of the crack.

To determine the function $\sigma_y^+(t)$ from boundary conditions (VIII.87) and formula (VIII.101), we will obtain a singular integral equation

$$\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2}\frac{\sqrt{t^2-t^2}\,\sigma_y^+(t)}{t-t_0}\,dt + R_0t_0^4 + R_1t_0^3 + R_2t_0^2 + R_3t_0 + R_4 = 0, \tag{VIII.102}$$

where $\lambda_1 < t_0 < \lambda_2$.

The solution of equation (VIII.102) that satisfies the condition of boundedness of contact stresses for $t_0 = \lambda_1$ and $t_0 = \lambda_2$ will have the form

$$\sigma_y^+(t_0) = \frac{\sqrt{(t_0-\lambda_1)(t_0-\lambda_2)}}{\sqrt{t_0^2-t^2}}(m_0t_0^3 + m_1t_0^2 + m_2t_0 + m_3), \tag{VIII.103}$$

where $\lambda_1 < t_0 < \lambda_2$.

The parameters λ_1 and λ_2 in the solution of (VIII.103) are determined from the conditions of boundedness of the solution of equation (VIII.102) for $t_0 = \lambda_1$ and $t_0 = \lambda_2$; the coefficients m_0, m_1, m_2, m_3 are expressed through the

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coefficients A_n and B_n (n = 0, 1, 2, 3) of functions (VIII.93) and (VIII.94). For the given forms of external load that act on the rod, the coefficients m_0, m_1, m_2, m_3 and parameters λ_1 and λ_2 will have the following form:

a) during pure deflection of a rod with a crack (Figure VIII.17)

$$m_0 = 0, \quad m_1 = 0, \quad m_2 = \frac{M}{J}; \quad m_3 = \frac{-2Ml}{3J},$$
$$\lambda_1 = -l, \quad \lambda_2 = -\frac{1}{3}l; \tag{VIII.104}$$

b) during deflection of a beam (rod) with a crack by uniformly distributed load (Figure VIII.18) /583

$$m_0 = \frac{q}{3J}, \quad m_1 = \frac{q}{6J}(\lambda_1 + \lambda_2),$$
$$m_2 = \frac{q}{24J}(3\lambda_1^2 + 2\lambda_1\lambda_2 + 3\lambda_2^2) + \frac{q}{2J}\left(L^2 - \frac{2h^2}{5} - \frac{l^2}{3}\right), \tag{VIII.105}$$
$$m_3 = \frac{q}{48J}(\lambda_1 + \lambda_2)(5\lambda_1^2 - 2\lambda_1\lambda_2 + 5\lambda_2^2) + \frac{q}{4J}(\lambda_1 + \lambda_2)\left(L^2 - \frac{2h^2}{5} - \frac{l^2}{3}\right),$$

where $\lambda_1 = -l$, and λ_2 is determined from the equation

$$35\lambda_2^4 - 20l\lambda_2^3 - 6l^2\lambda_2^2 - 4l^3\lambda_2 - 5l^4 + 24\left(L^2 - \frac{2h^2}{5}\right)(3\lambda_2^2 - 2l\lambda_2 - l^2) = 0. \tag{VIII.106}$$

By substituting relations (VIII.103) and (VIII.106) into formulas (VIII.97) and (VIII.98), we obtain for the examined forms of loading of a rod (beam) with a crack, the following functions:

a) for pure deflection of rod with crack (see Figure VIII.17)

$$\Phi(z) = \frac{M}{6J}(3z - 2l)\sqrt{\frac{z-\lambda_2}{z-l}} - \frac{M}{4J}z,$$
$$\Omega(z) = \frac{M}{6J}(3z - 2l)\sqrt{\frac{z-\lambda_2}{z-l}} + \frac{M}{4J}z, \tag{VIII.107}$$

where $\lambda_2 = -\frac{1}{3}l$;

b) for pure deflection of a beam (rod) with a crack by uniformly distributed load (Figure VIII.18)

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$$\Phi(z) = \frac{1}{2}\sqrt{\frac{z-\lambda_2}{z-l}}(m_0z^3 + m_1z^2 + m_2z + m_3) - \frac{q}{8J}\left[z^3 + \left(L^2 - \frac{7}{5}h^2\right)z + \frac{2}{3}h^3\right],$$

$$\Omega(z) = \frac{1}{2}\sqrt{\frac{z-\lambda_2}{z-l}}(m_0z^3 + m_1z^2 + m_2z + m_3) + \frac{q}{8J}\left[z^3 + \left(L^2 - \frac{7}{5}h^2\right)z + \frac{2}{3}h^3\right], \quad \text{(VIII.108)}$$

where the coefficients m_0, m_1, m_2, m_3 are determined by formulas (VIII.105) and the parameter λ_2 is found from equation (VIII.106).

By knowing the functions $\Phi(z)$ and $\Omega(z)$ and by using formulas (VIII.90) and (VIII.91), we easily find the components of the stress tensor in the vicinity of the examined crack. In particular, the stress components $\sigma_x(x, 0)$ $\tau_{xy}(x, 0)$, $\sigma_y(x, 0)$ along the Ox axis (see Figures VIII.17 and VIII.18), i.e., when $y = 0$ ($l < |x| < h$), will have the following form:

a) during pure deflection of rod with crack (Figure VIII.17) /584

$$\sigma_y(x, 0) = B\left(\frac{x}{l} - \frac{2}{3}\right)\sqrt{\frac{x-\lambda_2}{x-l}}, \qquad \tau_{xy}(x, 0) = 0,$$

$$\sigma_x(x, 0) = B\left(\frac{x}{l} - \frac{2}{3}\right)\sqrt{\frac{x-\lambda_2}{x-l}} - \frac{x}{l}B, \quad \text{(VIII.109)}$$

where $l < |x| < h$, $\lambda_2 = -\frac{1}{3}l$, $B = \frac{Ml}{J}$;

b) during deflection of rod (beam) with crack by uniformly distributed load (Figure VIII.18)

$$\sigma_x(x, 0) = \sigma_y(x, 0) - \frac{q}{2J}\left[x^3 + \left(L^2 - \frac{7h^2}{5}\right)x + \frac{2h^3}{3}\right],$$

$$\sigma_y(x, 0) = (m_0x^3 + m_1x^2 + m_2x + m_3)\sqrt{\frac{x-\lambda_2}{x-l}}, \quad \text{(VIII.110)}$$

$$\tau_{xy}(x, 0) = 0,$$

where $l < |x| < h$, λ_2 is found from equation (VIII.106) and m_0, m_1, m_2, m_3 are defined by formulas (VIII.105).

The contact stresses σ_y^+ for the examined problems are determined in accordance with formula (VIII.103).

In order to obtain a graphic representation of the rate of extinction of the stress field caused by the presence of the crack in a deflected rod, the graphs (solid lines) of change of the components of the stress tensor

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$\sigma_x(x, 0)$, $\sigma_y(x, 0)$, $\tau_{xy}(x, 0)$ are constructed in Figure VIII.19 on the basis of formulas (VIII.109) as functions of the distance x for $|x| \geqslant l$. For comparison, the broken line represents the change of stresses

$$\sigma_y^0(x, 0) = Bx/l, \qquad \sigma_x^0(x, 0) = 0, \qquad \tau_{xy}^0(x, 0) = 0$$

during pure deflection of such a rod, but without a crack. As we see, when $|x| > 2l$, the perturbed stress state can be assumed, for all practical purposes, to coincide with the unperturbed stress state determined by functions (VIII.93) and (VIII.94).

By substituting the values of stresses $\sigma_y(x, 0)$ from (VIII.109) and (VIII.110) into equation (VIII.31) and then, by accomplishing limit transition in the equations obtained for $x \to \pm l$, we will find the formulas for the determination of the magnitude of limit (critical) external loads for the problems shown on Figures VIII.17 and VIII.18.

$$M_* = 2\sqrt{\frac{3}{2}}\frac{\delta h^3}{l}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}, \tag{VIII.111}$$

$$q_* = \sqrt{\frac{l}{2(l-\lambda_2)}}\frac{64\delta h^3}{(\lambda_2+l)\left[5\lambda_2^2 - 2l\lambda_2 + 5l^2 + 12\left(L^2 - \frac{2h^2}{5}\right)\right]}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}, \tag{VIII.112}$$

where the parameter λ_2 is determined from equation (VIII.106).

Deflection of Rod with Rectilinear Crack Located in Zone of Tension Stresses[1]. When rectilinear isolated cracks are located in the region of tension stresses and are directed perpendicular to the lateral surfaces of a deflected rod (Figures VIII.20 and VIII.21), the edges of the crack will always be free of contact stresses. If we relate the examined rod with a noncentral crack to coordinate system xOy and place the Oy axis on the longitudinal axis of the rod and the Ox axis on the plane of the crack, the boundary conditions on the contour of the crack, located in the zone of tension stresses, will have the form /585

$$\sigma_y^+(x, 0) = \sigma_y^-(x, 0) = 0,$$
$$\tau_{xy}^+(x, 0) = \tau_{xy}^-(x, 0) = 0, \tag{VIII.113}$$

where $a \leqslant |x| \leqslant b$ (a, b are the abscissas of the ends of the crack).

[1]See V. V. Panasyuk, B. L. Lozovoy [4], and also B. L. Lozovoy, V. V. Panasyuk [2].

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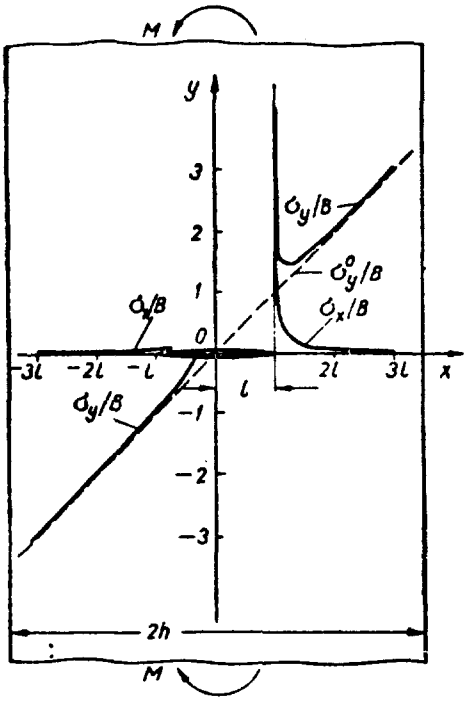


Figure VIII.19.

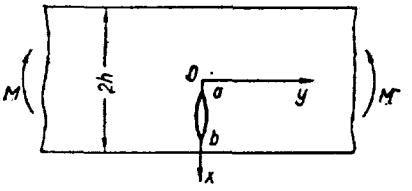


Figure VIII.20.

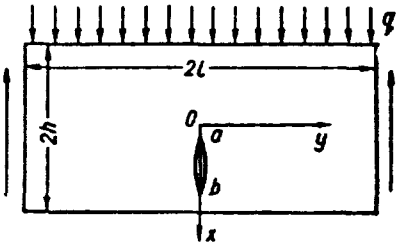


Figure VIII.21.

Far from the crack we will assume that the stress state in the rod is characterized by the functions (VIII.93) and (VIII.94), where the coefficients A_n and B_n (n = 0, 1, 2, 3) are defined by formulas (VIII.95) and (VIII.96) and the desired functions $\Phi(z)$ and $\Omega(z)$ for $z \to \infty$, satisfy conditions (VIII.100).

From (VIII.90), (VIII.91) and boundary conditions (VIII.113) it is easy to determine the functions of stresses $\Phi(z)$ and $\Omega(z)$ for the examined problems. Actually, by accomplishing limit transition in equations (VIII.90) and (VIII.91) to the contour of the crack, and by using boundary conditions (VIII.113), we arrive at the problem of linear conjugation for the boundary values of the functions $\Phi(z)$ and $\Omega(z)$:

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 0,$$
$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 0, \qquad \text{(VIII.114)}$$

where t is the abscissa of the points of the contour of the crack $(a \leqslant |t| \leqslant b)$.

By solving[1] equations (VIII.114) with consideration of the behavior of /586
functions $\Phi(z) + \Omega(z)$ and $\Phi(z) - \Omega(z)$ for $|z| \to \infty$ we obtain

[1]See N. I. Muskhelishvili [1].

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$$\Phi(z) = \frac{c_0 z^4 + c_1 z^3 + c_2 z^2 + c_3 z + c_4}{\sqrt{(z-a)(z-b)}} + \frac{1}{2}[\Phi_0(z) - \Omega_0(z)],$$
$$\Omega(z) = \frac{c_0 z^4 + c_1 z^3 + c_2 z^2 + c_3 z + c_4}{\sqrt{(z-a)(z-b)}} - \frac{1}{2}[\Phi_0(z) - \Omega_0(z)], \quad \text{(VIII.115)}$$

where c_0, c_1, c_2, c_3, c_4 are constants found from condition (VIII.100).

Recalling relations (VIII.93)-(VIII.96) from formulas (VIII.115), when conditions (VIII.114) are satisfied, we obtain the coefficients c_n (n = 0, 1, 2, 3, 4).

Thus, the stress functions for each of the examined problems will have the following form:

a) during pure deflection of rod with central crack (Figure VIII.20)

$$\Phi(z) = \frac{M}{16J} \cdot \frac{8z^2 - 4(a+b)z - (b-a)^2}{\sqrt{(z-a)(z-b)}} - \frac{M}{4J} z,$$
$$\Omega(z) = \frac{M}{16J} \frac{8z^2 - 4(a+b)z - (b-a)^2}{\sqrt{(z-a)(z-b)}} + \frac{M}{4J} z, \quad \text{(VIII.116)}$$

where $J = 4\delta h^3/3$ is the moment of inertia of the rod;

b) for deflection of rod with noncentral crack by uniformly distributed load (Figure VIII.21)

$$\Phi(z) = f_2(z) - \frac{q}{8J}\left[z^3 + \left(L^2 - \frac{7h^2}{5}\right)z + \frac{2h^3}{3}\right],$$
$$\Omega(z) = f_2(z) + \frac{q}{8J}\left[z^3 + \left(L^2 - \frac{7h^2}{5}\right)z + \frac{2h^3}{3}\right], \quad \text{(VIII.117)}$$

where

$$f_2(z) = \frac{q}{96J\sqrt{(z-a)(z-b)}}\left\{16z^4 + 8(a+b)z^3 + 2\left[12\left(L^2 - \frac{2h^2}{5}\right) - \right.\right.$$
$$\left. - (5a^2 + 6ab + 5b^2)\right]z^2 - 3(a+b)\left[(b-a)^2 + 4\left(L^2 - \frac{2h^2}{5}\right)\right]z -$$
$$\left. - \frac{(b-a)^2}{8}\left[24\left(L^2 - \frac{2h^2}{5}\right) + 13a^2 + 22ab + 13b^2\right]\right\}. \quad /587$$

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The stress components for the points located on the Ox axis, i.e., when y = 0, will have the following form:

a) for pure deflection of rod (Figure VIII.20)

$$\sigma_x(x,0)=\frac{M}{8J}\cdot\frac{8x^2-4(a+b)x-(b-a)^2}{\sqrt{(x-a)(x-b)}}-\frac{M}{J}x,$$

$$\sigma_y(x,0)=\frac{M}{8J}\cdot\frac{8x^2-4(a+b)x-(b-a)^2}{\sqrt{(x-a)(x-b)}},\qquad \tau_{xy}(x,0)=0 \qquad \text{(VIII.118)}$$

$$(x\geqslant b \text{ or } x\leqslant a);$$

b) for deflection of rod (beam) by uniformly distributed load (Figure VIII.21)

$$\sigma_x(x,0)=2f_2(x)-\frac{q}{2J}\left[x^3+\left(L^2-\frac{7h^2}{5}\right)x+\frac{2h^3}{3}\right],$$

$$\sigma_y(x,0)=2f_2(x),\qquad \tau_{xy}(x,0)=0, \qquad \text{(VIII.119)}$$

where $f_2(x)$ is defined by formulas (VIII.117) for z = x + iy and y = 0.

Recalling $\sigma_y(x, 0)$, defined by formulas (VIII.118) and (VIII.119), from the condition of limit state (VIII.31), which is conveniently represented in the form

$$\lim_{x\to\pm b}\left[\sqrt{|x-b|}\,\sigma_y^*(x,0)\right]=\frac{K}{\pi},$$

we obtain the magnitudes of limit forces M = M* and q = q_* for the problems shown in Figures VIII.20 and VIII.21:

$$M_*=\frac{4J}{3b+a}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}\qquad \left(J=\frac{4\delta h^3}{3}\right), \qquad \text{(VIII.120)}$$

$$q_*=\frac{192J}{35b^3+15ab^2+9a^2b+5a^3+24\left(L-\frac{2h^2}{5}\right)(a+3b)}\sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}, \qquad \text{(VIII.121)}$$

where 2l = b - a is the length of the crack; the other symbols are shown in Figures VIII.20 and VIII.21, respectively.

8. Limit State of Elastic Body Weakened by Stress Concentrator with Small Radius of Curvature

An interesting and unique theory of macrostresses, which permits the graphic and comparatively simple formulation of the criteria of the limit state

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of an elastic body weakened by stress concentrators with a small radius of curvature, is presented in the works of M. Ya. Leonov [2] and M. Ya. Leonov and K. N. Rusinko [1, 2]. The basic positions of the above approach are outlined in this section. /588

Macrodeformations and Macrostresses. We will examine a solid elastic body assuming it to be macrohomogeneous in the sense that the mechanical properties of any elementary volume bounded by a sphere of radius R, hypothetically cut from the above described body, and not extending to its surface, are identical for radius $R \geqslant \rho$, where 2ρ is some constant of the material, denoting the dimension of length[1], related to its structure. Moreover, we will assume that the rupture or plastic deformation within the body, enclosed within a sphere of radius ρ, do not depend on the character of distribution of stresses within this volume, but is determined by the deformation of the surface that bounds the above-described elementary volume. We will also assume that if cracks are not formed within the body, then the relative displacements of two points of the solid body, separated by a distance greater than 2ρ, are determined by the corresponding solutions of linear elasticity theory.

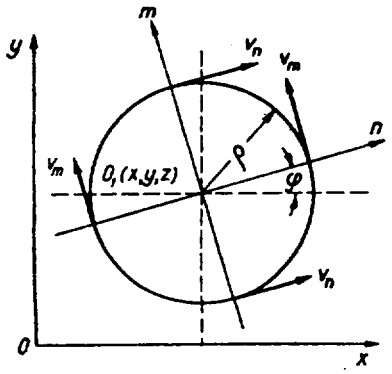


Figure VIII.22.

For such a body we will introduce certain relative elongations of the diameter of the sphere of radius ρ (macroelongations) and stresses (macrostresses) related to them by a linear law of stress.

We will call macroelongation $\varepsilon_n(x, y, z, \rho)$ the relative elongation of the corresponding diameter (n) (Figure VIII.22) of the sphere of radius ρ with its center at the point (x, y, z). By definition, macroelongation in the direction of the Ox axis (Figure VIII.22) can be represented in the form

$$\varepsilon_x(x, y, z, \varrho) = \frac{1}{2\varrho}[u(x+\varrho, y, z) - u(x-\varrho, y, z)], \qquad \text{(VIII.122)}$$

where $u(x \pm \rho, y, z)$ is elastic displacement in the direction of the Ox axis. Analogously, we will determine the macroelongations in the direction of the Oy and Oz axes.

---

[1]Moment elasticity theory (see Chapter VI) also yields, in addition to other material constants, a constant with the dimension of length. This constant reflects the "granularity" of the structure of a real hard body.

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We will define macroexpansion θ as the ratio of the increment (during deformation) ΔV of the volume of the material enclosed within the sphere of radius ρ, to its initial value ($V = 4/3\ \pi\rho^3$), i.e.,

$$\theta = \frac{3}{4\pi\varrho^3}\iint\limits_{(F)} \varepsilon_n dF_n, \tag{VIII.123}$$

where F is the surface of the sphere of radius ρ whose center is located at the point (x, y, z); dF_n is an element of surface F with normal n. /589

In view of the above definitions, formula (VIII.123) expresses the average elastic expansion within the sphere of radius ρ, i.e.,

$$\theta = \frac{1}{V}\iiint\limits_{(V)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z}\right) dV, \tag{VIII.124}$$

where u, v, w are the components of the vector of elastic displacements; V is volume bounded the sphere (F).

Under the effect of both surface and constant mass forces, elastic expansion represents a harmonic function. Therefore, by using the average harmonic function theorem, we find from (VIII.124)

$$\theta(x, y, z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z}, \tag{VIII.125}$$

i.e., macroexpansion does not depend on radius ρ of the examined sphere, but is determined by elastic expansion in the center of this sphere.

We will define macrodisplacement Γ_{mn} as change of the right angle between the rays (lines) that connect the points of the body located prior to deformation at the ends of the diameters of the sphere of radius ρ, parallel to m, n (Figure VIII.22).

For the plane (m, n), parallel, for instance, to the coordinate plane xOy, we will have

$$\Gamma_{mn} = \frac{1}{2\varrho}[v_m(x + \varrho\cos\varphi, y + \varrho\sin\varphi, z) - v_m(x - \varrho\cos\varphi, y - \varrho\sin\varphi, z)] + \\ + \frac{1}{2\varrho}[v_n(x - \varrho\sin\varphi, y + \varrho\cos\varphi, z) - v_n(x + \varrho\sin\varphi, y - \varrho\cos\varphi, z)], \tag{VIII.126}$$

where the projections of the elastic displacements in the directions of the m and n axes are, respectively,

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$$v_n = u\cos\varphi + v\sin\varphi, \qquad v_m = -u\sin\varphi + v\cos\varphi.$$

Here u and v are the projections of the vector of displacement onto coordinate axes Ox and Oy, respectively.

The component of normal macrostress on the plane to which direction (n) is normal is determined by the formula

$$s_n = 2G\varepsilon_n + \lambda\theta, \tag{VIII.127}$$

where the displacement modulus and Lame's constant are

$$G = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

Here E is Young's modulus; ν is Poisson's ratio.

In view of the fact that macroexpansion is not a function of the direction (n), the direction of maximum macroelongation coincides with the direction of maximum normal macrotension. /590

Macrotension of displacement (tangent) is the value

$$T_{mn} = G\Gamma_{mn}, \tag{VIII.128}$$

where Γ_{mn} is macrodisplacement defined by formula (VIII.126).

Macrotension Concentration Coefficient. Condition of Brittle Rupture. Let an unbounded body with stress concentrators in the form of holes or cracks be subjected at its infinitely distant points to tension by a field of uniform stresses of intensity s. Then the coefficient of concentration of normal macrostresses will be the ratio

$$k = \frac{s_n^{(\max)}}{s}, \tag{VIII.129}$$

where $s_n^{(\max)}$ is the maximum magnitude of macrotensions for the points (x, y, z) of the examined body.

The coordinates (x, y, z) of the center of the elementary sphere of radius ρ, for which macrostresses s_n achieve their maximum value, and direction (n) in which they act (Figure VIII.22), generally speaking, are unknown beforehand. To determine these values it is necessary to analyze expression (VIII.127) to the fullest extent.

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As before, we will denote through σ_0 the magnitude of resistance to rupture for a given material and we will find the intensity of external load (s) at which the greatest macrostress will achieve the magnitude σ_0. Such load is defined by formula

$$s_* = \frac{\sigma_0}{k}, \qquad \text{(VIII.130)}$$

where k is expressed by relation (VIII.129).

External load $s = s_*$ is defined as the limit or critical load. When the external load achieves the magnitude s_*, no new cracks can form within the body, and the existing cracks cannot spread. This condition of crack formation is called the macroscopic hypothesis of brittle rupture.

Elastic Plate Weakened by Elliptical Hole under Tension. We will determine the coefficient of concentration of normal macrotensions for an unbounded body (in the case of plane deformation) with a cylindrical cavity, the base of which is an ellipse with semiaxes a, b. We will assume that the surface of the cavity is free of external stresses, and that tension stresses of intensity s are applied at the infinitely distant points of the body, directed perpendicular to the large axis 2a of the elliptical hole.

The displacements in the elastic plate are defined by formulas (see §1, Chapter I)

$$2G(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \qquad \text{(VIII.131)}$$

During plane deformation $\varkappa = 3 - 4\nu$. The analytical functions in the range occupied by the body are /591

$$\varphi(z) = \frac{s}{4(a-b)}\left[2a - \sqrt{z^2 - a^2 + b^2} - (a+b)z\right],$$

$$\psi(z) = \frac{s}{2(a-b)}\left[\frac{a^2+b^2}{a-b}z - \frac{2ab}{a-b}\sqrt{z^2 - a^2 + b^2} - \frac{a(a^2+b^2)}{\sqrt{z^2 - a^2 + b^2}}\right], \qquad \text{(VIII.132)}$$

where $\lim z^{-1}\sqrt{z^2 - a^2 + b^2} = 1$ for $z \to \infty$.

The maximum macrostress develops in the elementary volumes bounded by spheres of radius ρ, with the center at the points $(a + \rho, 0)$ and $(-a - \rho, 0)$. For the above spheres, in accordance with formulas (VIII.125), (VIII.131) and (VIII.132), we obtain the following formula for the determination of macroexpansion

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$$2G\theta(a+\varrho, 0) = \frac{1-2\nu}{1-\eta}\left[\frac{2(1+\alpha)}{\sqrt{\eta^2+2\alpha+\alpha^2}} - (1+\eta)\right]s, \tag{VIII.133}$$

where

$$\alpha = \frac{\varrho}{a}, \qquad \eta = \frac{b}{a}.$$

Macroelongation for these spheres will be maximal:

$$\varepsilon_m = \frac{1}{\varrho} v(a+\varrho, \varrho).$$

Hence, and on the basis of relations (VIII.131) and (VIII.132), we obtain

$$2G\varepsilon_m = \frac{s}{2(1-\eta)}\left[-(1-2\nu)(1+\eta) + \frac{1+\eta^2}{1-\eta} + \right.$$
$$\left. + \frac{(1+\alpha)\sqrt{2}}{\sqrt{\eta^2+2\alpha+2\alpha_1}}\left(3-4\nu-\frac{2\eta}{1-\eta}+\frac{\eta^2-2\alpha-2\alpha^2}{2\alpha_1}\right)\right], \tag{VIII.134}$$

where

$$\alpha_1 = \sqrt{\left(\frac{\eta^2}{2}+\alpha\right)^2+\alpha^2(1+\alpha)^2}.$$

Thus, from (VIII.127), (VIII.133) and (VIII.134), the maximal tension macrostresses will be

$$s_m^{(\max)} = ks, \tag{VIII.135}$$

where the concentration coefficient is

$$k = \frac{\eta^2}{(1-\eta)^2} + \frac{2(1+\alpha)\nu}{(1-\eta)\sqrt{\eta^2+2\alpha+\alpha^2}} +$$
$$+ \frac{1+\alpha}{(1-\eta)\sqrt{2(\eta^2+2\alpha+2\alpha_1)}}\left(3-4\nu-\frac{2\eta}{1-\eta}+\frac{\eta^2-2\alpha-2\alpha^2}{2\alpha_1}\right) \tag{VIII.136}$$

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Assuming in (VIII.136) b = η = 0, we obtain the coefficient of concentration of macrostresses during tension of an elastic plate with a slit of length 2a: /592

$$k = (1+\alpha)\left[\frac{2\nu}{\sqrt{2\alpha+\alpha^2}} + \frac{(3-4\nu)\alpha_2 - (1+\alpha)}{2\alpha_2\sqrt{\alpha(1+\alpha_2)}}\right], \qquad \text{(VIII.137)}$$

where

$$\alpha_2 = \sqrt{\alpha^2+2\alpha+2}.$$

If, in the right hand side of equation (VIII.136), we proceed to the limit for η → 1, we obtain the coefficient k for an unbounded plate with a round hole under uniaxial tension

$$k = \frac{2\nu\alpha^2}{(1+\alpha)^2(1+2\alpha+2\alpha^2)} + \frac{3+11\alpha+25\alpha^2+40\alpha^3+42\alpha^4+24\alpha^5+8\alpha^6}{(1+2\alpha+2\alpha^2)^3}. \qquad \text{(VIII.138)}$$

If semiaxis a of the elliptical hole is small in comparison with the structure parameter ρ (i.e., α → ∞), then we obtain from (VIII.136) k = 1.

Thus, holes and slits, the dimensions of which are small in comparison with structure parameter ρ, do not cause the concentration of macrostresses. This, of course, is to be expected.

Let us consider a macroscopic elliptical hole for the case where the semiaxes (a, b) are much greater than structural parameter ρ (i.e., α → 0). Disregarding in formula (VIII.136) the values of α that are small in comparison with unity and η, we obtain formula[1] (VIII.2):

$$k = 1 + \frac{2a}{b} \qquad (a \gg \varrho;\ b \gg \varrho). \qquad \text{(VIII.139)}$$

Thus, in the case of a macroscopic hole, the coefficient of concentration of macrostresses coincides with the coefficient of concentration of elastic stresses given by classical elasticity theory.

The coefficient of concentration of perfect (elastic) stresses for a fixed ratio a/b is independent of the absolute dimensions of the semiaxes of the ellipse. In particular, for a circular cavity, k = 3, regardless of the magnitude of the radius of the round hole. At the same time, the coefficient of concentration of macrostresses k in the case of unlimited decrease of the dimensions of the hole approaches the value for a solid body, namely k = 1.

---

[1] In the case under examination, the elastic plate is subjected to tension by forces s = const, perpendicular to the 2a axis of the elliptical hole.

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We will examine the case of a macroscopic slit ($a \gg \rho$; $\eta = 0$). Formula (VIII.137) (after discarding the small ρ/a of higher order) can be converted to the form

$$k = \frac{\beta_\nu}{2}\sqrt{\frac{a}{\rho}}, \qquad \text{(VIII.140)}$$

where

$$\beta_\nu = \left[2\left(1 - \frac{\sqrt{2}}{\sqrt{1+\sqrt{2}}}\right)\nu + \frac{3\sqrt{2}-1}{2\sqrt{1+\sqrt{2}}}\right]\sqrt{2}. \qquad \text{(VIII.141)}$$

Certain values of the coefficient k (VIII.136) for a slit ($\eta = 0$), round hole ($\eta = 1.0$) and three intermediate values of η, equal to 0.25, 0.50 and 0.75 for an elliptical hole, are presented in the table. The coefficient k was calculated for Poisson ratios ν equal to 0.15, 0.20, 0.25 and 0.30. /593

ν	a/ρ								Hole
	0	1	2	3	5	10	15	25	
0,15	1	1,183	1,396	1,585	1,911	2,551	3,061	3,885	Slit
0,20	1	1,188	1,403	1,595	1,924	2,571	3,084	3,917	(η=0)
0,25	1	1,192	1,411	1,604	1,937	2,590	3,108	3,948	
0,30	1	1,197	1,418	1,614	1,950	2,610	3,133	3,979	
0,15	1	1,200	1,424	1,623	1,967	2,651	3,199	4,049	Elliptical
0,20	1	1,205	1,433	1,633	1,979	2,665	3,213	4,061	(η=0,25)
0,25	1	1,210	1,441	1,644	1,992	2,680	3,227	4,073	
0,30	1	1,216	1,450	1,654	2,005	2,695	3,241	4,084	
0,15	1	1,215	1,450	1,661	2,029	2,707	3,153	3,682	Elliptical
0,20	1	1,220	1,458	1,670	2,037	2,712	3,157	3,684	(η=0,5)
0,25	1	1,226	1,466	1,678	2,045	2,717	3,160	3,686	
0,30	1	1,232	1,474	1,686	2,052	2,722	3,164	3,688	
0,15	1	1,226	1,476	1,702	2,064	2,600	2,875	3,149	Elliptical
0,20	1	1,232	1,483	1,708	2,068	2,602	2,876	3,149	(η=0,75)
0,25	1	1,237	1,490	1,713	2,072	2,603	2,877	3,150	
0,30	1	1,244	1,494	1,723	2,077	2,606	2,878	3,150	
0,15	1	1,240	1,500	1,725	2,041	2,422	2,610	2,741	Round
0,20	1	1,244	1,505	1,728	2,043	2,423	2,610	2,741	(η=1,0)
0,25	1	1,249	1,510	1,731	2,044	2,423	2,610	2,741	
0,30	1	1,254	1,514	1,735	2,046	2,424	2,610	2,741	

Tr. Note: Commas indicate decimal points.

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Figure VIII.23 shows the graphs of k as a function of the ratio a/ρ for certain values of η. For comparison, the coefficients of concentration of perfect (elastic) stresses given by classical elasticity theory, for which the values of the coefficients of concentration of macrostresses approach asymptotically during the unbounded increase of semiaxes a and b, are also represented in the figure by the broken lines. In the case of tension of a body weakened by a slit ($\eta = 0$), the coefficient of concentration of macrostresses obviously has no asymptote.

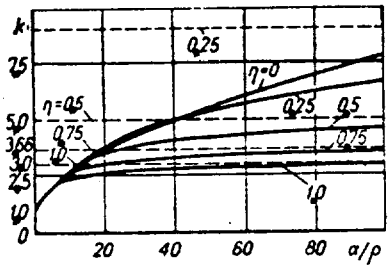


Figure VIII.23.

Determination of Limit Load. Relationship between Parameter ρ and Other Material Constants. For an elastic plate weakened by an elliptical hole and located in a homogeneous stress state at infinity, the stress concentration coefficients k are determined for various ratios $\eta = b/a$ by formulas (VIII.136)-(VIII.141). By using these formulas and relation (VIII.130), it is easy to determine the magnitude of limit load $s = s_*$ for the problem at hand. For instance, in the case of an elastic plate with a macroscopic slit, we have on the basis of formulas (VIII.130) and (VIII.140), /594

$$s_* = \frac{2\sigma_0}{\beta_\nu}\sqrt{\frac{\rho}{a}}, \qquad \text{(VIII.142)}$$

where β_γ is defined by formula (VIII.141).

The relationship obtained between limit stress and the dimension a of the crack coincides with an accuracy up to a constant factor with Griffith's formula

$$s_* = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)a}}. \qquad \text{(VIII.143)}$$

We will require that limit load (VIII.142) coincides identically with limit load (VIII.143). We will obtain the relationship between structural parameter ρ and other material constants:

$$\rho = \frac{\beta_\nu^2}{2\pi(1-\nu^2)}\cdot\frac{E\gamma}{\sigma_0^2} \qquad \text{(VIII.144)}$$

or

$$\rho = \mu^*\frac{E\gamma}{\sigma_0^2}, \qquad \text{(VIII.145)}$$

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where $\mu^* = \frac{\beta_\nu^2}{2\pi(1-\nu^2)}$ is a numerical factor equal to 0.386, 0.386, 0.402, 0.421, respectively, for values of ν equal to 0.15, 0.20, 0.25, 0.30; γ is the effective surface energy of the material.

§9. Brittle Rupture of Material[1]

On the basis of analysis of the limit equilibrium state of a plane (brittle plate) with defects of the arbitrarily oriented sharp-pointed cavity-crack or hypocycloid hole types, a diagram of limit loads can be constructed for the case where the body is subjected to plane biaxial tension -- compression. This diagram can be used as the criterion for determining the beginning of brittle or quasibrittle rupture of solid bodies weakened by variously oriented defects of the above-mentioned types (stress concentrators with small radius of curvature).

Biaxial Tension -- Compression of Elastic Plate with Hole in the Form of a Hypocycloid. Let us consider an elastic plane xOy (plate of unit thickness), weakened by a hole in the form of a hypocycloid, i.e., by a hole whose contour L is described in parametric form by equations /595

$$x = A\left(\cos\vartheta + \frac{1}{n}\cos n\vartheta\right), \qquad y = A\left(\sin\vartheta - \frac{1}{n}\sin n\vartheta\right), \tag{VIII.146}$$

where $0 \leqslant \vartheta \leqslant 2$; $A > 0$; n is a whole positive number (n = 1, 2, 3 ...).

The contours for n = 1 and n = 2 are represented in Figures VIII.24 and VIII.25, respectively, by formulas (VIII.146). The vertices of the contour of a hypocycloidal hole, as follows from formulas (VIII.146), are angular cusp points, and contour L, for a fixed n, has the cusp point n + 1.

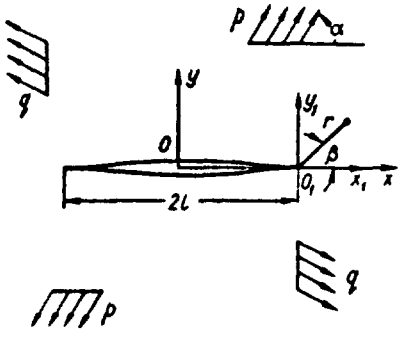


Figure VIII.24

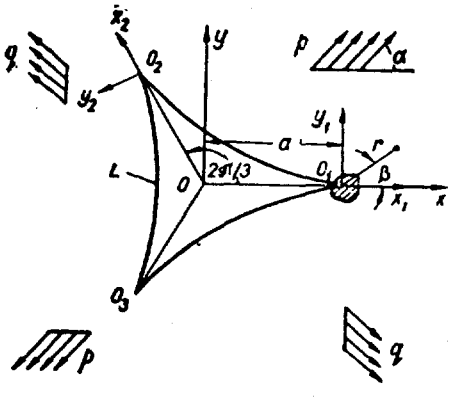


Figure VIII.25.

[1] See V. V. Panasyuk [4].

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Let an elastic plate with a hole in the form of a hypocycloid be subjected to tension-compression by monotonically increasing stresses p and q ($q/p = \eta_0$), acting in mutually perpendicular directions, and applied at the infinitely distant points of the plate, where stresses p are directed at angle α to the Ox axis (see Figures VIII.24 and VIII.25). The contour of the hole is assumed to be free of external forces. We are required to determine the values of limit stresses $p = p_*$ and $q = q_*$, whereupon one of the vertices of the given hole goes into the state of limit equilibrium and it becomes possible for a rupture crack in the vicinity of this vertex to spread.

It is necessary first of all to determine the field of elastic stresses in the vicinity of the angular points of the examined plate for the given external action (Figures VIII.24 and VIII.25), and then, by using limit state equation (VIII.31), to determine the magnitude of the limit load. Here, however, in the general case of angular points on the contour bounded by the examined range, the determination of the field of elastic stresses, i.e., the solution of the corresponding problem of elasticity theory, is quite difficult. /596
In the case where there are angular cusp points on the contour of the given infinite range containing a hole, as is the case of the given problem (Figures VIII.24 and VIII.25), the solution of the required problem of elasticity theory for such a range can be found, for instance, from the solution of the corresponding problem of elasticity theory for a range bounded by a contour with a continuously changing tangent and which depends on certain parameters, if, in such a solution, the parameters (characterizing the smooth contour under consideration) approach given limit values such that it will be possible to obtain at the limit a contour with angular points of the specified type; for instance, by the method used for solving the problem of elasticity theory for an elastic plate with a rectilinear slit (crack) from the solution of the corresponding problem for a plate with an elliptical hole[1].

We will construct the solution of the problem of elasticity theory for the case where an elastic plate is weakened by a hole in the form of a hypotrochoid, and where biaxial stress field (p, q) acts at its infinitely distant points. For this purpose we will examine the function

$$z = \omega(\zeta) = A\left(\zeta + \frac{m}{\zeta^n}\right); \qquad \text{(VIII.147)}$$

here $z = x + iy$ is a complex variable in the plane $z(xOy)$, and $\zeta = \xi + i\eta = \rho e^{i\vartheta}$ is a complex variable in the plane $\zeta(\xi O\eta)$; $A > 0$; $0 \leqslant m \leqslant 1/n$; n is a whole positive number.

The function (VIII.147) conformally maps the range located in plane z outside of the hypotrochoid on the exterior of the unit circle in plane ζ. It is easy to show also that when $m = 1/n$, function (VIII.147) conformally maps the

[1]See §2, Chapter II.

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exterior of the hypocycloid in plane z onto the exterior of the unit circle in plane ζ, whereupon contour L of the hypocycloid in plane z is found from equation (VIII.147) for m = $1/n$, $A=\frac{na}{n+1}$, $a>0$ and $|\zeta|=1$.

Thus, if with a given stress field at the infinitely distant points of an elastic plate containing a hole in the form of a hypotrochoid $0 \leqslant m < 1/n$ will be the solution of the problem of elasticity theory, then, by assuming in this solution that $m \to 1/n$, $A = na/n + 1$, we will obtain the solution of the corresponding problem of elasticity theory for a plate with a hole in the form of a hypocycloid.

Since contour L of the hypocycloid (and also of the hypotrochoid) is assumed to be free of external forces, then, for the determination of stress functions, we will have the condition[1]

$$\varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = 0 \text{ on L.} \qquad \text{(VIII.148)}$$

On the basis of complex potentials $\phi_1(z)$ and $\psi_1(z)$, we determine[2] the stress components

$$\sigma_x + \sigma_y = 2[\varphi_1'(z) + \overline{\varphi_1'(z)}], \quad \sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\varphi_1''(z) + \psi_1'(z)]. \qquad \text{(VIII.149)}$$

In transformed range ζ, contour condition (VIII.148) can be represented in the form /597

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = 0,$$

where $\sigma = e^{i\vartheta}$, $0 < \vartheta < 2\pi$;

$$\varphi(\zeta) = \varphi_1[\omega(\zeta)], \qquad \psi(z) = \psi_1[\omega(z)],$$
$$\varphi_1'(z) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \qquad \psi_1'(z) = \frac{\psi'(z)}{\omega'(z)}. \qquad \text{(VIII.150)}$$

For an elastic plate with a hole, where main stresses p and q act at its intinitely distant points, and the contour of the hole is free of external forces, the functions[3] are

$$\varphi(\zeta) = A\Gamma\zeta + \varphi_0(\zeta), \qquad \psi(\zeta) = A\Gamma'\zeta + \psi_0(\zeta); \qquad \text{(VIII.151)}$$

[1]See §1, Chapter I.
[2]See §1, Chapter I.
[3]See §1, Chapter II.

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here $\phi_0(\zeta)$ and $\psi_0(\zeta)$ are holomorphic functions in the range $|\zeta| \geqslant 1$; $\varphi_0(\infty) =$ the constants are

$$\Gamma = \bar{\Gamma} = \frac{1}{4}(p+q), \qquad \Gamma' = -\frac{1}{2}(p-q)\,e^{-2i\alpha}, \tag{VIII.152}$$

where α is the angle between the Ox axis and the direction of main stress p.

We will assume that in formula (VIII.147) $0 < m < 1/n$. In this case the contour of the hole will represent a hypotrochoid, i.e., there are no angular points on the contour of the hole. Consequently, to determine functions $\phi(\zeta)$ and $\psi(\zeta)$ from contour condition (VIII.150), we may use N. I. Muskhelishvili's method, outlined in §1, Chapter I. Consequently, we find readily stress functions $\phi(\zeta)$ and $\psi(\zeta)$ for an elastic plate with a hole in the form of a hypotrochoid $0 < mn < 1$. Accomplishing limit transition in these functions for $mn \to 1$ $(m = 1/n)$, we obtain the desired functions for an infinite elastic plate with a hole in the form of a hypocycloid, where the forces p and q are applied at infinitely distant points of this plate:

$$\begin{aligned}
\varphi(\zeta) &= A\Gamma\zeta - G_0^*(\zeta) - A\left(\frac{\Gamma}{n\zeta^n} + \frac{\bar{\Gamma}^1}{\zeta}\right), \\
\psi(\zeta) &= A\Gamma'\zeta + G_\infty^*(\zeta) - \frac{\zeta^n\left(1+\frac{1}{n}\,\zeta^{n+1}\right)}{\zeta^{n+1}-1}\varphi_0'(\zeta) - \\
&\quad - A\Gamma\left[\frac{1}{\zeta} + \frac{(n+1)\,\zeta^n}{n\,(\zeta^{n+1}-1)}\right];
\end{aligned} \tag{VIII.153}$$

here

$$\varphi_0(\zeta) = -G_0^*(\zeta) - A\left(\frac{\Gamma}{n\zeta^n} + \frac{\bar{\Gamma}'}{\zeta}\right);$$

/598

$$G_0^*(\zeta) = \begin{cases} -\frac{1}{n}\left[\frac{\bar{a}_1}{\zeta^{n-2}} + 2\frac{\bar{a}_2}{\zeta^{n-3}} + \ldots + \frac{(n-2)\,\bar{a}_{n-2}}{\zeta}\right] \text{ for } n > 2, \\ 0 \text{ for } n < 2; \end{cases} \tag{VIII.154}$$

$$G_\infty^*(\zeta) = \begin{cases} -\frac{1}{n}\left[a_1\zeta^{n-2} + 2a_2\zeta^{n-3} + \ldots + (n-1)\,a_{n-1}\right] \text{ for } n \geqslant 2, \\ 0 \text{ for } n < 2; \end{cases}$$

$$\varphi_0(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \frac{a_3}{\zeta^3} + \ldots \text{ for } |\zeta| \geqslant 1, \tag{VIII.155}$$

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where the coefficients a_1, a_2, a_3, ... are found by comparing the expressions of the right and left sides in formulas (VIII.154) and (VIII.155).

Stresses in Vicinity of Angular Points. To determine the principal part of the stress tensor components in the vicinity of the angular points of the contour of the given range (Figure VIII.25) with a hypocycloidal hole, it is necessary to bear in mind the following. Angular points O_j (j = 1, 2, 3, ...) on the boundary of the range are cusp points. In the small vicinity of such points, the stress tensor components in a polar coordinate system with the origin placed at angular point O_j are determined for the case of the plane stress state by the following relations:

$$\begin{aligned}
\sigma_r &= \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left(5\cos\frac{\beta}{2}-\cos\frac{3}{2}\beta\right)+k_{2,j}\left(-5\sin\frac{\beta}{2}+3\sin\frac{3}{2}\beta\right)\right\}+0(1),\\
\sigma_\beta &= \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left(3\cos\frac{\beta}{2}+\cos\frac{3}{2}\beta\right)-3k_{2,j}\left(\sin\frac{\beta}{2}+\sin\frac{3}{2}\beta\right)\right\}+0(1),\\
\tau_{r\beta} &= \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left(\sin\frac{\beta}{2}+\sin\frac{3}{2}\beta\right)+k_{2,j}\left(\cos\frac{\beta}{2}+3\cos\frac{3}{2}\beta\right)\right\}+0(1),
\end{aligned} \qquad \text{(VIII.156)}$$

where j = 1, 2, 3 is the number of the angular point; 0(1) is the regular part of the stress component for $r \to 0$; $k_{1,j}$ and $k_{2,j}$ are coefficients of intensity (concentration) of stresses in the vicinity of the angular point, which were found[1] from the equation

$$k_{1,j}\sqrt{\frac{2}{r}}\cos\frac{\beta}{2}-k_{2,j}\sqrt{\frac{2}{r}}\sin\frac{\beta}{2}+0(1)=4\mathrm{Re}\Phi_j(z_j), \qquad \text{(VIII.157)}$$

where $\Phi_j(z_j)$ is a function equal to

$$\Phi_j(z_j)=\frac{\varphi'(\zeta)}{\omega'(\zeta)},$$

related to the local system of polar coordinates (r, β), with the origin at the angular point O_j; $z_j = re^{i\beta}$. This function can be represented in the form

$$\Phi_j(z_j)=\Phi(z_{0j}+z_je^{i\vartheta_j})=\frac{\varphi'(\sigma_j+\zeta_je^{i\vartheta})}{\omega'(\sigma_j+\zeta_je^{i\vartheta})}, \qquad \text{(VIII.158)}$$

where j = 1, 2, 3...; $z_j = re^{i\beta}$; $\zeta_j = \rho_1 e^{i\lambda}$; $r \ll a$; $\rho_1 \ll 1$; z_{0j} is the affix /599

[1]See V. V. Panasyuk, L. T. Berezhnitskiy [1], Si, Paris, Erdogan [1].

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of the origin of the polar coordinate system in the plane z, and ϑ_j is the cusp angle of the polar axis of this system relative to the Ox axis.

Thus, by using formulas (VIII.153), (VIII.157) and (VIII.158), we can find the values of components $k_{1,j}$ and $k_{2,j}$ (j= 1, 2, 3,...) for an elastic plane (plate) with a hypocycloidal hole (in particular, with a rectilinear crack) under biaxial tension (Figures VIII.24 and VIII.25).

By way of examples, we will consider the following problems: 1) the case of a rectilinear cavity (crack) and 2) the case of a hole in the form of a hypocycloid with three vertices.

For the first problem (Figure VIII.24) it is necessary to assume in formulas (VIII.153) that n = 1. In this case formulas (VIII.153) will acquire simple form and in accordance with expressions (VIII.157) and (VIII.158), we will obtain

$$\begin{aligned} k_{1,1} &= p\sqrt{l}\,(\sin^2\alpha + \eta_0\cos^2\alpha) \quad (l = a), \\ k_{2,1} &= p\,(1-\eta_0)\sqrt{l}\,\sin\alpha\cos\alpha \qquad (\eta_0 = q/p), \end{aligned} \tag{VIII.159}$$

where l is the half length of the crack (slit).

For the second problem (Figure VIII.25) it is necessary to assume in formulas (VIII.147) and (VIII.153) that n = 2:

$$\begin{aligned} \varphi(\zeta) &= \frac{2}{3}a\left[\Gamma\left(\zeta - \frac{1}{2\zeta^2}\right) - \frac{\bar{\Gamma}'}{\zeta}\right]; \\ \Phi(z) &= \frac{\varphi'(z)}{\omega'(\zeta)} = \frac{1}{\zeta^3-1}\left[(\zeta^3+1)\Gamma + \zeta\bar{\Gamma}'\right]; \\ z &= \omega(\zeta) = \frac{2}{3}a\left(\zeta + \frac{1}{2\zeta^2}\right). \end{aligned} \tag{VIII.160}$$

Hence, and on the basis of formula (VIII.158), we obtain

$$\Phi_j(r,\beta) = \frac{1}{3}\left(2\Gamma + e^{i\vartheta_j}\bar{\Gamma}'\right)\sqrt{\frac{a}{r}}\,e^{-i\frac{\beta}{2}} + 0(1). \tag{VIII.161}$$

where j = 1, 2, 3.

By using formula (VIII.161) and expression (VIII.152), we obtain

$$\Phi_j(r,\beta) = \frac{1}{6}\sqrt{\frac{a}{r}}\left[p + q - (p-q)\,e^{i(2\alpha+\vartheta_j)}\right]e^{-i\frac{\beta}{2}} + 0(1). \tag{VIII.162}$$

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By substituting the function (VIII.162) into equation (VIII.157) and comparing coefficients for identical harmonics, we obtain

$$k_{1,j} = \frac{\sqrt{2a}}{3}[p + q - (p - q)\cos(2\alpha + \vartheta_j)],$$
$$k_{2,j} = \frac{\sqrt{2a}}{3}(p - q)\sin(2\alpha + \vartheta_j), \qquad \text{(VIII.163)}$$

where angles ϑ_j (j = 1, 2, 3) have, for vertices O_1, O_2 and O_3, respectively, the following values: $\vartheta_1 = 0$, $\vartheta_2 = 2/3\pi$ and $\vartheta_3 = 2/3\ \pi$.

Determination of Limit Load. Let external forces p and q, applied at infinitely distant points of a plate with a hypocycloidal hole (Figure VIII.25) /600 increase monotonically in proportion to some parameter. We will determine the values of limit loads $p = p_*$ and $q = q_*$ ($q/p = \eta_0$, $\eta_0 = \text{const}$), i.e., external stresses p_* and q_* whereupon it is possible for a rupture crack to develop (spread) from the angular points of the given hole.

To determine the magnitude of stresses p_* and q_* for an elastic plate weakened by a hole with angular cusp points, we obtain the following equations:

$$p_* = \min\{p_{*j}\}, \qquad q_* = \eta_0 p_* \qquad \left(\eta_0 = \frac{q}{p}\right),$$
$$\lim_{r\to 0}\{\sqrt{r}\sigma_\beta(r, \beta_*, p_{*j}, \eta_0)\} = \frac{K}{\pi}. \qquad \text{(VIII.164)}$$

Here j = 1, 2, 3 is the number of the angular points; K is the coupling modulus[1], which is expressed through Young's modulus E, Poisson's ratio ν and effective surface energy γ of the material of the plate as follows: $K = \sqrt{\frac{\pi E\gamma}{1+\nu^2}}$ for plane deformation and $K = \sqrt{\pi E\gamma}$ for the generalized plane stress state;

$$\beta_* = \pm 2\arcsin\sqrt{\frac{6n_j^2 + 1 - \sqrt{8n_j^2+1}}{2(9n_j^2+1)}} \text{ for } k_{1,j} > 0,$$
$$\beta_* = \pm 2\arcsin\sqrt{\frac{6n_j^2 + 1 + \sqrt{8n_j^2+1}}{2(9n_j^2+1)}} \text{ for } k_{2,j} < 0, \qquad \text{(VIII.165)}$$

where "plus" corresponds to the values $k_{2,j} < 0$ and "minus" to the values $k_{2,j} > 0$; the parameter $n_j = k_{2,j}/k_{1,j}$.

[1]See G. I. Barenblatt [1].

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By using equations (VIII.164), (VIII.165) and expressions (VIII.159), (VIII.163), we can determine the magnitude of limit stresses for a plate containing a sharp-pointed slit (Figure VIII.24):

$$p_* = Rf(\alpha, \eta_0), \qquad q_* = \eta_0 Rf(\alpha, \eta_0);$$

here

$$f(\alpha, \eta_0) = \left\{ \cos^2\frac{\beta_*}{2}\left[\cos\frac{\beta_*}{2}(\sin^2\alpha + \eta_0\cos^2\alpha) - 3(1-\eta_0)\sin\alpha\cos\alpha\sin\frac{\beta_*}{2}\right]\right\}^{-1}.$$

$$R = \frac{K\sqrt{2}}{\pi\sqrt{l}}: \qquad l = a; \qquad \eta_0 = \frac{q}{p}, \tag{VIII.166}$$

where the values of angles β_* are defined by formulas (VIII.165), if we substitute in them

$$n_j = \frac{k_{2,1}}{k_{1,1}} = \frac{(1-\eta_0)\sin\alpha\cos\alpha}{\sin^2\alpha + \eta_0\cos^2\alpha};$$

the coefficients $k_{1,1}$ and $k_{2,1}$ are defined by formulas (VIII.159).

For a plate with a hole in the form of a hypocycloid with three vertices /601
(Figure VIII.25), on the basis of equations (VIII.164), (VIII.165) and equalities (VIII.163), we find

$$p_* = \min\{p_{*,j}\}, \qquad q_* = \eta_0 p_* \qquad (j = 1, 2, 3),$$

$$p_{*,j} = R_1 f_j(\alpha, \eta_0, \beta_*, \vartheta_j), \qquad \eta_0 = \frac{q}{p}, \qquad R_1 = \frac{\sqrt{2}K}{\pi\sqrt{a}}, \tag{VIII.167}$$

where

$$f_j(\alpha, \eta_0, \beta_*, \nu_j) = 6\sqrt{2}\left\{[1 + \eta_0 - (1-\eta_0)\cos(2\alpha + \vartheta_j)]\left(3\cos\frac{\beta_*}{2} + \cos\frac{3}{2}\beta_*\right) - \right.$$

$$\left. - 3(1-\eta_0)\sin(2\alpha + \vartheta_j)\left(\sin\frac{\beta_*}{2} + \sin\frac{3}{2}\beta_*\right)\right\}^{-1} \tag{VIII.167}$$

and the parameter β_* is defined by formulas (VIII.165) and (VIII.163).

Diagram of Limit Stresses. If the parameters η_0, α and a are given, then, on the basis of formulas (VIII.166) or (VIII.167), we can calculate the magnitude of limit stresses $p = p_*$ and $q = q_*$ in each specific case. However, it

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is better to construct, on the basis of these formulas, the diagrams of limit (in the sense of strength) stresses for a body subjected to the plane biaxial stress state.

Before constructing such diagrams, we will note the following. The rupture of brittle bodies, as we know, is related to the development of forces during the process of their deformation, under which defects of the sharp-pointed cavity-crack type go into the state of limit equilibrium and, consequently, during small perturbations of the field of external stresses, it is possible for them to spread through the cross section of the body. To determine the conditions of such a state we can use formulas of the type (VIII.166) or (VIII.167).

Actually, we will assume that a brittle body contains internal defects of the sharp-pointed cavity-crack type, the characteristic linear dimension of which is equal to a, and such defects are variously oriented and scattered through the entire volume of the body such that they can be assumed to be isolated from each other. If such a body is subjected to the plane stress state due to the effect of monotonically increasing (principal) stresses p and q ($q/p = \eta_0$), then, among the variously oriented isolated defects -- cracks, within the deformed body, there should exist such (the most dangerous for the given η_0) orientation for which limit stresses $p = p_*^{(\min)}$ and $q = q_*^{(\min)}$ acquire their minimal values in comparison with the limit values of p_* and q_* for different orientation of such defects.

As external stresses p and q achieve the values $p_*^{(\min)}$ and $q_*^{(\min)}$ in the given brittle body containing sharp-pointed crack-like cavities, it becomes possible for the given defect to develop (spread) in a more dangerous orientation. This signifies that when the external stresses exceed somewhat the values $p_*^{(\min)}$ and $q_*^{(\min)}$, the body may be ruptured. Therefore, from the point of view of the strength of a brittle body, weakened by defects of the sharp- /602
pointed crack-cavity type (or sharp-pointed holes) stresses $p_*^{(\min)}$ and $q_*^{(\min)}$ are the maximum tolerable stresses under the conditions of the plane stress state of the body. Consequently, the curve that determines the change of stresses $p_*^{(\min)}$ and $q_*^{(\min)}$ as a function of the parameter η_0, i.e., as a function of the form of the plane stress state, will represent the diagram of the limit (tolerable from the point of view of the strength of the body) stresses.

The orientation of the defect for which, in accordance with formulas (VIII.167) or (VIII.166), we obtain the minimum stresses p_* and q_*, it is defined by some angle $\alpha = \alpha_*$ ($0 \leqslant \alpha \leqslant \pi/2$). For such a value of angle $\alpha = \alpha_*$, the function $f(\alpha, \eta, \beta_*, \vartheta_j)$ acquires (for the given η_0) its maximal value in comparison with the other values of angle α ($0 \leqslant \alpha \leqslant \pi/2$). By constructing the

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graph of change, for instance, of the function $f_1(\alpha, \eta_0, \beta_*, 0)$ for a given η_0 where α acquires the values in the range $0 \leqslant \alpha \leqslant \pi/2$, we may determine the angle $\alpha = \alpha_*$. Then, by using formulas (VIII.167), we can find the expressions

$$q_*^{(min)} = \eta_0 R_1 f_1(\alpha_*, \eta_0, \beta_*, 0), \qquad p_*^{(min)} = R_1 f_1(\alpha_*, \eta_0, \beta_*, 0), \tag{VIII.168}$$

where the function $f(\alpha, \eta_0, \beta_*, 0)$ is represented by formula (VIII.167) for $\vartheta_j = 0$.

In the partial case where the body is subjected to unaxial tension by stresses p ($q = 0$, $\eta_0 = 0$), we find, on the basis of formulas (VIII.167),

$$p_*^{(min)} = R_1 f_1(\alpha_*, 0) = 1.03 R_1; \qquad q_* = 0. \tag{VIII.169}$$

The value R_1 in formulas (VIII.167)-(VIII.169) is

$$R_1 = \frac{\sqrt{2}K}{\pi\sqrt{a}},$$

where a is the characteristic linear dimension of a sharp-pointed crack cavity of a hypocycloidal hole in the structure of the given body. Since the value for the given material, under the given conditions (temperature, surrounding medium, character of heterogeneity of structure, etc) is a constant value, then, under the very same conditions, the value R_1 can also be assumed to be constant. When a brittle body containing the given type of defect is under uniaxial tension, the stress $p_*^{(min)}$ represents the average technical strength σ_t of the given material under uniaxial tension. Thus

$$\sigma_в = 1.03 R_1. \tag{VIII.170}$$

On the basis of the above examples, we may calculate, using formulas (VIII.167), (VIII.168) and (VIII.170), the value $p_*^{(min)}/\sigma_t$ and $q_*^{(min)}/\sigma_t$ for several values of the parameter η_0 ($-\infty \leqslant \eta_0 \leqslant \infty$), and then, on the basis of these data, we may construct the diagram of limit (rupture) stresses for the case when the body -- plate with defects of the sharp-pointed crack cavity type, /603 is in the plane stress state.

By calculating on the basis of formulas (VIII.167) the dimensionless stresses $p_*^{(min)}/\sigma_t$ and $q_*^{(min)}/\sigma_t$ for the values of the parameter η_0 ($-\infty \leqslant \eta_0 \leqslant \infty$), we can construct the diagram of limit (rupture) stresses for the brittle rupture of a body with defects of the hypocycloidal type, where the latter is subjected to the biaxial plane stress state. In particular, for the case when

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the body is weakened by sharp-pointed holes in the form of hypocycloids with three vertices[1], the diagram of limit stresses, with respect to strength, constructed by formulas (VIII.167), is represented by curve 1 in Figure VIII.26. We will note here that for a body weakened by defects of the hypocycloidal hole type with three vertices, the values R_1 and σ_t in formula (VIII.167) differs, generally speaking, from the analogous values for a body containing defects of the narrow crack cavity type.

Comparison of Results of Theory and Experiment. The question of the choice of the criterion of strength during the brittle rupture of solid bodies located in the biaxial stress state has not yet found final solution. The hypothesis of maximum normal stresses[2] and Griffith's theory[3] must be regarded as the most common criteria of strength.

Griffith's theory of rupture of a brittle body under biaxial tension-compression by principal stresses p and q is based, as we know, on the analysis of elastic stresses near an elongated elliptical hole in a plate subjected to the plane stress state. Here it is assumed that the radius of curvature of such a hole at its vertex is small, but finite. Moreover, it is assumed that a real brittle body also contains cavities of all orientations, and rupture of the body will occur when the greatest tension stress on the contour of the most dangerously oriented cavity reaches the magnitude σ_t, the value of brittle strength of the material. In his works [1, 2] A. A. Griffith formulas the following criteria of brittle rupture of a body in the plane stress state: 1) if $3p + q > 0$, rupture occurs when $p = \sigma_t$; 2) if $3p + q < 0$, rupture occurs when $(p - q)^2 + 8\sigma_t(p + q) = 0$. The graphic interpretation of these equations is shown in Figure VIII.26 in the form of curve 2. The conditions of brittle rupture in accordance with the hypothesis of maximum normal stresses, are expressed through the equalities $p = \sigma_t$ or $q = \sigma_t$ (line 3).

We will compare the diagrams of brittle rupture shown in Figure VIII.26 with experimental data obtained during the testing of brittle bodies. First let us consider some experimental results obtained[4] during the rupture of tubular cast iron samples subjected to the plane stress state, which was created by the application of axial tension-compression q and internal pressure /604 p in various ratios $q/p = \eta_0$.

As we know, cast iron is the material with relatively limited plasticity, and during a certain amount of thermal treatment numerous (arbitrarily oriented through the volume of the body) graphite inclusions exist within its structure in the form of thin plates. The latter possess negligibly low strength to

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[1] See V. V. Panasyuk [4], where the corresponding values of stresses $p_*^{(\min)}$ and $q_*^{(\min)}$ are presented.
[2] See N. N. Davidenkov and A. N. Stavrogin [1].
[3] See A. A. Griffith [2].
[4] See J. Cornet and R. C. Grassi [1].

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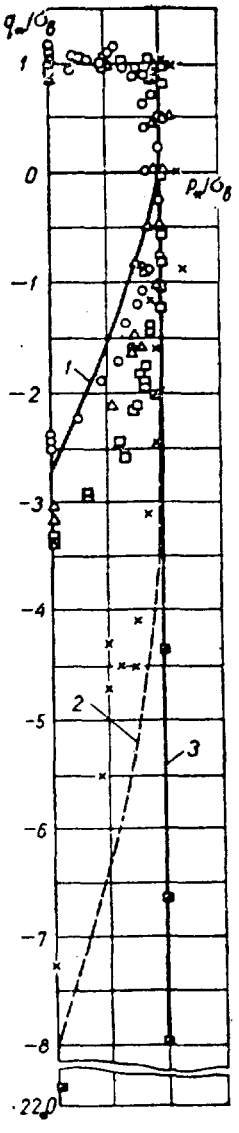


Figure VIII.26.

rupture in comparison with the analogous strength of the ferrite base of cast iron, and conseqnetly, they can be regarded (in the first approximation) as defects of the sharp-pointed crack cavity type in the basic structure of the material. Thus, cast irons with laminar graphite inclusions represent (in the first approximation) a likeness of a real object that satisfies the conditions of the model of an ideally elastic body, formulated above, with sharp-pointed crack cavities. Therefore, the results of the testing of such cast iron samples under the plane stress state are quite important from the point of view of comparison of the results of theory and experiment.

Figure VIII.26 shows the results of tests conducted by J. Cornet and R. C. Grassi [1] on the rupture of tubular cast iron samples in the plane stress state, where the circles pertain to samples made of modified cast iron, for which $\sigma_t = 345.3 \cdot 10^6$ n/m^2, and the triangles and squares pertain to samples made of gray cast irons, the technical strengths of which are, respectively, $228.6 \cdot 10^6$ and $185.4 \cdot 10^6$ n/m^2. As we see, the results of the experiments agree quite well with the results of theory, i.e., with diagram 1 on this figure. For such materials, the ratio of the magnitude of the technical strength during uniaxial tension ($k = \sigma_c/\sigma_t$), established by the tests of J. Cornet and R. C. Grassi [1], varies within the range 2.5-3.3. The theoretical coefficient k from diagram 1 is 2.7, and the k calculated on the basis of Griffith's theory is 8.

Figure VIII.26 represents, in the form of crosses and half-shaded boxes, the experimental data obtained by N. N. Davidenkov and A. N. Stavrogin [1] during tests for the rupture of tubular gypsum and glass samples in the plane stress state. As we see, these data agree quite satisfactorily with the theoretical diagrams of rupture in the first quadrant, i.e., when /605 $q \geqslant 0$ and $p \geqslant 0$, and deviate from them in the second quadrant, i.e., when $q < 0$ and $p \geqslant 0$. Here the rupture of the glass samples in the second quadrant is best described by the diagram of large normal stresses (although in this case the mechanism of rupture under conditions close to pure compression is not clear), and the rupture of gypsum samples in the second quadrant falls within the range between diagrams 1 and 2 in Figure VIII.26. According to the data of N. N. Davidenkov and A. N. Stavrogin [1], $\sigma_t = 4.1 \cdot 10^6$ n/m^2 and $k = 7.1$ for gypsum samples; for glass samples $\sigma_t = 39.2 \cdot 10^6$ n/m^2 and $k = 22$.

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Comparison of the results of theory and experiment show that the general principles of rupture of brittle materials in the plane stress state, as established theoretically on the basis of the model of an ideally elastic body with sharp-pointed crack cavities, are quite satisfactorily verified experimentally. Moreover, for brittle materials with clearly expressed defective structure, in the form of crack slits, as is the case, for instance, of cast irons with laminar graphite inclusions (in the first approximation), theoretical diagram 1 of rupture (Figure VIII.26) agrees with the experimental data in the quantitative sense as well, considering that this diagram characterizes the state of initial rupture of a body with the more dangerous defects of the sharp-pointed crack cavity. Therefore diagram 1 can be regarded as the lower bound of strength of a brittle body in the plane stress state.

The distances between the theoretical (diagram 1, Figure VIII.26) and experimental values of strength of a brittle body of the gypsum and glass types where such bodies are subjected to the plane stress state for $q \ll 0$ and $p \geqslant 0$, are probably attributed to the effect of the forces of friction between the contacting edges of the spreading defect, and in certain cases, to the shape of the vertex of such defects. Consideration of these factors will obviously make it possible to construct more universal diagrams of the limit state of brittle bodies under plane tension-compression, and to describe more completely the mechanism of rupture of brittle bodies of various physico-chemical nature. Furthermore, on the basis of the comparisons of the results of theory and experiment presented herein, we may conclude that the theoretical formulation developed in this section describes the behavior of brittle bodies of varying nature under the conditions of the plane stress state in agreement with the experiment, if the principal stresses are $p > 0$ and $q > 0$. In this case we obtain good verification of the fact that the rupture of brittle bodies is controlled by the development of defects of the sharp-pointed crack cavity type in the structure of the deformed material. It is characteristic that in the specified range of the plane stress state ($p \geqslant 0$; $q \geqslant 0$), the diagrams established for the rupture of brittle bodies coincides, for all practical purposes, with the widely known hypothesis of maximum normal (tension) stresses[1] used in engineering practice.

[1]See also V. I. Mossakovskiy, M. T. Rybka [1].

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CHAPTER IX. STRESS DISTRIBUTION NEAR HOLES IN GENERAL CASE OF PLANE NONLINEAR PROBLEM OF ELASTICITY THEORY

Abstract. This chapter is devoted to the formulation of the general nonlinear plane problem of the theory of elasticity. Two methods of solutions are discussed. The first method is based on the works of Adkins, Green and Zerna. The principal resolving equation systems are presented for the plane strain and the plane stress state. The second method is based on the works of L. A. Tolokonnikov. The formulation in complex variables is given of the problem on the stress concentration around holes and the solution of some particular problems is obtained.

Much attention has been focused during the last 12 years on the plane /608
nonlinear problem of elasticity theory in the general statement, i.e., nonlinear physically and geometrically.

Among the first works pertaining to this problem are the foreign studies of J. E. Adkins, A. E. Green, R. T. Shield [1], J. E. Adkins, A. E. Green, G. G. Nicholas [1], A. E. Green and W. Zerna [1], J. E. Adkins, A. E. Green [1]. Later, this problem was analyzed in national literature in the works of G. N. Savin and Yu. I. Koyfman [1], Yu. I. Koyfman [1-5].

The basis of the statement of the problems of A. E. Green, J. E. Adkins, G. F. Smith, R. S. Rivlin [1] and their successors is the possibility of representing stress potential in an isotropic body as the function of algebraic invariance of the deformation tensor. In turning to the solution of partial problems, the potential is considered to be a polynomial of some degree in relation to the algebraic invariance. Thus, the experimental problem of determining the mechanical properties of a material reduces to the determination of certain constants.

This general statement of problems resulted in such a complex system of resolution equations that their solution by a single method turned out to be the approximate small parameter method, or perturbation method, which makes it possible to construct theoretically the approximations for any order on the basis of classical solutions of single-type problems as approximations of the first order.

There is also a somewhat different general statement of problems of nonlinear elasticity theory (plane deformation); this approach is used in the works of L. A. Tolokonnikov [1-5] and is developed by V. G. Gromov [1-4], G. S. Taras'yev and L. A. Tolokonnikov [2] and others. In this statement, the deformation invariants that are suitable for the processing of experimental data which permit the determination of the properties of change in volume and properties of change in shape of an element of the body, are singled out. It is further assumed that the mechanical properties of the body are defined by

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curves that represent the laws of change in volume. The method of series approximations remains, as in the first statement, the uniquely suitable method of solving the resolution equation.

The solutions of several problems concerning stress concentration near a round hole, or hole of different shape, show that the second approximation, following the classical solution, is the effect primarily of geometrical nonlinearity, i.e., change in position and shape of the elements of the body. /609
Physical nonlinearity[1] is expressed by approximations of an even higher order.

We will discuss below certain results of analyses of problems of stress concentration in the above-mentioned statements.

§1. Basic Equations of Nonlinear Plane Elasticity Theory

Some Relations of Elasticity Theory in Curvilinear Coordinates. The location of points of a three-dimensional body in an initial undeformed and unstressed state at the moment of time t = 0 is defined by their coordinates[2] x^i (i = 1, 2, 3) in the rectangular Cartesian coordinate system, or by arbitrary curvilinear coordinates θ^i, related to x^i by relations

$$x^i = x^i(\theta^1, \theta^2, \theta^3). \tag{IX.1}$$

The location of points of a deformed body is defined by Cartesian coordinates y^i, which are functions of initial coordinates x^i and time t:

$$y^i = y^i(x^1, x^2, x^3, t). \tag{IX.2}$$

We will assume that curvilinear coordinate system θ^i is "frozen" to the examined body and is deformed along with it. Due to deformation of the body, the curvilinear coordinate system θ^i of the initial state is transformed to curvilinear coordinate system θ^i of the deformed state at moment of time t. The covariant and contravariant metric tensor components in coordinate system θ^i of the initial state will be denoted, respectively, through g_{ik}, g^{ik}

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[1] The analysis of the effect of physical nonlinearity of a material on stress distribution near holes is discussed in a special (IV) chapter of this monograph.

[2] The index i (i = 1, 2, 3) of coordinates x, θ and y, as we know, can be placed either above or below. However, due to the fact that the differentials of these coordinates are transformed as contravariant components of the corresponding vector, the indices of these coordinates are used as superscripts. We will obey this law in the following discussion.

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(i, k = 1, 2, 3), and the analogous values in the coordinate system of the deformed state at moment of time t will be denoted through G_{ik}, G^{ik}, whereupon

$$\det(g_{ij}) = g, \qquad \det(G_{ij}) = G.$$

The function of the specific energy of deformation for a homogeneous and isotropic elastic material in the nondeformed state is a function of the three algebraic deformation invariants I_1, I_2, I_3, i.e.,

$$W = W(I_1, I_2, I_3), \tag{IX.3}$$

where /610

$$I_1 = g^{ij}G_{ij}; \qquad I_2 = I_3 g_{ij}G^{ij}; \qquad I_3 = \frac{G}{g}. \tag{IX.4}$$

In this case the physical law of deformation of an ideally elastic material can be written in the following form:

$$\tau^{ij} = \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_1} g^{ij} + \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_2} B^{ij} + 2\sqrt{I_3}\frac{\partial W}{\partial I_3} G^{ij}. \tag{IX.5}$$

Here

$$B^{ij} = g^{ij}I_1 - g^{ir}g^{js}G_{rs}; \tag{IX.6}$$

τ^{ij} are contravariant stress tensor components, calculated per unit area of the deformed body and related to the curvilinear coordinate system of the deformed state.

The equilibrium equations in the absence of volumetric forces can be represented in one of the following[1] forms:

$$\vec{T}_{i,i} = \frac{\partial \vec{T}_i}{\partial \theta^i} = 0 \quad \text{or} \quad \nabla_i \tau^{ij} = 0, \tag{IX.7}$$

where

$$\vec{T}_i = \sqrt{G} \cdot \tau^{ij} \cdot \vec{G}_j;$$

[1]For brevity we will denote the partial derivatives with respect to the coordinates by the comma.

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$\vec{G}_j$ are vectors of the covariant base of the deformed state; ∇_i is the covariant derivative with respect to coordinates θ^i of the deformed state.

In plane nonlinear elasticity theory, as in its linear variant, basically two types of equilibrium of an elastic body are analyzed -- plane deformation and plane stress state.

Plane Stress State. We will assume that the body under examination represents in the nondeformed state a thin plate made of a homogeneous isotropic material, bounded by planes $x^3 = \pm h_0$. This plate is subjected to great deformations that are symmetrical with respect to the middle plane $x^3 = 0$, which becomes middle plane $y^3 = 0$ of the deformed body. Here, due to deformation, the flat bases of plate $x^3 = \pm h_0$ are transformed into the surfaces

$$y^3 = \pm h(y^1, y^2). \tag{IX.8}$$

Considering deformation to be homogeneous through thickness, such a form of deformation of a thin plate can be described approximately by the following relations:

$$y^\alpha = y^\alpha(x^1, x^2, t), \qquad y^3 = \lambda x^3 \qquad (\alpha = 1, 2), \tag{IX.9}$$

where λ is a function of the y^α (or x^α) coordinates.

We will introduce fixed coordinate system θ^i such that in the deformed state /611

$$y^\alpha = y^\alpha(\theta^1, \theta^2, t), \qquad y^3 = \theta^3 \qquad (\alpha = 1, 2). \tag{IX.10}$$

By selecting the θ^i coordinates in this manner, the metric tensor components of the deformed state will acquire the form

$$\begin{aligned} &G_{\alpha\beta} = A_{\alpha\beta}, \quad G_{33} = 1, \quad G_{\alpha 3} = 0, \\ &G^{\alpha\beta} = A^{\alpha\beta}, \quad G^{33} = 1, \quad G^{\alpha 3} = 0, \\ &G = A, \quad A = |A_{\alpha\beta}| \quad (\alpha, \beta = 1, 2), \end{aligned} \tag{IX.11}$$

where $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant components, respectively, of the metric tensor and are functions of the θ^α coordinates in the middle plate $y^3 = 0$ of the deformed body.

We will write the formula for the vector $\vec{t}_h$ of stresses acting on the bases of the plate:

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$$y^3 = \pm h(\theta^1, \theta^2).$$

The vector of stresses acting on the surface, with normal unit vector $\vec{n} = n_i \vec{G}^i$, is expressed through the stress tensor components by the formula

$$\vec{t} = n_i \tau^{ij} \vec{G}_j. \tag{IX.12}$$

The n_i (i = 1, 2, 3) components of the unit vector $\vec{n}$ of the normal to surface

$$y^3 = \pm h(\theta^1, \theta^2)$$

will have the form

$$(n_1, n_2, n_3) = k_1(-h_{,1}, -h_{,2}, 1), \tag{IX.13}$$

where

$$h_{,\alpha} = \frac{\partial h(\theta^1, \theta^2)}{\partial \theta^\alpha}; \qquad k_1 = (A^{\alpha\beta} h_{,\alpha} h_{,\beta} + 1)^{-\frac{1}{2}} \qquad (\alpha, \beta = 1, 2).$$

Consequently, we obtain from formula (IX.12)

$$\vec{t}_h = e^\alpha \vec{G}_\alpha + e^3 \vec{G}_3 \qquad (\alpha = 1, 2); \tag{IX.14}$$

here

$$e^\alpha = k_1(\tau^{\alpha 3} - \tau^{\alpha\beta} h_{,\beta}); \qquad e^3 = k_1(\tau^{33} - \tau^{3\beta} h_{,\beta}). \tag{IX.15}$$

If the bases of the plate are free of external forces, then $\vec{t}_h = 0$. Consequently

$$\tau^{\alpha 3} - \tau^{\alpha\beta} h_{,\beta} = 0; \qquad \tau^{33} - \tau^{3\alpha} h_{,\alpha} = 0. \tag{IX.16}$$

By discarding $\tau^{\alpha 3}$ from relations (IX.16), we obtain /612

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$$\tau^{33} - \tau^{\alpha\beta} h_{,\alpha} h_{,\beta} = 0 \tag{IX.17}$$

when

$$y^3 = \pm h(\theta^1, \theta^2).$$

For a thin plate we may assume that the value $h_{,\alpha}$ is small; hence, as follws from formula (IX.17), the τ^{33} component on the bases of the plate is a magnitude of the second order of smallness in comparison with the $\tau^{\alpha\beta}$ components. Considering this fact and the symmetrical character of deformation, we may assume, with a sufficient degree accuracy, that

$$\tau^{33} = 0. \tag{IX.18}$$

in the entire plate.

The case of elastic equilibrium of a thin plate subjected to deformations that are symmetrical with respect to the middle plane, when the bases of the plate are free of external load, and when the stress tensor component τ^{33} is equal to zero in the entire plate, will be defined as the plane stress state.

The equilibrium equations of the plane stress state in the absence of volumetric forces will be of the form

$$\nabla_\alpha n^{\alpha\beta} = 0 \qquad (\alpha, \beta = 1, 2), \tag{IX.19}$$

where the integral values of the stress tensor components are

$$n^{\alpha\beta} = \int_{-h}^{+h} \tau^{\alpha\beta} dy^3. \tag{IX.20}$$

From equilibrium equations (IX.19) we find the Airy stress function $U(\theta^1, \theta^2)$, which is related to stress components $n^{\alpha\beta}$ as follows:

$$n^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\varrho}\nabla_{\gamma\varrho} U \qquad (\alpha, \beta, \gamma, \varrho = 1, 2), \tag{IX.21}$$

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or

$$\nabla_{\alpha\beta} U = \epsilon_{\alpha\gamma}\epsilon_{\beta\varrho} n^{\gamma\varrho}, \tag{IX.22}$$

where

$$\epsilon^{\alpha\gamma}\sqrt{A} = \frac{\epsilon_{\alpha\gamma}}{\sqrt{A}} \begin{cases} 1, & \alpha = 1, \quad \gamma = 2; \\ -1, & \alpha = 2, \quad \gamma = 1; \\ 0, & \alpha = \gamma; \end{cases} \tag{IX.23}$$

$$\epsilon_{\alpha\gamma}\epsilon^{\alpha\gamma} = \delta^{\beta}_{\gamma}.$$

The principal vector $\vec{P}$ and principal moment $\vec{M}$ of the forces acting on an element of an arbitrary profile in the plane $y^3 = 0$ can also be expressed through stress function U: /613

$$\vec{P} = -\epsilon^{\varrho\beta} U_{,\varrho} \vec{G}_{\beta} + \vec{P}_0,$$
$$\vec{M} = (U - R^{\beta} U_{,\beta}) \vec{G}^3 + \vec{M}_0 \qquad (\varrho, \beta = 1, 2), \tag{IX.24}$$

where $\vec{P}_0$ and $\vec{M}_0$ are arbitrary constant vectors; R^{β} are components of the vector radius of the points of the profile.

If some curve AB is the boundary curve of the body and is free of external load, then it follows from the formula for $\vec{P}$ that the conditions

$$U_{,1} = U_{,2} = 0. \tag{IX.25}$$

will be satisfied at all points of the curve.

Let us proceed to determine the resolution equation system of the plane stress state. We see from formulas (IX.9) and (IX.10) that Cartesian coordinates x^i are related to curvilinear coordinates θ^i of the nondeformed state:

$$x^{\alpha} = x^{\alpha}(\theta^1, \theta^2), \qquad x^3 = \frac{\theta^3}{\lambda(\theta^1, \theta^2)} \qquad (\alpha = 1, 2). \tag{IX.26}$$

Then the metric tensor components of the nondeformed state are

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$$g_{\alpha\beta} = a_{\alpha\beta}, \quad g_{33} = \frac{1}{\lambda^2}, \quad g_{\alpha 3} = 0, \quad y^3 = 0,$$
$$g^{\alpha\beta} = a^{\alpha\beta}, \quad g^{33} = \lambda^2, \quad g^{\alpha 3} = 0, \tag{IX.27}$$
$$g = \frac{a}{\lambda^2}, \quad a = |a_{\alpha\beta}| \quad (\alpha, \beta = 1, 2),$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the metric tensor components of the curvilinear coordinate system θ^{α} in the plane $x^3 = 0$ of the nondeformed body.

By substituting into formulas (IX.4) the expressions for the metric tensor components of the initial and final states (IX.27) and (IX.11), we find that the deformation invariants are

$$I_1 = \lambda^2 + a^{\alpha\beta}A_{\alpha\beta}, \quad I_2 = (\lambda^2 a_{\alpha\beta}A^{\alpha\beta} + 1)\frac{A}{a},$$
$$I_3 = \lambda^2 \frac{A}{a}; \quad y^3 = 0, \tag{IX.28}$$

where

$$I_3 - \lambda^2 I_2 + \lambda^4 I_1 - \lambda^6 = 0.$$

The B^{ij} components in (IX.6) will have the form

$$B^{\alpha\beta} = \lambda^2 a^{\alpha\beta} + A^{\alpha\beta}\frac{A}{a}, \quad B^{33} = \lambda^2 (I_1 - \lambda^2), \quad y^3 = 0. \tag{IX.29}$$

From linear elasticity law (IX.5), recalling relations (IX.11), (IX.27) and (IX.29), we obtain /614

$$\tau^{\alpha\beta} = \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right)a^{\alpha\beta} + \left(\frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_2} \cdot \frac{A}{a} + P\right)A^{\alpha\beta}, \tag{IX.30}$$
$$y^3 = 0,$$

$$\tau^{33} = \lambda^2 \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_1} + \lambda^2 (I_1 - \lambda^2)\frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_2} + P = 0, \tag{IX.31}$$

where

$$P = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}. \tag{IX.32}$$

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The value P can be discarded from relation (IX.30) by using equation (IX.31):

$$\tau^{\alpha\beta} = \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right) a^{\alpha\beta} + \left[\left(\frac{A}{a} + \lambda^4 - \lambda^2 I_1\right)\frac{\partial W}{\partial I_2} - \lambda^2 \frac{\partial W}{\partial I_1}\right]\frac{2}{\sqrt{I_3}} A^{\alpha\beta}. \qquad \text{(IX.33)}$$

We find from (IX.20)

$$n^{\alpha\beta} = \int_{-h}^{+h} \tau^{\alpha\beta} dy^3 = 2h\tau^{\alpha\beta} = 2\lambda h_0 \tau^{\alpha\beta} \qquad (y^3 = 0). \qquad \text{(IX.34)}$$

By substituting the $n^{\alpha\beta}$ components through stress function U in accordance with formulas (IX.21), and $\tau^{\alpha\beta}$ components in accordance with formulas (IX.33), we convert relations (IX.34) to the form

$$\nabla_{\alpha\beta} U = \mathcal{H} a_{\alpha\beta} + K A_{\alpha\beta} \qquad (\alpha, \beta = 1, 2), \qquad \text{(IX.35)}$$

where

$$\mathcal{H} = 4h_0 \frac{\sqrt{I_3}}{\lambda}\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right), \qquad \text{(IX.36)}$$

$$K = -4h_0 \frac{\lambda}{\sqrt{I_3}}\left[\lambda^2 \frac{\partial W}{\partial I_1} + \left(\lambda^2 I_1 - \lambda^4 - \frac{I_3}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right]. \qquad \text{(IX.37)}$$

Equations (IX.35), together with equation (IX.31), form the resolution equation system of the plane stress state, from which stress function U, two components u and v of displacement in the middle plane and the λ function are found.

In deriving the resolution system, we may use, instead of invariants I_k, deformation tensor invariants J_k, which are related to I_k by the relations

$$J_1 = I_1 - 3, \qquad J_2 = I_2 - 2I_1 + 3, \qquad J_3 = I_3 - I_2 + I_1 - 1. \qquad \text{(IX.38)}$$

Then the resolution equation system can be represented as follows: /615

$$\nabla_{\alpha\beta} U = \mathcal{H} a_{\alpha\beta} + K A_{\alpha\beta}, \qquad \text{(IX.39)}$$

$$\lambda^2 \frac{\partial W}{\partial J_1} + \lambda^2 (J_1 + 1 - \lambda^2)\frac{\partial W}{\partial J_2} + [(1 - \lambda^2)(J_1 + 1 - \lambda^2) + J_2 + J_3]\frac{\partial W}{\partial J_3} = 0,$$

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where

$$\mathcal{H} = 4h_0 \frac{\sqrt{I_3}}{\lambda}\left[\frac{\partial W}{\partial J_1} + (\lambda^2 - 2)\frac{\partial W}{\partial J_2} - (\lambda^2 - 1)\frac{\partial W}{\partial J_3}\right], \tag{IX.40}$$

$$K = 4h_0 \frac{\sqrt{I_3}}{\lambda}\left[\frac{\partial W}{\partial J_2} + (\lambda^2 - 1)\frac{\partial W}{\partial J_3}\right] \tag{IX.41}$$

In deriving these expressions, the value P is retained in formulas (IX.30) and (IX.31).

We will introduce the complex coordinates of the nondeformed and deformed states:

$$\eta = x^1 + ix^2, \qquad \bar{\eta} = x^1 - ix^2;$$
$$z = y^1 + iy^2, \qquad \bar{z} = y^1 - iy^2. \tag{IX.42}$$

If we denote the displacement components along the x^α (α = 1, 2) axis through u and v, the relationship between the (z, $\bar{z}$) and (η, $\bar{\eta}$) coordinates can be represented in the form

$$z = \eta + D, \qquad \bar{z} = \bar{\eta} + \bar{D}, \tag{IX.43}$$

where D = u + iv is the complex displacement function.

Suppose the θ^α coordinates in a deformed body coincide with the (z, $\bar{z}$) coordinates:

$$\theta^1 = z^1 = z, \qquad \theta^2 = z^2 = \bar{z}. \tag{IX.44}$$

In this case the metric tensor components $A_{\alpha\beta}$, $A^{\alpha\beta}$, $a_{\alpha\beta}$, $a^{\alpha\beta}$ will be of the form

$$A_{11} = \bar{A}_{22} = 0, \qquad A_{12} = \frac{1}{2}, \qquad \sqrt{A} = \frac{i}{2},$$
$$A^{11} = \bar{A}^{22} = 0, \qquad A^{12} = 2; \tag{IX.45}$$

$$a_{11} = \bar{a}_{22} = \frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right), \qquad a^{11} = \bar{a}^{22} = \frac{a_{11}}{a}, \qquad a^{12} = -\frac{a_{12}}{a},$$

$$a_{12} = \frac{1}{2}\left(1 - \frac{\partial D}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\cdot\frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}\right) = \frac{\lambda}{2\sqrt{I_3}} + \frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z},$$

$$\sqrt{a} = \frac{i}{2}\left(1 - \frac{\partial D}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\cdot\frac{\partial \bar{D}}{\partial \bar{z}} - \frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}\right) = \frac{i}{2}\cdot\frac{\lambda}{\sqrt{I_3}}. \tag{IX.46}$$

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The expressions for the invariants I_k and J_k will be the following: /616

$$I_1 = \lambda^2 + a^{12}, \qquad I_2 = -\frac{\lambda^2 a_{12}}{a} - \frac{1}{4a}, \qquad I_3 = -\frac{\lambda^2}{4a};$$

$$J_1 = \lambda^2 - 3 - \frac{a_{12}}{a} = \lambda^2 + \frac{2\sqrt{I_3}}{\lambda} + 4\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z} - 3, \tag{IX.47}$$

$$J_2 = 3 - 2\lambda^2 - (\lambda^2 - 2)\frac{a_{12}}{a} - \frac{1}{4a} = 3 - 2\lambda^2 + \frac{I_3}{\lambda^2} +$$
$$+ 2(\lambda^2 - 2)\left[\frac{\sqrt{I_3}}{\lambda} + 2\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}\right],$$
$$J_3 = (\lambda^2 - 1)\left(1 + \frac{a_{12}}{a} - \frac{1}{4a}\right) = (\lambda^2 - 1)\left[\left(1 - \frac{\sqrt{I_3}}{\lambda}\right)^2 - 4\frac{I_3}{\lambda^2}\cdot\frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}\right]. \tag{IX.48}$$

Since the components $A_{\alpha\beta}$, $A^{\alpha\beta}$ are constants, then Kristoffel's symbols are equal to zero and the covariant derivatives in the deformed body are converted to partial derivatives. Consequently, recalling relations (IX.45)-(IX.48), the resolution equation system (IX.39) in complex coordinates $(z, \bar{z})$ can be represented in the form

$$\frac{\partial^2 U}{\partial z^2} = \mathcal{H}\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right),$$
$$2\frac{\partial^2 U}{\partial z \partial \bar{z}} = \mathcal{H}\left(1 - \frac{\partial D}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\cdot\frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}\right) + K,$$
$$\frac{\partial W}{\partial J_1} + 2\left(\frac{\sqrt{I_3}}{\lambda} - 1\right)\frac{\partial W}{\partial J_2} + \left(\frac{\sqrt{I_3}}{\lambda} - 1\right)^2\frac{\partial W}{\partial J_3} + \tag{IX.49}$$
$$+ 4\frac{I_3}{\lambda^2}\left(\frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3}\right)\frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z} = 0,$$

where H and K are defined by formulas (IX.40) and (IX.41). This system can be used for determining the functions U, D and λ.

As follows from formulas (IX.21), the stress components related to complex coordinate system $(z, \bar{z})$ are:

$$n^{11} = \bar{n}^{22} = -4\frac{\partial^2 U}{\partial \bar{z}^2}; \qquad n^{12} = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}. \tag{IX.50}$$

In turn, the stress components that are related to Cartesian coordinate system y^α, are also related to the $n^{\alpha\beta}$ components as follows:

$$n^{\alpha\beta} = \frac{\partial z^\alpha}{\partial y^\nu}\cdot\frac{\partial z^\beta}{\partial y^\mu}\sigma^{\nu\mu}. \tag{IX.51}$$

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Consequently, /617

$$n^{11} = \overline{n^{22}} = \sigma^{11} - \sigma^{22} + i2\sigma^{12}, \qquad n^{12} = \sigma^{11} + \sigma^{22}. \tag{IX.52}$$

By introducing the definitions $\sigma^{11} = \sigma_x$, $\sigma^{22} = \sigma_y$, $\sigma^{12} = \tau_{xy}$, we find from formulas (IX.50) and (IX.52)

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 4\frac{\partial^2 U}{\partial z^2}, \qquad \sigma_x + \sigma_y = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}. \tag{IX.53}$$

We will write, in complex coordinates, expressions (IX.24) for the principal vector and principal moment of forces. If we denote the components of principal vector P along the y^α axes through (X, Y), then, in coordinate system $(z, \bar{z})$,

$$\vec{P} = (X + iY)\vec{G}_1 + (X - iY)\vec{G}_2 = P\vec{G}_1 + \overline{P\vec{G}_2}. \tag{IX.54}$$

By comparing relations (IX.24) and (IX.54) recalling that $\sqrt{A} = i/2$, we find

$$P = X + iY = -2i\left(\frac{\partial U}{\partial \bar{z}}\right) + C_0. \tag{IX.55}$$

Analogously, for the magnitude of the moments, we find from formula (IX.24)

$$M = \left(U - z\frac{\partial U}{\partial z} - \bar{z}\frac{\partial U}{\partial \bar{z}}\right) + C^*, \tag{IX.56}$$

where C_0 and C* are arbitrary constants.

Plane Deformation. If a body is deformed such that all of its points experience displacements that are parallel to the plane $x^3 = 0$ and do not depend on x^3, such deformation is called plane deformation. We will select the axes of the y^α coordinate system such that the x^3 and y^3 axes will be parallel and the planes $x^3 = 0$ and $y^3 = 0$ will coincide. Then plane deformation will be described by the relations

$$y^\alpha = y^\alpha(x^1, x^2, t), \qquad y^3 = x^3. \tag{IX.57}$$

We will introduce coordinate system θ^i:

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$$y^\alpha = y^\alpha(\theta^1, \theta^2, t), \qquad y^3 = \theta^3;$$
$$x^\alpha = x^\alpha(\theta^1, \theta^2), \qquad x^3 = \theta^3. \tag{IX.58}$$

Consequently, for stress tensor components τ^{ij}, we obtain, from deformation law (IX.5), the following expressions :

$$\tau^{\alpha\beta} = \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial J_1} - \frac{\partial W}{\partial J_2}\right)a^{\alpha\beta} + 2\sqrt{I_3}\,\frac{\partial W}{\partial J_2}\,A^{\alpha\beta}; \tag{IX.59}$$

$$\tau^{33} = \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2}\right); \qquad \tau^{\alpha 3} = 0 \qquad (\alpha, \beta = 1, 2). \tag{IX.60}$$

Since, as follows from formulas (IX.59) and (IX.60), the $\tau^{\alpha\beta}$, τ^{33} components depend only on the θ^α components in the plane $y^3 = 0$, the equilibrium equations in the absence of volumetric forces are converted to the form /618

$$\nabla_\alpha \tau^{\alpha\beta} = 0. \tag{IX.61}$$

As in the case of the plane stress state, we introduce stress function $U(\theta^1, \theta^2)$, which satisfies these equations, whereupon

$$\nabla_{\alpha\beta} U = \epsilon_{\alpha\gamma}\epsilon_{\beta\varrho}\tau^{\gamma\varrho}. \tag{IX.62}$$

The expressions for the principal vector and principal moment of forces acting on an arbitrary arc in the $y^3 = 0$ plane are defined by formulas (IX.55) and (IX.56).

After replacing the $\tau^{\alpha\beta}$ components in the left hand sides of relations (IX.59) by the stress function, we obtain the resolution equation system for plane deformation, from which the stress function $U(\theta^1, \theta^2)$ and displacement components in the plane are determined.

In the complex coordinates of the deformed state, the resolution equation system in the case of plane deformation has the following form:

$$\frac{\partial^2 U}{\partial z^2} = 2\sqrt{I_3}\left(\frac{\partial W}{\partial J_1} - \frac{\partial W}{\partial J_2}\right)\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right); \tag{IX.63}$$

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} = \frac{\partial W}{\partial J_1} + (\sqrt{I_3} - 1)\frac{\partial W}{\partial J_2} + 2\sqrt{I_3}\left(\frac{\partial W}{\partial J_1} - \frac{\partial W}{\partial J_2}\right)\frac{\partial D}{\partial \bar{z}}\cdot\frac{\partial \bar{D}}{\partial z}.$$

[1]These expressions can be found from the corresponding expressions for the case of the plane stress state when $\lambda = 1$.

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Integration of Resolution Systems (IX.49) and (IX.63). We will analyze the integration of resolution equation system (IX.49) for the plane stress state. The method of integration should be selected such that it will be possible to exploit to the fullest extent the results of classical (linear) theory. For this purpose we will use the method perturbation theory (small parameter method) and represent the functions $U(z, \bar{z})$, $D(z, \bar{z})$, $\lambda(z, \bar{z})$ in the following form[1]:

$$U = {}^0H\varepsilon\,(U^{(1)} + \varepsilon U^{(2)} + \varepsilon^2 U^{(3)} + \ldots), \tag{IX.64}$$

$$D = \varepsilon D^{(1)} + \varepsilon^2 D^{(2)} + \varepsilon^3 D^{(3)} + \ldots, \tag{IX.65}$$

$$\lambda = 1 + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \ldots, \tag{IX.66}$$

where ε is the characteristic small parameter; ${}^0H$ is a constant, the numerical value of which will be specified below.

Since the functions $D(z, \bar{z})$ and $\lambda(z, \bar{z})$ are represented in the form of series with respect to parameter ε, then, from formulas (IX.48), we find for invariants J_k:

$$J_1 = \varepsilon J_1^{(1)} + \varepsilon^2 J_1^{(2)} + \ldots, \qquad J_2 = \varepsilon^2 J_2^{(1)} + \ldots, \qquad J_3 = \varepsilon^3 J_3^{(1)} + \ldots. \tag{IX.67}$$

Here /619

$$\begin{aligned}
J_1^{(1)} &= 2\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} + \lambda^{(1)}\right),\\
J_1^{(2)} &= 2\left(\frac{\partial D^{(2)}}{\partial z} + \frac{\partial \bar{D}^{(2)}}{\partial \bar{z}} + \lambda^{(2)}\right) + 2\left[\left(\frac{\partial D^{(1)}}{\partial z}\right)^2 + \left(\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)^2 + \right.\\
&\qquad \left. + \frac{\partial D^{(1)}}{\partial z}\cdot\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} + 3\,\frac{\partial D^{(1)}}{\partial \bar{z}}\cdot\frac{\partial \bar{D}^{(1)}}{\partial z}\right] + \lambda^{(1)2},
\end{aligned} \tag{IX.68}$$

$$J_2^{(1)} = \left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)^2 - 4\,\frac{\partial D^{(1)}}{\partial \bar{z}}\cdot\frac{\partial \bar{D}^{(1)}}{\partial z} + 4\lambda^{(1)}\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)$$

and

$$\begin{aligned}
\frac{\sqrt{I_2}}{\lambda} &= 1 + \varepsilon\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right) + \varepsilon^2\left[\frac{\partial D^{(2)}}{\partial z} + \frac{\partial \bar{D}^{(2)}}{\partial \bar{z}} + \left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)^2 - \right.\\
&\qquad \left. - \frac{\partial D^{(1)}}{\partial z}\cdot\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} + \frac{\partial \bar{D}^{(1)}}{\partial z}\cdot\frac{\partial D^{(1)}}{\partial \bar{z}}\right] \ldots.
\end{aligned} \tag{IX.69}$$

[1] The applicability of this method to the equations of nonlinear elasticity theory was discussed by F. Stoppeli [1, 2].

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Resolution equation system (IX.49) and (IX.63) contains functions $\partial W/\partial J_k$, which characterize the mechanical properties of the material. For the approximate integration of the equation system, we will use the expansion of these functions into the Taylor series in the vicinity of the initial (nondeformed) state, i.e., for $J_k = 0$. Recalling formulas (IX.67), this expansion can be represented in the form

$$\frac{\partial W}{\partial J_k} = \left[\frac{\partial W}{\partial J_k}\right]_0 + \varepsilon J_1^{(1)}\left[\frac{\partial^2 W}{\partial J_2 \partial J_k}\right]_0 + \varepsilon^2\left\{J_1^{(2)}\left[\frac{\partial^2 W}{\partial J_1 \partial J_k}\right]_0 + J_2^{(1)}\left[\frac{\partial^2 W}{\partial J_2 \partial J_k}\right]_0 + \frac{1}{2}(J_1^{(1)})^2\left[\frac{\partial^3 W}{\partial J_1^2 \partial J_k}\right]_0\right\} + 0(\varepsilon^3), \tag{IX.70}$$

where the symbol $[\ ]_0$ denotes the value of the function when its arguments have the null value.

Since there are no stresses in the nondeformed, then

$$\left[\frac{\partial W}{\partial J_1}\right]_0 = 0. \tag{IX.71}$$

In linear elasticity theory, the Lame constants are given by the following formulas:

$$\lambda = 4\left\{\left[\frac{\partial W}{\partial J_2}\right]_0 + \left[\frac{\partial^2 W}{\partial J_1^2}\right]_0\right\}, \qquad \mu = -2\left[\frac{\partial W}{\partial J_2}\right]_0. \tag{IX.72}$$

The constant in (IX.64) will be defined as follows:

$${}^0H = \mathcal{H}|_{\varepsilon=0} = -4h_0\left[\frac{\partial W}{\partial J_2}\right] = 2h_0\mu. \tag{IX.73}$$

By substituting expansions (IX.64)-(IX.70) into equation system (IX.49) and equating the coefficients for identical degrees of ε, we obtain /620

$$\begin{gathered}
\frac{\partial^2 U^{(j)}(z,\bar{z})}{\partial z^2} + \frac{\partial \bar{D}^{(j)}(z,\bar{z})}{\partial z} = S_1^{(j)}(z,\bar{z}), \\
2\frac{\partial^2 U^{(j)}(z,\bar{z})}{\partial z \partial \bar{z}} + (2c_1+1)\left(\frac{\partial D^{(j)}(z,\bar{z})}{\partial z} + \frac{\partial \bar{D}^{(j)}(z,\bar{z})}{\partial \bar{z}}\right) + 2(c_1+1)\lambda^{(j)}(z,\bar{z}) = \\
= S_2^{(j)}(z,\bar{z}), \\
(c_1+1)\left(\frac{\partial D^{(j)}(z,\bar{z})}{\partial z} + \frac{\partial \bar{D}^{(j)}(z,\bar{z})}{\partial \bar{z}}\right) + c_1\lambda^{(j)}(z,\bar{z}) = S_3^{(j)}(z,\bar{z}),
\end{gathered} \tag{IX.74}$$

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where

$$S_i^{(1)} = 0 \qquad (i = 1, 2, 3); \tag{IX.75}$$

$$S_1^{(2)}(z, \bar{z}) = \frac{\partial \bar{D}^{(1)}}{\partial z}\left[2(c_1 - c_2)\frac{\partial D^{(1)}}{\partial z} + (2c_1 - 2c_2 - 1)\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} + \right.$$

$$\left. + 2(c_1 - c_2 - c_3 + 1)\lambda^{(1)}\right]; \tag{IX.76}$$

$$S_2^{(2)}(z, \bar{z}) = (2c_1 + 1)\frac{\partial D^{(1)}}{\partial z}\cdot\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} - (2c_1 + 3c_2 + 2c_4 + 1)\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)^2 -$$

$$- (c_1 + 4c_2 + 2c_4 + 1)\lambda^{(1)2} - (6c_1 - 4c_2 - 1)\frac{\partial D^{(1)}}{\partial \bar{z}}\cdot\frac{\partial \bar{D}^{(1)}}{\partial z} -$$

$$- 2(5c_2 + c_3 + 2c_4)\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)\lambda^{(1)}; \tag{IX.77}$$

$$S_3^{(2)}(z, \bar{z}) = (c_1 + 1)\frac{\partial D^{(1)}}{\partial z}\cdot\frac{\partial \bar{D}^{(1)}}{\partial \bar{z}} - \frac{1}{2}(2c_1 + 5c_2 + c_3 + 2c_4 + 2)\left(\frac{\partial D^{(1)}}{\partial z} + \right.$$

$$\left. + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right)^2 - \frac{1}{2}(c_1 + 2c_4)\lambda^{(1)2} - (3c_1 - 2c_2 - 2c_3 + 3)\frac{\partial D^{(1)}}{\partial \bar{z}}\cdot\frac{\partial \bar{D}^{(1)}}{\partial z} -$$

$$- 2(2c_2 + c_4)\lambda^{(1)}\left(\frac{\partial D^{(1)}}{\partial z} + \frac{\partial \bar{D}^{(1)}}{\partial \bar{z}}\right); \tag{IX.78}$$

$$c_1 = -\frac{2}{\mu}\left[\frac{\partial^2 W}{\partial J_1^2}\right]_0, \qquad c_2 = -\frac{2}{\mu}\left[\frac{\partial^2 W}{\partial J_1 \partial J_2}\right]_0,$$

$$c_3 = -\frac{2}{\mu}\left[\frac{\partial W}{\partial J_3}\right]_0, \qquad c_4 = -\frac{2}{\mu}\left[\frac{\partial^3 W}{\partial J_1^3}\right]_0. \tag{IX.79}$$

Thus, expressions (IX.76)-(IX.78) for the terms of the second order will include three new elastic material constants.

After integrating systems (IX.74) we find that the general solution $U_0^{(j)}$, $D_0^{(j)}$, $\lambda_0^{(j)}$ of the homogeneous systems corresponding to systems (IX.74) has the following form[1]: /621

[1]By discarding from the second and third equations of (IX.74) $\lambda_0^{(j)}$, differentiating the first equation for $\partial^2/\partial z^2$, and the second for $\partial^2/\partial z \partial \bar{z}$, and excluding the derivatives of $D_0^{(j)}$, it can be proved that

$$\frac{\partial^4 U_0^{(i)}}{\partial z^2 \partial \bar{z}^2} = 0.$$

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$$U_0^{(j)}(z, \bar{z}) = \bar{z}\varphi^{(j)}(z) + z\overline{\varphi^{(j)}(z)} + \chi^{(j)}(z) + \overline{\chi^{(j)}(z)}, \tag{IX.80}$$

$$D_0^{(j)}(z, \bar{z}) = k\varphi^{(j)}(z) - z\overline{\varphi^{(j)'}(z)} - \overline{\psi^{(j)}(z)}, \tag{IX.81}$$

$$\lambda_0^{(j)}(z, \bar{z}) = \frac{1}{2}(k-3)[\varphi^{(j)'}(z) + \overline{\varphi^{(j)'}(z)}], \tag{IX.82}$$

where $\phi^{(j)}(z)$, $\chi^{(j)}(z)$ $\left(\psi^{(j)}(z) = \chi^{(j)'}(z) = \frac{d\chi^{(j)}}{dz}\right)$ are complex potential functions of the j-th order;

$$k = \frac{5c_1 + 2}{3c_1 + 2} = \frac{3 - \nu}{1 + \nu}; \tag{IX.83}$$

ν is Poisson's ratio.

The general solution of heterogeneous systems (IX.74) can be writtein in the following form[1]:

$$\frac{\partial U^{(j)}(z, \bar{z})}{\partial z} = \varphi^{(j)}(z) + z\overline{\varphi^{(j)'}(z)} + \overline{\psi^{(j)}(z)} - F_1^{(j)}(z, \bar{z}), \tag{IX.84}$$

$$D^{(j)}(z, \bar{z}) = k\varphi^{(j)}(z) - z\bar{\varphi}^{(j)'}(\bar{z}) - \bar{\psi}^{(j)}(\bar{z}) - F_2^{(j)}(z, \bar{z}), \tag{IX.85}$$

$$\lambda^{(j)}(z, \bar{z}) = \frac{1}{2}(k-3)[\varphi^{(j)'}(z) + \bar{\varphi}^{(j)'}(\bar{z})] - F_3^{(j)}(z, \bar{z}), \tag{IX.86}$$

where the functions $F_i^{(j)}$ are related to the partial solution of heterogeneous systems (IX.74), whereupon $F_i^{(1)} = 0$ (i = 1, 2), and the values $F_1^{(2)}$ and $F_2^{(2)}$ will be given below.

For the terms of the second order (j = 1), expressions (IX.84)-(IX.86) are known Kolosov-Muskhelishvili functions of the plane problem of linear elasticity theory.

The solution of resolving equation system (IX.63) for plane deformation will also be found in the form (IX.64) and (IX.65), where we assume

$$^0H = -2\left[\frac{\partial W}{\partial J_2}\right]_0 = \mu. \tag{IX.87}$$

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[1]We write the expressions for $\partial U^{(j)}/\partial\bar{z}$, since the vector of forces (IX.55) is expressed through this function and it is more convenient for application than is the function $U^{(j)}$.

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Proceeding analogously as in the case of the plane stress state, we find from system (IX.63), for each approximation. /622

$$\frac{\partial^2 U^{(j)}(z,\bar{z})}{\partial z^2} + \frac{\partial \overline{D}^{(j)}(z,\bar{z})}{\partial z} = S_1^{*(j)}(z,\bar{z}),$$
$$2\frac{\partial^2 U^{(j)}(z,\bar{z})}{\partial z \partial \bar{z}} + (2c_1+1)\left(\frac{\partial D^{(j)}(z,\bar{z})}{\partial z} + \frac{\partial \overline{D}^{(j)}(z,\bar{z})}{\partial \bar{z}}\right) = S_2^{*j}(z,\bar{z}), \quad \text{(IX.88)}$$

where $S_{i_*}^{*(1)} = 0$; $S_i^{*(2)}$ can be found from formulas (IX.76) and (IX.77), assuming[1] $\lambda^{(j)} = 0$ and $c_3 = 0$.

After integrating equations (IX.88), we find

$$\frac{\partial U^{(j)}(z,\bar{z})}{\partial \bar{z}} = \varphi^{(j)}(z) + z\overline{\varphi^{(j)'}(z)} + \overline{\psi^{(j)}(z)} - F_1^{*(j)}(z,\bar{z}), \quad \text{(IX.89)}$$

$$D^{(j)}(z,\bar{z}) = k\varphi^{(j)}(z) - z\overline{\varphi^{(j)'}(z)} - \overline{\psi^{(j)}(z)} - F_2^{*(j)}(z,\bar{z}), \quad \text{(IX.90)}$$

where

$$k = \frac{2c_1 - 1}{2c_1 + 1} = 3 - 4\nu. \quad \text{(IX.91)}$$

In the following analysis we will examine only the nonlinear effects of the second order.

Basic Relations of Plane Nonlinear Elasticity Theory for Terms of Second Order. We see from formulas (IX.84), (IX.85) and (IX.89) that the structure of the relations for the terms of the second order, both for the case of the plane stress state and of plane deformation, is identical, and therefore they can be combined, and these relations for terms of the second order of plane nonlinear elasticity theory are represented in the form[2]:

$$\frac{\partial U^{(2)}(z,\bar{z})}{\partial \bar{z}} = \varphi^2(z) + z\overline{\varphi^{(2)'}(z)} + \overline{\psi^{(2)}(z)} - F_1(z,\bar{z}), \quad \text{(IX.92)}$$

$$D^{(2)}(z,\bar{z}) = k\varphi^{(2)}(z) - z\overline{\varphi^{(2)'}(z)} - \overline{\psi^{(2)}(z)} - F_2(z,\bar{z}), \quad \text{(IX.93)}$$

where

$$F_1(z,\bar{z}) = \gamma\Gamma(z,\bar{z}) - \frac{B_3}{k}\overline{\varphi^{(1)'}(z)}\,D^{(1)}(z,\bar{z}) - k_3 z\left[\overline{\varphi^{(1)'}(z)}\right]^2 - \quad \text{(IX.94)}$$
$$- k_1\int \overline{\varphi^{(1)'}(z)}\,\overline{\psi^{(1)'}(z)}\,d\bar{z} - k_2\int [\varphi^{(1)'}(z)]^2\,dz;$$

[1] In the case of plane deformation, $J_3 = 0$.

[2] f, p. 622.

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$$F_2(z,\bar{z}) = \gamma\Lambda(z,\bar{z}) + \frac{B_3}{k}\overline{\varphi^{(1)'}(z)}\,D^{(1)}(z,\bar{z}) - k_3'z\,[\overline{\varphi^{(1)'}(z)}]^2 - \\ - k_1'\int\overline{\varphi^{(1)'}(z)}\,\overline{\psi^{(1)'}(z)}\,d\bar{z} - k_2'\int[\varphi^{(1)'}(z)]^2\,dz; \tag{IX.95}$$

$$\Gamma(z,\bar{z}) = \left(D^{(1)}\frac{\partial}{\partial z} + \bar{D}^{(1)}\frac{\partial}{\partial\bar{z}}\right)\frac{\partial U^{(1)}}{\partial\bar{z}}; \qquad \Lambda(z,\bar{z}) = \left(D^{(1)}\frac{\partial}{\partial z} + \bar{D}^{(1)}\frac{\partial}{\partial\bar{z}}\right)D^{(1)}. \tag{IX.96}$$

Constants in the function $F_1(z, \bar{z})$, $F_2(z, \bar{z})$ are expressed through the elastic constants of the material: /623

$$k = \frac{2m+3q}{2m+q}; \qquad \gamma = \frac{B_1}{k+1}; \qquad m = c_1 + 1;$$

$$B_1 = \frac{1}{2m+q}[6m + 7q - 4c_2 - 4(q+1)c_3];$$

$$B_2 = \frac{1}{(2m+q)^3}\{[4(m+q)^2 - 3q^2](2m+q) + 12q(3q+4)c_2 - \\ - 12q^2(q+1)c_3 - 8c_4\}; \tag{IX.97}$$

$$B_3 = B_1 - \frac{2B_2}{k+1},$$

where

$$q = \begin{cases} -1 \text{ for plane deformation} \\ c_1 \text{ for the plane stress state} \end{cases}$$

The constants k_r, k_r' (r = 1, 2, 3) can be found by a more convenient mehtod, but such that the following equalities are satisfied[1]:

$$k_1 + k_1' = B_1'; \qquad kk_2 - k_2' = B_4;$$

$$k_3 - k_2 = \frac{B_3}{k} - B_1; \qquad kk_3' + k_2' = kB_1' - B_3 - B_4, \tag{IX.98}$$

where

$$B_1' = B_1 - (k+1); \qquad B_2' = B_2 - \frac{1}{2}(k+1)^2;$$

$$B_1' = B_1 + \frac{1}{2}B_1'; \qquad B_4 = \frac{1}{2}B_1' - B_2'. \tag{IX.99}$$

[1]For greater detail see A. E. Green, J. E. Adkins [1], J. E. Adkins, A. E. Green, G. G. Nicholas [1].

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In particular, we may discard the integral terms from formula (IX.94), assuming

$$k_1 = k_2 = 0 \tag{IX.100}$$

and we may introduce the corresponding values of the constants k_r' (r = 1, 2, 3) into formulas (IX.95).

Assuming

$$k_1' = k_2' = 0 \tag{IX.101}$$

and placing the corresponding values of the constants k_r (r = 1, 2, 3) into expression (IX.94) for the function $F_1(z, \bar{z})$, we may discard the integral terms from the function $F_2(z, \bar{z})$ (IX.95). /624

To find the terms of the second power in the case of an incompressible material with the Mooney form of energy function

$$W = A_1(I_1 - 3) + A_2(I_2 - 3) \tag{IX.102}$$

it is necessary to use in formulas (IX.97) limit transition: $c_1 \to \infty$; $c_2 \to \infty$; $\frac{c_1}{c_2} = 1$; $\frac{c_4}{c_1^3} \to 0$;

$$\left[\frac{\partial W}{\partial J_2}\right]_0 = -(A_1 + A_2); \qquad \left[\frac{\partial W}{\partial J_3}\right] = -(A_1 + 2A_2); \qquad c_3 = \frac{A_1 + 2A_2}{A_1 + A_2}. \tag{IX.103}$$

Here the constants (IX.97)-(IX.99) will have the following form:

1) in the case of plane deformation

$$k = B_1 = B_2 = -B_1' = -B_2' = 1; \qquad B_1'' = B_4 = \gamma = \frac{1}{2}; \qquad B_3 = 0; \tag{IX.104}$$

2) in the case of plane stress state

$$k = \frac{5}{3}; \qquad B_1 = \frac{5 + \varrho_*}{3(1 + \varrho_*)}; \qquad B_2 = \frac{21 + 17\varrho_*}{9(1 + \varrho_*)};$$

$$B_3 = -\frac{1 + 13\varrho_*}{12(1 + \varrho_*)}; \qquad B_4 = \frac{13 + 9\varrho_*}{18(1 + \varrho_*)}; \tag{IX.105}$$

$$\gamma = \frac{5 + \varrho_*}{8(1 + \varrho_*)}; \quad B_1' = -\frac{3 + 7\varrho_*}{3(1 + \varrho_*)}; \quad B_1'' = \frac{7 - 5\varrho_*}{6(1 + \varrho_*)}; \quad B_2' = -\frac{11 + 15\varrho_*}{9(1 + \varrho_*)},$$

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where $\rho_* = A_2/A_1$ is a constant that characterizes the ratio of Mooney's constants in (IX.102).

The formulas for stress components, modulus (magnitude) of principal vector P (with an accuracy up to terms of the second order) acquire the form

$$n^{11} = -4\,{}^0H\varepsilon\left[\frac{\partial^2 U^{(1)}}{\partial z^2} + \varepsilon\frac{\partial^2 U^{(2)}}{\partial \bar{z}^2}\right]; \qquad n^{12} = 4\,{}^0H\varepsilon\left[\frac{\partial^2 U^{(1)}}{\partial z\partial\bar{z}} + \varepsilon\frac{\partial^2 U^{(2)}}{\partial z\partial\bar{z}}\right]; \tag{IX.106}$$

$$P = -2i\,{}^0H\varepsilon\left[\frac{\partial U^{(1)}}{\partial\bar{z}} + \varepsilon\frac{\partial U^{(2)}}{\partial\bar{z}}\right] + C. \tag{IX.107}$$

For the solution of problems in the case where the boundary of the range is given in the nondeformed state, it is necessary to use formulas that express magnitudes which characterize the stress and deformation states of the body, in the form of functions of coordinates $(\eta, \bar{\eta})$. To derive these formulas it is necessary to remember that displacements u + iv can be represented as functions /625
of coordinates $(z, \bar{z})$ or $(\eta, \bar{\eta})$. In this connection we may write

$$u + iv = D(z, \bar{z}) = D_1(\eta, \bar{\eta}) \tag{IX.108}$$

or

$$\varepsilon D^{(1)}(z, \bar{z}) + \varepsilon^2 D^{(2)}(z, \bar{z}) + \ldots = \varepsilon D_1^{(1)}(\eta, \bar{\eta}) + \varepsilon^2 D_1^{(2)}(\eta, \bar{\eta}) + \ldots. \tag{IX.109}$$

If we expand the functions $D^{(i)}(z, \bar{z})$ into Taylor's series in the vicinity of the point $(\eta, \bar{\eta})$ recalling that

$$z - \eta = D_1(\eta, \bar{\eta}) = \varepsilon D_1^{(1)}(\eta, \bar{\eta}) + \varepsilon^2 D_1^{(2)}(\eta, \bar{\eta}) + \ldots, \tag{IX.110}$$

we obtain

$$D^{(k)}(z, \bar{z}) = D^{(k)}(\eta, \bar{\eta}) + \varepsilon\left\{\left[D_1^{(1)}(\eta, \bar{\eta})\frac{\partial}{\partial\eta} + \bar{D}_1^{(1)}(\eta, \bar{\eta})\frac{\partial}{\partial\bar{\eta}}\right]D^k(\eta, \bar{\eta})\right\} + \ldots. \tag{IX.111}$$

By substituting (IX.111) into the left side of relation (IX.109) and equating the terms for identical degrees of ε, we obtain

$$D_1^{(1)}(\eta, \bar{\eta}) = D^{(1)}(\eta, \bar{\eta}), \tag{IX.112}$$

$$D_1^{(2)}(\eta, \bar{\eta}) = D^{(2)}(\eta, \bar{\eta}) + \Lambda(\eta, \bar{\eta}), \tag{IX.113}$$

where the Λ function is given by formula (IX.96).

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Applying Taylor's expansion to functions $U^{(j)}(z, \bar{z})$, $\partial U^{(j)}/\partial \bar{z}$, we find

$$U^{(1)}(z, \bar{z}) = U^{(1)}(\eta, \bar{\eta}); \qquad \frac{\partial U^{(1)}(z, \bar{z})}{\partial \bar{z}} = \frac{\partial U^{(1)}(\eta, \bar{\eta})}{\partial \bar{\eta}}, \tag{IX.114}$$

$$U^{(2)}(z, \bar{z}) = U^{(2)}(\eta, \bar{\eta}) + \left(D^{(1)} \frac{\partial}{\partial \eta} + \bar{D}^{(1)} \frac{\partial}{\partial \bar{\eta}}\right) U^{(1)}(\eta, \bar{\eta});$$

$$\frac{\partial U^{(2)}(z, \bar{z})}{\partial \bar{z}} = \frac{\partial U^{(2)}(\eta, \bar{\eta})}{\partial \bar{\eta}} + \Gamma(\eta, \bar{\eta}), \tag{IX.115}$$

where the Γ function is given by formula (IX.96).

Thus, as follows from (IX.114), the form of the terms of the second order that correspond to linear theory do not depend on the choice of the coordinate system, which, of course, is to be expected.

By comparing formulas (IX.113), (IX.114) with relations (IX.92), (IX.93), we see that in changing to the coordinates of the nondeformed state $(\eta, \bar{\eta})$, it is sufficient in the formulas for the terms of the second order to substitute in the right hand sides of relations (IX.92) and (IX.93) the coordinates $(z, \bar{z})$ by coordinates $(\eta, \bar{\eta})$ and the constant γ by the constant γ', equal to $\gamma - 1$.

We will represent the functions $F_1(z, \bar{z})$ and $F_2(z, \bar{z})$ in the form /626

$$F_1(z, \bar{z}) = \gamma [(\overline{z\Phi^{(1)'}(z)} + \overline{\Psi^{(1)}(z)}) \bar{D}^{(1)} + (\overline{\delta\Phi^{(1)}(z)} + \Phi^{(1)}(z)) D^{(1)}] -$$
$$- k_3 z [\overline{\Phi^{(1)}(z)}]^2 - k_1 \int \overline{\Phi^{(1)}(z)}\, \overline{\Psi^{(1)}(z)}\, d\bar{z} - k_2 \int [\Phi^{(1)}(z)]^2 dz, \tag{IX.116}$$

$$F_2(z, \bar{z}) = -\gamma [(\overline{z\Phi^{(1)}(z)} + \overline{\Psi^{(1)}(z)}) \bar{D}^{(1)} + (\overline{\delta\Phi^{(1)}(z)} - k\Phi^{(1)}(z)) D^{(1)}] -$$
$$- k_3' z [\overline{\Phi^{(1)}(z)}]^2 - k_1' \int \overline{\Phi^{(1)}(z)}\, \overline{\Psi^{(1)}(z)}\, d\bar{z} - k_2' \int [\Phi^{(1)}(z)]^2 dz, \tag{IX.117}$$

where

$$\frac{d\varphi^{(1)}}{dz} = \varphi^{(1)'}(z) = \Phi^{(1)}(z); \qquad \psi^{(1)'}(z) = \Psi^{(1)}(z);$$
$$\delta = 1 - \frac{B_3}{k\gamma}. \tag{IX.118}$$

By denoting

$$F_1(z,\bar{z}) = F_1(z,\bar{z},\gamma,\delta), \qquad F_2(z,\bar{z}) = F_2(z,\bar{z},\gamma,\delta),$$

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we represent the formulas for the terms of the second order in coordinates $(\eta, \bar{\eta})$ in the form

$$\frac{\partial U^{(2)}(z,\bar{z})}{\partial \bar{z}} = \varphi^{(2)}(\eta) + \eta\overline{\varphi^{(2)'}(\eta)} + \overline{\psi^{(2)}(\eta)} - F_1(\eta,\bar{\eta},\gamma',\delta'); \tag{IX.119}$$

$$D_1^{(2)}(\eta,\bar{\eta}) = k\varphi^{(2)}(\eta) - \eta\overline{\varphi^{(2)'}(\eta)} - \overline{\psi^{(2)}(\eta)} - F_2(\eta,\bar{\eta},\gamma'\delta'). \tag{IX.120}$$

Here

$$\gamma' = \gamma - 1; \quad \delta' = 1 - \frac{B_3}{k\gamma'} = \frac{\gamma\delta - 1}{\gamma - 1}. \tag{IX.121}$$

We will express stress components $n^{\alpha\beta}$ through coordinates $(\eta, \bar{\eta})$. By using transformation formulas

$$\frac{\partial}{\partial z} = \frac{1}{\Delta_1}\left(\frac{\partial \bar{z}}{\partial \eta}\cdot\frac{\partial}{\partial \eta} - \frac{\partial \bar{z}}{\partial \eta}\cdot\frac{\partial}{\partial \eta}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{\Delta_1}\left(\frac{\partial z}{\partial \eta}\cdot\frac{\partial}{\partial \eta} - \frac{\partial z}{\partial \eta}\cdot\frac{\partial}{\partial \eta}\right), \tag{IX.122}$$

where

$$\Delta_1 = \frac{\partial z}{\partial \eta}\cdot\frac{\partial \bar{z}}{\partial \bar{\eta}} - \frac{\partial z}{\partial \bar{\eta}}\cdot\frac{\partial \bar{z}}{\partial \eta},$$

recalling the expansions for D (IX.111), we find from relations (IX.106)

$$\begin{aligned} n^{11} = -4^0 H\varepsilon\left\{\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial U^{(1)}(z,\bar{z})}{\partial \bar{z}}\right) + \varepsilon\left[\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial U^{(2)}(z,\bar{z})}{\partial \bar{z}}\right) - \frac{\partial D^{(1)}}{\partial \eta}\cdot\frac{\partial}{\partial \eta}\left(\frac{\partial U^{(1)}}{\partial \bar{z}}\right) - \right.\right. \\ \left.\left. - \frac{\partial \bar{D}^{(1)}}{\partial \eta}\cdot\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial U^{(1)}}{\partial \bar{z}}\right)\right]\right\}, \\ n^{12} = 4^0 H\varepsilon\left\{\frac{\partial}{\partial \eta}\left(\frac{\partial U^{(1)}(z,\bar{z})}{\partial z}\right) + \varepsilon\left[\frac{\partial}{\partial \eta}\left(\frac{\partial U^{(2)}(z,\bar{z})}{\partial \bar{z}} - \frac{\partial D^{(1)}}{\partial \eta}\cdot\frac{\partial}{\partial \eta}\left(\frac{\partial U^{(1)}}{\partial z}\right) - \right.\right. \\ \left.\left. - \frac{\partial \bar{D}^{(1)}}{\partial \eta}\cdot\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial U^{(1)}}{\partial z}\right)\right]\right\}. \end{aligned} \tag{IX.123}$$

We will represent the stress state components, magnitudes of resultant vector and resultant moment of forces, through complex potentials of the first and second orders. By substituting in formulas (IX.106) expression (IX.92), and in formulas (IX.123), the expression (IX.119), taking the dimensionless small parameter ε such that

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$$2^0 H\varepsilon = \begin{cases} 1 & \kappa/\mathcal{M}^2 \text{ for plane deformation} \\ 1 & \kappa/\mathcal{M} \text{ for plane stress state} \end{cases}$$

we obtain the following expressions for the stress state components:

$$\sigma_x + \sigma_y = 2\left\{2\,\mathrm{Re}\,\Phi^{(1)}(z) + \frac{1}{2^0 H}\,[2\mathrm{Re}\,\Phi^{(2)}(z) + f(z,\bar{z})]\right\}, \tag{IX.124}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left\{\bar{z}\Phi^{(1)\prime}(z) + \Psi^{(1)}(z) + \right.$$

$$\left. + \frac{1}{2^0 H}\,[\bar{z}\Phi^{(2)\prime}(z) + \Psi^{(2)}(z) + f_1(z,\bar{z})]\right\}, \tag{IX.125}$$

where

$$f(z,z) = \gamma\left\{[\bar{z}\Phi^{(1)\prime}(z) + \Psi^{(1)}(z)]\,[z\,\overline{\Phi^{(1)\prime}(z)} + \overline{\Psi^{(1)}(z)}] + \right.$$

$$+ (1 - k\delta)\,\Phi^{(1)}(z)\,\overline{\Phi^{(1)}(z)} + \left(\frac{k_2}{\gamma} - k\right)[(\Phi^{(1)}(z))^2 +$$

$$\left. + (\overline{\Phi^{(1)}(z)})^2] - 2\delta_1\,\mathrm{Re}\,[\Phi^{(1)\prime}(z)\,D^{(1)}(z,\bar{z})]\right\}; \tag{IX.126}$$

$$f_1(z,\bar{z}) = \gamma\{(\bar{z}\Phi^{(1)\prime}(z) + \Psi^{(1)}(z))\,[2\overline{\Phi^{(1)}(z)} +$$

$$+ (\delta - k)\,\Phi^{(1)}(z)] - \delta_1(\bar{z}\Phi^{(1)\prime\prime}(z) + \Psi^{(1)\prime}(z))\,D^{(1)}(z,\bar{z}) -$$

$$- \delta_2\Phi^{(1)\prime}(z)\,\bar{D}^{(1)}(z,\bar{z})\} + [k_1\Psi^{(1)}(z) + 2k_3\bar{z}\Phi^{(1)\prime}(z)]\,\Phi^{(1)}(z). \tag{IX.127}$$

In the case of the solution of problems in coordinates $(z, \bar{z})$ in functions $f(z, \bar{z})$ and $f_1(z, \bar{z})$, it is necessary to assume

$$\delta_1 = 1; \qquad \delta_2 = \delta. \tag{IX.128}$$

If, however, the problems are solved in coordinates $(\eta, \bar{\eta})$, then, in formulas (IX.124)-(IX.127) it is necessary to substitute $(z, \bar{z})$ by $(\eta, \bar{\eta})$ $\Phi^{(2)}(z)$ and $\Psi^{(2)}(z)$ by $\Phi_1^{(2)}(\eta)$ and $\Psi_1^{(2)}(\eta)$ and assume

$$\delta_1 = 1 - \frac{1}{\gamma}; \qquad \delta_2 = \delta - \frac{1}{\gamma}. \tag{IX.129}$$

We will represent the magnitude of resultant vector (IX.55)

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$$P = X + iY = \int_A^B (X_n + iY_n)\, ds = -2i\left[\frac{\partial U}{\partial \bar{z}}\right]_A^B \tag{IX.130}$$

and the magnitude[1] of resultant moment (IX.56) /628

$$M = \left[U - 2\mathrm{Re}\left(z\frac{\partial U}{\partial z}\right)\right]_A^B \tag{IX.131}$$

in the form of the following series:

$$\begin{aligned} P &= 2^0 H\varepsilon\,(P^{(1)} + \varepsilon P^{(2)} + \ldots). \\ M &= 2^0 H\varepsilon\,(M^{(1)} + \varepsilon M^{(2)} + \ldots). \end{aligned} \tag{IX.132}$$

By substituting the expansion

$$U = {}^0 H\varepsilon\,(U^{(1)} + \varepsilon U^{(2)} + \ldots)$$

into the right hand sides of formulas (IX.130) and (IX.131) and comparing them with formulas (IX.132), we obtain

$$P^{(k)} = -i\left[\frac{\partial U^{(k)}}{\partial \bar{z}}\right]_A^B; \qquad 2M^{(k)} = \left[U^{(k)} - 2\mathrm{Re}\left(z\frac{\partial U^{(k)}}{\partial z}\right)\right]_A^B. \tag{IX.133}$$

We will express the terms of the second order $P^{(2)}$ and $M^{(2)}$ through complex potential functions. For this purpose we will determine a priori the function $U^{(2)}(z, \bar{z})$. By integrating the expression for $\partial U^{(2)}/\partial z$, we obtain

$$\begin{aligned} U^{(2)} = 2\mathrm{Re}\Big\{ &\overline{z\varphi^{(2)}(z)} + \chi^{(2)}(z) - \frac{B_3 - kB_1}{k}\int \Phi^{(1)}(z)\,\Psi^{(1)}(z)\,dz + \\ &+ k_1\left[z\int \Phi^{(1)}(z)\,\Psi^{(1)}(z)\,dz - \int z\Phi^{(1)}(z)\,\Psi^{(1)}(z)\,dz\right] + \\ &+ k_2\bar{z}\int [\Phi^{(1)}(z)]^2\,dz\Big\} + \gamma D^{(1)}(z,\bar{z})\,\bar{D}^{(1)}(z,\bar{z}) + (B_3 - kB_1)\,\varphi^{(1)}(z)\,\overline{\varphi^{(1)}(z)}, \end{aligned} \tag{IX.134}$$

where

$$\chi^{(2)}(z) = \int \psi^{(2)}(z)\,dz.$$

[1] The function $U(z, \bar{z})$ is a real function of complex arguments (z and $\bar{z}$) or (η, $\bar{\eta}$).

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If we introduce in formulas (IX.133) expressions (IX.92) and (IX.134) for $\partial U^{(2)}/\partial\bar{z}$ and $U^{(2)}$, we find

$$P^{(2)} = -i[\varphi^{(2)}(z) + z\overline{\varphi^{(2)\prime}(z)} + \overline{\psi^{(2)}(z)}]_A^B + i[F_1(z,\bar{z},\gamma,\delta)]_A^B; \tag{IX.135}$$

$$M^{(2)} = \operatorname{Re}[\chi^{(2)}(z) - z\bar{z}\varphi^{(2)\prime}(z) - z\psi^{(2)}(z)]_A^B - \operatorname{Re}\left[\frac{B_3 - kB_1}{k}\int \Phi^{(1)}(z)\Psi^{(1)}(z)\,dz + \right.$$

$$+ k_1\int z\Phi^{(1)}(z)\Psi^{(1)}(z)\,dz + z(k_3\bar{z}\{\Phi^{(1)}(z)\}^2 + F_4(z,\bar{z}))]_A^B +$$

$$+ \frac{1}{2}\left[\gamma D^{(1)}(z,\bar{z})\bar{D}^{(1)}(\bar{z},z) + (B_3 - kB_1)\varphi^{(1)}(z)\overline{\varphi^{(1)}(z)}\right]_A^B, \tag{IX.136}$$

where

$$F_4(z,\bar{z}) = \gamma\Gamma(z,\bar{z}) + \frac{B_3}{k}\overline{\Phi^{(1)}(z)}\,D^{(1)}(z,\bar{z}).$$

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We will assume that region S is a finite singly-connected region. In this case the holomorphic functions $\phi^{(j)}(z)$, $\psi^{(j)}(z)$, $\chi^{(j)}(z)$, $\Phi^{(j)}(z)$, $\Psi^{(j)}(z)$ are single-valued in S. Moreover, since the functions

$$F_1(z) = \Phi^{(1)}(z)\Psi^{(1)}(z), \quad F_2(z) = [\Phi^{(1)}(z)]^2,$$
$$F_3(z) = \Phi^{(1)}(z)\psi^{(1)}(z), \quad F_4(z) = zF_1(z) \tag{IX.137}$$

are also holomorphic (as derivatives of holomorphic functions), then the integrals

$$T_k(z) = \int_{z_0}^{z} F_k(z')\,dz' \quad (k = 1,2,3,4) \tag{IX.138}$$

in region S are unique functions of their upper boundary. Therefore, in the case of a closed contour where the points A and B coincide, it follows from formulas (IX.135), (IX.136) and (IX.116) that

$$P^{(j)} = M^{(j)} = 0 \qquad (j = 1,2).$$

Consequently, if the closed contour bounds a finite singly-connected region, then

$$X = Y = M = 0. \tag{IX.139}$$

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If region S is multiply-connected, but displacement components and stress components in it are single-valued, then, for closed contour L_m, which belongs to the region, formulas (IX.135) and (IX.136) acquire the following form[1]:

$$P_m^{(2)} = P_{L_m}^{(2)} = -i\left[\varphi^{(2)}(z) + \overline{\psi^{(2)}(z)}\right]_{L_m} - i\left[k_1\overline{T_1(z)} + k_2 T_2(z)\right]_{L_m}; \tag{IX.140}$$

$$M_m^{(2)} = M_{L_m}^{(2)} = \operatorname{Re}\left[\chi^{(2)}(z) - z\psi^{(2)}(z)\right]_{L_m} - \operatorname{Re}\left[\frac{B_3 - kB_1}{k} T_3(z) + k_1 T_4(z)\right]_{L_m} +$$

$$+ \frac{1}{2}(B_3 - kB_1)\left[\varphi^{(1)}(z)\,\overline{\varphi^{(1)}(z)}\right]_{L_m}. \tag{IX.141}$$

Since the relation for $P^{(2)}$ does not contain constant γ, it is sufficient in formula (IX.140) to substitute $(z, \bar{z})$ by $(\eta, \bar{\eta})$ to find the corresponding expression in coordinates $(\eta, \bar{\eta})$; the analogous result can be found for $M^{(2)}$ by using functions $U^{(2)}(z, \bar{z})$ and $\partial U^{(2)}/\partial\bar{z}$, expressed through coordinates $(\eta, \bar{\eta})$.

Analysis of Complex Potentials of Second Order[2]. We will examine a problem of the degree of definition of potentials of the first and second orders. In the linear approximation, for a given stress state, the functions $\phi^{(1)}(z)$ and $\psi^{(1)}(z)$ are determined, respectively, with an accuracy up to the terms /630

$$C^{(1)} iz + a_1^{(1)}; \quad a_2^{(1)}. \tag{IX.142}$$

Here displacements $D^{(1)}(z, \bar{z})$ are determined with an accuracy up to the expression

$$(k+1)\,C^{(1)} iz + k\alpha_1^{(1)} - \bar{\alpha}_2^{(1)},$$

which characterizes rigid displacement of the body as a whole.

If potentials $\phi^{(k)}(z)$ and $\psi^{(1)}(z)$ are determined with an accuracy up to terms (IX.142), then functions $f(z, \bar{z})$ and $f_1(z, \bar{z})$ in the formulas for the stress state components, are determined with an accuracy up to the terms, respectively

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[1] In this case the functions $\phi^{(j)'}(z)$, $\psi^{(j)'}(z)$ are single-valued, as follows from the formulas for stresses (IX.124) and (IX.125).

[2] See G. N. Savin and Yu. I. Koyfman [1], and also Yu. I. Koyfman [5].

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$$\gamma\{\alpha_3 C^{(1)2} + i\alpha_4 C^{(1)}\overline{[\Phi^{(1)}(z) - \Phi^{(1)}(z)]} -$$

$$- 2\delta_1 \operatorname{Re}(\Phi^{(1)'}(z)[(k+1)C^{(1)}iz + k\alpha_1^{(1)} - \bar{\alpha}_2^{(1)}])\};$$

$$\gamma\{(z\Phi^{(1)'}(z) + \Psi^{(1)}(z))(\delta - k - 2)C^{(1)}i - \delta_1(z\Phi^{(1)'}(z) + \quad \text{(IX.143)}$$

$$+ \Psi^{(1)'}(z))[(k+1)C^{(1)}iz + k\alpha_1^{(1)} - \bar{\alpha}_2^{(1)}] + \delta_2\Phi^{(1)'}(z)[(k+1)C^{(1)}i\bar{z} -$$

$$- k\bar{\alpha}_1^{(1)} + \alpha_2^{(1)}]\} + [k_1\Psi^{(1)}(z) + 2k_3\bar{z}\Phi^{(1)'}(z)]C^{(1)}i,$$

where α_3, α_4 are complex constants. Consequently, functions $f(z, \bar{z})$, $f_1(z, \bar{z})$ will be defined under the condition

$$C^{(1)} = 0; \quad k\alpha_1^{(1)} - \bar{\alpha}_2^{(1)} = 0, \qquad \text{(IX.144)}$$

i.e., when displacements of the first order are completely defined.

Thus, at a given stress state, in contrast to linear theory, arbitrariness in the selection of complex potentials of the first order is decreased. When both stresses and displacements are given, it is possible to assign arbitrarily only one of the constants $\alpha_1^{(1)}$ or $\alpha_2^{(1)}$. For instance, if the origin of the coordinate system is located in the region occupied by the body, then the appropriate choice of $\alpha_1^{(1)}$ or $\alpha_2^{(1)}$ can be refined such that

$$\varphi^{(1)}(0) = 0 \text{ or } \psi^{(1)}(0) = 0. \qquad \text{(IX.145)}$$

Hence both functions $\phi^{(1)}(z)$ and $\psi^{(1)}(z)$ are completely defined.

If conditions (IX.144) are satisfied, then from formulas (IX.124) and (IX.125) we find that in the given stress state the potentials of the second order $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ are defined with an accuracy up to the terms, respectively, /631

$$C^{(2)}iz + \alpha_1^{(2)}; \quad \alpha_2^{(2)}, \qquad \text{(IX.146)}$$

where $C^{(2)}$ is real and $\alpha_1^{(2)}$, $\alpha_2^{(2)}$ are complex constants.

Since the function $D^{(1)}(z, \bar{z})$ is defined, it is natural to require that the function $D^{(2)}(z, \bar{z})$ also be completely defined, and to use potentials

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$\phi^{(j)}(z)$ and $\psi^{(j)}(z)$ $(j = 1, 2)$, which, in the given stress state, completely define displacements $D^{(j)}(z, \bar{z})$. The integrals

$$T_1(z) = \int^z \Phi^{(1)}(z)\,\Psi^{(1)}(z)\,dz; \quad T_2(z) = \int^z [\Phi^{(1)}(z)]^2\,dz,$$

in function $F_j(z, \bar{z}, \gamma, \delta)$ are defined with an accuracy up to the constants g_1, g_2. Accordingly, if potentials $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ are defined with an accuracy up to terms (IX.146), then the function $D^{(2)}(z, \bar{z})$ is defined with an accuracy up to the expression

$$(k+1)C^{(2)}iz + k\alpha_1^{(2)} - \bar{\alpha}_2^{(2)} + (k_1'\bar{g}_1 + k_2' g_2). \tag{IX.147}$$

When the function $D^{(2)}(z, \bar{z})$ is completely defined, it is obviously necessary that the conditions

$$C^{(2)} = 0; \quad k\alpha_1^{(2)} - \bar{\alpha}_2^{(2)} = -(k_1'\bar{g}_1 + k_2' g_2). \tag{IX.148}$$

be satisfied.

Since the four constants $\alpha_1^{(2)}$, $\alpha_2^{(2)}$, g_1, g_2 are related by one condition, it is possible to fix arbitrarily three of them. For instance, we may assume (when $k_1' \neq 0$; $k_1' \neq 0$):

$$\varphi^{(2)}(0) = 0; \quad \psi^{(2)}(0) = 0; \quad g_1 = 0 \tag{IX.149}$$

or

$$\varphi^{(2)}(0) = 0; \quad g_j = 0 \quad (j = 1,2), \text{ etc.}$$

If, however,

$$k_1' = k_2' = 0,$$

then we may fix only one of the constants, namely $\alpha_1^{(2)}$ or $\alpha_2^{(2)}$. Such arbitrariness in the selection of the functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ is also preserved in the case where the displacements are given.

Let region S occupied by the body (after deformation) be bounded by several simple closed contours L_1, L_2, ..., L_{m+1}, where contour L_{m+1} encompasses

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all others. We will determine the general form of the potentials of the second order for such a region by using coordinates $(z, \bar{z})$.

We will assume that the displacement and stress components are single-valued in region S. Consequently, functions $\Phi^{(1)}(z)$, $\Psi^{(1)}(z)$, $D^{(1)}(z, \bar{z})$, hence $f(z, \bar{z})$, $f_1(z, \bar{z})$, are also single-valued, where /632

$$\operatorname{Im} f(z,\bar{z}) = 0.$$

Since the functions $f(z, \bar{z})$ and $f_1(z, \bar{z})$ are single-valued, then the analysis of the possible multiple-connectedness of the potentials of the second order can be accomplished, as an outcome of formulas (IX.124) and (IX.125), analogously as was done in the linear theory. Therefore, we may write the general form of the potentials of the second order for region S (for the time being without consideration of the uniqueness of the function $D^{(2)}(z, \bar{z})$):

$$\varphi^{(2)}(z) = z\sum_{j=1}^{m} A_j^{(2)} \ln(z - z_j) + \sum_{j=1}^{m} B_j^{(2)} \ln(z - z_j) + \varphi_*^{(2)}(z), \tag{IX.150}$$

$$\psi^{(2)}(z) = \sum_{j=1}^{m} B_j^{*(2)} \ln(z - z_j) + \psi_*^{(2)}(z), \tag{IX.151}$$

where z_j are arbitrary points within contours L_j; $A_j^{(2)}$ are real, and $B_j^{(2)}$, $B_j^{*(2)}$ are complex constants; $\phi_*^{(2)}(z)$ and $\psi_*^{(2)}(z)$ are functions that are holomorphic in region S.

We will find the conditions which must be satisfied by functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ so that the function $D^{(2)}(z, \bar{z})$ will be single-valued. By substituting functions (IX.150) and (IX.151) into relation (IX.93), we obtain

$$[D^{(2)}(z,\bar{z})]_{L_j'} = -2\pi i\{(k+1)A_j^{(2)}z + kB_j^{(2)} + \bar{B}_j^{*(2)}\} + [k_1'\bar{T}_1(\bar{z}) + k_2' T_2(z)]_{L_j'}, \tag{IX.152}$$

where L_j' is some contour that encompasses contour L_j; contour L'_j is circuited in the clockwise direction.

From the condition of single-valuedness of displacements $[D^{(2)}(z, \bar{z})]_{L'_j} = 0$, we obtain

$$\begin{aligned} A_j^{(2)} &= 0; \\ kB_j^{(2)} + \bar{B}_j^{*(2)} &= -\frac{i}{2\pi}[k_1'\bar{T}_1(\bar{z}) + k_2'T_2(z)]_{L_j'}. \end{aligned} \tag{IX.153}$$

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We will express coefficients $B_j^{(2)}$ and $B_j^{*(2)}$ through the principal vector components of forces acting on the contour. We will assume that the solution of interest to us is regular (functions $\phi^{(k)}(z)$, $\phi^{(k)'}(z)$, $\psi^{(k)}(z)$ are extended continuously[1] to all points of boundary L of region S). In this case, to find the resultant vector $P_j^{(2)}$, we may use formula (IX.140), where contour L_j is the boundary. By substituting formulas (IX.150) and (IX.151) into relation (IX.140) recalling that $A_j^{(2)} = 0$, we obtain[2] /633

$$P_j^{(2)} = -2\pi(B_j^{(2)} - \bar{B}_j^{*(2)}) - i[k_1\overline{T_1(z)} + k_2 T_2(z)]_{L_j}. \qquad \text{(IX.154)}$$

From equations (IX.153) and (IX.154) we have

$$B_j^{(2)} = -\frac{X_j^{(2)} + iY_j^{(2)} - E_{1,j}}{2\pi(k+1)}; \quad B_j^{*(2)} = \frac{k(X_j^{(2)} - iY_j^{(2)}) + E_{2,j}}{2\pi(k+1)}. \qquad \text{(IX.155)}$$

where

$$E_{1,j} = -i[(k_1 + k_1')T_1(z) + (k_2 + k_2')\overline{T_2(z)}]_{L_j};$$
$$E_{2,j} = -i[(kk_1 - k_1')\overline{T_1(z)} + (kk_2 - k_2')T_2(z)]_{L_j}. \qquad \text{(IX.156)}$$

If we assume in formulas (IX.156)

$$k_1 = k_2 = 0 \qquad (\text{or } k_1' = k_2' = 0),$$

then we obtain

$$E_{1,j} = \bar{E}_{2,j}; \qquad kE_{1,j} = -\bar{E}_{2,j}. \qquad \text{(IX.157)}$$

Considering relations (IX.153) and (IX.155), functions (IX.150) and (IX.151) can be represented in the form

$$\varphi^{(2)}(z) = -\frac{1}{2\pi(k+1)}\sum_{j=1}^{m}(X_j^{(2)} + iY_j^{(2)} - E_{1,j})\ln(z - z_j) + \varphi_*^{(2)}(z), \qquad \text{(IX.158)}$$

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[1] See N. I. Muskhelishvili [1].

[2] It can then be assumed in formula (IX.153) that contour L'_j coincides with contour L_j.

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$$\psi^{(2)}(z) = \frac{1}{2\pi(k+1)}\sum_{j=1}^{m}[k(X_j^{(2)} - iY_j^{(2)}) + E_{2,j}]\ln(z - z_j) + \psi_*^{(2)}(z). \qquad \text{(IX.159)}$$

Obviously, functions $\phi_1^{(2)}(\eta)$ and $\psi_1^{(2)}(\eta)$ are of the same form.

We will determine the general form of complex potentials $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ for the case of an infinite region with boundary contours L_1, L_2, ..., ..., L_m. We write a circle of radius L_R whose center is located at the origin of the coordinate system (the origin of the coordinates is located outside of region S) such that contours L_j are located within L_R. Then, for all z lying outside of L_R, we will have

$$|z| > |z_j|$$

and

$$\ln(z - z_j) = \ln z + \varphi_{0*}^{(2)}(z), \qquad \text{(IX.160)}$$

where $\phi_{0*}^{(2)}(z)$ is a function that is holomorphic outside of circle L_R.

Consequently, formulas (IX.158) and (IX.159) will acquire the form /634

$$\varphi^{(2)}(z) = -\frac{X^{(2)} + iY^{(2)} - E_1}{2\pi(k+1)}\ln z + \varphi_{01}^{(2)}(z),$$
$$\psi^{(2)}(z) = \frac{k(X^{(2)} - iY^{(2)}) + E_2}{2\pi(k+1)}\ln z + \psi_{01}^{(2)}(z). \qquad \text{(IX.161)}$$

Here

$$X^{(2)} = \sum_{j=1}^{m} X_j^{(2)}; \qquad Y^{(2)} = \sum_{j=1}^{m} Y_j^{(2)}; \qquad E_i = \sum_{j=1}^{m} E_{i,j} \quad (i = 1, 2); \qquad \text{(IX.162)}$$

$\phi_{01}^{(2)}$, $\psi_{01}^{(2)}$ are functions that are holomorphic outside of L_R, with the exception, possibly, of the infinitely distant point.

We will determine the form of functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ that satisfy the condition of boundedness of stresses in the entire region S. Since, in this case, the potentials of the first order are of the form[1]

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[1]See N. I. Muskhelishvili [1].

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$$\varphi^{(1)}(z) = -\frac{X^{(1)} + iY^{(1)}}{2\pi(k+1)} \ln z + \Gamma^{(1)} z + \varphi_0^{(1)}(z),$$
$$\psi^{(1)}(z) = \frac{k(X^{(1)} - iY^{(1)})}{2\pi(k+1)} \ln z + \Gamma'^{(1)} z + \psi_0^{(1)}(z), \qquad \text{(IX.163)}$$

it follows from expressions (IX.126) and (IX.127) that

$$f(z, \bar{z}) = V_1 + 0\left(\frac{1}{z}\right),$$
$$f_1(z, \bar{z}) = V + 0\left(\frac{1}{z}\right) \qquad \text{(IX.164)}$$
(z outside of L_R).

where

$$V_1 = \gamma\left[\Gamma'^{(1)}\bar{\Gamma}'^{(1)} + (1 - k\delta)\Gamma^{(1)}\bar{\Gamma}^{(1)} + \left(\frac{k_2}{\gamma} - k\right)(\Gamma^{(1)2} + \bar{\Gamma}^{(1)2})\right],$$
$$V_2 = \gamma\Gamma'^{(1)}[2\bar{\Gamma}^{(1)} + (\delta - k)\Gamma^{(1)}] + k_1\Gamma'^{(1)}\Gamma^{(1)}. \qquad \text{(IX.165)}$$

In this connection, as the outcome of formulas (IX.124), (IX.125) and (IX.164), it is easy to show that under the condition of boundedness of stresses in region S

$$\varphi^{(2)}(z) = -\frac{X^{(2)} + iY^{(2)} - E_1}{2\pi(k+1)} \ln z + \Gamma^{(2)} z + \varphi_0^{(2)}(z),$$
$$\psi^{(2)}(z) = \frac{k(X^{(2)} - iY^{(2)}) + E_2}{2\pi(k+1)} \ln z + \Gamma'^{(2)} z + \psi_0^{(2)}(z), \qquad \text{(IX.166)}$$

where $\phi_0^{(2)}(z)$ and $\psi_0^{(2)}(z)$ are functions that are holomorphic outside of L_R, including the infinitely distant point.

To find constants $\Gamma^{(2)}$ and $\Gamma'^{(2)}$, we will substitute functions (IX.166) into formulas (IX.124) and (IX.125) for the stress state components. Recalling relations (IX.164) for $|z| \to \infty$, we obtain /635

$$\sigma_x + \sigma_y = 2[2\mathrm{Re}\,\Gamma^{(1)} + \varepsilon(2\,\mathrm{Re}\,\Gamma^{(2)} + V_1)],$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\Gamma'^{(1)} + \varepsilon(\Gamma'^{(2)} + V_2)]. \qquad \text{(IX.167)}$$

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If N_1 and N_2 are values of the principal stresses at infinity, and α is the angle which the principal axis, corresponding to N_1, constitutes with the Oy^1 axis, then, by comparing the equations

$$\sigma_x + \sigma_y = N_1 + N_2,$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = -(N_1 - N_2)\, e^{-2i\alpha} \tag{IX.168}$$

with formulas (IX.167), we find

$$\operatorname{Re}\Gamma^{(1)} = \frac{1}{4}(N_1 + N_2); \qquad \Gamma'^{(1)} = -\frac{1}{2}(N_1 - N_2)\, e^{-2i\alpha}; \tag{IX.169}$$

$$\operatorname{Re}\Gamma^{(2)} = -\frac{1}{2}V_1; \qquad \Gamma'^{(2)} = -V_2. \tag{IX.170}$$

We will assume that there is no rotation at infinity, i.e., when x^α and y^α are large with respect to modulus, deformation of a continuous medium is described by the relations

$$x^\alpha = c_{\alpha\beta} y^\beta \qquad (\alpha, \beta = 1, 2), \tag{IX.171}$$

where

$$c_{\alpha\beta} = c_{\beta\alpha}.$$

Consequently, we may write

$$\frac{\partial x^2}{\partial y^1} = \frac{\partial x^1}{\partial y^2}; \qquad \frac{\partial y^2}{\partial x^1} = \frac{\partial y^1}{\partial x^2} \tag{IX.172}$$

or

$$\frac{\partial \eta}{\partial z} = \frac{\partial \bar{\eta}}{\partial \bar{z}}; \quad \frac{\partial z}{\partial \eta} = \frac{\partial \bar{z}}{\partial \bar{\eta}}.$$

Since $z = D + \eta$, we find from relations (IX.172)

$$\frac{\partial D^{(k)}}{\partial z} = \frac{\partial \bar{D}^{(k)}}{\partial \bar{z}}, \qquad \frac{\partial D^{(k)}}{\partial \eta} = \frac{\partial \bar{D}^{(k)}}{\partial \bar{\eta}}. \tag{IX.173}$$

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Since for large (z)

$$\frac{\partial D^{(1)}(z, \bar{z})}{\partial z} = -\frac{k(X^{(1)} + iY^{(1)})}{2\pi(1+k)} \cdot \frac{1}{z} + (k\Gamma^{(1)} - \bar{\Gamma}^{(1)}) + 0\left(\frac{1}{z^2}\right),$$

we find from equations (IX.173) for $z \to \infty$, /636

$$k\bar{\Gamma}^{(1)} - \bar{\Gamma}^{(1)} = k\bar{\bar{\Gamma}}^{(1)} - \Gamma^{(1)},$$

i.e.,

$$\operatorname{Im} \Gamma^{(1)} = 0.$$

Analogously, the derivative is

$$\frac{\partial D^{(2)}}{\partial z} = -\frac{k[X^{(2)} + iY^{(2)} - E_1]}{2\pi(1+k)} \cdot \frac{1}{z} + (k\Gamma^{(2)} - \bar{\Gamma}^{(2)}) + V_3 + 0\left(\frac{1}{z^2}\right), \qquad \text{(IX.174)}$$

where V_3 is a real constant.

When $|z| \to \infty$, we find from relations (IX.173) and (IX.174)

$$\operatorname{Im} \Gamma^{(2)} = 0. \qquad \text{(IX.175)}$$

Since $\Gamma^{(1)}$ is a real constant, expressions (IX:165) can be represented in the form

$$\begin{aligned} V_1 &= \gamma\left\{\Gamma'^{(1)}\bar{\Gamma}'^{(1)} + \left[1 - k\delta + 2\left(\frac{k_2}{\gamma} - k\right)\right]\Gamma^{(1)2}\right\}, \\ V_2 &= [\gamma(2 + \delta - k) + k_1]\Gamma'^{(1)}\Gamma^{(1)}. \end{aligned} \qquad \text{(IX.176)}$$

Repeating the same considerations for analyzing the general form of complex potentials of the second order in coordinates of the nondeformed state $(\eta, \bar{\eta})$, it is easy to show that functions $\phi_1^{(2)}(\eta)$, $\psi_1^{(2)}(\eta)$ will have the very same form (IX.166) as functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$; here, since V_1 and V_2 do not depend on the constants δ_1 (IX.128), δ_2 (IX.129), constants $\Gamma^{(2)}$ and $\Gamma'^{(2)}$ for functions $\phi_1^{(2)}(\eta)$ and $\psi_1^{(2)}(\eta)$ will also be determined by formulas (IX.170) and (IX.176) (γ and δ are not replaced here by γ' and δ'.

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Thus

$$\varphi_1^{(2)}(\eta) = -\frac{X^{(2)} + iY^{(2)} - E_1}{2\pi(1+k)} \ln \eta + \Gamma^{(2)}\eta + \varphi_{1,0}^{(2)}(\eta),$$
$$\psi_1^{(2)}(\eta) = \frac{k(X^{(2)} - iY^{(2)}) + E_2}{2\pi(1+k)} \ln \eta + \Gamma'^{(2)} + \psi_{1,0}^{(2)}(\eta). \qquad \text{(IX.177)}$$

From Cauchy's theorem we have

$$\sum_{j=1}^{m} \left[\int f(z)\, dz\right]_{L_j} = \left[\int f(z)\, dz\right]_{L_{m+1}},$$

where contour L_{m+1} encompasses all contours L_j, while function f(z) is holomorphic. Therefore, in our case we will have[1]

$$\sum_{j=1}^{m} [T_i(z)]_{L_j} = [T_i(z)]_{L_R}, \qquad (i = 1, 2). \qquad \text{(IX.178)}$$

Here $L_{R'}$ is an arbitrary contour encompassing contour L_R (in the clockwise direction). /637

We will calculate $T_i(z)$ for z outside L_R. Since in this case

$$\Phi^{(1)}(z) = -\frac{X^{(1)} + iY^{(1)}}{2\pi(k+1)} \cdot \frac{1}{z} + \Gamma^{(1)} + \varphi_0^{(1)'}(z),$$
$$\Psi^{(1)}(z) = \frac{k(X^{(1)} - iY^{(1)})}{2\pi(k+1)} \cdot \frac{1}{z} + \Gamma^{(1)'} + \psi_0^{(1)'}(z),$$

then

$$T_1(z) = \int \Phi^{(1)}(z)\, \Psi^{(1)}(z)\, dz = \frac{k(X^{(1)} - iY^{(1)})\Gamma^{(1)} - (X^{(1)} + iY^{(1)})\Gamma^{(1)'}}{2\pi(k+1)} \ln z + f_{1,2}(z),$$
$$T_2(z) = \int \Phi^{(1)2}(z)\, dz = -\frac{(X^{(1)} + iY^{(1)})\Gamma^{(1)}}{\pi(k+1)} \ln z + f_{2,2}(z), \qquad \text{(IX.179)}$$

where $f_{i,2}(z)$ are functions that are holomorphic outside of L_R.

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[1]The functions $\Phi^{(1)}(z)$ and $\Psi^{(1)}(z)$ are holomorphic in the entire region S, and consequently, the functions $F_1(z)$ and $F_2(z)$ are also holomorphic in S (see formulas (IX.137).

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Thus, from formulas (IX.178), recalling relation (IX.179), we obtain

$$\sum_{j=1}^{m}[T_1(z)]_{L_j} = -i\frac{k(X^{(1)}-iY^{(1)})\Gamma^{(1)}-(X^{(1)}+iY^{(1)})\Gamma^{(1)'}}{(k+1)},$$
$$\sum_{j=1}^{m}[T_2(z)]_{L_j} = \frac{2i(X^{(1)}+iY^{(1)})\Gamma^{(1)}}{k+1}. \qquad \text{(IX.180)}$$

In this connection, the values E_1 and E_2, as follows from formulas (IX.156) and (IX.162), will be of the form:

$$E_1 = -\frac{1}{k+1}\{(k_1+k_1')[(X^{(1)}-iY^{(1)})\bar{\Gamma}^{(1)'}-k(X^{(1)}+iY^{(1)})\Gamma^{(1)}-$$
$$-2(k_2+k_2')(X^{(1)}+iY^{(1)})\Gamma^{(1)}\};$$
$$E_2 = -\frac{1}{k+1}\{(kk_1-k_1')[k(X^{(1)}-iY^{(1)})\Gamma^{(1)}-(X^{(1)}+iY^{(1)})\Gamma^{(1)'}]+ \qquad \text{(IX.181)}$$
$$+2(kk_2+k_2')(X^{(1)}-iY^{(1)})\Gamma^{(1)}\}.$$

From (IX.181) we see that the values E_1 and E_2 are transformed to zero only in the case where there are no stresses at infinity or the resultant vector of forces applied to the boundary $L = L_1 + L_2 + \ldots + L_m$ of region S is equal to zero.

Statement of Basic Boundary Problems[1]. In connection with the fact that in nonlinear elasticity theory various forms of both external load acting on a deformed contour and of the contour itself are possible, the basic boundary problems can be analyzed in several variants. We will formulate the statement of several variants of the basic problems of plane nonlinear elasticity theory. /638

First Basic Problem. To find elastic equilibrium in cases where:

the boundary L of region S and external load P(t) applied to it are given in the nondeformed state, where t is the affix of a point of the deformed contour (variant A);

boundary L of region S is given in the nondeformed state and external load P(t) is given on a deformed contour (variant B);

boundary L of region S is given in the nondeformed state and the external load P(t) acting in the deformed state is related to the nondeformed contour (variant C).

[1]See G. N. Savin and Yu. I. Koyfman [1] and also Yu. I. Koyfman [5].

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Second Basic Problem. To find elastic equilibrium in cases where:

displacements $g(t)$ of points of the boundary, the shape of which is known in the deformed state, are given (variant D);

displacements $g_1(t^*)$ of points of the boundary, the shape of which is known in the nondeformed state, where t^* is the affix of a point of the nondeformed contour (variant E).

Along with the first and second basic problems, we can, as in the case of classical elasticity theory, analyze various "contact" problems, and in particular, the problems of equilibrium of a composite plate. We will limit our examination in the given monograph to two possible variants of the statement, specifically to find elastic equilibrium of a composite plate (body) in the cases where:

boundary L of a body and the contours of the seam L_j of the component plates, and also load $P(t)|_L$ are given in the deformed state; there are no displacement jumps[1] on contours L_j (variant F);

boundary L of the body and contours of the seam L_j of the component plates are given in the nondeformed state and load $P(t)$ is given on the deformed boundary of the body; there are no displacement jumps on the contours of the seam (variant I).

Since the components of stresses and displacements of the second order are expressed through complex variable functions, the basic boundary problems in the second approximation are reduced, as in the linear theory, to boundary problems of the theory of complex variable functions. In this connection, we will use the methods of plane elasticity theory to solve the boundary problems in the second approximation, and we will assume that the solutions of the corresponding linear problems are known.

We write the general form of the boundary conditions for potentials of the first and second orders in the various variants of the basic boundary problems.

First Basic Problem. Variant A. We know from formula (IX.55) or (IX.130) /639 that the boundary condition is of the form:

$$2\frac{\partial U}{\partial \bar{z}} = i\int_0^s (X_n + iY_n)\,ds + C = iP(t) + C \text{ on } L, \qquad \text{(IX.182)}$$

[1]Cases where displacement jumps on the contours of contact are given are analyzed in variants G and H, discussed by Yu. I. Koyfman [5].

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where X_n, Y_n are components of the external load; L is a known contour of the deformed region; C is an arbitrary constant.

If we assume that outer load P(t) can be represented in the form of series (IX.132) by degrees of ε, then, as follows from relations (IX.64) and (IX.182), the boundary conditions for the terms of the first and second orders are

$$\frac{\partial U^{(1)}}{\partial \bar{z}} = i\int_0^s (X_n^{(1)} + iY_n^{(1)})\,ds + C = iP^{(1)}(t) + C, \tag{IX.183}$$

$$\frac{\partial U^{(2)}}{\partial \bar{z}} = i\int_0^s (X_n^{(2)} + iY_n^{(2)})\,ds = iP^{(2)}(t). \tag{IX.184}$$

By substituting expressions $\left.\frac{\partial U^{(k)}}{\partial \bar{z}}\right|_L$ by the boundary value of the corresponding combination of complex variable functions, we obtain the following boundary conditions for potentials of the first and second orders, respectively:

$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)'}(t)} + \overline{\psi^{(1)}(t)} = iP^{(1)}(t) + C, \tag{IX.185}$$

$$\varphi^{(2)}(t) + \overline{\varphi^{(2)'}(t)} + \overline{\psi^{(2)}(t)} - F_1(t, \bar{t}, \gamma, \delta) = iP^{(2)}(t). \tag{IX.186}$$

Variant B. Since we know here the form of the contour of the region in the nondeformed state, boundary condition (IX.182) should be related to known contour L*. For this purpose we will express functions P(t) and $2\frac{\partial U(z, \bar{z})}{\partial \bar{z}}$ through coordinates $(\eta, \bar{\eta})$ of the nondeformed state. From relations (IX.114) and (IX.115) we know that the function $2\frac{\partial U(z, \bar{z})}{\partial \bar{z}}$ can be represented in this case (with an accuracy up to terms of the second order) in the form

$$2\frac{\partial U(z, \bar{z})}{\partial \bar{z}} = 2^0 H\varepsilon\left\{\frac{\partial U^{(1)}(\eta, \bar{\eta})}{\partial \bar{\eta}} + \varepsilon\left[\frac{\partial U^{(2)}(\eta, \bar{\eta})}{\partial \bar{\eta}} + \Gamma(\eta, \bar{\eta})\right]\right\}. \tag{IX.187}$$

We will expand the function of external load $P(t) = 2^0 H\varepsilon[P^{(1)}(t) + \varepsilon P^{(2)}(t) + \ldots]$ into Taylor's series in the vicinity of arbitrary point t* of the nondeformed contour L*: /640

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$$P(t) = 2^0 He\{P^{(1)}(t^*) + D_1(t^*)P^{(1)'}(t^*) + \dots$$
$$\dots + \varepsilon[P^{(2)}(t^*) + D_1(t^*)P^{(2)'}(t^*) + \dots] + \varepsilon^2[\dots] + \dots =$$
$$= 2^0 He\{P^{(1)}(t^*) + \varepsilon[P^{(2)}(t^*) + D_1^{(1)}(t^*)P^{(1)'}(t^*)] + \dots\}. \quad \text{(IX.188)}$$

By substituting the corresponding expressions (IX.187) and (IX.188) into boundary condition (IX.182) and equating the terms for identical degrees of ε, we obtain

$$\frac{\partial U^{(1)}(\eta, \bar{\eta})}{\partial \bar{\eta}} = iP^{(1)}(t^*) + C, \quad \text{(IX.189)}$$

$$\frac{\partial U^{(2)}(\eta, \bar{\eta})}{\partial \bar{\eta}} + \Gamma(\eta, \bar{\eta}) = i[P^{(2)}(t^*) + D_1^{(1)}(t^*)P^{(1)'}(t^*)]. \quad \text{(IX.190)}$$

By replacing the left sides in formulas (IX.189) and (IX.190) by the boundary value from (IX.119), we will represent the boundary conditions for the potentials of the first and second orders (omitting the asterisk in the affix t*) in the form

$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)'}(t)} + \overline{\psi^{(1)}(t)} = iP^{(1)}(t) + C, \quad \text{(IX.191)}$$

$$\varphi_1^{(2)}(t) + t\overline{\varphi_1^{(2)'}(t)} + \overline{\psi_1^{(2)}(t)} - F_1^*(t, \bar{t}, \gamma', \delta') = iP^{(2)}(t), \quad \text{(IX.192)}$$

where

$$F_1^*(t, \bar{t}, \gamma', \delta') = F_1(t, \bar{t}, \gamma', \delta') + iD_1^{(1)}(t)P^{(1)'}(t). \quad \text{(IX.193)}$$

Variant C. Here the form of the undeformed boundary of the body is known, and the load[1] P(t) acting on the deformed contour, assumed to be related to the nondeformed contour, will have the given form $P_1(t^*)$. Consequently, by relating boundary condition

$$2\frac{\partial U(z, \bar{z})}{\partial \bar{z}} = iP(t) + C \text{ on } L$$

to contour L*, we obtain the following boundary conditions for the terms of the first and second orders, respectively:

[1]The form of load P(t) in this variant, generally speaking, is unknown and can be determined after the solution of the problem.

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$$\frac{\partial U^{(1)}(\eta, \bar{\eta})}{\partial \bar{\eta}} = iP_1^{(1)}(t^*) + C \text{ on } L^*, \tag{IX.194}$$

$$\frac{\partial U^{(2)}(\eta, \bar{\eta})}{\partial \bar{\eta}} + \Gamma(\eta, \bar{\eta}) = iP_1^{(2)}(t^*). \tag{IX.195}$$

We will find, as before, the boundary conditions for the potentials (on L^*) in /641 the form

$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)'}(t)} + \overline{\psi^{(1)}(t)} = iP_1^{(1)}(t^*) + C, \tag{IX.196}$$

$$\varphi_1^{(2)}(t) + t\overline{\varphi_1^{(2)'}(t)} + \overline{\psi_1^{(2)}(t)} - F_1(t, \bar{t}, \gamma', \delta') = iP_1^{(2)}(t^*). \tag{IX.197}$$

By comparing boundary conditions (IX.197) and (IX.186), we see that if the functions of the external load in these variants coincide with respect to shape, the potentials of the second order in variants A and C are determined simultaneously: for this purpose it is sufficient to substitute in potentials $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ the coordinate z by η and the constants γ, δ by γ', δ'.

It was pointed out above that with a given stress state and completely defined displacements it is possible to assign arbitrarily in functions $\phi^{(1)}(z)$, $\psi^{(1)}(z)$, only one of the constants $\alpha_1^{(1)}$ or $\alpha_2^{(1)}$. In boundary condition (IX.185), the constant $\alpha_1^{(1)}$ (or $\alpha_2^{(1)}$ can be given, by the appropriate selection, any value of constant C. Consequently, with a fixed value of C, functions $\phi^{(1)}(z)$ and $\psi^{(1)}(z)$ are completely defined, while constants $\phi^{(1)}(0)$ and $\psi^{(1)}(0)$ (if the origin of the coordinate system is placed in the region occupied by the body) must be found from the solution of the problem. The potentials $\phi^{(1)}(z)$ and $\psi^{(1)}(z)$ can also be fixed, assuming

$$\varphi^{(1)}(0) = 0 \quad (\text{or } \psi^{(1)}(0) = 0); \tag{IX.198}$$

where the constants C and $\psi^{(1)}(0)$ (or $\phi^{(1)}(0)$) are found from the solution of the problem.

If potentials $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ are defined with an accuracy up to terms (IX.146), then the function $\partial U^{(2)}/\partial\bar{z}$ will be defined with an accuracy up to the expression (for $k_1 \neq 0$, $k_2 \neq 0$) $\alpha_1^{(2)} + \alpha_2^{(2)} + k_1\bar{g}_1 + k_2 g_2$. Since constants k_1, k_2 are related to constants k_1', k_2' by relations (IX.98), then, if the function $\partial U^{(2)}/\partial\bar{z}$ is given and if conditions (IX.148) are satisfied, we may

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arbitrarily fix only two of the four constants $\alpha_j^{(2)}$ and g_j (j = 1, 2). For instance, we may assume $g_j = 0$ (or $\phi^{(2)}(0) = \psi^{(2)}(0) = 0$) and determine constants $\phi^{(2)}(0)$ and $\psi^{(2)}(0)$ or g_j from the solution of the problem. If, however $k_1 = k_2 = 0$ (or $k_1' = k_2' = 0$), then we may assign arbitrarily either constants g_j or the constant $\alpha_1^{(2)}$ ($\alpha_2^{(2)}$) and one of constants g_j (j = 1, 2):

Second Basic Problem. Variant D. In this case the boundary condition has the form

$$D(z, \bar{z}) = g(t) \text{ on } L.$$

Assuming that $g(t) = g^{(1)}(t) + \varepsilon g^{(2)}(t) + \ldots$, we obtain the following boundary conditions for the potentials of the first and second orders, respectively

$$k\varphi^{(1)}(t) - t\overline{\varphi^{(1)'}(t)} - \overline{\psi^{(1)}(t)} = 2^0 H g^{(1)}(t) \text{ on } L, \tag{IX.199}$$

$$k\varphi^{(2)}(t) - t\overline{\varphi^{(2)'}(t)} - \overline{\psi^{(2)}(t)} - F_2(t, \bar{t}, \gamma, \delta) = 2^0 H g^{(2)}(t). \tag{IX.200}$$

Variant E. From boundary condition $D_1(\eta, \bar{\eta}) = g_1(t)$ on L*, as before, we find /642

$$k\varphi^{(1)}(t) - t\overline{\varphi^{(1)'}(t)} - \overline{\psi^{(1)}(t)} = 2^0 H g_1^{(1)}(t) \text{ on } L^*, \tag{IX.201}$$

$$k\varphi_1^{(2)}(t) - t\overline{\varphi_1^{(2)'}(t)} - \overline{\psi_1^{(2)}(t)} - F_2(t, \bar{t}, \gamma', \delta') = 2^0 H g_1^{(2)}(t), \tag{IX.202}$$

where t is the affix of a point of the nondeformed contour L*.

By comparing boundary conditions (IX.200) and (IX.202), we see that the complex potentials in variants D and E are defined simultaneously (by substituting z by η and γ, and δ by γ', δ') in those cases where the functions g(t) and $g_1(t^*)$ coincide with respect to shape.

In solving the second basic problem, we fix potentials $\phi^{(j)}(z)$ and $\psi^{(j)}(z)$ (j = 1, 2) in the same manner as described above.

We will write the boundary conditions of the problem of elastic equilibrium of composite plates (bodies) in variants F and I. We will examine the plate $S_{(0)}$ containing holes, in some of which (or in all) are placed, without tension, elastic discs (plates) $S_{(j)}$, consisting of various materials. We will assume that the contours of the inserted discs and the corresponding holes are in contact without clearance and are soldered together. We will further assume that on the boundary of the (composite) body S thus obtained is applied some system of external forces. We will denote the boundary of such composite

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body S through L, where $L = L_{1,0} + L_{2,0} + \ldots + L_{m,0}$ is the set of contours of free (not filled with discs) holes and the outer contour of the composite plate, and the contours of the seam between the plate and the disc will be denoted through L_j.

Variant F. The boundary conditions and conjugation conditions of the problem are written as follows:

$$2\frac{\partial U}{\partial \bar{z}} = iP(t) + C \text{ on } L, \tag{IX.203}$$

$$\frac{\partial U_{(0)}}{\partial \bar{z}} = \frac{\partial U_{(j)}}{\partial \bar{z}} + C_{(j)}; \qquad D_{(0)}(z, \bar{z}) = D_{(j)}(z, \bar{z}) \text{ on } L_j, \tag{IX.204}$$

where $U_{(0)}$, $U_{(j)}$, $D_{(0)}$, $D_{(j)}$ is Airy's function and the displacement function, respectively, in region $S_{(0)}$ and in the region of the inserted plates $S_{(j)}$; $G|_{L_{k,0}} = C_{k,0}$ $(k = 1, 2, \ldots, m)$.

By combining functions $\partial U_{(j)}/\partial \bar{z}$ and $D_{(j)}(z, \bar{z})$ into series

$$\frac{\partial U_{(j)}}{\partial \bar{z}} = {}^{0}He_j\left(\frac{\partial U_{(j)}^{(1)}}{\partial \bar{z}} + \varepsilon_j \frac{\partial U_{(j)}^{(2)}}{\partial \bar{z}} + \ldots\right),$$

$$D_{(j)} = \varepsilon_j D_{(j)}^{(1)}(z, \bar{z}) + \varepsilon_j^2 D_{(j)}^{(2)}(z, \bar{z}) + \ldots,$$

substituting these expansions into boundary conditions (IX.203) and conditions of conjugation (IX.204), and also equating the terms of identical order of smallness, we obtain the following conditions for the terms of the first and second orders, respectively: /643

$$\frac{\partial U^{(1)}}{\partial \bar{z}} = iP^{(1)}(t) + C^* \text{ on } L; \tag{IX.205}$$

$$\frac{\partial U_{(0)}^{(1)}}{\partial \bar{z}} = \frac{\partial U_{(j)}^{(1)}}{\partial \bar{z}} + C_{(j)}^*; \qquad \varepsilon_0 D_{(0)}^{(1)}(z, \bar{z}) = \varepsilon_j D_{(j)}^{(1)}(z, \bar{z}) \text{ on } L_j; \tag{IX.206}$$

$$\frac{\partial U^{(2)}}{\partial \bar{z}} = iP^{(2)}(t) \text{ on } L; \tag{IX.207}$$

$$\varepsilon_0 \frac{\partial U_{(0)}^{(2)}}{\partial \bar{z}} = \varepsilon_j \frac{\partial U_{(j)}^{(2)}}{\partial \bar{z}}; \qquad \varepsilon_0^2 D_{(0)}^{(2)}(z, \bar{z}) = \varepsilon_j^2 D_{(j)}^{(2)}(z, \bar{z}) \text{ on } L_j. \tag{IX.208}$$

From conditions (IX.205)-(IX.208) we obtain the boundary conditions for the potentials of the first and second orders:

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$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)'}(t)} + \overline{\psi^{(1)}(t)} = iP^{(1)}(t) + C^{\bullet} \qquad \text{(IX.209)}$$

on L;

$$\varphi^{(1)}_{(0)}(t) + t\overline{\varphi^{(1)'}_{(0)}(t)} + \overline{\psi^{(1)}_{(0)}(t)} = \varphi^{(1)}_{(j)}(t) + t\overline{\varphi^{(1)'}_{(j)}(t)} + \overline{\psi^{(1)}_{(j)}(t)} + C^{\bullet}_{(j)},$$

$$k_{(0)}\varphi^{(1)}_{(0)}(t) - t\overline{\varphi^{(1)'}_{(0)}(t)} - \overline{\psi^{(1)}_{(0)}(t)} = m_j\left[k_{(j)}\varphi^{(1)}_{(j)}(t) - t\overline{\varphi^{(1)'}_{(j)}(t)} - \overline{\psi^{(1)}_{(j)}(t)}\right]$$

on L_j; (IX.210)

$$\varphi^{(2)}(t) + t\overline{\varphi^{(2)'}(t)} + \overline{\psi^{(2)}(t)} - F_1(t, \bar{t}, \gamma, \delta) = iP^{(2)}(t) \text{ on L}; \qquad \text{(IX.211)}$$

$$\varphi^{(2)}_{(0)}(t) + t\overline{\varphi^{(2)'}_{(0)}(t)} + \overline{\psi^{(2)}_{(0)}(t)} - F_{1(0)}(t, \bar{t}, \gamma_{(0)}, \delta_{(0)}) =$$

$$= m_j\left[\varphi^{(2)}_{(j)}(t) + t\overline{\varphi^{(2)'}_{(j)}(t)} + \overline{\psi^{(2)}_{(j)}(t)} - F_{1(j)}(t, \bar{t}, \gamma_{(j)}, \delta_{(j)})\right] \text{ on } L_j, \qquad \text{(IX.212)}$$

$$k_0\varphi^{(2)}_{(0)}(t) - t\overline{\varphi^{(2)'}_{(0)}(t)} - \overline{\psi^{(2)}_{(0)}(t)} - F_{2(0)}(t, \bar{t}, \gamma_{(0)}, \delta_{(0)}) =$$

$$= m_j^2\left[k_{(j)}\varphi^{(2)}_{(j)}(t) - t\overline{\varphi^{(2)'}_{(j)}(t)} - \overline{\psi^{(2)}_{(j)}(t)} - F_{2(j)}(t, \bar{t}, \gamma_{(j)}, \delta_{(j)})\right],$$

where $m_j = \frac{{}^0H_{(0)}}{{}^0H_{(j)}}$; $\varphi^{(i)}_{(j)}(t)$, $\psi^{(i)}_{(j)}(t)$, $F_{i(j)}(t, \bar{t}, \gamma_{(j)}, \delta_{(j)})$ are the boundary values of potentials $\phi^{(i)}_{(j)}(z)$, $\psi^{(i)}_{(j)}(z)$ and functions $F_{i(j)}(z, \bar{z}, \gamma_{(j)}, \delta_{(j)})$.

Variant I. Boundary conditions (IX.203) and conjugation conditions (IX.204) should be written on the outer contour of the body and on the contours of the seam, the shape of which is given in the initial (nondeformed) state. By combining, as before, the external load P(t) into Taylor's series in the vicinity of point t* of the undeformed outer contour L* and using the function $\partial U/\partial\bar{z}$, expressed through coordinates $(\eta, \bar{\eta})$, we obtain the following conditions for the terms of the first and second orders: /644

$$\frac{\partial U^{(1)}}{\partial\bar{\eta}} = iP^{(1)}(t) + C \text{ on } L^*, \qquad \text{(IX.213)}$$

$$\frac{\partial U^{(1)}_{(0)}}{\partial\bar{\eta}} = \frac{\partial U^{(1)}_{(j)}}{\partial\bar{\eta}} + C^{\bullet}_{(j)}; \qquad \varepsilon_{(0)}D^{(1)}_{1(0)}(\eta, \bar{\eta}) = \varepsilon_{(j)}D^{(1)}_{1(j)}(\eta, \bar{\eta}) \text{ on } L^*,$$

$$\frac{\partial U^{(2)}}{\partial\bar{\eta}} + \Gamma(\eta, \bar{\eta}) = i\left[P^{(2)}(t) + D^{(1)}_1 P^{(1)'}(t)\right] \text{ on } L^*;$$

$$\varepsilon_{(0)}\left[\frac{\partial U^{(2)}_{(0)}}{\partial\bar{\eta}} + \Gamma_{(0)}(\eta, \bar{\eta})\right] = \varepsilon_{(j)}\left[\frac{\partial U^{(2)}_{(j)}}{\partial\bar{\eta}} + \Gamma_{(j)}(\eta, \bar{\eta})\right], \qquad \text{(IX.214)}$$

$$\varepsilon^2_{(0)}D^{(2)}_{1(0)}(\eta, \bar{\eta}) = \varepsilon^2_{(j)}D^{(2)}_{1(j)}(\eta, \bar{\eta}) \text{ on } L^*_j.$$

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The boundary conditions for the potentials of the first order will be of the form (IX.209) and (IX.210) where t is the affix of the point of the nondeformed contour.

The conditions on L* for potentials of the second order are represented in the form

$$\varphi^{(2)}(t) + t\overline{\varphi^{(2)'}(t)} + \overline{\psi^{(2)}(t)} - F_1^*(t, \bar{t}, \gamma', \delta') = iP^{(2)}(t);$$

$$\varphi_{(0)}^{(2)}(t) + t\overline{\varphi_{(0)}^{(2)'}(t)} + \overline{\psi_{(0)}^{(2)}(t)} - F_{1(0)}(t, \bar{t}, \gamma'_{(0)}, \delta'_{(0)}) = \tag{IX.215}$$

$$= m_j[\varphi_{(j)}^{(2)}(t) + t\overline{\varphi_{(j)}^{(2)'}(t)} + \overline{\psi_{(j)}^{(2)}(t)} - F_{1(j)}(t, \bar{t}, \gamma'_{(j)}, \delta'_{(j)})] \text{ on } L^*;$$

$$k\varphi_{(0)}^{(2)}(t) - t\overline{\varphi_{(0)}^{(2)'}(t)} - \overline{\psi_{(0)}^{(2)}(t)} - F_{2(0)}(t, \bar{t}, \gamma'_{(0)}, \delta'_{(0)}) =$$

$$= m_j^2[k_{(j)}\varphi_{(j)}^{(2)}(t) - t\overline{\varphi_{(j)}^{(2)'}(t)} - \overline{\psi_{(j)}^{(2)}(t)} - F_{2(j)}(t, t, \gamma'_{(j)}, \delta'_{(j)})] \text{ on } L_j^*. \tag{IX.216}$$

We will introduce the expanded form of the boundary conditions for the potentials of the second order in the case where region S represents an infinite plane weakened by some curvilinear hole. We will plate the origin of coordinate system y^α within the hole and assume that the origin of the coordinate system is located within the hole prior to deformation.

The potentials of the second order are of the form (IX.161)

$$\varphi^{(2)}(z) = -\frac{X^{(2)} + iY^{(2)} - E_1}{2\pi(k+1)} \ln z + \Gamma^{(2)} z + \varphi_0^{(2)}(z);$$

$$\psi^{(2)}(z) = \frac{k(X^{(2)} - iY^{(2)}) + E_2}{2\pi(k+1)} \ln z + \Gamma^{(2)'} z + \psi_0^{(2)}(z).$$

Assuming /645

$$k_1 = k_2 = 0,$$

we find from condition (IX.186) for variant A

$$\varphi_0^{(2)}(t) + t\overline{\varphi_0^{(2)'}(t)} + \overline{\psi_0^{(2)}(t)} = k_0(t, \bar{t}, \gamma, \delta) \text{ on } L, \tag{IX.217}$$

where L is the contour of the hole;

$$K_0(t, \bar{t}, \gamma, \delta) = iP^{(2)}(t) + F_1(t, \bar{t}, \gamma, \delta) - 2\Gamma^{(2)}t - \overline{\Gamma}^{(2)'}\bar{t} +$$

$$+ \frac{X^{(2)} - iY^{(2)} - \bar{E}_1}{2\pi(k+1)} \cdot \frac{t}{\bar{t}} - \frac{E_1}{2\pi(k+1)} \ln(t\bar{t}) + \frac{X^{(2)} + iY^{(2)}}{2\pi(k+1)} (\ln t - k \ln \bar{t}). \tag{IX.218}$$

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To determine the boundary condition of variant B, it is sufficient to replace in condition (IX.217), as shown above, the function $K_0(t, \bar{t}, \gamma, \delta)$ by the function

$$K_0^*(t, \bar{t}, \gamma', \delta') = K_0(t, \bar{t}, \gamma', \delta') + iD^{(1)}P^{(1)'}(t) \tag{IX.219}$$

and assume that t is the affix of a point of the nondeformed contour. Boundary problems of variants A and B are solved simultaneously by replacing γ, δ by γ', δ'.

We readily see that the function $K_0(t, \bar{t}, \gamma, \delta)$ is a single-valued continuous function of the point t of contour L, i.e.,

$$[K_0(t, \bar{t}, \gamma, \delta)]_L = 0.$$

If we assume that the boundary condition is satisfied also in the first approximation ($P^{(2)}(t) = 0$), and that the resultant vector of forces acting on L is equal to zero, then the function $K_0(t, \bar{t}, \gamma, \delta)$ is simplified and acquires the following form:

$$K_0(t, \bar{t}, \gamma, \delta) = F_1(t, \bar{t}, \gamma, \delta) - 2\Gamma^{(2)}t - \bar{\Gamma}^{(2)'}\bar{t}. \tag{IX.220}$$

We will notice also that if contour L is free of load, then

$$K_0^*(t, \bar{t}, \gamma', \delta') = K_0(t, \bar{t}, \gamma', \delta'), \tag{IX.221}$$

and the boundary conditions of variants A, B and C are solved simultaneously.

To fix functions $\phi_0^{(2)}(z)$ and $\psi_0^{(2)}(z)$, we may assume (for $k_1 = k_2 = 0$)

$$\varphi_0^{(2)}(\infty) = 0 \quad (\text{or } \psi^{(2)}(\infty) = 0). \tag{IX.222}$$

We will write the boundary conditions for the potentials of the second order in the case of the second basic problem for an infinite plate with a hole in variant D. Assuming

$$k_1' = k_2' = 0,$$

from condition (IX.200) we find on L: /646

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$$k\varphi_0^{(2)}(t) - t\overline{\varphi_0^{(2)'}(t)} - \overline{\psi_0^{(2)}(t)} = g_1^{(2)}(t, \bar{t}, \gamma, \delta), \tag{IX.223}$$

where

$$g_1^{(2)}(t, \bar{t}, \gamma, \delta) = 2^0 H g^{(2)}(t) + F_2(t, \bar{t}, \gamma, \delta) + \frac{k(X^{(2)} + iY^{(2)} - E_1)}{2\pi(k+1)} \ln(t\bar{t}) - \\ - (k\Gamma^{(2)} - \bar{\Gamma}^{(2)})t + \bar{\Gamma}'^{(2)}\bar{t} - \frac{X^{(2)} - iY^{(2)} - \bar{E}_1}{2\pi(k+1)} \cdot \frac{t}{\bar{t}}. \tag{IX.224}$$

In the case of variant E the function $g_1^{(2)}(t, \bar{t}, \gamma, \delta)$ is replaced by the function $g_1^{(2)}(t, \bar{t}, \gamma', \delta')$, i.e., boundary conditions of variants D and E, as mentioned above, can be solved simultaneously. It follows from formula (IX.224) that $g(\bar{t}, t, \gamma, \delta)$ is a single-valued function, continuous on L.

Potentials $\phi^{(2)}(z)$ and $\psi_0^{(2)}(z)$ in the second basic problem are also fixed by conditions (IX.222) (for $k' = k_2' = 0$).

If we assume that the boundary conditions are satisfied due to the first approximation ($g^{(2)}(t) = 0$) and the resultant vector of forces is equal to zero, then the function is

$$g_1^{(2)}(t, \bar{t}, \gamma, \delta) = F_2(t, \bar{t}, \gamma, \delta) - (k\Gamma^{(2)} - \bar{\Gamma}^{(2)})t - \Gamma'^{(2)}\bar{t}. \tag{IX.225}$$

We will write the basic relations for the terms of the second order for conformal mapping. By introducing, as a function of the shape of the hole, mapping function $z = \omega(\zeta)$ or $\eta = \omega(\zeta)$, we obtain[1]

$$\frac{\partial U^{(2)}}{\partial \bar{z}} = \varphi^{(2)}(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi^{(2)'}(\zeta)} + \overline{\psi^{(2)}(\zeta)} - F_1(\zeta, \bar{\zeta}, \gamma, \delta), \tag{IX.226}$$

$$D^{(2)} = k\varphi^{(2)}(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi^{(2)'}(\zeta)} - \overline{\psi^{(2)}(\zeta)} = -F_2(\zeta, \bar{\zeta}, \gamma, \delta), \tag{IX.227}$$

where

$$F_1(\zeta, \bar{\zeta}, \gamma, \delta) = \gamma\left\{\left[\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\Phi^{(1)'}(\zeta)} + \overline{\Psi^{(1)}(\zeta)}\right]\bar{D}^{(1)} + [\Phi^{(1)}(\zeta) + \delta\overline{\Phi^{(1)}(\zeta)}]D^{(1)}\right\} - \\ - k_3\omega(\zeta)[\overline{\Phi^{(1)}(\zeta)}]^2 - k_1\int \overline{\Phi^{(1)}(\zeta)}\,\overline{\Psi^{(1)}(\zeta)}\,\overline{\omega'(\zeta)}\,d\bar{\zeta} - k_2\int [\Phi^{(1)}(\zeta)]^2\omega'(\zeta)\,d\zeta; \tag{IX.228}$$

[1]Here we denote $\phi^{(2)}(\zeta) = \phi^{(2)}[\omega(\zeta)]$, etc.

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$$F_2(\zeta, \bar{\zeta}, \gamma, \delta) = -\gamma\left\{\left[\frac{\omega(\zeta)}{\omega'(\zeta)}\overline{\Phi^{(1)'}(\zeta)} + \overline{\Psi^{(1)}(\zeta)}\right]\overline{D^{(1)}} + [\delta\overline{\Phi^{(1)}(\zeta)} - k\Phi^{(1)}(\zeta)]D^{(1)}\right\} -$$
$$- k_3'\omega(\zeta)[\overline{\Phi^{(1)}(\zeta)}]^2 - k_1'\int\overline{\Phi^{(1)}(\zeta)}\,\overline{\Psi^{(1)}(\zeta)}\,\overline{\omega'(\zeta)}\,d\bar{\zeta} - k_2'\int[\Phi^{(1)}(\zeta)]^2\omega'(\zeta)\,d\zeta. \quad \text{(IX.229)}$$

The formulas for the components of the stress state are /647

$$\sigma_\rho + \sigma_\phi = 2\left\{2\operatorname{Re}\Phi^{(1)}(\zeta) + \frac{1}{2^0H}[2\operatorname{Re}\Phi^{(2)}(\zeta) + f(\zeta, \bar{\zeta})]\right\},$$
$$\sigma_\rho - \sigma_\phi + 2i\tau_{\rho\phi} = 2\frac{\zeta^2\omega'(\zeta)}{\rho^2\omega'(\zeta)}\left\{\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\Phi^{(1)'}(\zeta) + \Psi^{(1)}(\zeta) + \right. \quad \text{(IX.230)}$$
$$\left. + \frac{1}{2^0H}\left[\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\Phi^{(2)'}(\zeta) + \Psi^{(2)}(\zeta) + f_1(\zeta, \bar{\zeta})\right]\right\},$$

where

$$f(\zeta, \bar{\zeta}) = \gamma\left\{\left[\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\Phi^{(1)'}(\zeta) + \Psi^{(1)}(\zeta)\right]\left[\frac{\omega(\zeta)}{\omega'(\zeta)}\overline{\Phi^{(1)'}(\zeta)} + \overline{\Psi^{(1)}(\zeta)}\right] + \right.$$
$$\left. + (1 - k\delta)\Phi^{(1)}(\zeta)\overline{\Phi^{(1)}(\zeta)} + \left(\frac{k_0}{\gamma} - k\right)[\Phi^{(1)^2}(\zeta) + \overline{\Phi}^{(1)^2}(\bar{\zeta})] - 2\delta_1\operatorname{Re}\Phi^{(1)'}(\zeta)\frac{D^{(1)}}{\overline{\omega'(\zeta)}}\right\};$$
$$f_1(\zeta, \bar{\zeta}) = \gamma\left\{\left[\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\Phi^{(1)'}(\zeta) + \Psi^{(1)}(\zeta)\right][2\overline{\Phi^{(1)}(\zeta)} + (\delta - k)\Phi^{(1)}(\zeta)] - \right. \quad \text{(IX.231)}$$
$$- \delta_1\frac{1}{\overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\frac{\partial}{\partial\zeta}\left(\frac{\Phi^{(1)'}(\zeta)}{\omega'(\zeta)}\right) + \Psi^{(1)'}(\zeta)\right]D^{(1)} - \delta_2\Phi^{(1)'}(\zeta)\frac{\bar{D}^{(1)}}{\overline{\omega'(\zeta)}}\Big\} +$$
$$+ \left[k_1\Psi^{(1)}(\zeta) + 2k_3\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\Phi^{(1)'}(\zeta)\right]\Phi^{(1)}(\zeta).$$

We will write the expressions for the complex potentials of the second order in the case of the mapping of an infinite plane with an arbitrary hole onto an infinte plane with a round hole of unit radius. The mapping function will be taken in the form

$$z = \omega(\zeta) = R\left(\zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \ldots\right),$$

where R, a_1, a_2, generally speaking, are complex constant coefficients.

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In this case

$$\varphi^{(2)}(\zeta) = -\frac{X^{(2)} + iY^{(2)} - E_1}{2\pi(k+1)} \ln\zeta + R\Gamma^{(2)}\zeta + \varphi_0^{(2)}(\zeta),$$
$$\psi^{(2)}(\zeta) = \frac{k(X^{(2)} - iY^{(2)}) + E_2}{2\pi(k+1)} \ln\zeta + R\Gamma'^{(2)}\zeta + \psi_0^{(2)}(\zeta). \quad \text{(IX.232)}$$

The boundary conditions for potentials $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$ in the case of the first basic problem (variant A) have, on the contour of unit circle γ_1, the following form

$$\varphi_0^{(2)}(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0^{(2)'}}\left(\frac{1}{\sigma}\right) + \overline{\psi_0^{(2)}}\left(\frac{1}{\sigma}\right) = K_0\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right), \quad \text{(IX.233)}$$

where /648

$$K_0\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) = iP^{(2)}(\sigma) + F_1\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) - \Gamma^{(2)}R\sigma - \overline{\Gamma}'^{(2)}\frac{R}{\sigma} +$$
$$+ \frac{X^{(2)} + iY^{(2)}}{2\pi}\ln\sigma + \frac{\sigma\omega(\sigma)}{\overline{\omega'(\sigma)}}\left[\frac{X^{(2)} - iY^{(2)} - \overline{E}_1}{2\pi(k+1)} - \Gamma^{(2)}\frac{R}{\sigma}\right]. \quad \text{(IX.234)}$$

In the second basic problem (variant D), we have

$$k\varphi_0^{(2)}(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0^{(2)'}}\left(\frac{1}{\sigma}\right) - \overline{\psi_0^{(2)}}\left(\frac{1}{\sigma}\right) = g_1^{(2)}\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right), \quad \text{(IX.235)}$$

where

$$g_1^{(2)}\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) = 2^0 Hg^{(2)}(\sigma) + F_2\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) -$$
$$- k\Gamma^{(2)}R\sigma + \overline{\Gamma}'^{(2)}\frac{R}{\sigma} + \frac{\sigma\omega(\sigma)}{\overline{\omega'(\sigma)}}\left[\Gamma^{(2)}\frac{R}{\sigma} - \frac{X^{(2)} - iY^{(2)} - \overline{E}_1}{2\pi(k+1)}\right]. \quad \text{(IX.236)}$$

The boundary conditions for variants B, C and D are obvious, and therefore we will not include them.

§2. Influence of Nonlinear Effects of Second Order on Stress Distribution near Holes

Round Hole[1]. Let the examined region represent, in the deformed (nondeformed) state, an infinite plane with a round hole of radius R. We will place

[1] See Yu. I. Koyfman [2, 5].

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the origin of the coordinate system $(z, \bar{z})$ and $(\eta, \bar{\eta})$ at the center of the hole and assume that the stress state at infinity is homogeneous, and that the contour of the hole is subjected to the effect of uniform normal pressure of intensity p. To determine the stress and deformation states near the hole it is necessary to solve the first basic problem with following boundary conditions[1]

$$2\frac{\partial U}{\partial \bar{z}}\Big|_L = -pt + c, \tag{IX.237}$$

where t is the affix of a point of contour L of the hole;

$$4\frac{\partial U}{\partial \bar{z}} = (N_1 + N_2)z - (N_1 - N_2)e^{2i\alpha}\bar{z} \quad (\text{for } |z| \to \infty). \tag{IX.238}$$

The potentials of the first order are /649

$$\begin{aligned}\varphi^{(1)}(z) &= \Gamma^{(1)}z - \bar{\Gamma}'^{(1)}\frac{R^2}{z},\\ \psi^{(1)}(z) &= \Gamma'^{(1)}z - (2\Gamma^{(1)} + p)\frac{R^2}{z} - \bar{\Gamma}'^{(1)}\frac{R^4}{z^3}.\end{aligned} \tag{IX.239}$$

The potentials of the second order are determined from boundary (on L) condition (IX.217)

$$\varphi_0^{(2)}(t) + t\overline{\varphi_0^{(2)'}(t)} + \overline{\psi_0(t)} = K_0(t, \bar{t}, \gamma, \delta), \tag{IX.240}$$

where, for variant A, we have

$$K_0(t, \bar{t}, \gamma, \delta) = F_1(t, \bar{t}, \gamma, \delta) - 2\Gamma^{(2)}t - \bar{\Gamma}'^{(2)}\bar{t}. \tag{IX.241}$$

By substituting functions (IX.239) into the expression for $F_1(z, \bar{z}, \gamma, \delta)$ and converting to the values on the boundary, we find from condition (IX.240), considering relations (IX.241) and (IX.163), that the complex potentials of the second order in variant A will be of the form:

[1]Since, in the nonlinear theory, we cannot use superpositioning of solutions, the problem is solved for the joint effect of the examined system of forces.

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$$\varphi^{(2)}(z) = \Gamma^{(2)} z + a_{-1}^{(2)} \frac{R^2}{z} + a_{-3}^{(2)} \frac{R^4}{z^3},$$
$$\psi^{(2)}(z) = \Gamma'^{(2)} z + b_{-1}^{(2)} \frac{R^2}{z} + b_{-3}^{(2)} \frac{R^4}{z^3} + b_{-5}^{(2)} \frac{R^6}{z^5}, \tag{IX.242}$$

where coefficients $a_j^{(2)}$ and $b_j^{(2)}$ (for $k_1 = k_2 = 0$) are

$$\begin{aligned}
a_{-1}^{(2)} &= \gamma [(k+1)(1-\delta)\Gamma^{(1)} + 2p]\,\bar{\Gamma}'^{(1)} - \bar{\Gamma}'^{(2)};\\
a_{-3}^{(2)} &= -\gamma (k+1)(\bar{\Gamma}'^{(1)})^2;\\
b_{-1}^{(2)} &= [\gamma(3+\delta)(k+1) - k_3](\Gamma^{(1)})^2 - \gamma(k+1)(1+\delta)\Gamma'^{(1)}\bar{\Gamma}'^{(1)} -\\
&\quad - 2\Gamma^{(2)} + \gamma(4+\delta-k)\Gamma^{(1)}p + \gamma p^2;\\
b_{-3}^{(2)} &= \gamma[2+\delta-k]\,p\bar{\Gamma}'^{(1)} - 2k_3\bar{\Gamma}'^{(1)}\Gamma^{(1)} - \bar{\Gamma}'^{(2)};\\
b_{-5}^{(2)} &= -[4\gamma(k+1) + k_3](\bar{\Gamma}'^{(1)})^2.
\end{aligned} \tag{IX.243}$$

In the case of variant B the function $K_0(t, \bar{t}, \gamma, \delta)$ in the boundary condition of the problem is replaced by function (IX.219):

$$K_0^*(t, \bar{t}, \gamma', \delta') = K_0(t, \bar{t}, \gamma', \delta') - p\left\{[(k+1)\Gamma^{(1)} + p]\,t - (k+1)\bar{\Gamma}'^{(1)}\frac{R^2}{t}\right\}. \tag{IX.244}$$

As the result, we find that the coefficients of the expansions of functions $\phi_1^{(2)}(\eta)$ and $\psi_1^{(2)}(\eta)$ are determined from relations (IX.243) by substituting γ, δ by γ', δ' and by adding to coefficients $a_{-1}^{(2)}$, $b_{-1}^{(2)}$, the terms, respectively /650

$$p(k+1)\bar{\Gamma}'^{(1)}; \qquad -p[(k+1)\Gamma^{(1)} + p]. \tag{IX.245}$$

The coefficients of expansion of the potentials of the second order are determined in the case of variant C in accordance with formulas (IX.243) by replacing γ, δ by γ', δ'.

Let us consider some particular cases. We will assume that the contour of a hole is free of external load; then functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ are defined by formulas (IX.242), (IX.243) if we assume in them[1] that $p = 0$. The

[1] The expressions for the coefficients of the functions $\phi^{(2)}(z)$ and $\psi^{(2)}(z)$ for this case were first presented in the work of J. E. Adkins, A. E. Green [1].

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coefficients of the potentials of the second order for variant B are found in this case by simple substitution of γ, δ by γ', δ' in formulas (IX.243).

We will write the formulas for the stress state components in variants A and B.

Variant A. Here we analyze the elastic equilibrium of a body which, for a given magnitude of principal stresses at infinity N_1 and N_2, represents a plane with a round hole of radius R. Obviously, in the initial state, the hole, generally speaking, differed from round, whereupon, for each pair of values N_1 and N_2, the original shape and dimensions of the hole differ.

We will introduce, in the deformed state, the polar coordinate system r, ϑ ($z = re^{i\vartheta}$). Then, by substituting functions $\phi^{(k)}(z)$, $\psi^{(k)}(z)$ in the relations for stress components (IX.124) and (IX.125), consdering the formulas for conversion from Cartesian to polar coordinates, we obtain for the points of the contour circle

$$\sigma_\vartheta = N_1 + N_2 - 2(N_1 - N_2)\cos 2\vartheta + \frac{\gamma}{4^0 H}[(2 - k\delta)(N_1 - N_2)^2 + (N_1 + N_2)^2 - 4(N_1^2 - N_2^2)\cos 2\vartheta + 4(N_1 - N_2)^2\cos 4\vartheta];$$
$$\sigma_r = 0. \qquad \text{(IX.246)}$$

Formula (IX.246) enables us to find the stress distribution σ_ϑ on the contour of a round hole in the case of uniaxial tension-compression, multifold tension-compression, and pure deflection. The stress concentration coefficients in these problems are: for uniaxial tension-compression ($N_2 = 0$, $N_1 = N$)

$$k^{(A)} = 3\left[1 + \frac{\gamma(11 - k\delta)}{12}\cdot\frac{N}{^0H}\right]; \qquad \text{(IX.247)}$$

for multifold tension-compression ($N_1 = N_2 = N$)

$$k^{(A)} = 2\left(1 + \frac{\gamma}{2}\cdot\frac{N}{^0H}\right); \qquad \text{(IX.248)}$$

for pure deflection ($N_1 = -N_2 = N$), /651

$$k^{(A)} = 4\left[1 + \frac{\gamma(6 - k\delta)}{4}\cdot\frac{N}{^0H}\right]. \qquad \text{(IX.249)}$$

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Variant B. Here we analyze the elastic equilibrium of a body which, in the initial (undeformed) state, represented a plane with a round hole of radius R. During the process of deformation, the shape and dimensions of the hole changed continuously. Here, to each pair of values N_1 and N_2 corresponds a deformed contour of a certain shape.

If we introduce, in the deformed state, polar coordinates r, ϑ ($\eta = re^{i\vartheta}$), then stress components on the deformed contour are

$$n^{12} = N_1 + N_2 - 2(N_1 - N_2)\cos 2\vartheta + \frac{\gamma - \frac{k+1}{2}}{4^0 H} \times$$
$$\times \left\{ \frac{\gamma}{\gamma - \frac{k+1}{2}} [(2 - k\delta)(N_1 - N_2)^2 + (N_1 + N_2)^2] - \right. \tag{IX.250}$$
$$\left. - 4(N_1^2 - N_2^2)\cos 2\vartheta + 4(N_1 - N_2)^2 \cos 4\vartheta \right\},$$

where $n^{12} = \sigma_r + \sigma_\vartheta$ is the sum of principal stresses.

We will notice that since component n^{12} is invariant in relation to the transformation of the coordinate system, then, in the curvilinear coordinate system (ρ, ϑ^*), which corresponds to the contour of the deformed hole, we find

$$n^{12} = \sigma_\rho + \sigma_{\vartheta^*}. \tag{IX.251}$$

Since, on contour $\sigma_\rho = 0$, then component n^{12} is a ring stress for points of the deformed contour.

The stress concentration coefficients on the deformed contour are:

for uniaxial tension-compression ($N_2 = 0$; $N_1 = N$)

$$k^{(B)} = 3\left\{1 + [\gamma(11 - k\delta) - 4(k+1)]\frac{N}{12^0 H}\right\}; \tag{IX.252}$$

for multifold tension-compression ($N_1 = N_2 = N$)

$$k^{(B)} = 2\left(1 + \frac{\gamma}{2} \cdot \frac{N}{^0 H}\right); \tag{IX.253}$$

for pure deflection ($N_1 = -N_2 = N$)

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$$k^{(B)} = 4\left\{1 + [\gamma(6 - k\delta) - 2(k+1)]\frac{N}{4^0H}\right\}. \qquad \text{(IX.254)}$$

On the basis of formulas (IX.246) and (IX.250) we can analyze the effect of nonlinear effects of the second order on the stress distribution around the contour of a round and initially round hole in a homogeneous stress state at infinity. /652

We will analyze stress distribution σ_ϑ on the contour in the case of uniaxial tension-compression for an incompressible material with energy function of the Mooney form (IX.102). Here the constants in formulas (IX.246) and (IX.250) are defined by relations (IX.104) and (IX.105).

Variant A. The stress component on the round contour is defined by the formulas

in the case of plane deformation

$$\sigma_\vartheta = N\left[1 - 2\cos 2\vartheta + \frac{N}{4\mu}(1 - 2\cos 2\vartheta + 2\cos 4\vartheta)\right]; \qquad \text{(IX.255)}$$

in the case of plane stress state

$$\sigma_\vartheta = N\left\{1 - 2\cos 2\vartheta + \frac{5+\varrho^*}{16(1+\varrho^*)}\cdot\frac{N}{^0H}\left[\frac{9-11\varrho^*}{3(5+\varrho^*)} - 2\cos 2\vartheta + 2\cos 4\vartheta\right]\right\}, \qquad \text{(IX.256)}$$

where

$$\varrho^* = \frac{A_2}{A_1}; \qquad {}^0H = 2h_0\mu.$$

Stress distribution σ_ϑ (IX.255) and (IX.256) at the various points of the contour is characterized by Table IX.1.

We will notice that in the given problem a finite round contour can be obtained by two ways: by tension (+N) of a plane with an oval hole along its small axis, or by compression (-N) along its large axis.

As follows from Table IX.1 the greater the forces of tension that must be applied to achieve deformation of the original contour to round contour (for an oval with a shorter small axis), the greater the stress concentration must be on the contour; when the compressive forces increase (for an oval that is extended to a greater degree along the large axis), the stress concentration on the contour decreases.

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TABLE IX.1

$\vartheta°$	Plane deformation	Plane stress state for (ρ = 1/19)
0	$-N\left(1-0,25\frac{N}{\mu}\right)$	$-N\left(1-0,166\frac{N}{{}^0H}\right)$
30	$-0,25\frac{N^2}{\mu}$	$-0,433\frac{N^2}{{}^0H}$
45	$N\left(1-0,25\frac{N}{\mu}\right)$	$N\left(1-0,433\frac{N}{{}^0H}\right)$
60	$2N\left(1+0,125\frac{N}{\mu}\right)$	$2N\left(1+0,083\frac{N}{{}^0H}\right)$
90	$3N\left(1+0,417\frac{N}{\mu}\right)$	$3N\left(1+0,455\frac{N}{{}^0H}\right)$

TABLE IX.2

$\vartheta°$	Plane deformation	Plane stress state for (ρ = 1/19)
0	$-N\left(1-0,25\frac{N}{\mu}\right)$	$-N\left(1-0,166\frac{N}{{}^0H}\right)$
30	$0,75\frac{N^2}{\mu}$	$0,900\frac{N^2}{{}^0H}$
45	$N\left(1+0,75\frac{N}{\mu}\right)$	$N\left(1+0,900\frac{N}{{}^0H}\right)$
60	$2N\left(1+0,125\frac{N}{\mu}\right)$	$2N\left(1+0,083\frac{N}{{}^0H}\right)$
90	$3N\left(1-0,25\frac{N}{\mu}\right)$	$3N\left(1-0,433\frac{N}{{}^0H}\right)$

Tr. Note: Commas indicate decimal points.

Variant B. Here we analyze the stress concentration on the contour of a hole which was initially round. Because of deformation, the round hole was /653
transformed into an oval, extended along the Oy^1 axis under tension and along the Oy^2 axis under compression.

The ring stress on the deformed contour of the hole is defined[1] by the formulas:

in the case of plane deformation

$$n^{12} = N\left[1 - 2\cos 2\vartheta + \frac{N}{4\mu}(1 + 2\cos 2\vartheta - 2\cos 4\vartheta)\right]; \qquad \text{(IX.257)}$$

in the case of the plane stress state

$$n^{12} = N\left[1 - 2\cos 2\vartheta + \frac{17 + 29\varrho^*}{48(1+\varrho^*)}\cdot\frac{N}{{}^0H}\left(\frac{9 - 11\varrho^*}{17 + 29\varrho^*} + 2\cos 2\vartheta - 2\cos 4\vartheta\right)\right]. \qquad \text{(IX.258)}$$

In formulas (IX.257) and (IX.258) ϑ is the polar angle in the nondeformed plane.

Stress distribution σ_ϑ on a nondeformed contour is characterized by Table IX.2.

The difference between the results in variants A and B is attributed to the change of the shape of the contour during deformation. As the forces of

[1] Formulas (IX.255) and (IX.257) were first derived in the work of J. E. Adkins, A. E. Green, R. T. Shield [1].

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tension increase, the original contour of the hole becomes more and more flattened along the Oy^1 axis, and naturally, stress concentration at the point $\vartheta = \pi/2$ decreases. Under compression, the hole becomes flattened along the Oy^2 axis, with the result that stress concentration at the point $\vartheta = \pi/2$ increases. These results bear out the fact that the terms of the second order for an incompressible material take into account only the geometrical[1] nonlinearity of the problem.

Such analysis can also be accomplished in the case of multifold tension-compression and pure deflection. The results of this analysis, for an incompressible material with the Mooney form of energy function (IX.102) for $A_2:A_1 = 1:19$ can be characterized by Table IX.3 and the stress-strain diagrams of stresses σ_ϑ/N in Figures IX.1 and IX.2.

TABLE IX.3

Stress state	Plane Deformation		Plane stress state		Linear theory
	Variant A	Variant B	Variant A	Variant B	
Uniaxial tension	$3\left(1+0{,}417\frac{N}{\mu}\right)$	$3\left(1-0{,}250\frac{N}{\mu}\right)$	$3\left(1+0{,}455\frac{N}{\sigma_H}\right)$	$3\left(1-0{,}433\frac{N}{\sigma_H}\right)$	3
Uniaxial compression	$3\left(1-0{,}417\frac{N}{\mu}\right)$	$3\left(1+0{,}250\frac{N}{\mu}\right)$	$3\left(1-0{,}455\frac{N}{\sigma_H}\right)$	$3\left(1+0{,}433\frac{N}{\sigma_H}\right)$	3
Multifold tension	$2\left(1+0{,}250\frac{N}{\mu}\right)$	$2\left(1+0{,}250\frac{N}{\mu}\right)$	$2\left(1+0{,}300\frac{N}{\sigma_H}\right)$	$2\left(1+0{,}300\frac{N}{\sigma_H}\right)$	2
Multifold compression	$2\left(1-0{,}250\frac{N}{\mu}\right)$	$2\left(1-0{,}250\frac{N}{\mu}\right)$	$2\left(1-0{,}300\frac{N}{\sigma_H}\right)$	$2\left(1-0{,}300\frac{N}{\sigma_H}\right)$	2
Pure defection	$4\left(1+0{,}625\frac{N}{\mu}\right)$	$4\left(1+0{,}375\frac{N}{\mu}\right)$	$4\left(1+0{,}616\frac{N}{\sigma_H}\right)$	$4\left(1+0{,}716\frac{N}{\sigma_H}\right)$	4

Tr. Note: Commas indicate decimal points.

The results obtained can be formulated briefly as follows. /654

1. In nonlinear theory the stress concentration coefficient depends on the initial and final shapes of the contour of the hole, the form and magnitude of load at "infinity," elastic properties of the material, and form of elastic equilibrium (plane deformation or plane stress state).

2. Deviation of the magnitude of the stress concentration coefficient from the stress concentration coefficient given by linear theory is considerable

[1]We will also arrive at this very same conclusion below, in §3, as the outcome of several different starting points.

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in many problems. In the case of the homogeneous stress state at "infinity," the greatest deviation occurs in the problem of pure deflection.

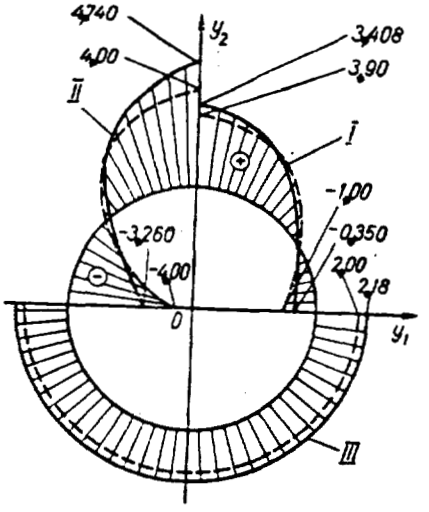


Figure IX.1.

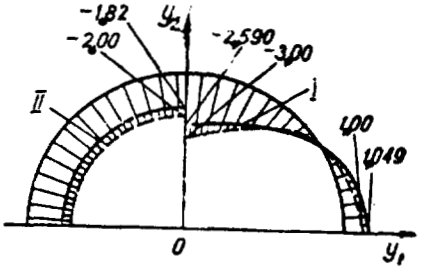


Figure IX.2.

By using the potential of the first (IX.239) and second (IX.242) orders, we can write out the formulas for the stress components and for the other partial cases. For instance, in the problem of multifold tension-compression of a plate with a round hole, the contour of which is subjected to the effect of uniform normal load with potentials of the second order, defined by formulas (IX.242) and (IX.243) for /655

$$\Gamma^{(1)} = \frac{N}{2}, \qquad \Gamma'^{(1)} = 0, \tag{IX.259}$$

for variant A

$$\varphi^{(2)}(z) = \Gamma^{(2)} z; \qquad \psi^{(2)}(z) = b_{-1}^{(2)} \frac{R^2}{z}. \tag{IX.260}$$

The curves of stresses σ_ϑ/N on the contour of a round hole are shown in Figure IX.1 for $N/^0H = 0.3$ in the plane stress state, where I is uniaxial tension; II is pure deflection; III is multifold tension; the broken curve corresponds to linear theory.

The curves of stresses σ_ϑ/N on the contour of a round hole are illustrated in Figure IX.2 for $N/^0H = 0.3$ in the plane stress state, where I is uniaxial tension; II is multifold compression; the broken curve corresponds to linear theory.

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For variant B

$$\varphi_1^{(2)}(\eta) = \Gamma^{(2)}\eta; \qquad \psi_1^{(2)}(\eta) = b_{1,1}^{(2)}\frac{R_1^2}{\eta}. \tag{IX.261}$$

In formulas (IX.260) and (IX.261) R is the radius of a round hole in the deformed plate; R_1 is the radius of a round hole in the initial state;

$$b_1^{(2)} = \frac{\gamma}{2}(N+p)[2(N+p)+(k\div\delta)N];$$
$$b_{1,1}^{(2)} = (\gamma-2)(N+p)^2 + \frac{1}{2}[\gamma(k+\delta)-2(k-1)](N+p)N. \tag{IX.262}$$

If we introduce in the deformed and initial states polar coordinate systems r, ϑ and r*, ϑ*, then, for stress components, we obtain the formulas[1]:

for variant A

$$\sigma_r^{\mathrm{I}} = N-(N+p)\frac{R^2}{r^2}\left[1-\frac{\gamma(N+p)}{2^0H}\left(1-\frac{R^2}{r^2}\right)\right],$$
$$\sigma_\vartheta^{\mathrm{I}} = N+(N+p)\frac{R^2}{r^2}\left[1-\frac{\gamma(N+p)}{2^0H}\left(1-3\frac{R^2}{r^2}\right)\right]; \tag{IX.263}$$

for variant B

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$$\sigma_{r^*}^{\mathrm{II}} = N-(N+p)\frac{R_1^2}{r^{*2}}\left[1-\frac{(\gamma-2)(N+p)}{2^0H}\left(1-\frac{R_1^2}{r^{*2}}\right)\right],$$
$$\sigma_{\vartheta^*}^{\mathrm{II}} = N+(N+p)\frac{R_1^2}{r^{*2}}\left[1-\frac{(\gamma-2)(N+p)}{2^0H}\left(1-\frac{R_1^2}{r^{*2}}-\frac{2\gamma R_1^2}{(\gamma-2)r^{*2}}\right)\right]. \tag{IX.264}$$

Since in this problem displacement component u_r is not a function of angle ϑ, and component u_ϑ is equal to zero, then during deformation the round contour retains its shape and only magnitude of the radius of the hole R varies. We will notice, however, that in the case of variant A, to each pair of values N and p corresponds a certain magnitude of initial radius, and in the case of variant B, finite radius of the hole.

Since deformation in the given problem is axisymmetric, σ_{r*}, $\sigma_{\vartheta*}$ coincide with the stress components in the polar coordinate system of the deformed state,

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[1]See Yu. I. Koyfman [3].

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but are expressed through the initial values of the radius of the hole and vector radius r. The formulas for the stress components for points of the contour of the circle have the form

$$\sigma_r^{\mathrm{I}} = \sigma_r^{\mathrm{II}} = -p; \quad . \quad \sigma_\phi^{\mathrm{I}} = \sigma_\phi^{\mathrm{II}} = N + (N+p)\left[1 + \frac{\gamma(N+p)}{{}^0H}\right]. \tag{IX.265}$$

Assuming in formulas (IX.263) and (IX.264) p = 0 or N = 0, we find the expression for the stress state components for the problem of uniform normal pressure on the contour of a round hole in a plane, or of multifold tension-compression of a plate with a round hole[1].

Elliptical Hole[2]. We will assume that the contour of an elliptical hole is subjected to the effect of uniform normal pressure and that the stress state at infinity is homogeneous.

The potentials of the second order for this problem are given in §2, Chapter II:

$$\begin{aligned} \varphi^{(1)}(\zeta) &= \Gamma^{(1)} R\zeta - \frac{RT_1}{\zeta}, \\ \psi^{(1)}(\zeta) &= \Gamma'^{(1)} R\zeta - \frac{R\zeta}{\zeta^2 - m} T_2 - \frac{R\bar{\Gamma}'^{(1)}}{\zeta(\zeta^2 - m)}, \end{aligned} \tag{IX.266}$$

where m and R are the parameters of the mapping function

$$\begin{aligned} z &= \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \\ \eta &= \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \end{aligned} \tag{IX.267}$$

or

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$$m = \frac{a-b}{a+b}; \qquad R = \frac{a+b}{2};$$

a and b are the semiaxes of the ellipse;

$$T_1 = m\Gamma^{(1)} + \bar{\Gamma}'^{(1)} + pm; \qquad T_2 = (1+m^2)(2\Gamma^{(1)} + p) + m\bar{\Gamma}'^{(1)}.$$

---

[1]The formulas for variant A in the first of these problems were first presented in the work of J. E. Adkins, A. E. Green [1].
[2]See Yu. I. Koyfman [2, 3, 5] and also G. Lianis [1].

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The potentials of the second order are defined by formulas (IX.232), which, for the given problem, acquire the form

$$\varphi^{(2)}(\zeta) = \Gamma^{(2)} R\zeta + \varphi_0^{(2)}(\zeta),$$
$$\psi^{(2)}(\zeta) = \Gamma'^{(2)} R\zeta + \psi_0^{(2)}(\zeta). \tag{IX.268}$$

Functions $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$ should be found from boundary condition (IX.233).

The functional expressions for the determination of the potentials can be written in analogy with the linear[1] theory in the form

$$\varphi_0^{(2)}(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1}\left[\frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\sigma^2}\,\overline{\varphi}_0^{(2)'}\left(\frac{1}{\sigma}\right) - K_0\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right)\right]\frac{d\sigma}{\sigma-\zeta} + \varphi_0^{(2)}(\infty),$$
$$\psi_0^{(2)}(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1}\left[\sigma\,\frac{1+m\sigma^2}{\sigma^2-m}\,\varphi_0^{(2)'}(\sigma) - \overline{K}_0\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right)\right]\frac{d\sigma}{\sigma-\zeta} + \psi_0^{(2)}(\infty), \tag{IX.269}$$

where

$$K_0\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) = F_1\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) - R\Gamma^{(2)}\left[\sigma + \frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\right] - \overline{\Gamma}'^{(2)}\frac{R}{\sigma}. \tag{IX.270}$$

In solving the problem in variant B, the K_0 function is replaced by the function

$$K_0^*\left(\sigma, \frac{1}{\sigma}, \gamma', \delta'\right) = K_0\left(\sigma, \frac{1}{\sigma}, \gamma', \delta'\right) - pD^{(1)}, \tag{IX.271}$$

where displacements of the first order for the points of the contour are

$$D^{(1)}(\sigma) = R\left(T_3\sigma - T_4\frac{1}{\sigma}\right); \tag{IX.272}$$

here

$$T_3 = (k+1)\Gamma^{(1)} + p; \qquad T_4 = (k+1)(m\Gamma^{(1)} + \overline{\Gamma}'^{(1)}) + kmp.$$

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[1] See formulas (I.37).

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From equations (IX.269), after elemental calculations, we obtain functions $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$, and then potentials of the second order, finally, can be written in the following (variant A) form:

$$\varphi^{(2)}(\zeta) = R\left[\Gamma^{(2)}\zeta + \frac{M_1 + M_2\zeta^2}{\zeta(\zeta^2 - m)}\right], \tag{IX.273}$$

$$\psi^{(2)}(\zeta) = R\left[\Gamma'^{(2)}\zeta + \frac{M_3 + M_4\zeta_2 + M_5\zeta^4}{\zeta(\zeta^2 - m)^2}\right] - \zeta\frac{1 + m\zeta^2}{\zeta^2 - m}\varphi_0^{(2)'}(\zeta), \qquad \underline{/658}$$

where the coefficients of these functions for $k_1 = k_2 = 0$ are

$$\begin{aligned}
M_1 &= -\gamma\{m\bar{\Gamma}'^{(1)}T_3 + T_4(T_1 - m\delta\Gamma^{(1)})\} + k_3m^2(\Gamma^{(1)})^2 \div m(\bar{\Gamma}'^{(2)} + m\Gamma^{(2)});\\
M_2 &= \gamma[(\bar{\Gamma}'^{(1)} + T_1 + m\Gamma^{(1)})T_3 - (1+\delta)\Gamma^{(1)}T_4] - k_3m(\Gamma^{(1)})^2 - (\bar{\Gamma}'^{(2)} + m\Gamma^{(2)});\\
M_3 &= \gamma[(m\Gamma^{(1)} - \delta T_1)mT_3 - \bar{\Gamma}'^{(1)}T_4] - k_3T_1^2 - m^2\Gamma^{(2)};\\
M_4 &= \gamma\{(\delta T_1 - m\Gamma^{(1)})T_3 - T_4T_5 + (\bar{\Gamma}'^{(1)} - m^2\Gamma'^{(1)})T_3 \div m[(m\Gamma^{(1)} + \\
&\quad + T_1)\delta T_4 - (1+\delta)T_3\Gamma^{(1)}\} - k_3[(2\Gamma^{(1)} + mT_1)T_1 - m^3(\Gamma^{(1)})^2] + \\
&\quad + m(3+m^2)\Gamma^{(2)};\\
M_5 &= \gamma[2mT_3\Gamma'^{(1)} + T_3T_5 + (1+\delta)\Gamma^{(1)}T_3 - \Gamma'^{(1)}T_4 - \delta\bar{T}_4(m\Gamma^{(1)} + T_1)] - \\
&\quad - k_3[2m(T_1 + m\Gamma^{(1)}) + \Gamma^{(1)}]\Gamma^{(1)} - (2+m^2)\Gamma^{(2)};\\
T_5 &= (1-m^2)(2\Gamma^{(1)} + p) - m(\Gamma'^{(1)} + \bar{\Gamma}'^{(1)}).
\end{aligned} \tag{IX.274}$$

In the case of variant B

$$\begin{aligned}
\varphi_1^{(2)}(\zeta) &= R\Gamma^{(2)}\zeta + \varphi_{1,0}^{(2)}(\zeta) = R\left[\Gamma^{(2)}\zeta + \frac{M_1^* + M_2^*\zeta^2}{\zeta(\zeta^2 - m)} + \frac{T_4p}{\zeta}\right],\\
\psi_1^{(2)}(\zeta) &= R\left[\Gamma'^{(2)}\zeta + \frac{M_3^* + M_4^*\zeta^2 + M_5^*\zeta^4}{\zeta(\zeta^2 - m)^2} - \frac{T_3p}{\zeta}\right] - \zeta\frac{1 + m\zeta^2}{\zeta^2 - m}\varphi_{1,0}^{(2)'}(\zeta).
\end{aligned} \tag{IX.275}$$

Here the components M_i^* are determined on the basis of formulas (IX.273) by replacing in them γ, δ by γ', δ'.

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By using the potentials of the second order (IX.273) or (IX.274), we can analyze certain partial problems. By way of example we introduce the formula for stress σ_ϑ on the contour of a free hole, i.e., when p = 0, in the homogeneous stress state N_1 and N_2 at infinity for $\alpha = 0$:

$$\sigma_\vartheta = \frac{1}{d}\left\{(N_1+N_2)+2(N_1-N_2)(m-\cos 2\vartheta)+\frac{1}{\upsilon H}\left[R_1+R_2\cos 2\vartheta+\right.\right.$$
$$\left.\left.+\frac{1}{d}(R_3+R_4\cos 2\vartheta+R_5\cos 4\vartheta)+\frac{1}{d^2}(R_6+R_7\cos 2\vartheta+R_8\cos 4\vartheta+R_9\cos 6\vartheta)\right]\right\}, \quad \text{(IX.276)}$$

where /659

$$R_1 = 2\left\{(1-m^2)\Gamma^{(2)}+\frac{m^2}{16}(N_1+N_2)^2[\gamma(1-\delta)(k+1)-k_3]-\right.$$
$$\left.-\frac{m\gamma}{8}(3-k\delta)(N_1^2-N_2^2)\right\}+\frac{1}{16}\gamma(1-k\delta)[(1+m^2)(N_1+N_2)^2+4d_1],$$
$$R_2 = \frac{\gamma}{4}[2(N_1^2-N_2^2)-mk(N_1-N_2)^2], \quad \text{(IX.277)}$$
$$R_3 = \gamma\left\{\delta_1 m(k+1)d_2-\frac{k}{2}\left[\frac{(N_1+N_2)^2}{4}((1-m^2)^2-2m^2)+\right.\right.$$
$$\left.\left.+2m(N_1^2-N_2^2)+m^2d_1\right]-(1-m^2)d_1+\frac{3}{2}(N_1-N_2)^2+\frac{(1-m^2)^2}{4}(N_1+N_2)^2\right\},$$
$$R_4 = -\frac{\gamma}{2}\left\{\delta_1(k+1)(1+m^2)d_2-(k-2)[(1-m^2)\times\right.$$
$$\left.\times(N_1^2-N_2^2)+2m(N_1-N_2)^2]\right\},$$
$$R_5 = \frac{\gamma}{4}[2(N_1-N_2)^2-km^2(N_1+N_2)^2-2kd_1],$$
$$R_6 = \frac{1}{2}\gamma(k+1)m^2(3+m^2)d_1,$$
$$R_7 = -\frac{1}{2}\gamma(k+1)m(3+4m^2)d_1,$$
$$R_8 = \frac{1}{2}\gamma(1+k)(1+3m^2)d_1,$$
$$R_9 = -\frac{1}{2}\gamma m(k+1)d_1;$$
$$d_1 = (N_1-N_2)^2-m(N_1^2-N_2^2),$$
$$d_2 = (N_1^2-N_2^2)-m(N_1+N_2)^2.$$

In the case of variant A, it is necessary to assume in these formulas that $\delta_1 = 1$.

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In analyzing variant B it is assumed that the region under examination represented in the nondeformed state a plane with an elliptical hole which was characterized by the parameter $m = (a - b)/(a + b)$. Conformal mapping in this case is related to curvilinear coordinates ρ^*, ϑ^* of the initial state.

During deformation, the shape of the contour changed, but since component n^{12} is invariant, then, in the curvilinear coordinate system ρ, ϑ, related to the deformed contour, we may write

$$n^{12} = \sigma^{II}_{\rho^*} + \sigma^{II}_{\vartheta^*} = \sigma^{II}_{\rho} + \sigma^{II}_{\vartheta^*}. \tag{IX.278}$$

Since there is no external load on the contour, then /660

$$\sigma^{II}_{\rho} = 0; \quad n^{12} = \sigma^{II}_{\vartheta}. \tag{IX.279}$$

The component σ^{II}_{ϑ} is defined by formula (IX.276), if in components R_4, R_3 we assume $\delta_1 = 1 - 1/\gamma$ and in components R_6, ..., R_9, replace γ by $\gamma - 1$.

As the outcome of formula (IX.276), we may determine the stresses on the contour for the following problems: 1) for the biaxial stress state ($N_1 \neq N_2$); 2) for tension-compression along the large axis ($N_2 = 0$); 3) for tension-compression along the small axis ($N_1 = 0$); 4) for multifold tension-compression ($N_1 = N_2$); 5) for pure deflection ($N_1 = -N_2$).

By way of example we will analyze the stress distribution around the contour of an elliptical hole for tension-compression along the large axis in the case of plane deformation of an incompressible material with the Mooney form of energy function when $m = 1/3$.

Variant A. The values of component σ^{I}_{ϑ} at the various points of the elliptical contour are presented below (ϑ is the polar angle in plane ζ of the deformed state):

ϑ°	σ^{I}_{ϑ}
0	$-N\left(1-0{,}250\frac{N}{\mu}\right)$
30	$0{,}71N\left(1-0{,}271\frac{N}{\mu}\right)$
45	$1{,}4N\left(1+0{,}029\frac{N}{\mu}\right)$
60	$1{,}77N\left(1+0{,}149\frac{N}{\mu}\right)$
90	$2N\left(1+0{,}219\frac{N}{\mu}\right)$

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These results show that as in the case of a round hole (see formulas (IX.255) and (IX.256)), the nonlinearity of the problem results in an increase in the concentration coefficient as the forces of tension increase, and to a decrease in the concentration coefficient as the compressive forces increase.

Variant B. Due to deformation, the elliptical hole becomes flattened along the large axis under tension and along the small axis under compression. The values of stresses σ_{ϑ}^{II} at the various points of the deformed contour are given below (ϑ is the polar angle in plane ζ of the nondeformed state):

$\vartheta°$	σ_{ϑ}^{II}
0	$-N\left(1-0{,}250\frac{N}{\mu}\right)$
30	$0{,}71N\left(1+0{,}671\frac{N}{\mu}\right)$
45	$1{,}4N\left(1+0{,}114\frac{N}{\mu}\right)$
60	$1{,}77N\left(1-0{,}619\frac{N}{\mu}\right)$
90	$2N\left(1-0{,}156\frac{N}{\mu}\right)$

A decrease in stress concentration at the point $\vartheta = 90°$ with tension in comparison with variant A and also in the case of a round hole is attributed to the fact that in this variant the magnitude of stress concentration is determined on a contour that is extended more along the large axis of the hole than in variant A. Under compression the shape of the hole approaches more closely to circular and therefore the stress concentration increases. /661

The values of the stress concentration coefficient on the contour of an elliptical hole (variant A) or initially elliptical (variant B) hole during the plane deformation of an incompressible material with the Mooney form of energy function for

$$m = \frac{a-b}{a+b} = \frac{1}{3}$$

in the case of plane deformation of an incompressible material for certain partial problems, are presented in Table IX.4.

Thus, the data in Table IX.4 illustrate the effect of geometric nonlinearity, which distorts the shape of the contour during deformation, on the magnitude of stress concentration.

Round Hole with Reinforced Edge[1]. Let the examined region represent a plane with a round hole of radius R in which is soldered a round and wide ring

[1]See G. N. Savin and Yu. I. Koyfman [1, 2].

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with internal radius R_1. We will assume that the stress state at infinity is homogeneous and that the inside contour of the ring is free of external forces.

The problem can be examined in two variants, namely F and I. In analyzing variant F it is assumed that the contour of seam L and in the internal contour of the ring L_1 are circular in the deformed state; in variant I it is assumed that these contours are round in the initial state. /662

TABLE IX.4

Form of Load	Variant	Tension-compression along major axis	Tension-compression along minor axis	Multifold tension compression
Tension	А	$2\left(1+0{,}219\frac{N}{\mu}\right)$	$5\left(1+0{,}800\frac{N}{\mu}\right)$	$4\left(1+0{,}687\frac{N}{\mu}\right)$
	Б	$2\ 1-0{,}156\frac{N}{\mu}\Big)$	$5\left(1-0{,}400\frac{N}{\mu}\right)$	$4\left(1-0{,}062\frac{N}{\mu}\right)$
Compression	А	$2\left(1-0{,}219\frac{N}{\mu}\right)$	$5\left(1-0{,}800\frac{N}{\mu}\right)$	$4\left(1-0{,}687\frac{N}{\mu}\right)$
	Б	$2\left(1+0{,}156\frac{N}{\mu}\right)$	$5\left(1+0{,}400\frac{N}{\mu}\right)$	$4\left(1+0{,}062\frac{N}{\mu}\right)$
Linear Theory	—	2	5	4

Tr. Note: Commas indicate decimal points.

The boundary conditions for the potentials of the second order in variant F (see formulas (IX.209)-(IX.212)) are written as follows:

$$\varphi_{(1)}^{(2)}(t)+t\overline{\varphi_{(1)}^{(2)'}(t)}+\overline{\psi_{(1)}^{(2)}(t)}-F_{1(1)}(t,\bar{t},\gamma_{(1)},\delta_{(1)})=0 \text{ on } L_1; \tag{IX.280}$$

$$\varphi_{(1)}^{(2)}(t)+t\overline{\varphi_{(1)}^{(2)'}(t)}+\overline{\psi_{(1)}^{(2)}(t)}-F_{1(1)}(t,\bar{t},\gamma_{(1)},\delta_{(1)})=$$

$$=m\left[\varphi_{(0)}^{(2)}(t)+t\overline{\varphi_{(0)}^{(2)'}(t)}+\overline{\psi_{(1)}^{2)}(t)}-F_{1(0)}(t,\bar{t},\gamma_{(0)},\delta_{(0)})\right],$$

$$k\varphi_{(1)}^{(2)}(t)-t\overline{\varphi_{(1)}^{(2)'}(t)}-\overline{\psi_{(1)}^{(2)}(t)}-F_{2(1)}(t,\bar{t},\gamma_{(1)}\delta_{(1)})=$$

$$=m^2\left[k_{(0)}\varphi_{(0)}^{(2)}(t)-t\overline{\varphi_{(0)}^{(2)'}(t)}-\overline{\psi_{(0)}^{(2)}(t)}-F_{2(0)}(t,\bar{t},\gamma_{(0)},\delta_{(0)})\right] \text{ on } L, \tag{IX.281}$$

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where $\phi^{(2)}_{(0)}(t)$, $\psi^{(2)}_{(0)}(t)$ and $\phi^{(2)}_{(1)}(t)$, $\psi^{(2)}_{(1)}(t)$ are the boundary values of the potentials of the second order for regions occupied by the plane and ring, respectively; $m = \frac{{}^0H_{(1)}}{{}^0H_{(0)}}$.

Since the internal contour of the ring is free of external forces, i.e., on L_1 we have the condition $p(t) = 0$, then

$$F^{\bullet}_{1(1)}(t, \bar{t}, \gamma'_{(1)}, \delta'_{(1)}) = F_{1(1)}(t, \bar{t}, \gamma'_{(1)}, \delta'_{(1)}) \text{ on } L_1.$$

Consequently, by comparing the boundary conditions of variants F and I (IX.211), (IX.212) and (IX.215), (IX.216), we see that in the problem under consideration the boundary conditions of variant I differ from boundary conditions (IX.280) and (IX.281) only in that the constants γ, δ in functions $F_1(t, \bar{t}, \gamma, \delta)$ and $F_2(t, \bar{t}, \gamma, \delta)$ are replaced by γ', δ', and t is the affix of contours L and L_1 in the initial state. Thus, the potentials of the second order in both variants of the given problem are determined simultaneously from boundary conditions (IX.280) and (IX.281).

The potentials of the first order are

$$\varphi^{(1)}_{(0)}(z) = \Gamma^{(1)}z + \alpha^{(1)}_{-1}\frac{R^2}{z}, \tag{IX.282}$$

$$\psi^{(1)}_{(0)}(z) = \Gamma'^{(1)}z + \beta^{(1)}_{-1}\frac{R^2}{z} + \beta^{(1)}_{-3}\frac{R^4}{z^3};$$

$$\varphi^{(1)}_{(1)}(z) = a^{(1)}_3\frac{z^3}{R^2} + a^{(1)}_1 z + a^{(1)}_{-1}\frac{R^2}{z},$$

$$\psi^{(1)}_{(1)}(z) = b^{(1)}_1 z + b^{(1)}_{-1}\frac{R^2}{z} + b^{(1)}_{-3}\frac{R^4}{z^3}. \tag{IX.283}$$

By substituting these potentials into functions $F_1(z, \bar{z}, \gamma, \delta)$ and $F_2(z, \bar{z}, \gamma, \delta)$, we find from boundary conditions (IX.280) and (IX.281), by the method of complex Fourier series, /66

$$\varphi^{(2)}_{(0)}(z) = \Gamma^{(2)}z + \sum_{n=1}^{2}\alpha^{(2)}_{-(2n-1)}\frac{R^{2n}}{z^{2n-1}},$$

$$\psi^{(2)}_{(0)}(z) = \Gamma'^{(2)}z + \sum_{n=1}^{3}\beta^{(2)}_{-(2n-1)}\frac{R^{2n}}{z^{2n-1}}; \tag{IX.284}$$

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$$\varphi_{(1)}^{(2)}(z) = \sum_{n=-2}^{2} a_{2n+1}^{(2)} \frac{z^{2n+1}}{R^{2n}}, \quad \psi_{(1)}^{(2)}(z) = \sum_{n=-2}^{2} b_{2n-1}^{(2)} \frac{z^{2n-1}}{R^{2(n-1)}}. \tag{IX.285}$$

The coefficients of the potentials of the second order are defined by the following relations (for $k_1 = k_2 = 0$):

$$a_{-3}^{(2)} = \frac{p^3}{d_0}\{[(1+mk_{(0)})A_{-3}' + p^3 D_1]T_0 + 5p^3(1+mk_{(0)})(1-p^2)D_3\},$$

$$a_{-1}^{(2)} = \frac{p}{d_1}\{[(1+mk_{(0)})A_{-1}' + pD_2]T_1 + 3p(1+mk_{(0)})(1-p^2)D_4\},$$

$$a_1^{(2)} = \frac{1}{d_2}\{mA_1 + A_{1,1} - m^2(Q_1 + Q_{1,1}) + (1-m)pA_1' + m^2(k_{(0)}+1)\Gamma^{(2)}\},$$

$$a_3^{(2)} = \frac{1}{T_1}[D_4 - (1-m)(1-p^2)a_{-1}'], \quad a_5^{(2)} = \frac{1}{T_0}[D_3 - 3(1-m)(1-p^2)a_{-3}^{(2)}];$$

$$b_{-5}^{(2)} = p^2[A_5'p^3 + 3a_{-3}^{(2)} - p^8 a_5^{(2)}], \quad b_{-3}^{(2)} = p^2[pA_3' + a_{-1}^{(2)} - p^4 a_3^{(2)}],$$

$$b_{-1}^{(2)} = p[A_1' - 2pa_1^{(2)}], \quad b_1^{(2)} = A_{-1} - a_{-1}^{(2)} - 3a_3^{(2)} + m[\Gamma'^{(2)} + \alpha_{-1}^{(2)} - Q_{-1}], \tag{IX.286}$$

$$b_3^{(2)} = A_{-3} - a_{-3}^{(2)} - 5a_5^{(2)} + m(\alpha_{-3}^{(2)} - Q_{-3});$$

$$\alpha_{-1}^{(2)} = \frac{(k_{(1)}+1)a_{-1}^{(2)} - D_5}{m(1+mk_{(1)})}, \quad \alpha_{-3}^{(2)} = \frac{(k_{(1)}+1)a_{-3}^{(2)} - D_6}{m(1+mk_{(1)})};$$

$$\beta_{-1}^{(2)} = \frac{1}{m}[2a_1^{(2)} + b_{-1}^{(2)} - m(2\Gamma^{(2)} - Q_1) - A_1],$$

$$\beta_{-3}^{(2)} = \frac{1}{m}[a_3^{(2)} - a_{-1}^{(2)} + b_{-3}^{(2)} + m(\alpha_{-1}^{(2)} + Q_3) - A_3],$$

$$\beta_{-5}^{(2)} = \frac{1}{m}[a^{(2)} - 3a_{-3}^{(2)} + b_{-5}^{(2)} + m(3\alpha_{-3}^{(2)} + Q_5) - A_5],$$

where /664

$$\begin{aligned}
d_0 &= 15p^6(1-p^2)^2(1-m)(1+mk_{(0)}) + [p^6(k_{(1)} - mk_{(0)}) + (1+mk_{(0)})]T_0,\\
d_1 &= 3p^2(1-p^2)^2(1-m)(1+mk_{(0)}) + [p^2(k_{(1)} - mk_{(0)}) + (1+mk_{(0)})]T_1,\\
d_2 &= 2[p^2(1-m) + m] + k_{(1)} - 1;\\
T_0 &= m + k_{(1)} + (1-m)p^{10}, \quad T_1 = m + k_{(1)} + (1-m)p^6;\\
D_1 &= m^2[k_{(0)}Q_{-3} - Q_{-3,1}] + A_{-3,1} - mk_{(0)}A_{-3},\\
D_2 &= m^2[k_{(0)}Q_{-1} - Q_{-1,1}] + A_{-1,1} - mk_{(0)}A_{-1} - m^2(k_{(0)}+1)\Gamma'^{(2)},
\end{aligned} \tag{IX.287}$$

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$$D_3 = mA_5 + A_{5,1} - m^2(Q_5 + Q_{5,1}) + (1-m)p^5A'_5,$$
$$D_4 = mA_3 + A_{3,1} - m^2(Q_3 + Q_{3,1}) + (1-m)p^3A'_3,$$
$$D_5 = A_{-1} + A_{-1,1} - m(Q_{-1} + mQ_{-1,1}) + m(1-m)\Gamma'^{(2)},$$
$$D_6 = A_{-3} + A_{-3,1} - m(Q_{-3} + mQ_{-3,1});$$
$$A'_5 = \gamma_{(1)}(M'_4V'_{-1} + S'_2V'_3) - k_{3(1)}p^{-3}(a^{(1)}_{-1})^2,$$
$$A'_3 = \gamma_{(1)}[(M'_4 + S'_2)V'_1 + M'_2V'_{-1} + S'_0V'_3] + 2k_{3(1)}p^{-1}a^{(1}_1a^{(1)}_{-1},$$
$$A'_1 = \gamma_{(1)}[(M'_2 + S'_0)V'_1 + (M'_0 + S'_2)V'_{-1} + (M'_4 + S'_{-2})V'_3] - $$
$$- k_{3(1)}p[(a^{(1)}_1)^2 - 6a^{(1)}_3a^{(1)}_{-1}],$$
$$A'_{-1} = \gamma_{(1)}[(M'_0 + S'_{-2})V'_1 + S'_0V'_{-1} + M'_2V'_3] - 6k_{3(1)}p^3a^{(1)}_1a^{(1)}_3,$$
$$A'_{-3} = \gamma_{(1)}[S'_{-2}V'_{-1} + M'_0V'_3] - 9k_{3(1)}p^5(a^{(1)}_3)^2,$$
$$A_{5,1} = \gamma_{(1)}[S_2V_3 - M_4V_{-1}] - k'_{3(1)}(a^{(1)}_{-1})^2 + \frac{3}{5}k'_{1(1)}a^{(1)}_{-1}b^{(1)}_{-3} - \frac{9}{5}k'_{2(1)}(a^{(1)}_3),^2 \qquad \text{(IX.288)}$$

$$A_{3,1} = \gamma_{(1)}[S_0V_3 - M_2V_{-1} + (S_2 - M_4)V_1] + 2k'_{3(1)}a^{(1)}_1a^{(1)}_{-1} -$$
$$- \frac{1}{3}k'_{1(1)}(3a^{(1)}_1b^{(1)}_{-3} - a^{(1)}_{-1}b^{(1)}_{-1}) - 2k'_{2(1)}a^{(1)}_1a^{(1)}_3,$$
$$A_{1,1} = \gamma_{(1)}[(S_{-2} - M_4)V_3 + (S_2 - M_0)V_{-1} + (S_0 - M_2)V_1] - k'_{3(1)}[(a^{(1)}_1)^2 -$$
$$- 6a^{(1)}_3a^{(1)}_{-1}] - k'_{1(1)}[a^{(1)}_{-1}b^{(1)}_1 + a^{(1)}_1b^{(1)}_{-1} + 9a^{(1)}_3b^{(1)}_{-3}] - k^1_{2(1)}[(a^{(1)}_1)^2 - 6a^{(1)}_3a^{(1)}_{-1}],$$
$$A_{-1,1} = \gamma_{(1)}[-M_2V_3 + S_0V_{-1} + (S_{-2} - M_0)V_1] - 6k'_{3(1)}a^{(1)}_1a^{(1)}_3 -$$
$$- k'_{1(1)}(a^{(1)}_1b^{(1)}_1 - 3a^{(1)}_3b^{(1)}_{-1}) - 2k'_{2(1)}a^{(1)}_{-1}a^{(1)}_1,$$
$$A_{-3,1} = \gamma_{(1)}[-M_0V_3 + S_{-2}V_{-1}] - 9k'_{3(1)}(a^{(1)}_3)^2 - k'_{1(1)}a^{(1)}_3b^{(1)}_1 + \frac{k'_{2(1)}}{3}(a^{(1)}_{-1})^2;$$

$$S'_0 = (1 + \delta_{(1)})a^{(1)}_1, \qquad S_2 = 3p^2a^{(1)}_3 - \delta_{(1)}p^{-2}a^{(1)}_{-1}, \qquad \text{/665}$$

$$S_{-2} = 3\delta_{(1)}p^2a^{(1)}_3 - p^{-2}a^{(1)}_{-1}, \quad S_0 = (k_{(1)} - \delta_{(1)})a^{(1)}_1,$$
$$S'_2 = 3k_{(1)}a^{(1)}_3 + \delta_{(1)}a^{(1)}_{-1}, \quad S'_{-2} = -(3\delta_{(1)}a^{(1)}_3 + k_{(1)}a^{(1)}_{-1});$$
$$M'_0 = 6p^2a^{(1)}_3 + b^{(1)}_1, \qquad M'_2 = -p^{-2}b^{(1)}_{-1},$$
$$M'_4 = p^{-2}(2a^{(1)}_{-1} - 3p^{-2}b^{(1)}_{-3});$$
$$V'_3 = k_{(1)}p^3a^{(1)}_3 + p^{-1}a^{(1)}_{-1} - p^{-3}b^{(1)}_{-3}, \qquad \text{(IX.289)}$$
$$V'_1 = (k_{(1)} - 1)pa^{(1)}_1 - p^{-1}b^{(1)}_{-1},$$
$$V'_{-1} = k_{(1)}p^{-1}a^{(1)}_{-1} - 3p^3a^{(1)}_3 - pb^{(1)}_1; \qquad p = \frac{R_1}{R}.$$

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The values A_{2n+1}, V_r, M_j are found from A'_{2n+1}, V'_r, M'_j, respectively, for $p = 1$; the values Q_{2n+1}, $Q_{2n+1,1}$ are found from A_{2n+1}, $A_{2n+1,1}$, respectively if we substitute the elastic constants of the ring by the elastic constants of the plate, and the coefficients $a^{(1)}_{2n+1}$, $b^{(1)}_{2n-1}$ ($n = 1, 0, -1$) by 0, $\Gamma^{(1)}$, $\alpha^{(1)}_{-1}$, $\Gamma'^{(1)}$, $\beta^{(1)}_{-1}$, $\beta^{(1)}_{-3}$, respectively.

As shown above, to find the coefficients of the potentials of the second order in variant I, it is sufficient to replace in formulas (IX.286)-(IX.289) the constants γ, δ by γ', δ' and assume that $p = R^*_1/R^*$, where R^*_1, R^* are the internal radius of the ring and the radius of the contour of the seam, respectively, in the initial state.

We will notice that by assuming in formulas (IX.286)-(IX.289) $p = R_1/R = 0$, we obtain the coefficients for the potentials of the second order for the problem of elastic equilibrium of a plate with a round hole, into which is soldered a heavy disc of a different type of material.

In the polar coordinate system of the deformed state, the stress components on the contour of the seam in the given problem are of the form

$$\sigma_\vartheta = 2\Gamma^{(1)} - \beta^{(1)}_{-1} + (\Gamma'^{(1)} - 3\beta^{(1)}_{-3})\cos 2\vartheta + \frac{1}{2^0 H_{(0)}}[B_1 + B_2\cos 2\vartheta + B_3\cos 4\vartheta],$$

$$\tau_{r\vartheta} = (\Gamma'^{(1)} + 3\beta^{(1)}_{-3} - 2\alpha^{(1)}_{-1})\sin 2\vartheta + \frac{1}{2^0 H_{(0)}}[B_7\sin 2\vartheta + B_8\sin 4\vartheta], \qquad \text{(IX.290)}$$

$$\sigma_r = 2\Gamma^{(1)} + \beta^{(1)}_{-1} + (3\beta^{(1)}_{-3} - \Gamma'^{(1)} - 4\alpha^{(1)}_{-1})\cos 2\vartheta + \frac{1}{2^0 H_{(0)}}[B_4 + B_5\cos 2\vartheta + B_6\cos 4\vartheta],$$

where

$$B_1 = X_1 + X'_1, \quad B_2 = X_2 + X'_2, \quad B_3 = X_3 + X'_3,$$

$$B_4 = X_1 - X'_1, \quad B_5 = X_2 - X'_2, \quad B_6 = X_3 - X'_3;$$

$$X_1 = \gamma_{(0)}\{(5 - k_{(0)}\delta_{(0)} - 4\delta_{1(0)})\alpha^{(1)2}_{-1} + 9\beta^{(1)2}_{-3} + \beta^{(1)2}_{-1} + 4(\delta_{1(0)} - 3)\alpha^{(1)}_{-1}\beta^{(1)}_{-3}\},$$

$$X_2 = -2\alpha^{(2)}_{-1} + \gamma_{(0)}\{4(\delta_{1(0)} - 1)\alpha^{(1)}_{-1}\beta^{(1)}_{-1} + 6\beta^{(1)}_{-1}\beta^{(1)}_{-3} - 2\Gamma'^{(1)}\beta^{(1)}_{-1} + \qquad \text{/666}$$

$$+ 2[2k_{(0)} - (1 - k_{(0)}\delta_{(0)}) - 2\delta_{1(0)}(k_{(0)} - 1)]\Gamma^{(1)}\alpha^{(1)}_{-1}\},$$

$$X_3 = -6\alpha^{(2)}_{-3} + \gamma_{(0)}\{4(1 + \delta_{1(0)})\Gamma'^{(1)}\alpha^{(1)}_{-1} - 6\Gamma'^{(1)}\beta^{(1)}_{-3} - 2k_{(0)}(1 + \delta_{1(0)})\alpha^{(1)2}_{-1}\};$$

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$$
\begin{aligned}
X_1' &= -\beta_{-1}^{(2)} + \gamma_{(0)}\{-2\alpha_{-1}^{(1)}V_\alpha - (2+\delta_{(0)}-k_{(0)})\Gamma^{(1)}\beta_{-1}^{(1)} - (\delta_{(0)}-k_{(0)})\alpha_{-1}^{(1)}\Gamma'^{(1)} - \\
&\quad - \delta_{1(0)}[6V_\beta V_\varepsilon + 2\beta_{-1}^{(1)}V_\gamma] - 2\delta_{2(0)}\alpha_{-1}^{(1)}V_\delta\}, \\
X_2' &= 2\alpha_{-1}^{(2)} - 3\beta_{-3}^{(2)} + \gamma_{(0)}\{(2+\delta_{(0)}-k_{(0)})(\alpha_{-1}^{(1)}\beta_{-1}^{(1)} + \Gamma^{(1}V_\alpha) - \\
&\quad - \delta_{1(0)}[6V_\gamma V_\varepsilon + 2\beta_{-1}^{(1)}V_\delta + 2\beta_{-1}^{(1)}V_\beta] - 2\delta_{2(0)}\alpha_{-1}^{(1)}V_\gamma\} + 4k_{3(0)}\Gamma^{(1)}\alpha_{-1}^{(1)}, \\
X_3' &= 12\alpha_{-3}^{(2)} - 5\beta_{-5}^{(2)} + \gamma_0\{(k_{(0)}-\delta_{(0)})\alpha_{-1}^{(1)}V_\alpha - 2\Gamma'^{(1)}\alpha_{-1}^{(1)} - 6\delta_{1(0)}V_\delta V_\varepsilon - \\
&\quad - 2\delta_{2(0)}\alpha_{-1}^{(1)}V_\beta\} - 4k_{3(0)}\alpha_{-1}^{(1)2}; \\
B_7 &= 3\beta_{-3}^{(2)} - 2\alpha_{-1}^{(2)} + \gamma_{(0)}\{(2-\delta_{(0}+k_{(0)})\alpha_{-1}^{(1)}\beta_{-1}^{(1)} - (2+\delta_{(0)}-k_{(0)})\Gamma^{1)}V_\alpha - \\
&\quad - \delta_{1(0)}[2\beta_{-1}^{(1)}V_\beta - 6V_\varepsilon V_\gamma - 2\beta_{-1}^{(1)}V_\delta] + 2\delta_{2(0)}\alpha_{-1}^{(1)}V_\gamma\} - 4k_{3(0)}\Gamma^{(1)}\alpha_{-1}^{1}, \\
B_8 &= 5\beta_{-5}^{(2)} - 12\alpha_{-3}^{(2)} + \gamma_{(0)}\{(\delta_{(0)}-k_{(0)})\alpha_{-1}^{(1)}V_\alpha - 2\Gamma'^{(1)}\alpha_{-1}^{(1)} + \\
&\quad + 6V_\delta V_\varepsilon\delta_{1(0)} + 2\delta_{2(0)}\alpha_{-1}^{(1)}V_\beta\} + 4k_{3(0)}\alpha_{-1}^{(1)2}; \\
V_\alpha &= 2\alpha_{-1}^{(1)} - 3\beta_{-3}^{(1)}, \quad V_\beta = \alpha_{-1}^{(1)} - \beta_{-3}^{(1)}, \\
V_\gamma &= (k_{(0)}-1)\Gamma^{(1)} - \beta_{-1}^{(1)}, \quad V_\delta = k_{(0)}\alpha_{-1}^{(1)} - \Gamma'^{(1)}, \\
V_\varepsilon &= 2\beta_{-3}^{(1)} - \alpha_{-1}^{(1)}.
\end{aligned}
\tag{IX.291}
$$

In variant F it is necessary to assume

$$\delta_{1(0)} = 1; \qquad \delta_{2(0)} = \delta_{(0)}.$$

In variant I it is necessary to assume

$$\delta_{1(0)} = 1 - \frac{1}{\gamma_{(0)}}, \qquad \delta_{2(0)} = \delta_{(0)} - \frac{1}{\gamma_{(0)}}.$$

On the basis of formulas (IX.290), (IX.291) and (IX.284)-(IX.289) we may determine the stress state in the points of the contour of the seam, both for uniaxial tension-compression ($N_2 = 0$, $N_1 = N$), and for multifold tension-compression ($N_1 = N_2 = N$), and also for pure displacement ($N_1 = -N_2 = N$) for both compressible and incompressible materials.

Hole with Soldered, Absolutely Rigid Inclusion (Ring or Disc). We will assume that region S under examination represents an infinite plate with a hole which possesses one axis of symmetry. We will assume that an absolutely rigid inclusion, in the form of a ring or continuous disc is placed (without clearance) into this hole. The rigid inclusion is soldered along the contour of contact to the surrounding material of the elastic plate. We will assume also that plane S, at sufficiently distant points from the hole, is under the effect

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of external forces N_1 = const and N_2 = const, i.e., in the biaxial stress state, /667 where forces N_1 act at infinity along the axis of symmetry of the hole, and forces N_2, perpendicular to this axis. The rigid inclusion is free of external forces, except for the forces of interaction with the surrounding material of the elastic plate. We will analyze the equilibrium near this absolutely rigid inclusion[1].

Under these assumptions there can be no rotation of the rigid inclusion, and therefore, on seam contour L, the following conditions[2] should be satisfied:

$$D(z, \bar{z}) = g(t) = 0 \tag{IX.292}$$

or

$$D^{(k)}(z, \bar{z}) = g^{(k)}(t) = 0. \tag{IX.293}$$

Since, in the given problem, the principal vector of forces acting on both the seam contour and on the contour of the hole is equal to zero, then $X^{(k)} = Y^{(k)} = 0$ (k = 1, 2) and as follows from formulas (IX.181), $E_k = 0$. Then if we introduce the mapping function

$$z = \omega(\zeta) = R\left(\zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \ldots\right), \tag{IX.294}$$

the potentials of the second order will be of the form

$$\varphi^{(2)}(\zeta) = \Gamma^{(2)} R\zeta + \varphi_0^{(2)}(\zeta),$$
$$\psi^{(2)} = \Gamma'^{(2)} R\zeta + \psi_0^{(2)}(\zeta). \tag{IX.295}$$

The functions $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$ are determined from boundary condition (IX.235) of the second basic problem:

$$k\varphi_0^{(2)}(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_0^{(2)'}}\left(\frac{1}{\sigma}\right) - \overline{\psi_0^{(2)}}\left(\frac{1}{\sigma}\right) = g_1\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) \text{ on } \gamma_1, \tag{IX.296}$$

where γ_1 is a circle of unit radius; $\sigma = e^{i\vartheta}$;

---

[1]See Yu. I. Koyfman [4], G. N. Savin and Yu. I. Koyfman [2].
[2]Since the contour of the hole is not deformed, the problem is analyzed in only one variant.

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$$g_1\left(\sigma, \frac{1}{\alpha}, \gamma, \delta\right) = F_2\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) - k\Gamma^{(2)} R\sigma + \overline{\Gamma}'^{(2)} \frac{R}{\sigma} + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} R\Gamma^{(2)}. \qquad \text{(IX.297)}$$

Assuming in the expressions for the function $F_2(\zeta, \overline{\zeta}, \gamma, \delta)$ (IX.299), $k_1' = k_2' = 0$, and considering that on L $D^{(1)} = 0$, we obtain

$$F_2\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right) = -k_3'\omega(\sigma)\left[\overline{\Phi}^{(1)}\left(\frac{1}{\sigma}\right)\right]^2 \text{ on } \gamma_1. \qquad \text{(IX.298)}$$

As in linear theory[1], the functions $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$ are found from functional equations /668

$$\varphi_0^{(2)}(\zeta) = -\frac{1}{2k\pi i}\int_{\gamma_1}\left[\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi}_0^{(2)'}\left(\frac{1}{\sigma}\right) + g_1\left(\sigma, \frac{1}{\sigma}, \gamma, \delta\right)\right]\frac{d\sigma}{\sigma - \zeta} + \varphi_0^{(2)}(\infty),$$

$$\psi_0^{(2)}(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1}\left[\overline{\frac{\omega(\sigma)}{\omega'(\sigma)}}\varphi_0^{(2)'}(\sigma) + \overline{g}_1\left(\frac{1}{\sigma}, \sigma, \gamma, \delta\right)\right]\frac{d\sigma}{\sigma - \zeta} + \psi_0^{(2)}(\infty). \qquad \text{(IX.299)}$$

We will examine in greater detail a partial case, specifically, we will determine the potentials of the second order for the case where the contour of the hole is elliptical. The potentials of the second order in the given problem, under the given assumptions, are of the form

$$\varphi^{(1)}(\zeta) = \Gamma^{(1)} R\zeta + (\Gamma^{(1)} m + \Gamma'^{(1)})\frac{R}{k\zeta},$$

$$\psi^{(1)}(\zeta) = \Gamma'^{(1)} R\zeta + \Gamma^{(1)} R\left(\frac{k}{\zeta} - \frac{1+m}{\zeta^2 - m}\zeta\right) + (\Gamma^{(1)} m + \Gamma'^{(1)})\frac{1 + m\zeta^2}{\zeta^2 - m}\cdot\frac{R}{k\zeta}, \qquad \text{(IX.300)}$$

where m and R are parameters of mapping function $z = R\left(\zeta + \frac{m}{\zeta}\right)$.

From functional equation (IX.299) we find the potentials

$$\varphi^{(2)}(\zeta) = \Gamma^{(2)} R\zeta + (\Gamma^{(2)} m + \Gamma'^{(2)} - k_3' m\Gamma^{(1)2})\frac{R}{k\zeta},$$

$$\psi^{(2)}(\zeta) = \Gamma'^{(2} R\zeta + \Gamma^{(2)} R\left(\frac{k}{\zeta} - \frac{1+m^2}{\zeta^2 - m}\zeta\right) + \left[\Gamma^{(2)} m + \Gamma'^{(2)} - \right. \qquad \text{(IX.301)}$$

$$\left. - k_3' m\Gamma^{(1)2}\right]\frac{1 + m\zeta^2}{\zeta^2 - m}\cdot\frac{R}{k\zeta} + Rk_3'\left[\frac{(\Gamma^{(1)}\zeta^2 - \alpha)^2}{\zeta(\zeta^2 - m)^2} - \right.$$

$$\left. - m\frac{2\zeta^3\Gamma^{(1)}(\alpha - m\Gamma^{(1)}) + \zeta(\Gamma^{(1)2} m^2 - \alpha^2)}{(\zeta^2 - m)^3}\right],$$

[1]See §1, Chapter I.

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where

$$\alpha = \frac{1}{k}\left(\Gamma^{(1)}m + \Gamma'^{(1)}\right).$$

Assuming in formulas (IX.301) m = 0, we obtain the potentials of the second order for a round hole of radius (z = Rζ)

$$\varphi^{(2)}(\zeta) = \Gamma^{(2)}R\zeta + \alpha_{-1}^{(2)}\frac{R}{\zeta}, \quad \psi^{(2)}(\zeta) = \Gamma'^{(2)}R\zeta + \beta_{-1}^{(2)}\frac{R}{\zeta} + \beta_{-3}^{(2)}\frac{R}{\zeta^3} + \beta_{-5}^{(2)}\frac{R}{\zeta^5}, \tag{IX.302}$$

where

$$\alpha_{-1}^{(2)} = \frac{\Gamma'^{(2)}}{k}, \qquad \beta_{-1}^{(2)} = (k-1)\,\Gamma^{(2)} + k_3'\Gamma^{(1)2},$$

$$\beta_{-3}^{(2)} = \frac{1}{k}[\Gamma'^{(2)} - 2k_3'\Gamma^{(1)}\Gamma'^{(1)}], \qquad \beta_{-5}^{(2)} = \frac{k_3'}{k^2}\Gamma'^{(1)2}.$$

By substituting the expressions obtained into the formulas for the stress state components, we can analyze the stress concentration in the general case, i.e., for any elliptical hole. Here we will analyze only the stress concentration in the case of plane deformation of an incompressible material. /669

The stress components on the contour of the seam for an elliptical hole are defined by the formulas

$$\sigma_\vartheta = \frac{1}{2}\left\{N_1 + N_2 + \frac{2}{d}(N_1 - N_2)(\cos 2\vartheta - m) - \frac{(N_1 - N_2)^2}{8\mu}\left[1 - \frac{6(1-\cos 4\vartheta)}{d^2}\right]\right\},$$

$$\sigma_\rho = \frac{1}{2}\left\{N_1 + N_2 + \frac{2}{d}(N_1 - N_2)(\cos 2\vartheta - m) - \frac{(N_1 - N_2)^2}{8\mu}\left[1 + \frac{2(1-\cos 4\vartheta)}{d^2}\right]\right\}, \tag{IX.303}$$

$$\tau_{\rho\vartheta} = -\frac{(N_1 - N_2)}{d}\sin 2\vartheta,$$

where

$$d = 1 - 2m\cos 2\vartheta + m^2.$$

By assuming in (IX.303) m = 0, we obtain the formulas for a round hole

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$$\sigma_\vartheta = \frac{1}{2}\left\{N_1 + N_2) + 2(N_1 - N_2)\cos 2\vartheta + \frac{(N_1 - N_2)^2}{8\mu}[5 - 6\cos 4\vartheta]\right\},$$

$$\sigma_r = \frac{1}{2}\left\{(N_1 + N_2) + 2(N_1 - N_2)\cos\vartheta - \frac{(N_1 - N_2)^2}{8\mu}[3 - 2\cos 2\vartheta]\right\}, \qquad \text{(IX.304)}$$

$$\tau_{r\vartheta} = -(N_1 - N_2)\sin 2\vartheta.$$

The values of the stress concentration coefficients σ_ϑ/N for round and elliptical (m = 1/3) holes in certain partial cases of load at infinity during the plane deformation of an incompressible material with the Mooney form of energy function are presented below:

Uniaxial tension-compression	Round hole Pure displacement	Multifold tension-compression
$1.5\left(1 \mp 0.041\frac{N}{\mu}\right)$	$2\left(1 \pm 0.125\frac{N}{\mu}\right)$	1
Tension-compression along large axis	**Elliptical hole** **Tension-compression along small axis**	**Multifold tension-compression**
$2\left(1 \mp 0.03\frac{N}{\mu}\right)$	$1.25\left(1 \mp 0.05\frac{N}{\mu}\right)$	1

In the problems under examination the difference between the maximum stresses on the contour of the seam and those calculated on the basis of the formulas of classical linear theory is slight. This is obviously related to the fact that the contour of the hole, during deformation of the plane, remains the same. /670

Round Hole Reinforced by Thin (Round) Elastic Rod. Consider an infinite plane with a round hole, the contour of which is reinforced by a thin elastic rod of constant cross section[1]. We will assume that one of the principal axes of inertia of each cross section of the reinforcing ring lies in the plane of the plate and that the plate and ring come into contact along the contour of the axis of the ring. Consequently, the reinforcing ring is regarded as an elastic line that operates only by tension and deflection. We will also assume that the elastic equilibrium of the plate (or large body during plane deformation) is described by the equations of plane nonlinear theory, and that elastic equilibrium of the ring is described by the equations of the theory of small deformations of thin linear-elastic curvilinear rods. We will place the origin of the coordinate system $(z, \bar{z})$ at the center of the hole.

[1]See Yu. I. Koyfman [1], G. N. Savin and Yu. I. Koyfman [2].

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On the contour of contact L of the plate and ring the following conditions are satisfied:

$$u = u_0, \quad v = v_0, \quad X_n = X_{n,0}, \quad Y_n = Y_{n,0},$$

or

$$u + iv = u_0 + iv_0,$$

$$i\int_0^s (X_n + iY_n)\,ds = i\int_0^s (X_{n,0} + iY_{n,0})\,ds + C, \tag{IX.305}$$

where u, v, X_n, Y_n are displacement and stress components along the axes of the Cartesian coordinate system for plates; u_0, v_0, $X_{n,0}$, $Y_{n,0}$ are the analogous values for the axis of the reinforcing ring.

Hooke's law for the reinforcing ring is taken in the form

$$\beta_0 = \frac{Q_1}{G_1} + \frac{M}{\rho G_1}, \qquad \frac{d\theta}{ds} = \frac{M}{G_2} + \frac{\beta_0}{\rho}, \tag{IX.306}$$

$\beta_0 = \beta_0(s)$ is relative elongation of an element of the axis of the ring; $\theta = \theta(s)$ is the magnitude of rotation of an element of the axis due to deformation; ρ is the radius of curvature of the ring after deformation; G_1, G_2 are rigidities to tension and deflection; Q_1 is normal force; M is deflecting moment in the cross section of the ring.

The relationship between normal force Q_1, shear force Q_2, deflecting moment M and load acting on the ring is defined by formulas presented in the work of M. P. Sheremet'yev [1] or in the work of Yu. I. Koyfman [1].

Complex displacement of the axis of the ring can be represented in the form[1] /671

$$u_0 + iv_0 = \int_0^s e^{i\alpha}(i\beta_0 - \theta)\,ds + C_0,$$

where α is the angle formed by the normal with the Oy^1 axis.

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[1]See Yu. I. Koyfman [1].

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Since it is assumed that deformations of the rod satisfy (geometrically) linear elasticity theory, then the form of these relations is not changed if we relate the values in them to the length of the arc of the deformed axis.

Suppose that the projections of the forces for the plate and ring are represented by series with respect to parameter ε:

$$X_n + iY_n = 2^0 H\varepsilon\,[(X_n^{(1)} + iY_n^{(1)}) + \varepsilon\,(X_n^{(2)} + iY_n^{(2)}) + \ldots],$$
$$X_{n,0} + iY_{n,0} = 2^0 H\varepsilon\,[(X_{n,0}^{(1)} + iY_{n,0}^{(1)}) + \varepsilon\,(X_{n,0}^{(2)} + iY_{n,0}^{(2)}) + \ldots]. \qquad \text{(IX.307)}$$

Since Q_1 and M are expressed through the projections of the forces, they, along with the values β_0 and θ, are represented in the form of the same series. Therefore we may write

$$D_0 = u_0 + iv_0 = 2^0 H\varepsilon\,[D_0^{(1)} + \varepsilon D_0^{(2)} + \ldots] + C_0, \qquad \text{(IX.308)}$$

where

$$D_0^k = \int_0^s e^{i\alpha}\,(i\beta_0^{(k)} - \theta^{(k)})\,ds.$$

Since, on the other hand,

$$D = u + iv = \varepsilon D^{(1)} + \varepsilon^2 D^{(2)} + \ldots,$$

the boundary conditions (IX.305) for terms of the first and second orders can be represented in the following form:

$$D^{(1)} = 2^0 H \int_0^s e^{i\alpha}\,(i\beta_0^{(1)} - \theta^{(1)})\,ds + C_0,$$
$$i\int_0^s (X_n^{(1)} + iY_n^{(1)})\,ds = i\int_0^s (X_{n,0}^{(1)} + iY_{n,0}^{(1)})\,ds + C; \qquad \text{(IX.309)}$$
$$D^{(2)} = 2^0 H \int_0^s e^{i\alpha}\,(i\beta_0^{(2)} - \theta^{(2)})\,ds,$$

$$i\int_0^s (X_n^{(2)} + iY_n^{(2)})\,ds = i\int_0^s (X_{n,0}^{(2)} + iY_{n,0}^{(2)})\,ds \text{ on } L. \qquad \text{(IX.310)}$$

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By expressing the left hand sides of the conditions (IX.310) through the boundary values of a combination of complex variable functions, we obtain the boundary conditions for the determination of the potentials of the first and second orders. In particular, for potentials of the second order /672

$$\varphi^{(2)}(t)+t\overline{\varphi^{(2)'}(t)}+\overline{\psi^{(2)}(t)}-F_1(t,\bar{t},\gamma,\delta)=i\int_0^s(X_{n,0}^{(2)}+iY_{n,0}^{(2)})\,ds,$$

$$k\varphi^{(2)}(t)-t\overline{\varphi^{(2)'}(t)}-\overline{\psi^{(2)}(t)}-F_2(t,\bar{t},\gamma,\delta)=2^0H\int_0^s e^{i\alpha}(i\beta_0^{(2)}-\theta^{(2)})\,ds, \tag{IX.311}$$

where t is the affix of a point of the deformed contour.

In solving the problem in the coordinates of the nondeformed state, it is necessary in boundary conditions (IX.311) to substitute the constants γ, δ by γ', δ'; t in this case is the affix of a point of the nondeformed contour.

We will assume that the stress state at infinity is homogeneous. The potentials of the first order for the plate are

$$\varphi^{(1)}(z)=\Gamma^{(1)}z+\alpha_{-1}^{(1)}\frac{R^2}{z},$$

$$\psi^{(1)}(z)=\Gamma'^{(1)}z+\beta_{-1}^{(1)}\frac{R^2}{z}+\beta_{-3}^{(1)}\frac{R^4}{z^3}. \tag{IX.312}$$

From (IX.312) we find the potentials of the second order

$$\varphi^{(2)}(z)=\Gamma^{(2)}z+\alpha_{-1}^{(2)}\frac{R^2}{z}+\alpha_{-3}^{(2)}\frac{R^6}{z^3},$$

$$\psi^{(2)}(z)=\Gamma'^{(2)}z+\beta_{-1}^{(2)}\frac{R^2}{z}+\beta_{-3}^{(2)}\frac{R^4}{z^3}+\beta_{-5}^{(2)}\frac{R^6}{z^5}, \tag{IX.313}$$

where

$$\alpha_{-1}^{(2)}=\frac{1}{d_1}\{3^0HR(3G_2-R^2G_1)(T_3+T_{3,1})+[4^0H^2R^4+3^0HR(R^2G_1+$$

$$+G_2)](T_{-1}-\Gamma'^{(2)})+[12G_1G_2+{}^0HR(R^2G_1+9G_2)](T_{-1,1}+\Gamma'^{(2)})\},$$

$$\alpha_{-3}^{(2)}=\frac{1}{d_2}\{5^0HR(15G_2-R^2G_1)(T_5+T_{5,1})+[4^0H^2R^4+5(R^2G_1+9G_2)^0HR]T_3+$$

$$+3[80G_1G_2+{}^0HR(R^2G_1+25G_2)]T_{-3,1}\};$$

$$\beta_{-1}^{(2)}=\frac{2^0HR^3(T_1-2\Gamma^{(2)})+(R^2G_1+G_2)[(k-1)\Gamma^{(2)}-T_{1,1}]}{2^0HR^3+R^2G_1+G_2},$$

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$$\beta_{-3}^{(2)} = \frac{1}{d_1}\{(\Gamma'^{(2)} + T_{-1,1})[12G_1G_2 + 6^0HRG_2 + 2^0HR^3G_1] +$$

$$+ (T_1 - \Gamma'^{(2)})[4^0HR^3 + 3(R^2G_1 + G_2) + k(3G_2 - R^2G_1)]^0HR -$$

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$$- 3(4kG_1G_2 - 2^0HRG_2 + 2^0HR^3G_1)\,T_{3,1} + {}^0HR[4^0HR^3 -$$

$$- 3(R^2G_1 - 3G_2) + k(R^2G_1 + 9G_2)]\,T_3\}, \qquad \text{(IX.314)}$$

$$\beta_{-5}^{(2)} = \frac{1}{d_2}\{12\,[60G_1G + {}^0HR(R^2G_1 + 15G_2)]\,T_{-3,1} + [12^0HR^3 +$$

$$+ 15(R^2G_1 + 9G_2) + 3k(15G_2 - R^2G_1)]^0 HRT_{-3} - 20[12kG_1G_2 +$$

$$+ {}^0HR(R^2G_1 - 9G_2)]\,T_{5,1} + {}^0HR[4^0HR^3 - 15(R^2G_1 - 15G_2) +$$

$$+ 3k(R^2G_1 + 25G_2)]\,T_5\};$$

$$d_1 = 4^0H^2R^4 + 3^0HR(3k+1)G_2 + {}^0HR^3(k+3)G_1 + 12kG_1G_2,$$

$$d_2 = 4^0H^2R^4 + 15^0HR(5k+3)G_2 + {}^0HR^3(3k+5)G_1 + 240kG_1G_2; \qquad \text{(IX.315)}$$

$$V_0 = 2\alpha_{-1}^{(1)} - 3\beta_{-3}^{(1)}, \qquad V_1 = k\alpha_{-1}^{(1)} - \Gamma'^{(1)},$$

$$V_2 = \alpha_{-1}^{(1)} - \beta_{-3}^{(1)}, \qquad V_3 = (k-1)\Gamma^{(1)} - \beta_{-1}^{(1)};$$

$$T_5 = \gamma[V_0V_1 - \delta\alpha_{-1}^{(1)}V_2] - k_3\alpha_{-1}^{(1)2} + \frac{3}{5}k_1\alpha_{-1}^{(1)}\beta_{-3}^{(1)},$$

$$T_3 = \gamma[V_0V_3 - \beta_{-1}^{(1)}V_1 + (1+\delta)\Gamma^{(1)}V_2 - \delta\alpha_{-1}^{(1)}V_3] + 2k_3\alpha_{-1}^{(1)}\Gamma^{(1)} -$$

$$- \frac{k_1}{3}(3\Gamma^{(1)}\beta_{-3}^{(1)} - \alpha_{-1}^{(1)}\beta_{-1}^{(1)}),$$

$$T_1 = \gamma[V_2V_0 - \beta_{-1}^{(1)}V_3 + \Gamma'^{(1)}V_1 + (1+\delta)\Gamma^{(1)}V_3 - \delta\alpha_{-1}^{(1)}V_1 - \alpha_{-1}^{(1)}V_2] -$$

$$- (k_3 + k_2)\Gamma^{(1)2} - k_1(\Gamma^{(1)}\beta_{-1}^{(1)} + \alpha_{-1}^{(1)}\Gamma'^{(1)}),$$

$$T_{-1} = \gamma[(\Gamma'^{(1)} - \alpha_{-1}^{(1)})V_3 - \beta_{-1}^{(1)}V_2 + (1+\delta)\Gamma^{(1)}V_1] - k_1\Gamma^{(1)}\Gamma'^{(1)} - 2k_2\Gamma^{(1)}\alpha_{-1}^{(1)},$$

$$T_{-3} = \gamma[\Gamma'^{(1)}V_2 - \alpha_{-1}^{(1)}V_1] + \frac{1}{3}k_2\alpha_{-1}^{(1)2},$$

$$T_{5,1} = \gamma[\delta\alpha_{-1}^{(1)}V_2 - V_1V_0] - k_3'\alpha_{-1}^{(1)2} + \frac{3}{5}k_1'\alpha_{-1}^{(1)}\beta_{-3}^{(1)},$$

$$T_{3,1} = \gamma[(\delta\alpha_{-1}^{(1)} - V_0)V_3 + \beta_{-1}^{(1)}V_1 - (\delta - k)\Gamma^{(1)}V_2] + 2k_3'\Gamma^{(1)}\alpha_{-1}^{(1)} -$$

$$- \frac{k_1'}{2}(3\Gamma^{(1)}\beta_{-3}^{(1)} - \alpha_{-1}^{(1)}\beta_{-1}^{(1)}), \qquad \text{(IX.316)}$$

$$T_{1,1} = \gamma[-(V_0 + k\alpha_{-1}^{(1)})V_2 + \beta_{-1}^{(1)}V_3 - (\delta - k)\Gamma^{(1)}V_3 + (\delta\alpha_{-1}^{(1)} - \Gamma'^{(1)})V_1] -$$

$$- (k_3' + k_2')\Gamma^{(1)2} - k_1'(\Gamma^{(1)}\beta_{-1}^{(1)} + \Gamma'^{(1)}\alpha_{-1}^{(1)}),$$

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$$T_{-1,1} = \gamma\left[\beta_{-1}^{(1)} V_2 - (\Gamma'^{(1)} + k\alpha_{-1}^{(1)})\, V_3 - (\delta - k)\,\Gamma^{(1)} V_1\right] - k_1'\Gamma^{(1)}\Gamma'^{(1)} - \\ - 2k_2'\Gamma^{(1)}\alpha_{-1}^{(1)},$$

$$T_{-3,1} = -\gamma\left[\Gamma'^{(1)} V_2 + k\alpha_{-1}^{(1)} V_1\right] + \frac{1}{3} k_2'\alpha_{-1}^{(1)2}.$$

The values $\alpha_{-1}^{(1)}$, $\beta_{-1}^{(1)}$, $\beta_{-3}^{(1)}$ can be found from formulas (IX.314), assuming $T_{2n+1} = T_{2n+1,1} = 0$ and by substituting $\Gamma^{(2)}$, $\Gamma'^{(2)}$ by $\Gamma^{(1)}$, $\Gamma'^{(1)}$.

Thus, the potentials of the first and second orders are completely defined.

We will assume that for sufficiently thin rods we may disregard the effect of rigidity to deflection, i.e., we may assume $G_2 = 0$; then we find the expressions for the coefficients of functions $\phi^{(k)}(z)$ and $\psi^{(k)}(z)$ (IX.312) and (IX.313) in the form

$$\alpha_{-1}^{(1)} = -\frac{(1+2\Delta)\,\Gamma'^{(1)}}{1+(k+3)\,\Delta}, \qquad \beta_{-1}^{(1)} = 2\,\frac{[(k-1)\,\Delta - 1]\,\Gamma^{(1)}}{1+2\Delta},$$

$$\beta_{-3}^{(1)} = \frac{[(k-1)\Delta - 1]\,\Gamma'^{(1)}}{1+(k+3)\,\Delta}, \qquad \text{(IX.317)}$$

$$\alpha_{-1}^{(1)} = \frac{T_{-1} - \Gamma'^{(2)} + \Delta\,[3\,(T_{-1} - T_3) + (T_{-1,1} - 3T_{3,1}) - 2\Gamma'^{(2)}]}{1+(k+3)\,\Delta},$$

$$\alpha_{-3}^{(2)} = \frac{T_{-3} + \Delta\,[5\,(T_{-3} - T_5) + (3T_{-3,1} - 5T_{5,1})]}{1+(3k+5)\,\Delta},$$

$$\beta_{-1}^{(2)} = \frac{1}{1+2\Delta}\{T_1 - 2\Gamma^{(2)} + 2\Delta\,[(k-1)\,\Gamma^{(2)} - T_{1,1}]\},$$

$$\beta_{-3}^{(2)} = \frac{T_{-1} + T_3 - \Gamma'^{(2)} + \Delta\,[(3-k)(T_{-1} - T_3) + 2\,(T_{-1,1} - 3T_{3,1}) + (k-1)\,\Gamma'^{(2)}]}{1+(k+3)\,\Delta},$$

$$\beta_{-5}^{(2)} = \frac{3T_{-3} + T_5 + \Delta\,[3\,(5-k)\,(T_{-3} - T_5) + 4\,(3T_{-3,1} - 5T_{5,1})]}{1+(3k+5)\,\Delta}; \qquad \text{(IX.318)}$$

$$\Delta = \frac{G_1}{4^0 HR}.$$

If we assume in these relations that $\Delta = 0$, we obtain the coefficients of the potentials of the first and second orders for the case of an absolutely flexible reinforcement, i.e., for the case a free hole.

As an example we will analyze the stress concentration along the contour of a round hole for plane deformation of an incompressible medium, the edge of

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which is reinforced by a thin ring[1], for a plate under uniaxial tension-compression. We will examine two cases.

1. Considering the ring to be sufficiently thin (b/R is small, where b is the thickness of the ring in the radial direction) we will disregard the effect of the rigidity of the ring to deflection and assume /675

$$G_2 = 0 \qquad \text{(IX.319)}$$

We will also assume that the linear-elastic material of the reinforcing ring possesses a much larger modulus E_0 (Young's modulus) of tension during small deformations than the corresponding modulus $E = 3\mu$ of the incompressible material of the surrounding medium, i.e., we will assume that $E_0/E \gg 1$. In our numerical calculations we will use

$$\Delta = \frac{3}{4} \cdot \frac{E_0}{E} \cdot \frac{b}{R} = 6. \qquad \text{(IX.320)}$$

2. We will assume that the ring represents a flexible nontensile thread. In this case $G_2 = 0$; $G_1 \to \infty$; $\Delta \to \infty$. With such a type of reinforcement the length of the axial line of the ring remains constant during deformation.

After determining the coefficients of the potentials of the first and second orders (IX.312), (IX.313) by formulas (IX.317), (IX.318) and substituting them into the corresponding formulas for the stress components, we obtain on the contour of the hole the values σ_ϑ presented in Table IX.5.

TABLE IX.5

$\vartheta°$	σ_ϑ (for case 1)	σ_ϑ (for case 2)
0	$-0{,}022N\left(1-5{,}91\frac{N}{\mu}\right)$	$0{,}133\frac{N^2}{\mu}$
30	$0{,}258N\left(1-0{,}271\frac{N}{\mu}\right)$	$0{,}25\,N\left(1-0{,}290\frac{N}{\mu}\right)$
45	$0{,}538N\left(1-0{,}298\frac{N}{\mu}\right)$	$0{,}50\,N\left(1-0{,}234\frac{N}{\mu}\right)$
60	$0{,}818N\left(1-0{,}165\frac{N}{\mu}\right)$	$0{,}75\,N\left(1-0{,}100\frac{N}{\mu}\right)$
90	$1{,}098N\left(1+0{,}003\frac{N}{\mu}\right)$	$1{,}00\,N\left(1+0{,}054\frac{N}{\mu}\right)$

Tr.Note: Commas indicate decimal points.

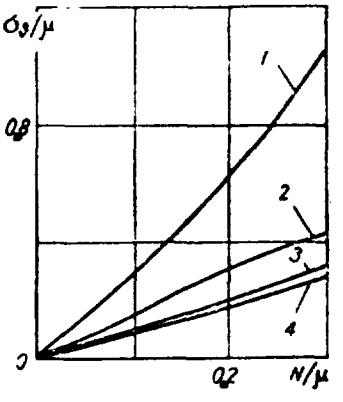


Figure IX.3.

[1]We will recall that the reinforcing ring is assumed to be an elastic line operating only on tension and deflection.

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As we see, the thin reinforcing ring, the rigidity of which is much greater than the rigidity of the surrounding material, decreases sharply the concentration of stresses on the contour of the round hole.

By comparing these results and the results obtained for an absolutely rigid reinforcing element, with the data for a free hole we see that the introduction of a reinforcing element that limits the mobility of the points of the contour of the hole, results in a reduction in the effect of the nonlinear corrections of the second order.

The graphs of $\sigma_\vartheta^{max}/\mu$ on the contour of a round hole in an incompressible medium are shown graphically in Figure IX.3 for plane deformation in the case of uniaxial tension for various types of reinforcement: 1, absolutely flexible ring; 2, absolute rigid ring; 3, flexible elastic ring when $E_0/E \cdot b/R = 8$; 4, flexible ring with nontensile line.

§3. Another Variant of the Solution of the Problem of Stress Distribution near Holes during Plane Deformation of Incompressible Material /676

Parameters of Plane Deformation. We will analyze two states of a body: natural (nondeformed) and deformed.

The natural state of a body is related to the rectilinear Cartesian coordinate system x, y.

The plane field of displacements will be given as a function of the coordinates of the points of the body in the natural state:

$$u = u(x, y), \qquad v = v(x, y). \tag{IX.321}$$

The deformed state of the vicinity of an arbitrary point of the body is characterized by the parameters of change of volume and shape.

Let λ_1, λ_2, λ_3 be the principal elongations, i.e., the elongations of the principal fibers. Then the invariant characteristic of change in volume during plane deformation is

$$\Delta = \lambda_1\lambda_2 - 1 \qquad (\lambda_3 = 1). \tag{IX.322}$$

Contour change will be characterized by two parameters from among three:

$$l_k = \frac{\lambda_k}{(1+\Delta)^{1/3}} \qquad (k = 1, 2, 3), \tag{IX.323}$$

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since

$$l_1 l_2 l_3 = 1. \tag{IX.324}$$

Parameters l_k are called elongations during contour change, although it is more convenient to use the logarithmic values:

$$\bar{l}_k = \ln l_k = \ln \lambda_k - \frac{1}{3} \ln (1 + \Delta). \tag{IX.325}$$

The deviator of logarithmic elongations $\bar{l}_k$ can be given by the principal values $\ln l_k$ and its principal directions are assumed to coincide with the principal directions of the deformation tensor. The second and third invariants of this deviator are expressed through the principal values by the formulas

$$\vartheta_i^2 = \frac{1}{3} (\bar{l}_1^2 + \bar{l}_2^2 + \bar{l}_3^2), \qquad \frac{\sqrt{2}}{2} \cos 3\beta = \frac{1}{\vartheta_i^3} \bar{l}_1 \bar{l}_2 \bar{l}_3. \tag{IX.326}$$

The values ϑ_i and β are defined as intensity and phase contour change. They retain the value of the characteristics of contour change during deformations of arbitrary magnitude. The principal values of the deviator are defined as

$$\bar{l}_k = \sqrt{2}\, \vartheta_i \cos \beta_k, \qquad \beta_1 = \beta, \qquad \beta_2 = \beta + \frac{2}{3}\pi, \qquad \beta_3 = \beta - \frac{2}{3}\pi, \tag{IX.327}$$

and therefore the phase of plane deformations of an incompressible material is /677 defined a priori:

$$\beta = \frac{\pi}{6}, \tag{IX.328}$$

while the principal elongations λ_k are defined by only one parameter, intensity of contour change:

$$\lambda_1 + \lambda_2 = 2 \operatorname{ch} \sqrt{1.5}\, \vartheta_i, \qquad \lambda_1 - \lambda_2 = 2 \operatorname{sh} \sqrt{1.5}\, \vartheta_i. \tag{IX.329}$$

Relations Between Stresses and Deformations. In the expression of the elementary effort of external forces, related to unit volume of the deformed body, the principal true stresses σ_k are generalized forces, if, as generalized

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coordinates of deformation, we use logarithmic elongations. These stresses and the parameters that determine the orientation of the principal directions, are used as the basic characteristics of the stress state. The symmetrical invariants of the tensor of true stresses in the principal axes are of the form

$$\sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3), \qquad \tau_i^2 = \frac{1}{3}[(\sigma_1 - \sigma)^2 + (\sigma_2 - \sigma)^2 + (\sigma_3 - \sigma)^2],$$

$$\cos 3\varphi = \frac{\sqrt{2}}{\tau_i^3}(\sigma_1 - \sigma)(\sigma_2 - \sigma)(\sigma_3 - \sigma). \tag{IX.330}$$

Here σ is hydrostatic stress; τ_i is intensity of stresses; ϕ is phase of stresses. Each of these values has a simple physical equivalent[1]. Analogously (IX.327), the principal stresses can be expressed through stress invariants:

$$\sigma_k = \sigma + \sqrt{2}\tau_i \cos\varphi_k, \qquad \varphi_1 = \varphi, \qquad \varphi_2 = \varphi + \frac{2}{3}\pi, \qquad \varphi_3 = \varphi - \frac{2}{3}\pi. \tag{IX.331}$$

We will assume in the following discussion that the principal directions of stresses in the deformation state under examination coincide with the principal directions of deformations, i.e., with the directions of the principal fibers. And if these principal directions are known, then, to establish the physical relationships between stresses and deformations, it is sufficient to determine experimentally the stress invariants as functions of deformation invariants:

$$\sigma = \sigma(\Delta, \vartheta_i, \beta), \qquad \tau_i = \tau_i(\Delta, \vartheta_i, \beta), \qquad \varphi = \varphi(\Delta, \vartheta_i, \beta). \tag{IX.332}$$

The question of how to express uniquely functions (IX.332) for the given material remains unanswered. There are several variants[2] of relations of the type (IX.332) that have been substantiated to some degree or other, experimentally. Let us examine one of them.

On the basis of Bridzhmen's tests [1, 2], we may postulate the law of three-dimensional deformation, the essence of which is that relative change in the volume of an element of the body is determined only by hydrostatic stress and is independent of the other two stress invariants, i.e., /678

$$\sigma = \sigma(\Delta). \tag{IX.333}$$

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[1]See L. A. Tolokonnikov [1].
[2]See V. V. Krylov [1, 2].

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It should be pointed out that for large deformations the magnitude of relative volumetric deformation remains quite small, in most cases, in comparison with deformations of contour change. Therefore, the law of volumetric deformation is often replaced by the condition of incompressibility

$$\Delta = 0, \tag{IX.334}$$

which reflects with a sufficient degree of accuracy the state of the material.

Since the elemental effort of the external forces per unit volume of the deformed body,

$$\delta A = \frac{\sigma}{1+\Delta}\delta\Delta + 3\tau_i[\cos(\varphi-\beta)\,\delta\vartheta_i + \vartheta_i \sin(\varphi-\beta)\,\delta\beta], \tag{IX.335}$$

is expended on changes of volume and contour, the outcome of (IX.333) and the conditions of the existence of the potential of internal forces is the dependence of intensity and phase of stresses only on contour change invariants:

$$\tau_i = \tau_i(\vartheta_i, \beta), \qquad \varphi = \varphi(\vartheta_i, \beta). \tag{IX.336}$$

Test data from axial compression and tension of cylindrical specimens, and also for pure deflection, indicate that the phase of true stresses and the phase of contour change coincide. Moreover, Davis' test data[1] show that for the intermediate states, the difference between these values, as L. A. Tolokonnikov [3] pointed out, will be slight. This enables us to introduce hypothesis of phase coincidence:

$$\varphi = \beta, \tag{IX.337}$$

which accurately reflects the state of the material not only at the extreme points of the range of change of β, but also at the center:

$$\beta = 0, \qquad \beta = \frac{\pi}{6}, \qquad \beta = \frac{\pi}{3} \qquad \left(0 \leqslant \beta \leqslant \frac{\pi}{3}\right). \tag{IX.338}$$

In this respect hypothesis (IX.337) has an advantage over the other such hypotheses. The consequence (IX.336) and (IX.337) of the above-mentioned condition of the existence of internal force potential is universality of representation of stress intensity as a function only of contour change intensity:

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[1] *Teoriya Plastichnosti* [Plasticity Theory], Translation Editor, Yu. N. Rabotonov Moscow, Foreign Literature Press, 1953.

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$$\tau_i = \tau_i(\vartheta_i). \tag{IX.339}$$

As the result we obtain the relations between stresses and deformations in the principal axes: /679

$$\sigma_k = \sigma(\Delta) + \frac{\tau_i(\vartheta_i)}{\vartheta_i}\bar{l}_k \qquad (k = 1, 2, 3). \tag{IX.340}$$

It is not difficult to convert to arbitrary axes. Considering (IX.328), we obtain $\phi = \pi/6$ and

$$\sigma_1 = \sigma + \sqrt{1{,}5}\,\tau_i, \qquad \sigma_2 = \sigma - \sqrt{1{,}5}\,\tau_i, \qquad \sigma_3 = \sigma. \tag{IX.341}$$

Thus, the determination of the physical properties of a plane-deformed body reduces to the experimental determination of the relationship (IX.339) between stress intensity and contour change intensity. This relationship is called the law of contour change.

Stress Function. The plane deformation state of the vicinity of an arbitrary point of a body and its orientation, as we know[1], are defined by four derivatives[2] $u_{,x} = \partial u/\partial x$, $u_{,y} = \partial u/\partial y$, $v_{,x} = \partial v/\partial x$, $v_{,y} = \partial v/\partial y$. For this purpose, however, we may use another group of four independent parameters, among which the deformation invariants are important. As such parameters we will use angle θ between the principal fiber and the x axis in the natural state, the principal elongations λ_k and the angle of rotation ω of the principal directions during the transition of the body from the natural state to the deformed state under examination. On the basis of the geometrical interpretation of the parameters θ, λ_1, λ_2 and ω, we find that they are sufficient for the determination of the deformed state and orientation of the vicinity of any point of the body. In particular, the following relationships can be established between the above-mentioned two groups of parameters:

$$\begin{aligned}
1 + u_{,x} &= \operatorname{ch}\frac{\vartheta}{2}\cos\omega + \operatorname{sh}\frac{\vartheta}{2}\cos(2\theta + \omega),\\
u_{,y} &= -\operatorname{ch}\frac{\vartheta}{2}\sin\omega + \operatorname{sh}\frac{\vartheta}{2}\sin(2\theta + \omega),\\
v_{,x} &= \operatorname{ch}\frac{\vartheta}{2}\sin\omega + \operatorname{sh}\frac{\vartheta}{2}\sin(2\theta + \omega),\\
1 + v_{,y} &= \operatorname{ch}\frac{\vartheta}{2}\cos\omega - \operatorname{sh}\frac{\vartheta}{2}\cos(2\theta + \omega).
\end{aligned} \tag{IX.342}$$

[1]See V. V. Novozhilov [1].
[2]For brevity we will denote these derivatives with a comma.

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Here displacements (IX.323) are used and the definition

$$\Im = 2\sqrt{1.5}\,\Im_i. \tag{IX.343}$$

is introduced. As the outcome of these relations and conditions of continuity of progressive displacements and rotations of the elements of the body, we have the relation /680

$$\begin{aligned}(\cos 2\theta \operatorname{sh}\Im)_{,yy} - (\cos 2\theta \operatorname{sh}\Im)_{,xx} - 2(\sin 2\theta \operatorname{sh}\Im)_{,xy} + (\operatorname{ch}\Im)_{,xx} + \\ + (\operatorname{ch}\Im)_{,yy} + 2\theta_{,y}(\operatorname{ch}\Im)_{,x} - 2\theta_{,x}(\operatorname{ch}\Im)_{,y} = 0,\end{aligned} \tag{IX.344}$$

which generalizes the classical condition of compatibility of plane deformations and preserves the geometrical value for deformations and displacements of arbitrary magnitude.

The general equations of equilibrium in the Lagrange variables, in application to the case of plane deformation of an incompressible material, as pointed out by L. A. Tolokonnikov [2], are greatly simplified. Considering (IX.342), the differential equations can be represented in the form of a homogeneous algebraic system relative to two differential operators over the stress invariants. The determinant of this system is a non-zero value, and therefore the uniquely possible trivial solution of the system represents a new form of equilibrium equations:

$$\begin{aligned}\left(\frac{\sigma}{\sqrt{1.5}} - f + \tau_i\cos 2\theta\right)_{,x} + (\tau_i \sin 2\theta)_{,y} = 0,\\ (\tau_i \sin 2\theta)_{,x} + \left(\frac{\sigma}{\sqrt{1.5}} - f - \tau_i\cos 2\theta\right)_{,y} = 0,\end{aligned} \tag{IX.345}$$

where

$$f = 2\sqrt{1.5}\int_0^{\Im_i} \tau_i(\Im_i)\,d\Im_i. \tag{IX.346}$$

The structure of equations (IX.345) duplicates the corresponding equations of the classical plane problem, and therefore, obviously, we may introduce stress functions U. By subordinating the stress functions to compatibility condition (IX.344), we arrive at a nonlinear differential equation[1] relative to this function that solves the problem.

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[1]This equation is not included here, since a compact formulation of the problem in complex coordinates will be given below.

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Let true normal $p\sigma_n$ and tangential $p\tau_n$ stresses be given at each point of the surface of a body. For convenience we introduce the constant factor p, which represents the stress that is characteristic for each problem. We will notice that the definition of the corresponding conditional stresses $p\lambda_\tau\sigma_n$ and $p\lambda_\tau\tau_n$, where λ_τ is -- a dimensionless value -- elongation of the contour fiber, a function of the coordinates of the points of the contour, does not complicate the solution of the problem.

Through α we will denote the angle between the direction of the outer normal to the boundary of the body and the direction of coordinate axis x. By determining the orientation of the boundary surface of the body relative to the principal directions, we can find[1] the expression of normal and tangential stresses on the contour surface through principal directions or symmetrical stress invariants: /681

$$\begin{aligned}\lambda_\tau^2 p\sigma_n &= \lambda_\tau^2\sigma - \sqrt{1{,}5}\,\tau_i\,[\operatorname{sh}\vartheta - \operatorname{ch}\vartheta\cos 2(\alpha-\theta)],\\ \lambda_\tau^2 p\tau_n &= -\sqrt{1{,}5}\,\tau_i\sin 2(\alpha-\theta),\end{aligned} \tag{IX.347}$$

hence

$$\lambda_\tau^2 = \operatorname{ch}\vartheta - \operatorname{sh}\vartheta\cos 2(\alpha-\theta). \tag{IX.348}$$

If we replace θ, τ_i and σ here by their expressions through the derivatives of stress function U

$$\begin{aligned}&\sin 2\theta = -\frac{p}{\tau_i}U_{,xy}, \qquad \cos 2\theta = \frac{1}{2}\frac{p}{\tau_i}(U_{,yy}-U_{,xx}),\\ &\tau_i^2 = p^2\left[U_{,xy}^2 + \frac{1}{4}(U_{,yy}-U_{,xx})^2\right], \quad \frac{\sigma}{\sqrt{1{,}5}} = \frac{1}{2}p(U_{,yy}+U_{,xx}) + f,\end{aligned} \tag{IX.349}$$

we find the formulation of the contour conditions for stress function U.

Statement of Problem in Complex Coordinates. In like manner as the classical plane problem of elasticity theory the solution of many problems of the theory of finite plane deformations can be found conveniently in complex coordinates

$$z = x + iy, \quad \bar{z} = x - iy.$$

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[1] See L. A. Tolokonnikov [5].

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Then, as we know

$$( )_{,x} = ( )_{,z} + ( )_{,\bar{z}}, \quad ( )_{,y} = i[( )_{,z} - ( )_{,\bar{z}}] \text{ etc.},$$

where the symbols $( )_{,z}$ and $( )_{,\bar{z}}$ denote differentiation with respect to z and $\bar{z}$, respectively. Relations (IX.349) are converted[1] to the form

$$\sin 2\theta = -i\frac{p}{\tau_i}(U_{,zz} - U_{,\bar{z}\bar{z}}), \quad \cos 2\theta = -\frac{p}{\tau_i}(U_{,zz} + U_{,\bar{z}\bar{z}}),$$
$$\tau_i^2 = 4p^2 U_{,zz} U_{,\bar{z}\bar{z}}, \qquad \frac{\sigma}{\sqrt{1.5}} = 2pU_{,z\bar{z}} + f. \tag{IX.350}$$

The resolution equation acquires the form

$$\left(\frac{\operatorname{sh}\vartheta}{\tau_i}U_{,zz}\right)_{,\bar{z}\bar{z}} + \left(\frac{\operatorname{sh}\vartheta}{\tau_i}U_{,\bar{z}\bar{z}}\right)_{,zz} + \frac{1}{p}(\operatorname{ch}\vartheta)_{,z\bar{z}} + p\frac{\operatorname{sh}\vartheta}{\tau_i}\frac{d\vartheta}{d\tau_i}(U_{,zz\bar{z}}U_{,\bar{z}\bar{z}z} - U_{,zzz}U_{,\bar{z}\bar{z}\bar{z}}) = 0. \tag{IX.351}$$

Here the intensity of contour change is assumed to be expressed by way of contour change law (IX.339) through stress intensity; the latter, in turn, is related to stress function U by relations (IX.350). The extreme complexity of equation (IX.351) is obvious. If, in particular, we use the assumptions of classical elasticity theory and replace $\operatorname{sh}\vartheta$ by ϑ, $\operatorname{ch}\vartheta$ by 1, assume $\tau_i = G$, where G is the displacement modulus, and disregard in equation (IX.351) the terms of the order of magnitudes of deformations in comparison with unity, then we arrive at a biharmonic equation[2]. /682

On the basis of relations (IX.347), we can find[3] a compact formulation of static boundary conditions for the stress function:

$$\frac{dU_z}{ds} = -\frac{i(\sigma_n - i\tau\operatorname{ch}\vartheta)}{2\sqrt{1.5}}e^{-i\alpha} + \frac{if}{2p}e^{-i\alpha} - \frac{p\tau_n}{\sqrt{1.5}}\cdot\frac{\operatorname{sh}\vartheta}{\tau_i}U_{,zz}e^{i\alpha}, \tag{IX.352}$$

where ds is an element of the arc of the boundary contour in the natural state. To formulate the geometrical boundary conditions it is necessary to know the physical properties of the material (IX.339) and to have the expressions of the displacement components through the stress function. Such relations are established in complex form in the works of V. G. Gromov [1], V. G. Gromov and L. A. Tolokonnikov [1]. If we introduce the complex displacement function D = u + iv, then

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[1] See L. A. Tolokonnikov [5].
[2] See L. A. Tolokonnikov [5].
[3] See V. G. Gromov [1]; V. G. Gromov and L. A. Tolokonnikov [1].

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$$D = -2p\int_{\tau_i}^{1} \text{th}\frac{\Theta}{2} U_{,\bar{z}\bar{z}}(1 + D_{,z})\,d\bar{z} + \xi(z), \tag{IX.353}$$

where $\xi(z)$ is an arbitrary function, determined from the law of conservation of mass during deformation:

$$D_{,z} + \overline{D}_{,\bar{z}} + D_{,z}\overline{D}_{,\bar{z}} - D_{,\bar{z}}\overline{D}_{,z} = 0. \tag{IX.354}$$

From (IX.353) we obtain the boundary conditions for the stress function when the displacements on the boundary of the body are given.

Such are the general relations of the nonlinear problem of elasticity theory for plane deformation.

Representation of Stress Function and Complex Displacement Function through Analytical Functions. The determination of the accurate solutions of most problems is fraught with insurmountable difficulties due to the nonlinearity of the basic relations. In order to construct the approximate solutions, however, we may use effectively the small parameter method.

We will examine physical nonlinearities of the type

$$\tau_i = G(\Theta + g\Theta^3 + \ldots); \quad G,\ g = \text{const.} \tag{IX.355}$$

Such, for instance, are the laws

$$\tau_i = G\,\text{th}\,\Theta = G\left(\Theta - \frac{1}{3}\Theta^3 + \ldots\right), \quad \tau_i = G\,\text{sh}\,\Theta = G\left(\Theta + \frac{1}{6}\Theta^3 + \ldots\right), \tag{IX.356}$$

which lead to contradictory deviations from the linear law of contour change during large deformations. The first of them approximates the experimental curves of many soft materials (nonferrous metals and their alloys), and the second approximates the experimental curves of highly elastic materials (resin, plastics). /683

We will assume that the expansion

$$U = U^{(0)} + U^{(1)}\varepsilon + U^{(2)}\varepsilon^2 + \ldots, \tag{IX.357}$$

is valid, where ε is a small parameter, the choice of which will be discussed below.

Then, on the basis of equation (IX.351), for the stress function and physical relations (IX.355), by way of ordinary operations that are essential

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to the classical small parameter method, we establish the differential equations for each of the approximations $U^{(k)}$ (k = 0, 1, 2...):

$$U^{(k)}_{,zzz\bar{z}} = F^{(k)} \quad (k = 0, 1, 2, \ldots). \tag{IX.358}$$

We write out the form of functions $F^{(k)}$ for the first three approximations:

$$F^{(0)} = 0, \quad F^{(1)} = -(U^{(0)}_{,zz}U^{(0)}_{,\bar{z}\bar{z}})_{zz} - \frac{1}{2}(U^{(0)}_{,zzz}U^{(0)}_{,zz} - U^{(0)}_{,zz}U^{(0)}_{,zzz}),$$

$$F^{(2)} = -2\,\mathrm{Re}\left[(U^{(0)}_{,zz}U^{(2)}_{,\bar{z}\bar{z}})_{,z\bar{z}} + \frac{1}{2}(U^{(0)}_{,zzz}U^{(1)}_{,\bar{z}\bar{z}} - U^{(0)}_{,zzz}U^{(0)}_{,\bar{z}\bar{z}}) + \left(\frac{1}{3} - 2g\right)(U^{(0)*}_{,zz}U^{(0)}_{,\bar{z}\bar{z}})_{,\bar{z}\bar{z}}\right]. \tag{IX.359}$$

The value g = -1/3 corresponds to the first and g = 1/6, to the second of the laws (IX.356). The function $U^{(0)}$ is biharmonic. Therefore we have for it the E. Gurs' representation through two analytical functions:

$$U^{(0)} = \mathrm{Re}(\bar{z}\varphi_0 + \chi_0). \tag{IX.360}$$

Now we can find sequentially the partial solutions of equation (IX.358):

$$U^{*(1)} = -\frac{1}{8}[(\bar{z}\varphi_0' + \psi_0)(z\bar{\varphi}_0' + \bar{\psi}_0) - \varphi_0\bar{\varphi}_0];$$

$$U^{*(2)} = -\frac{1}{4}\,\mathrm{Re}\,[(\bar{z}\varphi_0' + \psi_0)(z\bar{\varphi}_1' + \bar{\psi}_1) - \varphi_0\bar{\varphi}_1 + J], \tag{IX.361}$$

where

$$\psi_k = \chi_k' \equiv \frac{d\chi_k}{dz}; \quad \varphi_k' \equiv \frac{d\varphi_k}{dz};$$

$$-8J = [(\bar{z}\varphi_0' + \psi_0)^2(z\bar{\varphi}_0' + \bar{\psi}_0)]_{,z} - 2\varphi_0\bar{\varphi}_0'(\bar{z}\varphi_0' + \psi_0) - (z\bar{\varphi}_0' + \bar{\psi}_0) \times$$

$$\times [\bar{z}\varphi_0'^2 + 2\int \varphi_0^*\psi_0 dz] - \bar{\varphi}_0\int \varphi_0'^2 dz - 2\left(\frac{1}{3} - 8g\right)\int^{*} dz \int (\bar{z}\varphi_0^* + \psi_0')^2(z\bar{\varphi}_0^* + \bar{\psi}_0')dz. \tag{IX.362}$$

Functions $U^{(k)} - U^{*(k)}$ are biharmonic, and therefore, on the basis of E. Gurs' formula, we find /684

$$U^{(k)} = \mathrm{Re}(\bar{z}\varphi_k + \chi_k + U^{*(k)}) \quad (k = 1, 2, \ldots). \tag{IX.363}$$

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We further assume that the following expansions are valid

$$D = \varepsilon (D^{(0)} + D^{(1)}\varepsilon + D^{(2)}\varepsilon^2 + \ldots), \quad \xi = \varepsilon (\xi_0 + \xi_1\varepsilon + \xi_2\varepsilon^2 + \ldots). \tag{IX.364}$$

Then, from (IX.353) we find

$$D^{(k)} = -U^{(k)}_{,z} + \xi_k + D^{*k} \quad (k = 0, 1, 2, \ldots). \tag{IX.365}$$

If we limit ourselves to the first three approximations, then

$$\begin{gathered} D^{*(0)} = 0, \quad D^{*(1)} = -\int D^{(0)}_{,z} U^{(0)}_{,\bar{z}\bar{z}} d\bar{z}, \\ D^{*(2)} = -\int (D^{(0)}_{,z} U^{(1)}_{,\bar{z}\bar{z}} + D^{(1)}_{,z} U^{(0)}_{,\bar{z}\bar{z}})\, d\bar{z} + \left(\frac{1}{3} + 4g\right) \int U^{(0)}_{,zz} U^{(0)2}_{,\bar{z}\bar{z}} d\bar{z}. \end{gathered} \tag{IX.366}$$

By representing sequentially the stress functions $U^{(k)}$ and displacement functions $D^{(k)}$ through analytical functions, we obtain

$$\begin{aligned} D^{*(1)} &= \frac{1}{4}\left[-\varphi_0'(z\bar{\varphi}_0' + \bar{\psi}_0) + \frac{1}{2} z\bar{\varphi}_0'^2 + \int \bar{\varphi}_0'\bar{\psi}_0' d\bar{z}\right], \\ D^{*(2)} &= \frac{1}{4}\left[-\varphi_0'(z\bar{\varphi}_1' + \bar{\psi}_1) - \varphi_1'(z\bar{\varphi}_0' + \bar{\psi}_0) + z\bar{\varphi}_0'\bar{\varphi}_1' + \int (\bar{\varphi}_0'\bar{\psi}_1' + \bar{\varphi}_1'\bar{\psi}_0')\, d\bar{z}\right] + \\ &+ \frac{1}{2}(\bar{\varphi}_0' - \varphi_0') U^{*(1)}_{\bar{z}} + \frac{1}{4}\varphi_0' D^{*(1)} + \frac{1}{32}\left\{[\varphi_0'(z\bar{\varphi}_0' + \bar{\psi}_0')^2]_{,z} - \varphi_0\bar{\varphi}_0'^2 - \right. \\ &\left. - 2\int \varphi_0'^2 (z\bar{\varphi}_0'' + \bar{\psi}_0')\, d\bar{z} - 2\left(\frac{1}{3} - 8g\right)\int (z\bar{\varphi}_0'' + \psi_0')(z\bar{\varphi}_0'' + \bar{\psi}_0')^2 d\bar{z}\right\}. \end{aligned} \tag{IX.367}$$

From law of incompressibility (IX.354), recalling expansions (IX.364), (IX.365) and relations (IX.367), we find

$$\xi = \varphi_0, \; \xi_1 = \varphi_1 + \frac{1}{8}\int \varphi_0'^2 dz, \quad \xi_2 = \varphi_2 + \frac{1}{4}\int \varphi_0'\varphi_1' dz + \frac{1}{48}\int \varphi_0'^3 dz. \tag{IX.368}$$

Boundary Conditions for Functions $\phi_k(z)$ and $\psi_k(z)$. If, on the boundary of contour L, displacements D are given, then the boundary conditions for the analytical functions are written on the basis of (IX.363), (IX.365) and (IX.368) and in any approximation, in the form

$$\varphi_k(t^*) - t^*\overline{\varphi_k'(t^*)} - \overline{\psi_k(t^*)} = g_k(t^*) \quad (k = 0, 1, 2, \ldots), \tag{IX.369}$$

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where t* are affixes of the contour points in the initial nondeformed state;

$$g_0 = 2D|_L, \quad g_1 = 2\left[U_{\bar{z}}^{*(1)} - D^{*(1)} - \frac{1}{8}\int \varphi_0'^2 dz\right]\Big|_L,$$
$$g_2 = 2\left[U_{\bar{z}}^{*(2)} - D^{*(2)} - \frac{1}{4}\int \varphi_0'\varphi_1' dz - \frac{1}{48}\int \varphi_0'^3 dz\right]\Big|_L. \tag{IX.370}$$

When stresses $p\sigma_n$ and $p\tau_n$ on the contour are given, we have, in the case of physical laws (IX.355), /685

$$f = 2p\varepsilon U_{,zz}U_{,\bar{z}\bar{z}} + 0(\varepsilon^3), \tag{IX.371}$$

where p is the characteristic parameter of the given law. Therefore, relation (IX.352) yields

$$\frac{dU^{(k)}}{ds} = \Phi^{(k)} \quad (k = 0, 1, 2, \ldots). \tag{IX.372}$$

The functions $\Phi^{(k)}$ for the first three approximations have the form

$$\Phi^{(0)} = -\frac{i}{2\sqrt{1.5}}(\sigma_n - i\tau_n)e^{-i\alpha}, \quad \Phi^{(1)} = iU_{,zz}^{(0)}U_{,\bar{z}\bar{z}}^{(0)}e^{-i\alpha} - \frac{\tau_n}{\sqrt{1.5}}U_{,zz}^{(0)}e^{i\alpha},$$
$$\Phi^{(2)} = \left[2i\,\mathrm{Re}\,(U_{,zz}^{(0)}U_{,\bar{z}\bar{z}}^{(1)}) - \frac{\tau_n}{\sqrt{1.5}}U_{,zz}^{(0)}U_{,zz}^{(0)}\right]e^{-i\alpha} - \frac{\tau_n}{\sqrt{1.5}}U_{,zz}^{(1)}e^{i\alpha}. \tag{IX.373}$$

The boundary conditions for the analytical functions are

$$\varphi_k(t^*) + t^*\overline{\varphi_k'(t^*)} + \overline{\psi_k(t^*)} = f_k \quad (k = 0, 1, 2, \ldots), \tag{IX.374}$$

where

$$f_0 = \frac{1}{\sqrt{1.5}}\int (\sigma_n + i\tau_n)\,dt^*, \quad f_1 = -2\left[U_{,\bar{z}}^{*(1)}|_L + \right.$$
$$\left. + \int_L \left(U_{,zz}^{(0)}U_{,\bar{z}\bar{z}}^{(0)}dt^* + \frac{i}{\sqrt{1.5}}\tau_n U_{,\bar{z}\bar{z}}^{(0)}d\bar{t}^*\right)\right], \tag{IX.375}$$
$$f_2 = -2\left\{U_{,\bar{z}}^{*(2)}|_L + \int_L \left[2\,\mathrm{Re}\,(U_{,zz}^{(0)}U_{,zz}^{(1)}) - \frac{i\tau_n}{\sqrt{1.5}}U_{,zz}^{(0)}U_{,\bar{z}\bar{z}}^{(0)}\right]dt^* + \frac{i}{\sqrt{1.5}}\int_L \tau_n U_{,\bar{z}\bar{z}}^{(1)}d\bar{t}^*\right\}.$$

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Thus, the construction of the series approximations of both the static and geometrical boundary conditions reduces to the repeated solution of the basic plane problem of classical elasticity theory, and this solution can be found by one of N. I. Muskhelishvili's methods [1] (see, for instance, §1, Chapter I).

Uniqueness of Solution. We will examine, for simplicity, an infinite single-connected region, i.e., a plane weakened by some hole. The stress invariants, recalling (IX.350) and (IX.371), are represented in the form of series

$$\tau_i^2 = \tau^0 + \tau^{(1)}\varepsilon + \tau^{(2)}\varepsilon^2 + \ldots, \quad \sigma = \sigma^{(0)} + \sigma^{(1)}\varepsilon + \sigma^{(2)}\varepsilon^2 + \ldots, \tag{IX.376}$$

the coefficients of which are defined by formulas

$$\tau^{(0)} = 4p^2 U^{(0)}_{,zz} U^{(0)}_{,\bar{z}\bar{z}}, \quad \tau^{(1)} = 8p^2 \operatorname{Re}(U^{(0)}_{,zz} U^{(1)}_{,\bar{z}\bar{z}}),$$
$$\tau^{(2)} = 4p^2 [2 \operatorname{Re}(U^{(0)}_{,zz} U^{(2)}_{,\bar{z}\bar{z}}) + U^{(1)}_{,zz} U^{(1)}_{,\bar{z}\bar{z}}]. \tag{IX.377}$$

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$$\sigma^{(0)} = 2\sqrt{1.5}\, p U^{(0)}_{,\bar{z}\bar{z}}; \quad \sigma^{(1)} = 2\sqrt{1.5}\, p (U^{(1)}_{,\bar{z}\bar{z}} + U^{(0)}_{,zz} U^{(0)}_{,\bar{z}\bar{z}}),$$
$$\sigma^{(2)} = 2\sqrt{1.5}\, p [U^{(2)}_{,\bar{z}\bar{z}} + 2 \operatorname{Re}(U^{(0)}_{,zz} U^{(1)}_{,\bar{z}\bar{z}})].$$

The single-valuedness of displacements, single-valuedness and boundedness of stresses will obtain if for k = 0, 1, 2, ... the functions $U^{(k)}_{,zz}$, $U^{(k)}_{,\bar{z}\bar{z}}$ are unique bounded functions; $D^{(k)}$ are unique functions.

Recalling (IX.360), (IX.363) and (IX.365), we establish the general form of the functions

$$\varphi_k = B_k \ln z + \Gamma_k z + \varphi_k^*(z), \quad \psi_k = C_k \ln z + \Gamma_k' z + \psi_k^*(z), \tag{IX.378}$$

where B_k, Γ_k, C_k, Γ_k' are arbitrary constants; ϕ_k^*, ψ_k^* are holomorphic functions at the infinitely distant point. Here we discover sequentially that the partial solutions $U^{*(1)}$, $U^{*(2)}$, with an accuracy up to additive biharmonic terms, possess a unique form. In order to determine this form it is necessary in (IX.361) to replace the ambiguous functions ϕ_k, ψ_k by the unique functions

$$\varphi^{(k)} = B_k \ln \bar{z} + \varphi_k, \quad \psi^{(k)} = C_k \ln \bar{z} + \psi_k \quad (k = 0, 1, 2 \ldots) \tag{IX.379}$$

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and to use the identities

$$\ln z = \ln(z\bar{z}) - \ln\bar{z}, \quad \ln z \ln\bar{z} = \frac{1}{2}[\ln^2(z\bar{z}) - \ln^2 z - \ln^2\bar{z}]. \tag{IX.380}$$

The choice of functions $U^{*(1)}$, $U^{*(2)}$ in unique form leads to the corresponding unique forms of functions $D^{*(1)}$, $D^{*(2)}$ in all terms, with the exception of the integral terms. In other words, the functions ϕ_k, ψ_k in (IX.367) are replaced by $\phi^{(k)}$, $\psi^{(k)}$. Then, from the condition of uniqueness of $D^{(k)}$, we obtain

$$B_0 + \bar{C}_0 = 0, \quad B_1 + \bar{C}_1 = \bar{B}_0\bar{\Gamma}_0' + \Gamma_0(\bar{C}_0 - B_0),$$

$$B_2 + \bar{C}_2 = -\frac{1}{4}\Gamma_0\bar{B}_0\bar{\Gamma}_0' + \frac{1}{2}[\bar{B}_0\bar{\Gamma}_1' + \bar{B}_1\bar{\Gamma}_0' + \Gamma_0(\bar{C}_1 - B_1) + \Gamma_1(\bar{C}_0 - B_0)]. \tag{IX.381}$$

Furthermore,

$$B_k - \bar{C}_k = \frac{1}{2\pi i}[f_k]_L, \quad (k = 0, 1, 2, \ldots). \tag{IX.382}$$

Here the symbol $[f_k]_L$ denotes the increment of the function f_k during passage around the closed contour, which encompasses contour L in entirety.

The constants Γ_k and Γ_k' are determined by the stress state at infinity:

$$\Gamma_0 = \frac{N_1 + N_2}{4\sqrt{1.5p}}; \quad \Gamma_0' = \frac{N_1 - N_2}{2\sqrt{1.5p}}\, e^{-2i\theta^{(\infty)}},$$

$$\Gamma_1 = -\frac{1}{4}|\Gamma_0'|^2, \quad \Gamma_1' = \Gamma_2 = \Gamma_2' = 0, \tag{IX.383}$$

where $\pm N_1 = \sigma_1^{(\infty)}$, $\pm N_2 = \sigma_2^{(\infty)}$ are the principal directions at infinity; $\theta^{(\infty)}$ is the angle between the principal direction N_1 and the x axis. Here we assume that /687

$$\tau^{(1)(\infty)} = \tau^{(2)(\infty)} = \sigma^{(1)(\infty)} = \sigma^{(2)(\infty)} = 0. \tag{IX.384}$$

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For Γ_k (k = 0, 1, 2) we will use only the real parts, since we can show that their imaginary parts determine rotation of the body as a rigid whole.

The above relations enable us to construct the first three approximations[1] of the solutions of specific problems of stress concentration near arbitrary holes during the plane deformation of an incompressible material.

In the case where the shape of the hole differs from round, then, to solve the problem it is necessary to find a priori the functions that map conformally the range under examination onto the interior or exterior of the unit circle (see §1, Chapter I), i.e., the function

$$z = \omega(\zeta). \tag{IX.385}$$

The formulas for stress functions $U^{(k)}$ and displacement functions $D^{(k)}$, and also the boundary conditions (IX.369) and (IX.374) should be transformed to a new variable ζ, given by mapping function $\omega(\zeta)$ (IX.385). This can be done by using the relations[2]

$$\frac{\partial}{\partial z} = \frac{1}{\omega'(\zeta)} \cdot \frac{\partial}{\partial \zeta};$$
$$\frac{\partial^2}{\partial z^2} = \frac{1}{[\omega'(\zeta)]^2}\left[\frac{\partial^2}{\partial \zeta^2} - \frac{\omega''(\zeta)}{\omega'(\zeta)} \cdot \frac{\partial}{\partial \zeta}\right]. \tag{IX.386}$$

We will not discuss this aspect of the problem here, however, since the foregoing discussion gives the idea of the method of solving the problem of stress concentration near any hole[3], the contour of which has no angular points. To illustrate this idea we will examine the problem of stress concentration near a round hole. Here the case of the axisymmetrical stress state is characterized by the fact that it enables us to solve the stated problem to completion, i.e., permits the precise solution under any contour change law (IX.339). The value of such a precise solution is difficult to overestimate, since it can be used as the basis of evaluation of results obtained by various approximation methods.

Stresses Near Round Hole under Multifold Tension-Compression (Rigorous Solution). We will examine an elastic plane (under the conditions of plane deformation), in which, in the natural (unstressed) state, a round hole of

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[1]By using as the first approximation the solution of a single-type problem given by classical elasticity theory.
[2]See N. I. Muskhelishvili [1].
[3]See G. S. Taras'yev, L. A. Tolokonnikov [1], where the case of an incompressible material is examined. The case of a compressible material is examined by G. S. Taras'yev [1].

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radius R is made. This plane is under tension or compression "at infinity" by forces /688

$$N_1 = N_2 = \pm p. \tag{IX.387}$$

In (IX.387), "plus" corresponds to tension and "minus," to compression.

The contour of the hole will be regarded as free of external forces. We are required to determine the stress state near the hole under any contour change law (IX.339).

In the case of axisymmetric stress state (IX.387), the directions of the principal stresses can be selected arbitrarily. We will assume that the first principal direction is radial and the second, tangential.

In axial symmetry these stresses will be the principal stresses in each point of the plane and the solution of the problem will reduce to quadratures, independently of the law of contour change.

We will introduce a new variable $w = z\bar{z} = r^2$, where r is the vector radius of the points of the plane. We will assume that stress function U depends only on this vector radius r, i.e.,

$$U(z,\bar{z}) = \Phi(w). \tag{IX.388}$$

In this case equation (IX.351) can be twice integrated:

$$2p\frac{\operatorname{sh}\vartheta}{\tau_l}w^2\Phi'' + w\operatorname{ch}\vartheta = C_1 w + C_2, \tag{IX.389}$$

where C_1 and C_2 are integration constants.

From the third relation in (IX.350) and (IX.387) we obtain

$$\tau_l = \pm 2pw\Phi''. \tag{IX.390}$$

The physical meaning of the signs in (IX.390) is as follows: "plus" corresponds to selection of the tangential direction as the first principal direction of stresses and "minus," to the selection of the radial direction.

From (IX.389) and (IX.390) we find

$$\pm\vartheta = \ln\left(\frac{C_2}{w} + C_1\right), \tag{IX.391}$$

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i.e., due to (IX.387), the dependence of intensity ϑ on the coordinates of the points is completely defined under any law of contour change.

By expressing complex displacement function D through displacement vector components u_r and u_ϕ in polar coordinate system (r, ϕ), we obtain

$$D = \frac{r}{w^{\frac{1}{2}}}(u_r + iu_\varphi). \tag{IX.392}$$

Displacement vector $u_r + iu_\phi$ is a function only of w. Considering this /689 and (IX.392), we integrate relation (IX.353):

$$u_r = (C_1 w + C_2)^{\frac{1}{2}C_1} - r, \quad u_\varphi = 0. \tag{IX.393}$$

We see from (IX.393) that the assumption of axial symmetry leads to a completely defined displacement system. The constant C_1 in (IX.393) is determined from condition (IX.354).

Considering (IX.392) and (IX.393), we establish that relation (IX.354) will be satisfied identically only when C = 1, therefore

$$u_r = (r^2 + C_2)^{\frac{1}{2}} - r, \quad u_\varphi = 0 \tag{IX.394}$$

for any contour change law (IX.339). It follows from (IX.394) that

$$u_r = \frac{C_2}{2}\cdot\frac{1}{r} - \frac{C_2^2}{8}\cdot\frac{1}{r^3} + \ldots, \tag{IX.395}$$

where the first term is displacement during plane axisymmetric plastic flow in the case of small deformations. The constant C_2 is found from the boundary conditions. Relation (IX.352) can be converted[1] to the form

$$2\rho\Phi' + f = \frac{p\sigma_n}{\sqrt{1{,}5}}; \quad p\sigma_n = \text{const}, \quad p\tau_n = 0. \tag{IX.396}$$

From (IX.390), (IX.391) and (IX.396) we find

$$-\int_0^{\vartheta} \frac{\tau_i(\vartheta)}{e^{\pm\vartheta} - 1}\, d\vartheta + C = \frac{p\sigma_n}{\sqrt{1{,}5}}, \tag{IX.397}$$

[1]See V. G. Gromov [1].

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where C is the integration constant. Hence, for the above-selected distribution of principal stresses, we have[1]

$$\sigma_1 = \sqrt{1.5}\left(\int_0^{\Theta} \frac{\tau_i(\Theta)}{1-e^{-\Theta}}\,d\Theta + C\right), \quad \sigma_2 = \sigma_1 - 2\sqrt{1.5}\,\tau_i(\Theta). \tag{IX.398}$$

The complete solution of the problem requires the knowledge of the specific form of contour change law. For instance, by examining the physical nonlinearities (IX.356), we find for the first and second laws, respectively, of (IX.356)

$$\sigma_1 = \frac{\sqrt{1.5}}{2} G\left\{\ln\left[1 + \frac{1}{\left(1+\frac{C_2}{r^2}\right)^2}\right] - 2\arctan\left(1+\frac{C_2}{r^2}\right) + C\right\}; \tag{IX.399}$$

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$$\sigma_1 = \frac{\sqrt{1.5}}{2} G\left\{\ln\frac{1}{1+\frac{C_2}{r^2}} + \frac{1}{1+\frac{C_2}{r^2}} + C\right\}. \tag{IX.400}$$

For intensity of stresses we also have for the first and second laws, respectively, of (IX.356), the expressions

$$\tau_i = G\frac{1-\left(1+\frac{C_2}{r^2}\right)^2}{1+\left(1+\frac{C_2}{r^2}\right)^2} \text{ and } \tau_i = \frac{1}{2}G\frac{1-\left(1+\frac{C_2}{r^2}\right)^2}{1+\frac{C_2}{r^2}}. \tag{IX.401}$$

From the condition

$$\sigma_1 \to \sigma_1^{\infty} = \pm N_1 \quad \text{for} \quad r \to \infty \tag{IX.402}$$

we find the constant C for the first and second laws, respectively, of (IX.356)

$$\begin{aligned} C &= \pm\frac{2\varepsilon}{\sqrt{1.5}} - \ln 2 + \frac{\pi}{2}, \\ C &= \pm\frac{2\varepsilon}{\sqrt{1.5}} - 1, \end{aligned} \tag{IX.403}$$

[1]See V. G. Gromov [3].

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where

$$\varepsilon = \frac{p}{G}; \qquad p = N_1.$$

To determine the constant C_2 we will use the condition

$$\sigma_1 = 0 \quad \text{for } r = R. \tag{IX.404}$$

Relations (IX.399) and (IX.400) yield the equations for the determination of C_2:

$$\ln\left[1 + \frac{1}{(1+C)^2}\right] - 1\arctan(1+C) = \ln 2 - \frac{\pi}{2} - \varepsilon s, \tag{IX.405}$$

$$\ln\frac{1}{1+C} + \frac{1}{1+C} = 1 - \varepsilon s, \tag{IX.406}$$

where

$$C = \frac{C_2}{R^2}; \qquad s = \pm\frac{2}{\sqrt{1{,}5}}.$$

Equations (IX.405) and (IX.406) are transcendental. Their roots can be found (approximately) by some numerical method for calculated values of the parameter $\varepsilon = p/G$.

Let c_0 be one of the roots of equations (IX.405) and (IX.406); then, at the points of the contour of the hole

$$\sigma_2 = 2\sqrt{1{,}5}G\frac{(1+c_0)^2-1}{(1+c_0)^2+1}, \qquad \sigma_2 = \sqrt{1{,}5}G\frac{(1+c_0)-1}{(1+c_0)}. \tag{IX.407}$$

Hence, for concentration coefficients $k = \sigma_2/p$, we have the precise expressions /691 which we will use for comparing them with the approximate values obtained by different methods:

$$k_T = \frac{2\sqrt{1{,}5}}{\varepsilon}\cdot\frac{|(1+c_0)^2-1|}{(1+c_0)^2+1}, \qquad k_T = \frac{\sqrt{1{,}5}}{\varepsilon}\left|1 - \frac{1}{1+c_0}\right|, \tag{IX.408}$$

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where

$$\varepsilon = \frac{p}{G}; \qquad p = |\sigma^{(\infty)}| = N_1.$$

Stresses near Round Hole under Biaxial Stress-Compression (Approximate Solution). Let the stress state at infinity be

$$\sigma_1^{(\infty)} = \pm N_1, \qquad \sigma_2^{(\infty)} = \pm N_2, \tag{IX.409}$$

where N_1 and N_2 are given non-negative values; the signs in (IX.409) are defined as follows: "plus" corresponds to tension and "minus," to compression.

We will assume that the contour of the hole is free of external forces. It is convenient to introduce the parameters

$$s = \frac{1}{2\sqrt{1.5}}\left(\pm\frac{N_1}{p} \pm \frac{N_2}{p}\right), \qquad t = \frac{1}{2\sqrt{1.5}}\left(\pm\frac{N_1}{p} \mp \frac{N_2}{p}\right), \tag{IX.410}$$

which determine the stress state at infinity.

As mentioned earlier, the construction of the approximations of the stated problem reduces to the repeated solution of the first basic boundary problem of classical elasticity theory. Each such problem can be solved by one of N. I. Muskhelishvili's methods [1]. By using the most effective of Muskhelishvili's methods (see §1, Chapter I), based on the application of integrals of Cauchy's type, and relations for the first three approximations, L. A. Tolokonnikov and V. G. Gromov [1], and also V. G. Gromov [2-4] found the approximate solutions of the examined problem. On the basis of these solutions, the formulas for stresses on the contour of a hole were derived in the third approximation:

$$\sigma_1 = 0, \qquad \sigma_2 = 2\sqrt{1.5}p\,(\sigma_2^{(0)} + \sigma_2^{(1)}\varepsilon + \sigma_2^{(2)}\varepsilon^2), \tag{IX.411}$$

where

$$\sigma_2^{(0)} = s - 2t\cos 2\theta; \qquad \sigma_2^{(1)} = \frac{1}{4}(s^2 + t^2) + st\cos 2\theta - t^2\cos 4\theta; \tag{IX.412}$$

$$\sigma_2^{(2)} = \alpha + \beta\cos 2\theta + \gamma\cos 4\theta + \delta\cos 6\theta \qquad (0 < \theta < 2\pi);$$

$$\alpha = -\frac{1}{8}s\left(s^2 + \frac{81}{5}t^2\right); \quad \beta = \frac{1}{10}t\left(\frac{27}{4}s^2 + \frac{61}{7}t^2\right); \quad \gamma = \frac{139}{140}st^2; \quad \delta = -\frac{6}{10}t^3; \tag{IX.413}$$

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$$\alpha = \frac{1}{24} s\left(5s^2 + \frac{521}{5} t^2\right); \quad \beta = -\frac{1}{6} t\left(\frac{167}{20} s^2 + \frac{103}{35} t^2\right); \quad \gamma = \frac{383}{140} st^2; \quad -\delta = -\frac{13}{10} t^3$$

for the first and second laws, respectively, of (IX.356).

Hence the stresses on the contour of the hole can be represented in the form

$$\sigma_2 = 2\sqrt{1.5}p\left\{s + \frac{1}{4}(s^2 + t^2)\varepsilon + \alpha\varepsilon^2 + [-2t + ste + \beta\varepsilon^2]\cos 2\theta\right\} + \\ + 2\sqrt{1.5}p\varepsilon\{(-t^2 + \gamma\varepsilon)\cos 4\theta + \varepsilon\delta\cos 6\theta\}. \qquad \text{(IX.414)}$$

The first brace in (IX.414) is the refined value of that part of the stresses whose character of change is predicted by classical linear theory, and the second is the direct consequence of the consideration of geometrical and physical nonlinearity. These parts of the expressions are differentiated with respect to the character of the dependences on angle θ.

In conclusion we will examine certain partial cases.

Multifold Tension (Compression) will obtain when

$$\sigma_1^{(\infty)} = \sigma_2^{(\infty)} = \pm p \qquad (N_1 = N_2 = p). \qquad \text{(IX.415)}$$

From (IX.410) we find

$$s = \pm\frac{1}{\sqrt{1.5}}, \qquad t = 0. \qquad \text{(IX.416)}$$

The stresses on the contour of the hole for the first and second laws, respectively, of (IX.356) are

$$\sigma_1 = 0, \qquad \sigma_2 = \pm 2p\left(1 \pm \frac{\sqrt{6}}{12}\varepsilon - \frac{1}{12}\varepsilon^2\right); \\ \sigma_1 = 0, \qquad \sigma_2 = \pm 2p\left(1 \pm \frac{\sqrt{6}}{12}\varepsilon + \frac{5}{36}\varepsilon^2\right). \qquad \text{(IX.417)}$$

We see from (IX.417) that consideration of the second approximation leads to an increase in stresses for multifold tension and decrease during compression in comparison with the classical values, regardless of the law of contour change. The corrections of the third approximation have different signs for the first and second laws of (IX.356) and do not depend on the changing of the

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signs of the stresses at infinity. Consideration of these corrections leads to a decrease in stresses for the first law of (IX.356) and to an increase for the second.

For the concentration coefficients, we have, respectively,

$$k_{app} = 2\left(1 \pm \frac{\sqrt{6}}{12}\varepsilon - \frac{1}{12}\varepsilon^2\right),$$
$$k_{app} = 2\left(1 \pm \frac{\sqrt{6}}{12}\varepsilon + \frac{5}{36}\varepsilon^2\right). \qquad \text{(IX.418)}$$

We will compare the approximate values of k_{app} (IX.418) with their precise values k_{pr} (IX.408). The roots of equations (IX.405) and (IX.406) were calcu- /693 lated for certain values of the parameter $\varepsilon = p/G$, and then the values of concentration coefficients k_{pr} were calculated on the basis of formulas (IX.408) and of k_{app} on the basis of formulas (IX.418). The results of these calculations are presented in Table IX.6.

TABLE IX.6

ε	$\tau_i = G$ th ϑ			$\tau_i = G$ sh ϑ		
	c_0	$k_{пр}$ (IX. 408)	$k_{пр}$ (IX. 418)	c_0	$k_{пр}$ (IX. 408)	$k_{пр}$ (IX. 418)
0,06	0,052	2,022	2,023	0,052	2,025	2,025
0,12	0,108	2,048	2,048	0,108	2,059	2,055
0,18	0,169	2,069	2,069	0,168	2,082	2,084
0,24	0,236	2,088	2,090	0,234	2,118	2,117
0,30	0,310	2,106	2,109	0,305	2,216	2,151
0,36	0,390	2,179	2,125	0,382	2,256	2,184
0,42	0,480	2,183	2,142	0,466	2,294	2,219
0,48	0,581	2,188	2,158	0,556	2,332	2,260
0,54	0,696	2,190	2,172	0,655	2,379	2,302

Tr. Note: Commas indicate decimal points.

Pure displacement, directed at angle $\pi/4$ to the direction of the principal stresses at infinity, of intensity 2p, is found for

$$N_1 = -N_2 = \pm p. \qquad \text{(IX.419)}$$

In the case the parameters (IX.410) are

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$$s = 0, \qquad t = \pm \frac{1}{\sqrt{1{,}5}}.$$

The stresses on the contour of the hole are

$$\begin{gathered} \sigma_1 = 0, \\ \sigma_2 = 2\sqrt{1{,}5}p\left\{\frac{1}{4}t^2 s + (-2t + \beta\varepsilon^2)\cos 2\theta\right\} + \\ + 2\sqrt{1{,}5}p\varepsilon\{-t^2\cos 4\theta + \varepsilon\delta\cos 6\theta\}, \end{gathered} \tag{IX.420}$$

where

$$\beta = \frac{61}{70}t^3, \qquad \delta = -\frac{3}{5}t^3$$

and

$$\beta = -\frac{103}{210}t^3, \qquad \delta = -\frac{13}{10}t^3$$

for the first and second laws, respectively, of (IX.356).

Uniaxial Tension (Compression). We assume $N_2 = 0$. In this case

$$\pm N_1 = \sigma_1^{(\infty)} = \pm p \quad \left(s = \pm \frac{1}{2\sqrt{1{,}5}}\right). \tag{IX.421}$$

The stresses on the contour of the hole are /694

$$\begin{gathered} \sigma_1 = 0, \\ \sigma_2 = \pm p[1 \pm 0{,}204\varepsilon + \alpha\varepsilon^2 + (-2 \pm 0{,}408\varepsilon + \beta\varepsilon^2)\cos 2\theta] \pm \\ \pm p\varepsilon[(\mp 0{,}408 + \gamma\varepsilon)\cos 4\theta + \varepsilon\delta\cos 6\theta], \end{gathered} \tag{IX.422}$$

where, for the first and second laws, respectively, of (IX.356)

$$\begin{gathered} \alpha = -0{,}358, \quad \beta = 0{,}258, \quad \gamma = 0{,}165, \quad \delta = -0{,}100, \\ \alpha = +0{,}758, \quad \beta = -0{,}314, \quad \gamma = 0{,}456, \quad \delta = -0{,}217. \end{gathered} \tag{IX.423}$$

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The character of functions (IX.420) and (IX.422), or more accurately, of ratios σ_2/p, is shown in Figures IX.4 and IX.5, respectively, in which the solid curves correspond to linear theory, and the broken curves to nonlinear theory.

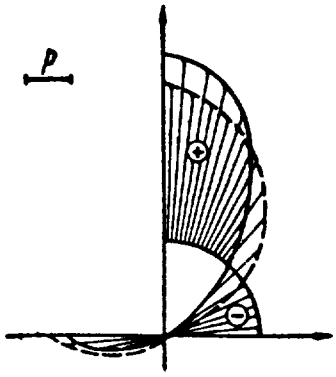


Figure IX.4.

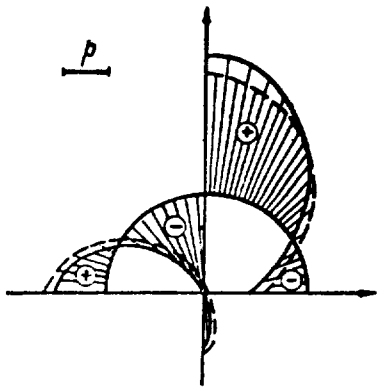


Figure IX.5.

From formula (IX.411), recalling the values of the parameters from (IX.413), and the fact that during the predominant effect of forces $N_1 = p > N_2$,

$$N_2 \neq 0, \quad t = \pm \frac{1}{\sqrt{1.5}} - s \qquad \left(-\frac{1}{\sqrt{1.5}} < s < \frac{1}{\sqrt{1.5}} \right),$$

where the top sign corresponds to tension and the bottom sign to compression, we find the formulas for the calculation of concentration coefficient $k = \sigma_2/p$ at the points of the contour of the hole, which, prior to deformation, were determined by the values of the angle $\theta = \pi/2$ and $\theta = 3\pi/2$, in the form

$$k = \frac{\sigma_2}{p} = 2\sqrt{1.5}\left| k^{(0)} + \frac{1}{2}k^{(1)}\varepsilon + \frac{1}{8}k^{(2)}\varepsilon^2 \right|, \qquad \text{(IX.424)}$$

where

$$k^{(0)} = -s \pm \frac{2}{\sqrt{1.5}}; \qquad k^{(1)} = s^2 \pm \frac{1}{\sqrt{1.5}}s - 1;$$

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$$k^{(2)} = -1\frac{24}{35}s^3 \pm 4\frac{3}{5}\cdot\frac{s^2}{\sqrt{1.5}} - 1\frac{17}{105}s \mp 1\frac{47}{105}\cdot\frac{1}{\sqrt{1.5}},$$

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or

$$k^{(2)} = 32\frac{29}{35}s^3 \mp 59\frac{2}{15}\cdot\frac{s^2}{\sqrt{1{,}5}} + 9\frac{31}{315}s \pm 9\frac{173}{315}\cdot\frac{1}{\sqrt{1{,}5}}. \qquad \text{(IX.425)}$$

The following evaluations are valid for $k^{(j)}$: if $\sigma_1^{(\infty)} < 0$, then $k^{(0)} < 0$;

$$\begin{aligned} &k^{(1)} < 0 \quad \text{for} \quad -\frac{\sqrt{7}-1}{\sqrt{6}} < s < 0; \\ &k^{(1)} > 0 \quad \text{for} \quad -\frac{1}{\sqrt{1{,}5}} < s < \frac{\sqrt{7}-1}{\sqrt{6}}; \\ &k^{(2)} > 0 \quad \text{for} \quad \tau_l = G\,\mathrm{th}\,\vartheta; \\ &k^{(2)} < 0 \quad \text{for} \quad \tau_l = G\,\mathrm{sh}\,\vartheta. \end{aligned} \qquad \text{(IX.426)}$$

If $\sigma_1^{(\infty)} > 0$, then $k^{(0)} > 0$;

$$\begin{aligned} &k^{(1)} < 0 \quad \text{for} \quad 0 < s < \frac{\sqrt{7}-1}{\sqrt{6}}; \\ &k^{(1)} > 0 \quad \text{for} \quad \frac{\sqrt{7}-1}{\sqrt{6}} < s < \frac{1}{\sqrt{1{,}5}}; \\ &k^{(2)} < 0 \quad \text{for} \quad \tau_l = G\,\mathrm{th}\,\vartheta; \\ &k^{(2)} > 0 \quad \text{for} \quad \tau_l = G\,\mathrm{sh}\,\vartheta. \end{aligned} \qquad \text{(IX.427)}$$

Hence consideration of the second approximation leads both to large and to small values of the concentration coefficient in comparison with its classical value. The magnitude and sign of $\sigma_2^{(\infty)}$ here are the deciding factors. For instance, when $\sigma_1^{(\infty)} > 0$, the coefficient $k^{(1)}$ increases monotonically from -1 to 1/3, whereupon $-1 < k^{(1)} < -1/2$ when $-p < \sigma_2^{(\infty)} < 0$ and

$$\begin{aligned} &-\frac{1}{2} < k^{(1)} < 0 \quad \text{for} \quad \sigma_2^{(\infty)} > 0 \quad \text{for} \quad \sigma_2^{(\infty)} < (\sqrt{7}-2)\,p; \\ &0 < k^{(1)} < \frac{1}{3} \quad \text{for} \quad \sigma_2^{(\infty)} > 0 \quad \text{for} \quad (\sqrt{7}-2)\,p < \sigma_2^{(\infty)} < p. \end{aligned} \qquad \text{(IX.428)}$$

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When $\sigma^{(\infty)} = (\sqrt{7} - 2)p$ the coefficient $k^{(1)} = 0$ and, as follows from (IX.424), in this case the second approximation for an incompressible medium will have no effect on the numerical value of the concentration coefficient given by classical linear theory.

Consideration of the third approximation always leads to a decrease in the coefficient of concentration for the first ($\tau_i = G\,\mathrm{th}\,\vartheta$) law of contour change from (IX.356) and to an increase for the second ($\tau_i = G\,\mathrm{sh}\,\vartheta$) law. /696

Recalling that geometrical nonlinearity is considered basically by the second approximation and that physical nonlinearity begins to be taken into account only in the third approximation, then the results obtained can be interpreted as the manifestation of qualitative properties of geometrical and physical nonlinearity.

The behavior of the coefficient $k^{(1)}$ is related to the purely geometrical effect, which is attributed to the influence of deformation of the contour. For instance, in the case of pure displacement (IX.420), the coefficient of concentration[1] is

$$k = 4(1 \mp 0{,}306\varepsilon - 0{,}090\varepsilon^2),$$
$$k = 4(1 \mp 0{,}306\varepsilon + 0{,}597\varepsilon^2); \qquad \text{(IX.429)}$$

and under uniaxial tension or compression (IX.422),

$$k = 3(1 \mp 0{,}204\varepsilon - 0{,}120\varepsilon^2),$$
$$k = 3(1 \mp 0{,}204\varepsilon + 0{,}582\varepsilon^2). \qquad \text{(IX.430)}$$

A decrease in the concentration during tension and increase during compression (in comparison with the classical value) are due to decrease during tension and increase during compression of the curvature of the contour (round prior to deformation) of the hole in the examined points $\theta = \pi/2$, $\theta = 3\pi/2$.

It should be noted that the stress σ_2 that is greatest in absolute value may appear not at the points $\theta = \pi/2$, $\theta = 3\pi/2$, but at the points $\theta = 0$, $\theta = \pi$. This, for instance, occurs in the case of pure displacement (IX.420). The general conditions under which stresses σ_2 at the points $\theta = \pi/2$, $\theta = 3\pi/2$ are greater in absolute value than the stresses at points $\theta = 0$, $\theta = \pi$ have the form:

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[1]The first expressions in formulas (IX.429)-(IX.431) correspond to the first law of (IX.356) and the second expressions, to the second law.

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$$s+\frac{1}{4}(s^2-3t)\,\varepsilon+(\alpha+\gamma)\,\varepsilon^2>0,\quad -2t+st\varepsilon+(\beta+\delta)\,\varepsilon^2<0$$

or

$$s+\frac{1}{4}(s^2-3t)\,\varepsilon+(\alpha+\gamma)\,\varepsilon^2<0,\qquad -2t+st\varepsilon+(\beta+\delta)\,\varepsilon^2>0. \tag{IX.431}$$

The independence of the concentration coefficients in the classical linear theory of the reversal of the signs of the stresses "at infinity" is caused by the disregarding of the effect of deformations of the elements of the contour of the holes on the stress distribution. As we know, this is one of the basic assumptions in classical elasticity theory.

Concluding Comments. In a more general case of physical nonlinearity (IX.355), proceeding from the classical solution /697

$$\varphi_0(z)=\frac{p+q}{4}z+\frac{p-q}{2}\cdot\frac{R^2}{z},$$
$$\psi_0(z)=-\frac{p-q}{2}z-\frac{p+q}{2}\cdot\frac{R^2}{z}+\frac{p-q}{2}\cdot\frac{R^4}{z^3}, \tag{IX.432}$$

which corresponds to the biaxial stress state at infinity $\sigma_x^{(\infty)}=\pm p$; $\sigma_y^{(\infty)}=\pm q$, as the first approximation, the approximate (theoretically with any degree of accuracy) solution of the problem of stress concentration near a round hole can be constructed.

Analyses show[1] that the boundary conditions, both in the forces and in the displacements, for all three contour change laws (IX.355) and (IX.356) are identical for the first two approximations, and therefore the first two approximations for all these three contour change laws (IX.355) and (IX.356) coincide. In other words, all the formulas that were derived in the preceding section, taken with an accuracy up to the second approximation (inclusively), i.e., with the retention in them of the terms with ε in the first power, will be valid for any of these three laws of contour change (IX.355) and (IX.356). The third approximation, which begins to take into account the effect of the physical nonlinearity of the problem will be different for each of the laws (IX.355) and (IX.356) and therefore the term $\sigma_2^{(2)}$ in formula (IX.411) will have the following form for law (IX.355):

[1]See V. G. Gromov [4].

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$$\sigma_2^{(2)} = \frac{1}{2}\left\{\left(\frac{1}{4} - \frac{4}{3}e\right)s^3 + \left(\frac{11}{2} - \frac{382}{15}e\right)st^2 - \left[\left(\frac{7}{4} - \frac{124}{15}e\right)s^2t + \right.\right.$$
$$\left.\left. + \left(\frac{3}{10} + \frac{572}{105}e\right)t^3\right]\cos 2\theta + \frac{1}{5}\left(23 - \frac{244}{7}e\right)st^2\cos 4\theta - \left(\frac{9}{4} - \frac{14}{5}e\right)t^3\cos 6\theta\right\}; \quad \text{(IX.433)}$$

where

$$e = \frac{1}{24} - g.$$

These stresses at the points of the contour of the hole, which prior to deformation were determined by the values $\theta = \pi/2$ and $\theta = \pi$, will have the form, respectively,

$$\sigma_2^{(2)} = \frac{1}{2}\left[\left(\frac{1}{4} - \frac{4}{3}e\right)s^3 \pm \left(\frac{7}{4} - \frac{124}{15}e\right)s^2t + \left(\frac{101}{10} - \right.\right.$$
$$\left.\left. - \frac{3406}{105}e\right)st^2 \mp \left(\frac{51}{20} - \frac{866}{105}e\right)t^3, \right. \quad \text{(IX.434)}$$

where the "plus" sign corresponds to the point $\theta = \pi/2$ and the "minus" sign to the point $\theta = \pi$.

During the multifold tension-compression by stresses (at infinity) of intensity 2p, the parameters are $t = 0$, $s = \pm 1/\sqrt{1.5}$, and therefore we find from (IX.434) /698

$$\sigma_2^{(2)} = \pm \frac{4}{9\sqrt{1.5}}\left(\frac{7}{48} + g\right). \quad \text{(IX.435)}$$

It follows from (IX.435) and (IX.411) that stresses σ_2 (IX.411) on the contour of the hole for $g < -7/48$, found with an accuracy up to the third approximation (inclusively), will be greater for materials with physical nonlinearity (IX.355) than stresses σ_2 found with an accuracy up to the second approximation (inclusively). When $g > -7/48$, we will have the opposite situation. When $g = g^* = -7/48 = -0.146$, the component is $\sigma_2^{(2)} = 0$ and therefore consideration of the third approximation will not transmit the correction to the values of the concentration coefficient to (IX.411), found with an accuracy up to the second approximation (inclusively).

Pure displacement at angle $\pi/4$ to the principal directions at infinity is found for $s = 0$ and $t = \pm 1/\sqrt{1.5}$. In this case, from (IX.334) we readily find

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$$\sigma_2^{(2)} = \mp \frac{1}{3\sqrt{1.5}}\left[\left(\frac{3}{10} + \frac{572}{105}e\right)\cos 2\theta + \left(\frac{9}{4} - \frac{14}{5}e\right)\cos 6\theta\right] \qquad \text{(IX.436)}$$

At the points $\theta = \pi/2$ and $\theta = \pi$, we obtain

$$\sigma_2^{(2)} = \mp \frac{1}{3\sqrt{1.5}}\left[\mp\left(-\frac{51}{20} + \frac{866}{105}e\right)\right]. \qquad \text{(IX.437)}$$

From the condition $\sigma_2^{(2)} = 0$ we find the "critical" value for $g^* = -0.268$. Here, as in (IX.434), the top sign in the brackets corresponds to the point $\theta = \pi/2$ and the bottom to the point $\theta = \pi$.

In the case of predominating effect of one of forces $\sigma_1^{(\infty)} = \pm N_1$ or $\sigma_2^{(\infty)} = \pm N_2$, for instance, $\sigma_1^{(\infty)} = N_1 > 0$, it follows from (IX.434) that

$$\sigma_2^{(2)} = \frac{1}{2}\left[\left(\frac{121}{20} - \frac{604}{35}e\right)s^3 \pm \left(-\frac{54}{5} + \frac{478}{15}e\right)\frac{s^2}{\sqrt{1.5}} + \right.$$
$$\left. + \left(\frac{49}{20} - \frac{808}{105}e\right)\frac{s}{1.5} \pm \left(\frac{51}{20} - \frac{866}{105}e\right)\frac{1}{(\sqrt{1.5})^3}\right]. \qquad \text{(IX.438)}$$

By equating $\sigma_2^{(2)}$ in (IX.438) to zero, we find the "critical" value

$$g^*(s) = \frac{1}{24} - \frac{21}{8}\cdot\frac{121x^3 - 216x^2 + 49x + 51}{906x^3 - 1673x^2 + 404x + 443}, \qquad \text{(IX.439)}$$

where

$$x = |s|\sqrt{1.5}.$$

From (IX.439) we see, recalling that s changes within the range $-1/\sqrt{1.5} < s < 1/\sqrt{1.5}$, that the function $g^*(s)$ is limited to the range /699

$$-0.268 < g^*(s) < -0.146. \qquad \text{(IX.440)}$$

From (IX.440) we conclude that for contour change laws (IX.355) with $g < -0.268$ and $g > -0.146$, the sign of correction $\sigma_2^{(2)}$ (IX.438) will be constant for any ratio of the stresses $\sigma_1^{(\infty)}$ and $\sigma_2^{(\infty)}$. In the first case (when $g < -0.268$), the sign of $\sigma_2^{(2)}$ is opposite to that of the prevailing stress at infinity, and in the second case (when $g > -0.146$), coincides with it. This leads either to an

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increase in the greatest stresses σ_2 (IX.411), or to a decrease. Therefore, nonlinearity of the type (IX.355) for $g < -0.268$ is naturally called "soft," and for $g > -0.146$, "hard." According to this classification, the first law of (IX.356) (for which $g = -1/3$) will be related to the "soft" category of nonlinearity, and the second law of (IX.356) (for which $g = +1/6$) to "hard" nonlinearity. In each individual case, i.e., for a given value of g, we may, by comparing this value g with the value g^* in (IX.439), make a judgement considering the effect of physical nonlinearity, or more precisely, of the third approximation, on stress concentration σ_2 (IX.411) near the hole. Thus, if $g < g^*$, then consideration of the third approximation will lead to a decrease in the greatest stress, and when $g > g^*$, to an increase. Obviously, when $g = g^*$, the stress $\sigma_2^{(2)} = 0$.

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CHAPTER X. STRESS DISTRIBUTION NEAR HOLES IN SHELLS

ABSTRACT. The formulation of problems is given on the stress distribution near curvilinear holes in shells. For stress concentration near holes, the basic equations and boundary conditions of the problem are considered. The solution is given of numerous problems on the stress concentration in the spherical and cylindrical shells near the circular, elliptic, square and triangular holes under different action of external forces applied to the shell.

§1. Statement of Problem[1]

Basic Equations of Problem in Differential Form. We will examine the stress state in a thin shell of constant thickness h, weakened by some hole, not too small, the contour of which represents spatial curve Γ, which does not possess angular points. We will relate the middle area of the examined shell to some isothermic, and generally speaking, nonself-conjugate system of curvilinear coordinates α and β (Figure X.1). /701

The equation of the middle area of the shell in these coordinates will be, in vector form,

$$R(\alpha, \beta) = \vec{i}_x f_1(\alpha, \beta) + \vec{i}_y f_2(\alpha, \beta) + \vec{i}_z f_3(\alpha, \beta), \tag{X.1}$$

where $f_1(\alpha, \beta) = x$, $f_2(\alpha, \beta) = y$, $f_3(\alpha, \beta) = z$ are given functions of parameters α and β.

In coordinate system α and β, the first and second quadratic forms, as we know, will have the form

$$\mathrm{I} \equiv ds^2 = \lambda^2(\alpha, \beta)(d\alpha^2 + d\beta^2), \tag{X.2}$$

$$\mathrm{II} \equiv -(d\vec{R}, d\vec{n}) = Dd\alpha^2 + 2D^* d\alpha d\beta + D^{**} d\beta^2. \tag{X.3}$$

The coefficients λ^2, D, D* and D** of these forms are found from the given equation of the middle area of the shell (X.1) by formulas

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[1] See G. N. Savin [1-6]; G. N. Savin, G. A. Van Fo Fy [1], G. N. Savin, G. A. Van Fo Fy, V. N. Buyvol [1-3]. A different approach to the investigation of the stress state near a round hole in a spherical shell is given by L. B. Imenitov [1-3].

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$$\lambda^2(\alpha, \beta) = \left(\frac{\partial \vec{R}}{\partial \alpha}, \frac{\partial \vec{R}}{\partial \alpha}\right) = \left(\frac{\partial \vec{R}}{\partial \beta}, \frac{\partial \vec{R}}{\partial \beta}\right); \tag{X.4}$$

$$D = \left(\vec{n}, \frac{\partial^2 \vec{R}}{\partial \alpha^2}\right), \qquad D^* = \left(\vec{n}, \frac{\partial^2 \vec{R}}{\partial \alpha \partial \beta}\right), \qquad D^{**} = \left(\vec{n}, \frac{\partial^2 \vec{R}}{\partial \beta^2}\right), \tag{X.5}$$

where $\vec{n}(\alpha, \beta)$ is the unit normal to the surface.

We will define the following:

$$\begin{aligned} T_\alpha^*, T_\beta^*;\ S_\alpha^*, S_\beta^* &- \text{tangential forces,} \\ Q_\alpha^*, Q_\beta^* &- \text{shear forces,} \\ H_\alpha^*, H_\beta^* &- \text{moments of torque,} \\ G_\alpha^*, G_\beta^* &- \text{bending moments,} \end{aligned} \tag{X.6}$$

which characterize the stress state in the shell with the given hole, under the effect of the given system of external forces and under the given boundary conditions. /702

The positive directions of the components are illustrated in Figure X.1. Moments H_α^* and H_β^*, G_α^* and G_β^* are shown in the form of vectors, where, when viewed from the positive side of the vector, the corresponding moment will tend to rotate in the clockwise direction.

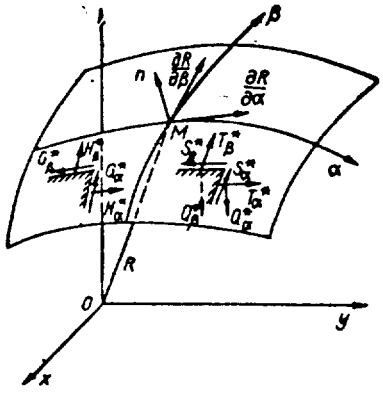


Figure X.1.

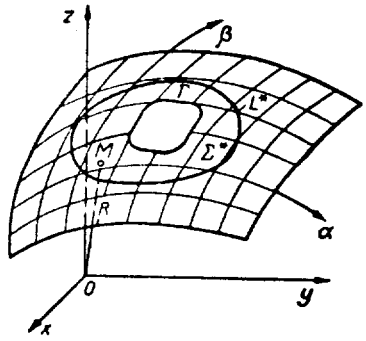


Figure X.2.

The stress state in the very same shell, but continuous (not weakened by a hole), acted upon by the very same system of external forces and subject to

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the very same boundary conditions, will be known as the basic stress state, and we will denote it in the following manner:

$$T_\alpha^0;\ T_\beta^0;\ S_\alpha^0;\ S_\beta^0;\ Q_\alpha^0;$$
$$Q_\beta^0;\ H_\alpha^0;\ H_\beta^0;\ G_\alpha^0;\ G_\beta^0. \tag{X.7}$$

We will represent components T_α^*, ..., G_β^* in the shell having a hole in the form

$$\begin{aligned} T_\alpha^* &= T_\alpha^0 + T_\alpha; \quad T_\beta^* = T_\beta^0 + T_\beta; \quad S_\alpha^* = S_\alpha^0 + S_\alpha; \\ S_\beta^* &= S_\beta^0 + S_\beta; \quad G_\alpha^* = G_\alpha^0 + G_\alpha; \quad G_\beta^* = G_\beta^0 + G_\beta; \\ H_\alpha^* &= H_\alpha^0 + H_\alpha; \quad H_\beta^* = H_\beta^0 + H_\beta; \quad Q_\alpha^* = Q_\alpha^0 + Q_\alpha; \\ & \qquad Q_\beta^* = Q_\beta^0 + Q_\beta. \end{aligned} \tag{X.8}$$

We see from (X.8) that the components

$$T_\alpha; T_\beta; S_\alpha; S_\beta; Q_\alpha; Q_\beta; H_\alpha; H_\beta; G_\alpha \text{ and } G_\beta \tag{X.9}$$

represent additional components of elastic forces and moments in the shell, produced by the presence of the hole within it and characterizing, particularly, the "concentration of stresses" in the given shell near the examined hole[1]. /703

The experimental analyses of the stress state in shells near holes with rather smooth contours indicate that perturbations in the stress state near the holes in a shell are of a local character, propagating to a comparatively limited zone Σ^* near the hole, bounded by contour L^* (Figure X.2).

The magnitude of this perturbation in the damping zone of the stress state near a round disc of thickness δ = 3 mm and diameter d = 15 mm, glued to the cylindrical shell (thickness h = 1 mm and diameter d = 40 mm), which simulates a round hole in a shell with an elastic ring under the simultaneous effect of internal pressure and axial tension of the cylindrical shell when $\varepsilon = 0$ and $\varepsilon = 0.5$, can be judged by Figure X.3. In the interest of graphical representation, the square grid of lines α = const and β = const is superimposed on these shells in the nondeformed state.

The very same pattern of the local character is also seen in the case of a spherical resin shell of diameter D = 250 mm and thickness h = 1 mm near a glued circular resin disc of thickness δ = 3 mm and diameter d = 25 mm under internal pressure (Figure X.4).

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[1]Such a breakdown of the stress state is possible in view of the linearity of the examined problem.

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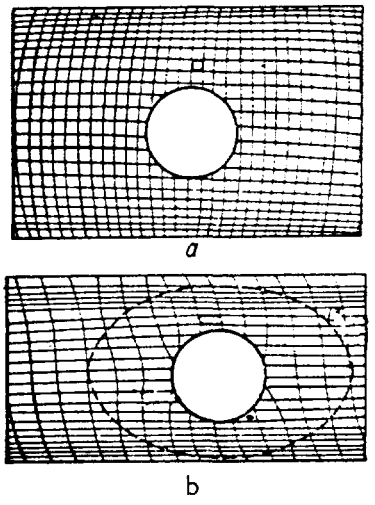


Figure X.3.

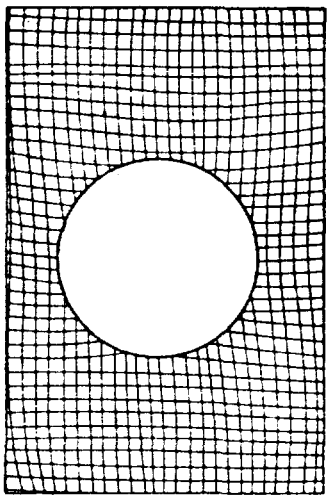


Figure X.4.

A more detailed description of the results of experiments can be found in the works of G. N. Savin [1-3].

Numerous experiments on the investigation of the stress state near round and elliptical holes in spherical and cylindrical shells lead to the conclusion that in also the general case, in a shell of positive and zero[1] gauss curvature, with a hole bounded by a sufficiently smooth contour, there may be such a local zone of the perturbed stress state Σ^*, bounded by contour L^* (see Figure X.2), on which, and especially, beyond which the components $T_\alpha, \ldots, G_\beta$ (X.9) will be, for all practical purposes, equal to zero, and components $T_\alpha^0, \ldots, G_\beta^0$ (X.7) and $T_\alpha^*, \ldots, G_\beta^*$ (X.6) will coincide. This indicates that components $T_\alpha, \ldots, G_\beta$ (X.9), which especially characterize the "concentration of stresses" in a shell near the examined hole, represent rapidly damping coordinate functions, and for their determination it is possible to employ the theory of stress states with a large variability index, coinciding with the theory of sloping shells (see A. L. Gol'denveyzer [1] and V. Z. Vlasov [1]). /704

Before proceeding to the establishment of the basic equations and boundary conditions for $T_\alpha, \ldots, G_\alpha$ (X.9), we will note that in view of the smallness of region Σ^*, we will represent (X.2) in the form

[1]See Yu. I. Volozhaninov, S. G. Shokot'ko [1], where perturbed zones near elliptical holes in cylindrical shells are analyzed by the photoelasticity method.

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$$I = ds^2 = \lambda_0^2 (d\alpha^2 + d\beta^2), \tag{X.10}$$

where λ_0 = const.

The substitution of ds^2 in region Σ^* by expression (X.10), which, in the case of a cylindrical shell, is an accurate expression, while in the theory of sloping shells it is used for an arbitrary shell, corresponds to the substitution of the noneuclidian metrics of the middle area of the shell by the euclidian metrics of a plane.

In the following analysis we will assume that the holes are of such dimensions that the dimensions of region Σ^*, bounded by contour L^*, are small and that the expression for ds^2 will be taken in the form (X.10).

We will abandon the old coordinate system (α, β) in favor of such an isothermic coordinate system (ρ, θ), one of whose coordinate lines $\rho = \rho_0$ = const will coincide with contour Γ of the examined hole in the shell. The vector equation of contour Γ is found by substituting in (X.1) α and β, determined by the function $\omega(\zeta)$ (X.11), by the new variables ρ and θ, assuming $\rho = \rho_0$:

$$\vec{R}(\varrho_0, \theta) = \vec{i}_x \varphi_1(\varrho_0, \theta) + \vec{i}_y \varphi_2(\varrho_0, \theta) + \vec{i}_z \varphi_3(\varrho_0, \theta).$$

On plane P of variables α and β, contours γ and l will correspond to contours Γ and L^* of the surface (Figure X.5).

We will examine holes in shells bounded by sufficiently smooth contours Γ, to which, in plane P, will coincide contours γ without angular points, the parametric equation of which can be given by the analytical function /705

$$\alpha + i\beta = \omega(\zeta) = a_1 \zeta + \frac{a_{-1}}{\zeta} + \frac{a_{-2}}{\zeta^2} + \ldots + \frac{a_{-n}}{\zeta^n}, \tag{X.11}$$

where $\zeta = \exp(\rho_0 + i\theta)$, and ρ_0, a_1, a_{-1}, ..., a_{-n} are constants. In other words, we will assume that analytical function (X.11) accomplishes the above-stated transformation of the coordinates. In the following discussion, a hole in a shell will be named according to the shape of curve γ on plane P (Figure X.5). We see from (X.11) that in the new coordinates ρ and θ the first quadratic form ds^2 (X.10) will be of the form

$$ds^2 = \mathcal{H}^2(\varrho, \theta)(d\varrho^2 + d\theta^2), \tag{X.12}$$

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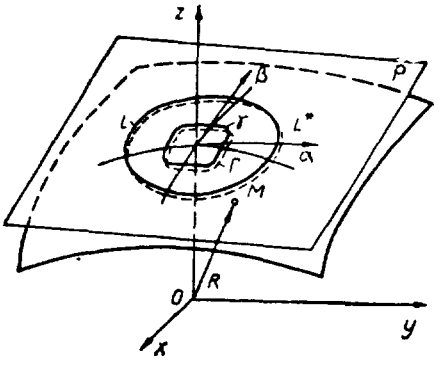


Figure X.5.

where

$$\mathcal{H}^2(\varrho, \theta) = \lambda_0^2 e^{2\varrho} \omega'(\zeta)\,\overline{\omega'(\zeta)}.$$

Contour l on plane P, corresponding to contour L*, can be substituted by the coordinate line $\rho = \rho_1$ closest to it.

To determine the components $T_\alpha, \ldots, G_\beta$ (X.9), which, in coordinates ρ and θ, will be denoted through

$$T_\theta; T_\varrho; \ldots, G_\varrho, \tag{X.13}$$

we will use the approximate theory of stress state with a large variability index[1].

Since each of stress states $T_\rho^*, \ldots, G_\rho^*$ (X.9) and $T_\rho^0, \ldots, G_\rho^0$ (X.7) satisfy the same equations of the theory of sloping shells for the same external forces and boundary conditions, then to determine components $T_\rho, \ldots, G_\theta$, we find, as readily seen from (X.8), a uniform equation system, which can be represented in the form[2]

$$\begin{aligned} \frac{Eh^3}{12(1-\nu^2)} \nabla^2\nabla^2\omega - \nabla_k^2\hat{\varphi} &= 0, \\ \frac{1}{Eh} \nabla^2\nabla^2\hat{\varphi} + \nabla_k^2\omega &= 0, \end{aligned} \tag{X.14}$$

where E is Young's modulus; ν is Poisson's ratio; h is the thickness of the shell. Operators ∇^2 and ∇_k^2 in isothermic coordinates ρ and θ have the form /706

$$\begin{aligned} \nabla^2 &= \frac{1}{\mathcal{H}^2}\left(\frac{\partial^2}{\partial\varrho^2} + \frac{\partial^2}{\partial\theta^2}\right), \\ \nabla_k^2 &= \frac{1}{\mathcal{H}^2}\left[\frac{\partial}{\partial\varrho}\left(\frac{1}{R_\theta}\cdot\frac{\partial}{\partial\varrho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R_{\varrho\theta}}\cdot\frac{\partial}{\partial\varrho}\right) + \frac{\partial}{\partial\varrho}\left(\frac{1}{R_{\varrho\theta}}\cdot\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R_\varrho}\cdot\frac{\partial}{\partial\theta}\right)\right], \end{aligned} \tag{X.15}$$

where $H^2(\rho, \theta)$ is the coefficient of form (X.12).

---

[1] See A. L. Gol'denveyzer [1], equations (20.8) and (20.9), p. 361.

[2] Since in the following the polar coordinates (r, ϕ) often will be satisfied, to avoid confusion, the symbol $\phi(\rho, \theta)$ is introduced for the stress functions.

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The values

$$\frac{1}{R_\varrho} = -\frac{D}{\mathcal{H}^2}, \qquad \frac{1}{R_\theta} = -\frac{D^{**}}{\mathcal{H}^2} \tag{X.16}$$

represent the curvatures of the normal cross sections of the middle area of the shell, extended along the coordinate lines ρ = const and θ = const, respectively.

The value

$$\frac{1}{R_{\varrho\theta}} = \frac{D^*}{\mathcal{H}^2} \tag{X.17}$$

characterizes the degree of nonself-conjugation of the isothermic coordinates ρ and θ. D, D* and D** are coefficients of form (X.3), written in isothermic coordinates ρ and θ. Deflection of the middle area of the shell $w(\rho, \theta)$ is assumed to be constant during the deflection of the shell in the direction of the negative normal.

The components of deflection deformation are determined through deflection $w(\rho, \theta)$ by formulas

$$\begin{aligned}
\varkappa_1 &= \frac{1}{\mathcal{H}} \cdot \frac{\partial}{\partial \varrho}\left(\frac{1}{\mathcal{H}} \cdot \frac{\partial w}{\partial \varrho}\right) + \frac{1}{\mathcal{H}^3} \cdot \frac{\partial \mathcal{H}}{\partial \theta} \cdot \frac{\partial w}{\partial \theta}, \\
\varkappa_2 &= \frac{1}{\mathcal{H}} \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\mathcal{H}} \cdot \frac{\partial w}{\partial \theta}\right) + \frac{1}{\mathcal{H}^3} \cdot \frac{\partial \mathcal{H}}{\partial \varrho} \cdot \frac{\partial w}{\partial \varrho}; \\
\tau^{(1)} &= \frac{1}{\mathcal{H}} \cdot \frac{\partial}{\partial \varrho}\left(\frac{1}{\mathcal{H}} \cdot \frac{\partial w}{\partial \theta}\right) - \frac{1}{\mathcal{H}^3} \cdot \frac{\partial \mathcal{H}}{\partial \theta} \cdot \frac{\partial w}{\partial \varrho}, \\
\tau^{(2)} &= -\frac{1}{\mathcal{H}} \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\mathcal{H}} \cdot \frac{\partial w}{\partial \varrho}\right) + \frac{1}{\mathcal{H}^3} \cdot \frac{\partial \mathcal{H}}{\partial \varrho} \cdot \frac{\partial w}{\partial \theta}.
\end{aligned} \tag{X.18}$$

The moments are determined by formulas

$$\begin{aligned}
G_\varrho &= -D(\varkappa_1 + \nu\varkappa_2), \qquad G_\theta = -D(\varkappa_2 + \nu\varkappa_1); \\
H_\varrho &= D(\tau^{(1)} + \nu\tau^{(2)}), \qquad H_\theta = D(\tau^{(2)} + \nu\tau^{(1)}),
\end{aligned} \tag{X.19}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$. is cylindrical rigidity.

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Components T_ρ, T_θ, S_ρ and S_θ are found through stress function $\Phi(\rho, \theta)$ by formulas /707

$$\begin{aligned}
T_\rho &= \frac{1}{\mathcal{H}}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\mathcal{H}}\cdot\frac{\partial\hat{\varphi}}{\partial\theta}\right)+\frac{1}{\mathcal{H}^3}\cdot\frac{\partial\mathcal{H}}{\partial\rho}\cdot\frac{\partial\hat{\varphi}}{\partial\rho},\\
T_\theta &= \frac{1}{\mathcal{H}}\cdot\frac{\partial}{\partial\rho}\left(\frac{1}{\mathcal{H}}\cdot\frac{\partial\hat{\varphi}}{\partial\rho}\right)+\frac{1}{\mathcal{H}^3}\cdot\frac{\partial\mathcal{H}}{\partial\theta}\cdot\frac{\partial\hat{\varphi}}{\partial\theta};\\
S_\rho &= -\frac{1}{\mathcal{H}}\cdot\frac{\partial}{\partial\theta}\cdot\left(\frac{1}{\mathcal{H}}\cdot\frac{\partial\hat{\varphi}}{\partial\rho}\right)+\frac{1}{\mathcal{H}^3}\cdot\frac{\partial\mathcal{H}}{\partial\rho}\cdot\frac{\partial\hat{\varphi}}{\partial\theta},\\
S_\theta &= \frac{1}{\mathcal{H}}\cdot\frac{\partial}{\partial\rho}\left(\frac{1}{\mathcal{H}}\cdot\frac{\partial\hat{\varphi}}{\partial\theta}\right)-\frac{1}{\mathcal{H}^3}\cdot\frac{\partial\mathcal{H}}{\partial\theta}\cdot\frac{\partial\hat{\varphi}}{\partial\rho}.
\end{aligned} \tag{X.20}$$

Shear forces are determined through the components (X.19) found by formulas

$$\begin{aligned}
Q_\rho &= \frac{1}{\mathcal{H}^2}\left[\frac{\partial}{\partial\rho}(\mathcal{H}G_\rho)-\frac{\partial\mathcal{H}}{\partial\theta}H_\rho+\frac{\partial}{\partial\theta}(\mathcal{H}H_\theta)-\frac{\partial\mathcal{H}}{\partial\rho}G_\theta\right],\\
Q_\theta &= -\frac{1}{\mathcal{H}^2}\left[\frac{\partial}{\partial\rho}(\mathcal{H}H_\rho)+\frac{\partial\mathcal{H}}{\partial\theta}G_\rho-\frac{\partial}{\partial\theta}(\mathcal{H}G_\theta)-\frac{\partial\mathcal{H}}{\partial\rho}H_\theta\right].
\end{aligned} \tag{X.21}$$

Equation system (X.14), by introducing the complex function

$$\Phi(\rho,\theta)=\frac{Eh^3}{\sqrt{12(1-\nu^2)}}w(\rho,\theta)+i\hat{\varphi}(\rho,\theta), \tag{X.22}$$

can be reduced to the equation

$$\nabla^2\nabla^2\Phi+i\frac{\sqrt{12(1-\nu^2)}}{h}\nabla_k^2\Phi=0. \tag{X.23}$$

Function (X.22) is a particular case of complex transformation introduced to the theory of shells by V. V. Novozhilov [1].

Basic Equations in Integral Form. We will introduce, following the example of I. N. Vekua [2], instead of the independent variables ρ and θ, new and conjugate independent variables:

$$\xi=\rho+i\theta;\qquad \eta=\rho-i\theta. \tag{X.24}$$

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Omitting the intermediate calculations[1], we obtain, instead of equation (X.23), its equivalent integral equation of the Volterra type:

$$V(\xi,\eta)+ie\left[\int_0^{\xi}K_1(\xi,\eta,t)V(t,\eta)\,dt+\int_0^{\eta}K_2(\eta,\xi,\tau)V(\xi,\tau)\,d\tau+\right.$$
$$\left.+\int_0^{\xi}\left\{\int_0^{\eta}K(\xi,\eta,t,\tau)V(t,\tau)\,d\tau\right\}dt\right]=\int_0^{\xi}\lambda_1(t,\eta)\psi_0(t)\,dt+ \qquad (X.25)$$
$$+\int_0^{\eta}\lambda_1^*(\xi,\tau)\psi_0^*(\tau)\,d\tau+\psi_1(\xi)+\psi_1^*(\eta).$$

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where

$$e=\frac{\sqrt{3(1-\nu^2)}}{8h}.$$

The function $V(\xi,\eta)=\Phi\left(\frac{\xi+\eta}{2},\ i\frac{\eta-\xi}{2}\right)$ is some regular solution of equation (X.23) in region $\Sigma+\bar{\Sigma}$ (Figure X.6), i.e., the solution is continuous along with its partial derivatives of the order $\leqslant 4$ in the above-specified region ($\xi\in\Sigma$; $\eta\in\bar{\Sigma}$). The functions $\psi_0(\xi)$, $\psi_0^*(\eta)$, $\psi_1(\xi)$ and $\psi_1^*(\eta)$ are holomorphic and should be determined from the corresponding boundary conditions and conditions at infinity. These functions, for the given function V(ξ, h), are determined uniquely if they satisfy conditions

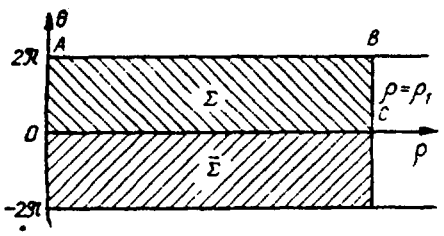


Figure X.6.

$$\psi_0(0)=\overline{\psi_0^*(0)};$$
$$\psi_1(0)=\overline{\psi_1^*(0)}. \qquad (X.26)$$

The functions $\lambda(\xi,\eta)$; $\lambda_1(\xi,\eta)$; $\lambda_1^*(\xi,\eta)$; $K(\xi,\eta,t,\tau)$; $K_1(\xi,\eta,t)$ and $K_2(\eta,\xi,\tau)$ in equation (X.25) are known and depend only on the form of shell and shape of hole:

$$\lambda(\xi,\eta)=\mathcal{H}^2(\rho,\theta)=\mathcal{H}^2\left(\frac{\xi+\eta}{2}+\frac{\eta-\xi}{2}\right);$$
$$\lambda_1(\xi,\eta)=\int_0^{\eta}\lambda(\xi,\tau)\,d\tau;\quad \lambda_1^*(\xi,\eta)=\int_0^{\xi}\lambda(t,\eta)\,dt;$$

[1] See G. N. Savin [3].

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$$K(\xi, \eta, t, \tau) = \lambda(t, \tau) B(t, \tau) + \int_{\tau}^{\eta} \left[D(t, \tau) \lambda(t, \tau_1) - A(t, \tau) \frac{\partial \lambda(t, \tau)}{\partial t} \right] d\tau_1 +$$

$$+ \int_{t}^{\xi} \left[E(t, \tau) \lambda(t_1, \tau) - C(t, \tau) \frac{\partial \lambda(t_1, \tau)}{\partial \tau} \right] dt_1 +$$

$$+ F(t, \tau) \int_{t}^{\xi} dt_1 \int_{\tau}^{\eta} \lambda(t_1, \tau_1) d\tau_1;$$

$$K_1(\xi, \eta, t) = C(t, \eta) \int_{t}^{\xi} \lambda(t_1, \eta) dt_1;$$

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$$K_2(\eta, \xi, \tau) = A(\xi, \tau) \int_{\tau}^{\eta} \lambda(\xi, \tau_1) d\tau_1;$$

$$\mathcal{H}^2 \nabla_k^2 \Phi = \frac{\partial^2 (AV)}{\partial \xi^2} + \frac{\partial^2 (BV)}{\partial \xi \partial \eta} + \frac{\partial^2 (CV)}{\partial \eta^2} +$$

$$+ \frac{\partial (DV)}{\partial \xi} + \frac{\partial (EV)}{\partial \eta} + FV.$$

By way of example, we introduce integral equation (X.25) for a spherical shell weakened by some hole:

$$V(\xi, \eta) = \psi_1(\xi) + \psi_1^*(\eta) + \int_0^{\xi} \lambda_1(t, \eta) \psi_1(t) dt +$$

$$+ \int_0^{\eta} \lambda_1^*(\xi, \tau) \psi_0^*(\tau) d\tau - i \frac{\sqrt{3(1-\nu^2)}}{2} \cdot \frac{1}{R_0 h} \int_0^{\xi} \left[\int_0^{\eta} \lambda(t, \tau) V(t, \tau) d\tau \right] dt. \qquad (X.27)$$

Here R_0 is the radius of the middle surface of the spherical shell. The function $\lambda(\xi, \eta) = R_0^2 e^{2\rho} \omega'(\zeta) \overline{\omega'(\zeta)}$, where the values

$$\rho = \frac{\xi + \eta}{2}; \quad \theta = i \frac{\eta - \xi}{2}; \quad \zeta = e^{\rho + i\theta}.$$

should be introduced into the right hand side.

Conditions of Uniqueness of Displacements. Boundary Conditions. The function $\Phi(\rho, \theta)$, which is the solution of equation (X.23) or equation (X.25), should also satisfy the vector equation

$$\oint_{\Gamma} \left\{ \frac{\omega}{2} \vec{e}_{\rho} + \varepsilon_{\theta} \vec{e}_{\theta} + [\vec{e}_{\theta}, \vec{\Omega}] \right\} \mathcal{H} d\theta = 0, \qquad (X.28)$$

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which insures the uniqueness of the displacement vector, where

$$\vec{\Omega} = \vec{\Omega}^0 + \int_{\rho_0 \rho} \vec{\Omega}_\theta d\theta; \quad \vec{\Omega}_\theta = \mathcal{H}(\varkappa_1 \vec{e}_\varrho + \tau^{(2)} \vec{e}_\theta + \zeta^* \vec{e}_n);$$

$$\zeta^* + \frac{1}{\mathcal{H}^2}\left(\varepsilon_\varrho \frac{\partial \mathcal{H}}{\partial \varrho} + \frac{1}{2} \cdot \frac{\partial(\mathcal{H}\omega)}{\partial \theta} - \frac{\partial(\mathcal{H}\varepsilon_\theta)}{\partial \varrho} + \frac{\omega}{2} \cdot \frac{\partial \mathcal{H}}{\partial \theta}\right).$$

Here ζ^* is not a complex variable. The expression under the integral in (X.28) is a function of deformation components ε_ρ, ε_θ, ω, $\varkappa_1$ and $\tau^{(2)}$ and vector radius $\vec{r}(p)$ of point P on the contour of the given hole.

If in expression (X.28) under the integral deformation components ε_ρ, ε_θ, ω, $\varkappa_1$ and $\tau^{(2)}$ are expressed in accordance with known formulas[1] through the function $\Phi(\rho, \theta)$, then we find three scalar relations. By substituting into these relations the values Φ (X.22) and separating the real and imaginary parts, we find for functions $\phi(\rho, \theta)$ and $w(\rho, \theta)$ six scalar relations that insure the uniqueness of displacement vector $\vec{u}$. /710

Thus, the solution of the stated problem is reduced to the integration of equation (X.23); here the solution must satisfy the conditions of uniqueness of displacements, specifically vector equation (X.28). Moreover, the solution must satisfy the boundary conditions on the contour of the hole as well as the so-called conditions at infinity, i.e., conditions at sufficiently distant parts of the shell from the hole, beyond zone Σ^*, bounded by contour L^*.

On contour Γ of the hole, i.e., when $\rho = \rho_0$, various conditions can be given: either external forces, displacements of points of the contour, external forces on one part of the contour and displacements on the other part of the contour, etc. We will consider the case where external forces $F_1(\rho_0, \theta)$, $F_2(\rho_0, \theta)$, $F_3(\rho_0, \theta)$ and $F_4(\rho_0, \theta)$ are given on the contour. External forces applied on the contour of hole Γ should be equal to the adduced forces:

$$\tilde{T} = F_1(\varrho_0, \theta); \quad \tilde{S} = F_2(\varrho_0, \theta); \quad \tilde{G} = F_3(\varrho_0, \theta); \quad \tilde{Q} = F_4(\varrho_0, \theta), \qquad (X.29)$$

where

$$\tilde{T} = T_\varrho^* + \frac{\sin 2\chi}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) H_\varrho^*; \quad \tilde{G} = G_\varrho^*; \quad \tilde{S} = S_\varrho^* + \frac{H_\varrho^*}{R_\Gamma}; \quad \tilde{Q} = Q_\varrho^* - \frac{\partial H_\varrho^*}{\partial S_\Gamma};$$

[1]See A. L. Gol'venveyzer [1].

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χ is the angle between contour Γ of the hole and the line on the middle surface of the shell whose normal curvature is $1/R_1$; R_Γ is the radius of curvature of the normal cross section of the middle surface along contour Γ. The derivative $\partial H_\rho^*/\partial S_\Gamma$ in $\tilde{Q}$ is found on arc S_Γ of contour Γ of the hole.

To solve the problem approximately the formulas for $\tilde{T}$ and $\tilde{S}$ can be simplified considerably, specifically by disregarding the terms of the order $1/R$ in them and, recalling formulas (X.8), by representing boundary conditions (X.29) in the form[1]

$$T_\varrho = F_1(\varrho_0, \theta) - T_\varrho^0; \quad S_{\varrho\theta} = F_2(\varrho, \theta) - S_{\varrho\theta}^0;$$
$$Q_\varrho - \frac{\partial H_{\varrho\theta}}{\partial S_\Gamma} = F_4(\varrho_0, \theta) - Q_\varrho^0 + \frac{\partial H_{\varrho\theta}^0}{\partial S_\Gamma}; \quad G_\varrho = F_3(\varrho_0, \theta) - G_\varrho^0. \tag{X.30}$$

Often the basic stress state (X.7) is momentless; then, assuming in (X.30) $H_{\rho\theta}^0 = Q_\rho^0 = G_\rho^0 = 0$, we obtain the boundary conditions

$$T_\varrho = F_1(\varrho_0, \theta) - T_\varrho^0; \quad S_{\varrho\theta} = F_2(\varrho_0, \theta) - S_{\varrho\theta}^0;$$
$$\tilde{\varrho} = F_4(\varrho_0, \theta); \quad G_\varrho = F_3(\varrho_0, \theta). \tag{X.31}$$

If the contour of the hole is free of external forces, then we must assume /711 in (X.31) $F_1 = F_2 = F_3 = F_4 = 0$:

$$T_\varrho = -T_\varrho^0; \quad S_{\varrho\theta} = -S_{\varrho\theta}^0; \quad G_\varrho = \tilde{Q}_\varrho = 0. \tag{X.32}$$

We will examine the boundary conditions for a hole in a shell, which is located under the conditions of the basic stress state caused by internal hydrostatic pressure $p_0 = \text{const}$. Here we will make the following assumptions: the hole of the shell is tightly covered by a specially constructed roof, which assures uniformity of internal pressure in the storage tank; to the contour of the hole this roof transmits from the internal pressure imparted to it only the shear force, and does not create any other force action on the region of the shell near the hole.

---

[1]Here, and in the following, we will use the basic assumptions of the theory of sloping shells and discard in (X.29) the terms with factors $1/R$. Moreover, we will denote $H_\rho = -H_\theta = H_{\rho\theta}$; $S_\rho = -S_\theta = -S_{\rho\theta}$.

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The boundary conditions in this case will be of the form

$$\widetilde{Q}_\rho = f^{(*)}(\theta);$$
$$T_\rho = -T_\rho^0; \quad S_{\rho\theta} = -S_{\rho\theta}^0; \quad G_\rho = -G_\rho^0. \tag{X.33}$$

The function $f^{(*)}(\theta)$ in conditions (X.33) characterizes the shear forces, which are transmitted by the roof to the contour of the hole. Generally speaking $f^{(*)}(\theta)$ can be an arbitrary integrated function, which should satisfy only one condition, specifically the following: that the resultant of these shear forces on contour Γ be equal to the force

$$Fp_0 = \oint f^{(*)}(\theta)\, ds = \int_0^{2\pi} f^{(*)}(\theta)\, \mathcal{H}(\rho_0, \theta)\, d\theta, \tag{X.34}$$

where F is the area of the hole in the shell; p_0 = const is internal hydrostatic pressure.

The boundary conditions for an arbitrary, and not simply sloping, shell, weakened by a hole whose edge is reinforced by a thin elastic ring, were found by N. P. Fleyshman [2, 3]. We will not write them out in the general case, but we will write them in the following section for the case of a sloping shell.

Let us turn now to the conditions "at infinity." Since components T_ρ, T_θ, ..., G_θ (X.13) by measure of distance from contour Γ of the hole, damped rapidly, the functions $w(\rho, \theta)$ and $\hat{\phi}(\rho, \theta)$ found from equation (X.23) and from the boundary conditions in one of forms (X.29)-(X.33) should satisfy these conditions, i.e., for $\rho \to \infty$, they should satisfy the following conditions

$$T_\rho = 0; \quad S_{\rho\theta} = 0; \quad G_\rho = 0 \text{ и } \widetilde{Q}_\rho = 0. \tag{X.35}$$

We will note that equations (X.23) and the relations related to it are found under the condition that the gauss curvature K of the middle surface is equal to zero. It is clear that the solution thus found for the problem of stress concentration in shells for which $K \neq 0$ will be more accurate as the zone Σ^* diminishes, i.e., the smaller the hole in the shell. However, as demonstrated /712 by theoretical and experimental studies, such as those conducted under the supervision of V. Z. Vlasov at TsNIIPS [Central Scientific Research Institute of Industrial Structures] [1], the approximate solutions based on the theory of sloping shells are in good agreement with the experiment for shells for which the ascent index f of the shell to its smallest dimension in the plate a will be $f/a < 1/5$.

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§2. Solution Methods

Concerning the Separation of Variables. We will notice that the solution $\Phi(\rho, \theta)$ (X.22) of equation (X.23) must be periodic with respect to θ, i.e.,

$$\Phi(\varrho, \theta) = \Phi(\varrho, \theta + 2\pi), \tag{X.36}$$

and therefore it is natural to represent it in the form of the following Fourier series:

$$\Phi(\varrho, \theta) = \sum_{k=0}^{\infty} [f_k(\varrho) \cos k\theta + g_k(\varrho) \sin k\theta]. \tag{X.37}$$

However, it is obvious after direct substitution of function (X.37) into (X.23) that for the arbitrary coordinate system (ρ, θ), given by function $\omega(\zeta)$ (X.11), that it is not possible to obtain the solution $\Phi(\rho, \theta)$ of equation (X.23) in the form (X.37) for all shells[1]. An exception is a spherical shell for particular forms of coordinate systems, and therefore we will proceed to the analysis of the possibility of separating the variables into equations for a spherical shell.

For a spherical shell

$$\frac{1}{R_\varrho} = \frac{1}{R_\theta} = \frac{1}{R}; \quad \frac{1}{R_{\varrho\theta}} = 0. \tag{X.38}$$

By substituting (X.38) into (X.15) and (X.23) we find the equation

$$\nabla^2\nabla^2\Phi + i\frac{\sqrt{12(1-\nu^2)}}{Rh}\nabla^2\Phi = 0, \tag{X.39}$$

the solution of which can be represented in the form of the sum

$$\Phi = \Phi_1 + \Phi_2, \tag{X.40}$$

where Φ_1 is the solution of Laplace's equation $\nabla^2\Phi_1 = 0$; Φ_2 is the solution of Helmholtz's equation

$$\nabla^2\Phi_2 + i\frac{\sqrt{12(1-\nu^2)}}{Rh}\Phi_2 = 0. \tag{X.41}$$

[1] By substituting (X.37) into (X.23) we obtain an infinite system of ordinary differential equations, which cannot be represented in convenient form.

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The variables in Laplace's equation are separated for any coordinate system given by mapping function $\omega(\zeta)$ (X.11), while the variables in Helmholtz's equation are separated only for polar and elliptical coordinate systems; the solution in the latter systems is represented through the Mathieu function. Certain problems for a spherical shell in an elliptical coordinate system are examined in the works of G. A. Van Fo Fy [1-4], G. N. Savin and G. A. Van Fo Fy [1, 2]. /713

The "Perturbation of Boundary Form" Method. The use of the method of separating the variables, as indicated above, can be successful only in an individual and extremely limited case, and therefore, for the solution of the examined problem, it is convenient to use approximation methods. In this section we will examine the "perturbations of boundary form" approximation method in the form first proposed by A. N. Guz' [2-7] for elliptical, square, and triangular holes with rounded corners. The analogous method for the plane problem of elasticity theory of an anisotropic medium was proposed earlier by S. G. Lekhnitskiy [1]. This method is extended in the work of A. N. Guz' [5] to double-connected regions, while in another work [8], several problems concerning stress concentration near certain curvilinear holes in a round cylindrical shell were examined. In the work of G. N. Savin and A. N. Guz' [1], the "perturbations of boundary form" method is used for the solution of problems of stress concentration near arbitrary holes whose contours have no angular points.

For convenience we will substitute the positive directions w, G_ρ, G_θ, Q_ρ and Q_θ by the opposite, and equation (X.23) we will relate to dimensionless coordinates. Then this equation becomes

$$\nabla^2\nabla^2\Phi - i\varkappa^2 R\nabla_k^2\Phi = 0, \qquad (X.42)$$

where

$$\Phi = w + in\hat{\varphi}; \quad n = \frac{\sqrt{12(1-\nu)}}{Eh^2}; \quad \varkappa = r_0\sqrt[4]{\frac{12(1-\nu^2)}{R^2h^2}};$$

r_0 is a real number characterizing the dimensions of the hole; R is the least radius of curvature.

We will examine the plane of variables to which the middle surface of the shell is related. In Figure X.7 (x, y) is a rectangular coordinate system; (r, θ) is a polar coordinate system[1]; (ρ, ϑ) is an orthogonal coordinate system;

[1]Here all coordinates are dimensionless, related to some magnitude r_0, which is characteristic of the examined hole. Thus, in the case of a round hole r_0 is the radius of the hole in plane P (Figure X.5), in the case of an elliptical hole $r_0 = (a + b)/2$, where a and b are the semiaxes of the ellipse (see Figures X.8 and X.9).

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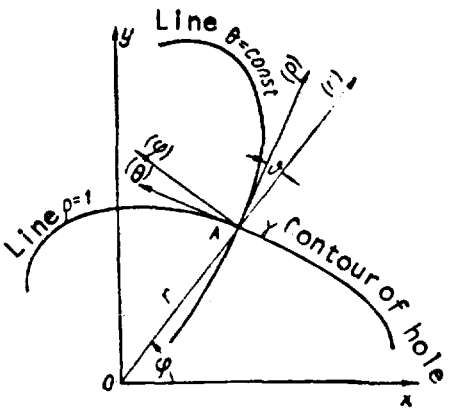


Figure X.7.

γ is the contour of the hole in plane P of variables (ρ , θ) to which the middle surface of the shell is related. Contour Γ in the shell corresponds to contour γ on plane P (Figure X.5). The (ρ) axis is directed along the external normal to contour γ (Figure X.7); ϑ is the angle by which coordinate system (ρ)A(θ) is rotated in relation to polar coordinate system (r)A(ϕ). We will introduce the formulas for the stress components and deformation state, expressed through function Φ in the polar coordinate system (r, ϕ). In accordance with the assumed positive directions, we obtain for displacements, forces, and moments

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$$T_r = \frac{1}{nr_0^2}\left(\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}\right)\operatorname{Im}\Phi(r,\varphi);\quad T_\varphi = \frac{1}{nr_0^2}\cdot\frac{\partial^2}{\partial r^2}\operatorname{Im}\Phi(r,\varphi);$$

$$S_{r\varphi} = -\frac{1}{nr_0^2}\cdot\frac{\partial^2}{\partial r\partial\varphi}\cdot\frac{\operatorname{Im}\Phi(r,\varphi)}{r};\quad G_r = -\frac{D}{r_0^2}\left[(1-\nu)\frac{\partial^2}{\partial r^2}+\nu\nabla^2\right]\operatorname{Re}\Phi(r,\varphi);$$

$$G_\varphi = -\frac{D}{r_0^2}\left[\nabla^2-(1-\nu)\frac{\partial^2}{\partial r^2}\right]\operatorname{Re}\Phi(r,\varphi);\quad H_{r\varphi} = -D\frac{1-\nu}{r_0^2}\cdot\frac{\partial^2}{\partial r\partial\varphi}\cdot\frac{\operatorname{Re}\Phi(r,\varphi)}{r}; \tag{X.43}$$

$$\tilde{Q}_r = -\frac{D}{r_0^3}\left[\frac{\partial}{\partial r}\nabla^2+\frac{1-\nu}{\tau}\cdot\frac{\partial^3}{\partial r\partial\varphi^2}\tau\cdot\frac{1}{r}\right]\operatorname{Re}\Phi(r,\varphi).$$

The variables u and v are found by way of integration of the following equation system:

$$\frac{\partial u}{\partial r} = -r_0\frac{\operatorname{Re}\Phi(r,\varphi)}{R_r}+\frac{1}{Ehnr_0}\left[\nabla^2-(1+\nu)\frac{\partial^2}{\partial r^2}\right]\operatorname{Im}\Phi(r,\varphi);$$

$$\frac{1}{r}\cdot\frac{\partial v}{\partial\varphi}+\frac{u}{r} = -r_0\frac{\operatorname{Re}\Phi(r,\varphi)}{R_\varphi}+\frac{1}{Ehnr_0}\left[(1+\nu)\frac{\partial^2}{\partial r^2}-\nu\nabla^2\right]\operatorname{Im}\Phi(r,\varphi); \tag{X.44}$$

$$\frac{1}{r}\cdot\frac{\partial u}{\partial\varphi}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = 2r_0\frac{\operatorname{Re}\Phi(r,\varphi)}{R_{r\varphi}}-2\frac{1+\nu}{Ehnr_0}\cdot\frac{\partial^2}{\partial r\partial\varphi}\frac{\operatorname{Im}\Phi(r,\varphi)}{r}.$$

Let the contour Γ of the hole in the shell (Figure X.5) have a form such that the function

$$z = \omega(\zeta);\quad \omega(\zeta) = \zeta+\varepsilon f(\zeta)\quad (z = re^{i\varphi};\quad \zeta = \varrho e^{i\theta}) \tag{X.45}$$

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maps conformally the infinite plane (ζ) with a round hole of unit radius on the infinite plane P with a hole whose contour is γ (Figure X.5).

The function $f(\zeta)$ depends on the form of Γ ($\varepsilon \ll 1$), and the roots of the equation $1 + \varepsilon f'(\zeta) = 0$ should lie in plane ζ within the circle of unit radius. Under these limitations the function $\omega(\zeta)$ maps conformally the exterior of the unit circle onto the exterior of the examined hole.

The values r, ϕ and ϑ (Figure X.7) on coordinate line ϑ = const and, in particular, on contour γ of the hole ($\rho = 1$) will be[1] /715

$$r = \sqrt{\zeta\bar{\zeta} + \varepsilon\left[\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}\right] + \varepsilon^2 f(\zeta)\overline{f(\zeta)}}\Big|_{\varrho=\text{const}};$$

$$\varphi = \arctan\frac{y}{x} = \arctan\frac{\sin\theta + \varepsilon\dfrac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}}{\cos\theta + \varepsilon\dfrac{f(\zeta) + \overline{f(\zeta)}}{2i\varrho}}\Bigg|_{\varrho=\text{const}}; \qquad (X.46)$$

$$e^{i\vartheta} = \frac{\zeta}{\varrho}\cdot\frac{\overline{\omega(\zeta)}}{|\omega(\zeta)|}\cdot\frac{\omega'(\zeta)}{|\omega'(\zeta)|}.$$

In order to have in each of the series approximations a boundary problem with separable variables, it is necessary that the solution of equation (X.42) in polar coordinate system (r, ϕ) yield to representation in the form

$$\Phi(r, \varphi) = \sum_{k=0}^{\infty}\left[f_k(r)\cos k\varphi + g_k(r)\sin k\varphi\right]. \qquad (X.47)$$

We will represent the solution of equation (X.42) in polar coordinate system (r, ϕ) and components of the stress and deformation states in curvilinear orthogonal coordinate system[2] (ρ, ϑ) in the form of series with respect to parameter ε:

$$\Phi(r, \varphi) = \sum_{j=0}^{\infty}\varepsilon^j\Phi_j(r, \varphi); \qquad (X.48)$$

$$T_\varrho = \sum_{j=0}^{\infty}\varepsilon^j T_\varrho^{(j)}; \quad T_\vartheta = \sum_{j=0}^{\infty}\varepsilon^j T_\vartheta^{(j)}; \quad S_{\varrho\vartheta} = \sum_{j=0}^{\infty}\varepsilon^j S_{\varrho\vartheta}^{(j)};$$

$$G_\varrho = \sum_{j=0}^{\infty}\varepsilon^j G_\varrho^{(j)}; \quad G_\vartheta = \sum_{j=0}^{\infty}\varepsilon^j G_\vartheta^{(j)}; \quad H_{\varrho\vartheta} = \sum_{j=0}^{\infty}\varepsilon^j \bar{H}_{\varrho\vartheta}^{(j)};$$

[1] See Chapter VI, where the derivation of these formulas is given.
[2] The coordinate line $\rho = 1$ coincides with contour γ of the hole on plane P (Figure X.5).

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$$\tilde{Q}_\varrho = \sum_{j=0}^{\infty} \varepsilon^{(j)} \tilde{Q}_\varrho^{(j)}; \quad w = \sum_{j=0}^{\infty} \varepsilon^j w_j;$$

$$\frac{\partial w}{\partial n} = \sum_{j=0}^{\infty} \varepsilon^{(j)} \left(\frac{\partial w}{\partial n}\right)^{(j)}; \quad u_\varrho = \sum_{j=0}^{\infty} \varepsilon^j u_\varrho^{(j)}; \tag{X.49}$$

$$u_\vartheta = \sum_{j=0}^{\infty} \varepsilon^j u_\vartheta^{(j)}.$$

After finding the solution of equation (X.42) in the form (X.47), we can find, in accordance with formulas (X.43) and (X.44), the components of the stress and deformation states in the polar coordinate system (r, ϕ). To determine the corresponding components in the curvilinear coordinate system (ρ, θ), we will use the formulas for the transformation of the components of the stress and deformation states for the case of the rotation of coordinate system (ρ, θ) in relation to coordinates (r, ϕ) by angle ϑ (Figure X.7): /716

$$\begin{aligned}
T_\varrho &= T_r \cos^2\vartheta + T_\varphi \sin^2\vartheta + 2S_{r\varphi} \sin\vartheta\cos\vartheta;\\
T_\vartheta &= T_r \sin^2\vartheta + T_\varphi \cos^2\vartheta - 2S_{r\varphi} \sin\vartheta\cos\vartheta,\\
S_{\varrho\vartheta} &= (T_\varphi - T_r)\sin\vartheta\cos\vartheta + S_{r\varphi}(\cos^2\vartheta - \sin^2\vartheta);\\
G_\varrho &= G_r \cos^2\vartheta + G_\vartheta \sin^2\vartheta + 2H_{r\varphi}\sin\vartheta\cos\vartheta;\\
G_\vartheta &= G_r \sin^2\vartheta + G_\varphi \cos^2\vartheta - 2H_{r\varphi}\sin\vartheta\cos\vartheta;\\
H_{\varrho\vartheta} &= (G_\varphi - G_r)\sin\vartheta\cos\vartheta + H_{r\varphi}(\cos^2\vartheta - \sin^2\vartheta);\\
\tilde{Q}_\varrho &= Q_r\cos\vartheta + Q_\varphi \sin\vartheta + \frac{\partial H_{\varrho\vartheta}}{\partial s};\\
u_\varrho &= u\cos\vartheta + v\sin\vartheta;\\
u_\vartheta &= -u\sin\vartheta + v\cos\vartheta;\\
\frac{\partial w}{\partial n} &= \frac{\partial w}{\partial r}\cos\vartheta + \frac{\partial w}{\partial\varphi}\cdot\frac{\sin\vartheta}{r}\\
&\quad (s - \text{arc}).
\end{aligned} \tag{X.50}$$

It is necessary to substitute in the expressions obtained the values r, ϕ and ϑ from (X.46).

By substituting (X.48) into equation (X.42), after first relating it to polar coordinate system (r, ϕ), we obtain the infinite chain of equations

$$\nabla^2\nabla^2\Phi_j(r,\varphi) - i\varkappa^2 R\nabla_k^2\Phi_j(r,\varphi) = 0. \tag{X.51}$$

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In accordance with (X.47) the solution of equation (X.51) becomes

$$\Phi_j(r, \varphi) = \sum_{k=0}^{\infty} f_{kj}(r)\cos k\varphi + g_{kj}(r)\sin k\varphi. \qquad (X.52)$$

Relations (X.46) are also expanded into series with respect to parameter ε, while expressions (X.50) are expanded into series as complex functions. By equating the coefficients for ε^j of the left and right sides of (X.50), we obtain

$$T_\varrho^{(j)} = T_r^{(j)} + \sum_{m=0}^{j-1}[L_1^{(j-m)}T_r^{(m)} + L_2^{(j-m)}(T_\varphi^{(m)} - T_r^{(m)}) + L_3^{(j-m)}S_{r\varphi}^{(m)}];$$

$$T_\theta^{(j)} = T_\varphi^{(j)} + \sum_{m=0}^{j-1}[L_1^{(j-m)}T_\varphi^{(m)} + L_2^{(j-m)}(T_r^{(m)} - T_\varphi^{(m)}) - L_3^{(j-m)}S_{r\varphi}^{(m)}];$$

$$S_{\varrho\theta}^{(j)} = S_{r\varphi}^{(j)} + \sum_{m=0}^{j-1}\left[(L_1^{(j-m)} - 2L_2^{(j-m)})S_{r\varphi}^{(m)} + \frac{L_3^{(j-m)}}{2}(T_\varphi^{(m)} - T_r^{(m)})\right]; \qquad /717$$

$$G_\varrho^{(j)} = G_r^{(j)} + \sum_{m=0}^{j-m}[L_1^{(j-m)}G_r^{(m)} + L_2^{(j-m)}(G_\varphi^{(m)} - G_r^{(m)}) + L_3^{(j-m)}H_{r\varphi}^{(m)}];$$

$$G_\theta^{(j)} = G_\varphi^{(j)} + \sum_{m=0}^{j-1}[L_1^{(j-m)}G_\varphi^{(m)} + L_2^{(j-m)}(G_2^{(m)} - G_\varphi^{(m)}) - L_3^{(j-m)}H_{r\varphi}^{(m)}];$$

$$H_{\varrho\theta}^{(j)} = H_{r\varphi}^{(j)} + \sum_{m=0}^{j-1}\left[(L_1^{(j-m)} - 2L_2^{(j-m)})H_{r\varphi}^{(m)} + \frac{L_3^{(j-m)}}{2}(G_\varphi^{(m)} - G_r^{(m)})\right];$$

$$\tilde{Q}_\varrho^{(j)} = \tilde{Q}_r^{(j)} - \frac{D}{r_0^3}\sum_{m=0}^{j-1}L_4^{(j-m)}\operatorname{Re}\Phi_m(\varrho, \theta);$$

$$w_j = \operatorname{Re}\Phi_j(\varrho, \theta) + \sum_{m=0}^{j-1}L_1^{(j-m)}\operatorname{Re}\Phi_m(\varrho, \theta);$$

$$\left(\frac{\partial w}{\partial n}\right)^{(j)} = \frac{1}{r_0}\cdot\frac{\partial}{\partial r}\operatorname{Re}\Phi_j(\varrho, \theta) + \frac{1}{r_0}\sum_{m=0}^{\infty}\left[L_5\frac{\partial}{\partial\varrho} + L_6\frac{1}{\varrho}\cdot\frac{\partial}{\partial\theta}\right]\operatorname{Re}\Phi_m(\varrho, \theta);$$

$$u_\varrho^{(j)} = u^{(j)} + \sum_{m=0}^{j-1}[L_5^{(j-m)}u^{(m)} + L_6^{(j-m)}v^{(m)}];$$

$$u_\theta^{(j)} = v^{(j)} + \sum_{m=0}^{j-1}[L_5^{(j-m)}v^{(m)} - L_6^{(j-m)}u^{(m)}].$$

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In order to find in (X.53) the values of the components of the stress and deformation states on the contour of the hole, it is necessary to assume in them that $\rho = 1$.

The values with the indices j and m found in the right hand sides of (X.53) are of the following form

$$T_r^{(m)} = \frac{1}{nr_0^2}\left(\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\theta^2} + \frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right)\operatorname{Im}\Phi_m(\varrho,\theta); \qquad T_\varphi^{(m)} = \frac{1}{nr_0^2}\cdot\frac{\partial^2}{\partial\varrho^2}\operatorname{Im}\Phi_m(\varrho,\theta);$$

$$S_{r\varphi}^{(m)} = -\frac{1}{nr_0^2}\cdot\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{\operatorname{Im}\Phi_m(\varrho,\theta)}{\varrho}; \quad G_r^{(m)} = -\frac{D}{r_0^2}\left[(1-\nu)\frac{\partial^2}{\partial\varrho^2} + \nu\nabla^2\right]\operatorname{Re}\Phi_m(\varrho,\theta); \tag{X.54}$$

$$G_\varphi^{(m)} = -\frac{D}{r_0^2}\left[\nabla^2 - (1-\nu)\frac{\partial^2}{\partial\varrho^2}\right]\operatorname{Re}\Phi_m(\varrho,\theta); \quad H_{r\varphi}^{(m)} = -D\frac{1-\nu}{r_0^2}\cdot\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{\operatorname{Re}\Phi_m(\varrho,\theta)}{\varrho}; \qquad /718$$

$$\tilde{Q}_r^{(m)} = -\frac{D}{r_0^3}\left[\frac{\partial}{\partial\varrho}\nabla^2 + \frac{1-\nu}{\varrho}\cdot\frac{\partial^2}{\partial\varrho\partial\theta^2}\cdot\frac{1}{\varrho}\right]\operatorname{Re}\Phi_m(\varrho,\theta).$$

Displacements $u^{(m)}$ and $v^{(m)}$ are determined by integrating the equation system

$$\frac{\partial u^{(m)}}{\partial\varrho} = -r_0\frac{\operatorname{Re}\Phi_m(\varrho,\theta)}{R_r'} + \frac{1}{Ehnr_0}\left[\nabla^2 - (1+\nu)\frac{\partial^2}{\partial\varrho^2}\right]\operatorname{Im}\Phi_m(\varrho,\theta);$$

$$\frac{1}{\varrho}\cdot\frac{\partial v^{(m)}}{\partial\theta} + \frac{u^{(m)}}{\varrho} = -r_0\frac{\operatorname{Re}\Phi_m(\varrho,\theta)}{R_\varphi'} + \frac{1}{Ehnr_0^2}\left[(1+\nu)\frac{\partial^2}{\partial\varrho^2} - \nu\nabla^2\right]\operatorname{Im}\Phi_m(\varrho,\theta); \tag{X.55}$$

$$\frac{1}{\varrho}\cdot\frac{\partial u^{(m)}}{\partial\theta} + \varrho\frac{\partial}{\partial\varrho}\left(\frac{v^{(m)}}{\varrho}\right) = 2r_0\frac{\operatorname{Re}\Phi_m(\varrho,\theta)}{R_{r\varphi}} - 2\frac{1+\nu}{Ehnr_0}\cdot\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{\operatorname{Im}\Phi_m(\varrho,\theta)}{\varrho},$$

where R_r', R_ϕ' and $R_{r\phi}'$ are equal to R_r, R_ϕ and $R_{r\phi}$, respectively, in which r is substituted by ρ and ϕ by θ. We will notice that (X.54) and (X.55) coincide with (X.43) and (X.44), respectively, if in the latter we substitute r by ρ and ϕ by θ, i.e., if we replace the letters r and ϕ in the corresponding formulas by ρ and θ.

In (X.53) there are six other differential operators $L_k^{(j-m)}$ (k = 1, 2, ..., ..., 6), the order of which is specified by the superscript. We will write in inverted form the operators $L_k^{(j-m)}$ which will be required for the solution of the problems in the zero, first and second approximations ($L_k^{(0)} \equiv 0$):

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$$L_1^{(1)} = \frac{\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)}{2\varrho}\cdot\frac{\partial}{\partial\varrho} + \left(\frac{f(\zeta) - \overline{f(\zeta)}}{2i\varrho}\cos\theta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}\sin\theta\right)\frac{\partial}{\partial\theta}; \quad L_2^{(1)} = 0;$$

$$L_1^{(2)} = \frac{[\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)]^2}{8\varrho^2}\cdot\frac{\partial^2}{\partial\varrho^2} + \frac{\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)}{\varrho^2}\left[\frac{f(\zeta) - \overline{f(\zeta)}}{4i}\cos\theta - \right.$$

$$\left. - \frac{f(\zeta) + \overline{f(\zeta)}}{4}\sin\theta\right]\frac{\partial^2}{\partial\varrho\partial\theta} + \frac{i[f^2(\zeta) + \overline{f^2(\zeta)}]\sin 2\theta - [f^2(\zeta) - \overline{f^2(\zeta)}]\sin 2\theta}{4i\varrho^2}\cdot\frac{\partial}{\partial\theta} -$$

$$- \frac{[\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)]^2}{8\varrho^3}\cdot\frac{\partial}{\partial\varrho} + \frac{2if(\zeta)\overline{f(\zeta)} - i[f^2(\zeta) + \overline{f^2(\zeta)}]\cos 2\theta - [f^2(\zeta) - \overline{f^2(\zeta)}]\sin 2\theta}{8i\varrho^3}\cdot\frac{\partial^2}{\partial\theta^2};$$

$$L_2^{(2)} = -\frac{1}{4}[f'(\zeta) - \overline{f'(\zeta)}]^2 - \frac{[\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)]^2 + 2\zeta\bar{\zeta}[\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)][f'(\zeta) - \overline{f'(\zeta)}]}{4\zeta^2\bar{\zeta}^2};$$

$$L_3^{(1)} = \frac{f'(\zeta) - \overline{f'(\zeta)}}{i} + \frac{\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)}{i\zeta\bar{\zeta}}; \qquad (X.56)$$

$$L_3^{(2)} = \frac{\zeta^2 f^2(\zeta) - \zeta^2\overline{f^2(\zeta)} + \zeta^2\bar{\zeta}^2[\overline{f'^2(\zeta)} - f'^2(\zeta)]}{2i\zeta^2\bar{\zeta}^2} + \left[\frac{\zeta\overline{f(\zeta)} - \bar{\zeta}f(\zeta)}{i\zeta\bar{\zeta}} + \right.$$

$$\left. + \frac{f'(\zeta) - \overline{f'(\zeta)}}{i}\right]\left[\frac{\zeta f(\zeta) + \bar{\zeta}\overline{f(\zeta)}}{2\varrho}\cdot\frac{\partial}{\partial\varrho} + \left(\frac{f(\zeta) - f(\zeta)}{2i\varrho}\cos\theta - \frac{f(\zeta) + \overline{f(\zeta)}}{2\varrho}\sin\theta\right)\frac{\partial}{\partial\theta}\right];$$

$$L_4^{(1)} = \left[L_1^{(1)}\frac{\partial}{\partial\varrho} + \frac{1}{2}L_3^{(1)}\frac{1}{\varrho}\cdot\frac{\partial}{\partial\theta}\right]\nabla^2 + \frac{1-\nu}{\varrho}\left[\left(\frac{\partial}{\partial\theta}L_1^{(1)} - \frac{f'(\zeta) + \overline{f'(\zeta)}}{2}\cdot\frac{\partial}{\partial\theta}\right)\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{1}{\varrho} + \right. \qquad /719$$

$$\left. + \frac{1}{2}\cdot\frac{\partial}{\partial\theta}L_3^{(1)}\left(\nabla^2 - 2\frac{\partial^2}{\partial\varrho^2}\right)\right]; \quad L_4^{(2)} = \left[\left(L_1^{(2)} - \frac{1}{2}L_2^{(2)}\right)\frac{\partial}{\partial\varrho} + \right.$$

$$\left. + \frac{1}{2}L_3^{(2)}\frac{1}{\varrho}\cdot\frac{\partial}{\partial\theta}\right]\nabla^2 + \frac{1-\nu}{\varrho}\left[\frac{3f'^2(\zeta) + 2f'(\zeta)\overline{f'(\zeta)} + 3\overline{f'^2(\zeta)}}{8}\cdot\frac{\partial}{\partial\theta} - \right.$$

$$\left. - \frac{f'(\zeta) + \overline{f'(\zeta)}}{2}\cdot\frac{\partial}{\partial\theta}L_1^{(1)} + \frac{\partial}{\partial\theta}(L_1^{(2)} - 2L_2^{(2)})\right]\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{1}{\varrho} + \frac{1}{2}\left[\frac{\partial}{\partial\theta}L_3^{(2)} - \right.$$

$$\left. - \frac{f'(\zeta) + \overline{f'(\zeta)}}{2}\cdot\frac{\partial}{\partial\theta}L_3^{(1)}\right]\left(\nabla^2 - 2\frac{\partial^2}{\partial\varrho^2}\right)\Bigg];$$

$$L_5^{(1)} = L_1^{(1)}; \quad L_6^{(1)} = \frac{1}{2}L_3^{(1)}; \quad L_5^{(2)} = L_1^{(2)} - \frac{1}{2}L_2^{(2)};$$

$$L_6^{(2)} = \frac{1}{2}L_3^{(2)}.$$

Using relation (X.46), we also expand the right hand sides of boundary conditions (X.29)-(X.33) into series with respect to ε.

The functions $\Phi_m(\rho, \theta)$ in (X.54) and (X.55) should be regarded as the solution of equation (X.50) in the m-th approximation, in which r is substituted by ρ and ϕ by θ. In the j-th approximation only the function $\Phi_j(\rho, \theta)$ is unknown, whereas the functions $\Phi_m(\rho, \theta)$ $(m < j)$ are known from the preceding approximations.

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By placing from (X.53) the expansions of the corresponding components into the examined boundary conditions, we obtain algebraic equation systems for the determination of the coefficients in the functions $f_{kj}(r)$ and $q_{kj}(r)$.

We see from (X.53)-(X.55) that in each of the following approximations (for each j), the problems are reduced formally to a series of boundary conditions for a round hole in plane ζ (in curvilinear coordinates ρ, θ). Thus, the problem for a noncanonic region (infinite plane with a curvilinear hole) can be solved if the total solution of the basic equation for the canonic region (infinite plane with a round hole) is known.

The universe form of function $f(\zeta)$ (X.45) for the various forms of holes is found from the function $\omega(\zeta)$ (X.45), which maps conformally the exterior of the unit circle onto the exterior of a hole of the form under consideration.

Any value, for instance T_θ^*, obtained in the n-th approximation will be regarded as follows:

$$T_\theta^*|_{n,\Gamma} = T_\theta^0|_\Gamma + \sum_{j=0}^{n} \varepsilon^j T_\varphi^j|_{\rho=1} + \sum_{j=0}^{n}\sum_{m=0}^{j-1} \varepsilon^j [L_1^{(j-m)} T_\varphi^{(m)} + L_2^{(j-m)}(T_r^{(m)}) - T_\varphi^{(m)}) - L_3^{(j-m)} S_{r\varphi}^{(m)}]|_{\rho=1}. \quad (X.57)$$

We will examine the application of the "boundary form perturbations" method /720 to the analysis of the stress state of shells weakened by a hole whose edge is reinforced by a thin elastic ring. The latter will be regarded as a material thread which resists stretching, bending and twisting. The boundary conditions for this case were found by N. P. Fleyshman [2]. Considering that one of the axes of inertia of the cross section of the reinforcing ring lies in the middle area of the shell, and using the basic assumptions of the theory of sloping shells, the boundary conditions can be written in the following form:

$$T_\rho|_{\rho=1} + T_\rho^{(0)}|_{\rho=1} = T_\rho^{(0)} - B\frac{\partial^3}{\partial s^3}\left[\frac{u_\theta + u_\theta^0}{R^*} - \frac{\partial}{\partial s}(u_\rho + u_\rho^0)\right]\Big|_{\rho=1} + \frac{E_1 F}{R^*}\left[\frac{\partial}{\partial s}(u_\theta + u_\theta^0) + \frac{u_\rho + u_\rho^0}{R^*}\right]\Big|_{\rho=1};$$

$$S_{\rho\theta}|_{\rho=1} + S_{\rho\theta}^0|_{\theta=1} = S_{\rho\theta}^{(0)} - \frac{B}{R^*}\frac{\partial^2}{\partial s^2}\left[\frac{u_\theta + u_\theta^0}{R^*} - \frac{\partial}{\partial s}(u_\rho + u_\rho^0)\right]\Big|_{\rho=1} - E_1 F\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}(u_\theta + u_\theta^0) + \frac{u_\rho + u_\theta^0}{R^*}\right]\Big|_{\rho=1};$$

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$$G_Q^0|_{Q=1} + G_Q^0|_{Q=1} = G_Q^{(0)} + \left[C\frac{\partial}{\partial s}\left(\frac{\partial^2}{\partial n \partial s} - \frac{1}{R^*}\cdot\frac{\partial}{\partial s}\right) - \right.$$
$$\left. - \frac{A}{R^*}\left(\frac{1}{R^*}\cdot\frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2}\right)\right](w + w^0)|_{Q=1};$$
$$\tilde{Q}_Q|_{Q=1} + \tilde{Q}_Q^0|_{Q=1} = \tilde{Q}_Q^{(0)} + \frac{\partial}{\partial s}\left[\frac{C}{R^*}\left(\frac{\partial^2}{\partial n \partial s} - \frac{1}{R^*}\cdot\frac{\partial}{\partial s}\right) + \right. \quad \text{(X.58)}$$
$$\left. + A\frac{\partial}{\partial s}\left(\frac{1}{R^*}\cdot\frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2}\right)\right](w + w^0)|_{Q=1}.$$

Here u_ρ^0, u_θ^0 and w are the components of the basic stress state; $T_\rho^{(0)}$, ..., $\tilde{Q}_\rho^{(0)}$ are external loads on the reinforcing ring; R* is the radius of curvature of the contour of the hole in the plane of variables (ρ, θ), to which is related the middle area of the shell; A, B and C are the rigidities of the ring to bending relative to two axes and to twisting; F is the area of cross section of the ring; E_1 and ν_1 is Young's modulus and Poisson's ratio for the material from which the ring is made.

By substituting (X.48) and (X.49) into (X.58) and combining the coefficients for ε^j, we obtain

$$T_Q^{(j)}|_{Q=1} + T_Q^{0(j)}|_{Q=1} = T_Q^{0(j)} - \frac{B}{r_0^4 Q^4}\cdot\frac{\partial^3}{\partial\theta^3}\left[v^{(j)} + \delta_j^0 v^0 - \frac{\partial}{\partial\theta}(u^{(j)} + \delta_j^0 u^0)\right]\Big|_{Q=1} +$$
$$+ \frac{E_1 F}{r_0^2 Q^2}\left[\frac{\partial}{\partial\theta}\left(v^{(j)} + \delta_j^0 v^0\right) + u^{(j)} + \delta_j^0 u^0\right]\Big|_{Q=1} + \sum_{m=0}^{j-1}[L_7^{(j-m)}(u^{(m)} + \delta_m^0 u^0) + \quad \text{/721}$$
$$+ L_8^{(j-m)}(v^{(m)} + \delta_m^0 v^0)]|_{Q=1};$$
$$S_{Q\theta}^{(j)}|_{Q=1} + S_{Q\theta}^{0(j)}|_{Q=1} = S_{Q\theta}^{(0)j} - \frac{B}{r_0^4 Q^4}\cdot\frac{\partial^2}{\partial\theta^2}[v^{(j)} + \delta_j^0 v^0 - \frac{\partial}{\partial\theta}(u^{(j)} +$$
$$+ \delta_j^0 u^0]|_{Q=1} - \frac{E_1 F}{r_0^2 Q^2}\cdot\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}(v^{(j)} + \delta_j^0 v^0) + u^{(j)} + \delta_j^0 u^0\right]|_{Q=1} + \quad \text{(X.59)}$$
$$+ \sum_{m=0}^{j-1}[L_9^{(j-m)}(u^{(m)} + \delta_m^0 u^0) + L_{10}^{(j-m)}(v^{(m)} + \delta_m^0 v^0)]|_{Q=1};$$
$$G_Q^{(j)}|_{Q=1} + G_Q^{0(j)}|_{Q=1} = G^{0(j)} + \frac{1}{r_0^3}\left[\frac{C}{Q}\cdot\frac{\partial^3}{\partial Q\partial\theta^2}\cdot\frac{1}{Q} - \frac{A}{Q^2}\left(\frac{\partial}{\partial Q} + \frac{1}{Q}\cdot\frac{\partial^2}{\partial\theta^2}\right)\right]\times$$
$$\times [\mathrm{Re}\,\Phi_j(Q,\theta) + \delta_j^0 w^0]|_{Q=1} + \frac{1}{r_0^3}\sum_{m=0}^{j-1} L_{11}^{(j-m)}[\mathrm{Re}\,\Phi_m(Q,\theta) + \delta_m^0 w^0]|_{Q=1};$$

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$$Q_{\varrho}^{(j)}|_{\varrho=1} + Q_{\varrho}^{0(j)}|_{\varrho=1} = \tilde{Q}_{\varrho}^{(0)j} + \frac{1}{r_0^4}\cdot\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\theta^2}\left[C\varrho\frac{\partial}{\partial\varrho}\cdot\frac{1}{\varrho} + \right.$$

$$\left. + A\left(\frac{\partial}{\partial\varrho} + \frac{1}{\varrho}\cdot\frac{\partial^2}{\partial\theta^2}\right)\right][\mathrm{Re}\,\Phi_j(\varrho,\theta) + \delta_j^0 w^0]|_{\varrho=1} +$$

$$+ \frac{1}{r_0^4}\sum_{m=0}^{j-1} L_{12}^{(j-m)}[\mathrm{Re}\,\Phi_m(\varrho,\theta) + \delta_m^0 w^0]|_{\varrho=1},$$

where $T_\rho^{0(j)}$, $S_{\rho\theta}^{0(j)}$, ..., $Q_\rho^{0(j)}$ are the coefficients of expansions of the components of the basic stress state; $T_\rho^{(0)(j)}$, ..., $\tilde{Q}_\rho^{(0)(j)}$ are the coefficients of expansions of the external forces acting on the ring; u^0, v^0, w^0 are the components of the basic deformation state in the polar coordinate system, in which r and ϕ are substituted by ρ and θ, respectively; $L_1^{(j-m)}$, ..., $L_{12}^{(j-m)}$ are differential operators; $\delta_k^n = \begin{cases} 1; & k = n, \\ 0; & k \neq n. \end{cases}$

From relations (X.53)-(X.55) and boundary conditions (X.59) we see that the problem is reduced to a series of boundary problems for a round hole in plane ζ.

We will examine the case where the forces on the contour of a curvilinear hole are known. Then the boundary conditions become

$$T_\varrho|_{\varrho=1} = g_1(\theta,\varepsilon) - T_\varrho^0|_{\varrho=1}; \quad S_{\varrho\theta}|_{\varrho=1} = g_2(\theta,\varepsilon) - S_{\varrho\theta}^0|_{\varrho=1};$$
$$G_\varrho|_{\varrho=1} = g_3(\theta,\varepsilon) - G_\varrho^0|_{\varrho=1}; \quad \tilde{Q}_\varrho|_{\varrho=1} = g_4(\theta,\varepsilon) - \tilde{Q}_\varrho^0|_{\varrho=1}. \tag{X.60}$$

By expanding the functions $g_1(\theta, \varepsilon)$, $g_2(\theta, \varepsilon)$, $g_3(\theta, \varepsilon)$, $g_4(\theta, \varepsilon)$ into series with respect to ε and substituting (X.49) into (X.60), we obtain, in the j-th approximation, /722

$$T_\varrho^{(j)}|_{\varrho=1} = \hat{g}_1^{(j)}(\theta); \quad S_{\varrho\theta}^{(j)}|_{\varrho=1} = \hat{g}_2^{(j)}(\theta);$$
$$G_\varrho^{(j)}|_{\varrho=1} = \hat{g}_3^{(j)}(\theta); \quad \tilde{Q}_\varrho^{(j)}|_{\varrho=1} = \hat{g}_4^{(j)}(\theta). \tag{X.61}$$

By substituting (X.53) in (X.61), we obtain the boundary conditions for the determination of the function $\Phi_j(r, \phi)$:

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$$\left(\frac{1}{\varrho}\cdot\frac{\partial^2}{\partial\theta^2}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right)\operatorname{Im}\Phi_j(\varrho,\theta)|_{\varrho=1}=-\sum_{m=0}^{j-1}\left[L_1^{(j-m)}\left(\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\theta^2}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right)+\right.$$
$$\left.+L_2^{(j-m)}\left(2\frac{\partial^2}{\partial\varrho^2}-\nabla^2\right)-L_3^{(j-m)}\right)\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{1}{\varrho}\Big]\operatorname{Im}\Phi_m(\varrho,\theta)|_{\varrho=1}+$$
$$+nr_0^2\hat{g}_1^{(j)}(\theta);$$
$$-\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{\operatorname{Im}\Phi_j(\varrho,\theta)}{\varrho}\Big|_{\varrho=1}=\sum_{m=0}^{j-1}\Big[(L_1^{(j-m)}-2L_2^{(j-m)})\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{1}{\varrho}+$$
$$+\frac{1}{2}L_3^{(j-m)}\left(\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\theta^2}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}-\frac{\partial^2}{\partial\varrho^2}\right)\Big]\operatorname{Im}\Phi_m(\varrho,\theta)|_{\varrho=1}+$$
$$+nr_0^2\hat{g}_2^{(j)}(\theta);\qquad\text{(X.62)}$$
$$-\left[(1-\nu)\frac{\partial^2}{\partial\varrho^2}+\nu\nabla^2\right]\operatorname{Re}\Phi_j(\varrho,\theta)|_{\varrho=1}=\sum_{m=0}^{j-1}\Big[L_1^{(j-m)}(1-\nu)\frac{\partial^2}{\partial\varrho^2}+$$
$$+\nu\nabla^2+L_2^{(j-m)}(1-\nu)\left(\nabla^2-2\frac{\partial^2}{\partial\varrho^2}\right)+(1-\nu)L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\theta}\cdot\frac{1}{\varrho}\Big]\times$$
$$\times\operatorname{Re}\Phi_m(\varrho,\theta)|_{\varrho=1}+\frac{r_0^2}{D}\hat{g}_3^{(j)}(\theta);$$
$$-\left[\frac{\partial}{\partial\varrho}\nabla^2+\frac{1-\nu}{\varrho}\cdot\frac{\partial^2}{\partial\varrho\partial\theta^2}\cdot\frac{1}{\varrho}\right]\operatorname{Re}\Phi_j(\varrho,\theta)|_{\varrho=1}=\sum_{m=0}^{j-1}L_4^{(j-m)}\operatorname{Re}\Phi_m(\varrho,\theta)|_{\varrho=1}+\frac{r_0^3}{D}\hat{g}_4^{(j)}(\theta).$$

Application of Method of Series Approximations with the Use of the Basic Equations in Integral Form. It is easy to see that in certain cases equation (X.25) with conditions (X.26), i.e., the basic equation system in integral form, is more convenient in applications than equation systems (X.14) or (X.23) in differential form. This is related to the fact that the solution $V(\xi, \eta)$ of integral equation (X.25) always exists and it can be found by the method of series approximations by using as the zero (initial) approximation the function /723

$$V_0(\xi,\eta)=\Phi_0\left(\frac{\xi+\eta}{2},\ i\frac{\eta-\xi}{2}\right)=\frac{Eh^2}{\sqrt{12(1-\nu^2)}}w_0+i\hat{\varphi}_0,\qquad\text{(X.63)}$$

where $\hat{\phi}_0$ is Airy's function of stresses for a plate weakened by the same kind of hole, under the very same basic stress state, and subject to the very same boundary conditions in relation to tangential components on the contour of the hole, as exist in the shell under examination, while the function w^0 is deflection of a plate weakened by the same kind of hole as in the shell under examination, found under the very same nontangential boundary conditions on the contour of the hole as in the given shell.

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For numerous more interesting cases the functions $\hat{\phi}_0$ and w^0 for plates weakened by a hole are known[1]; consequently[2], we also know the function $V_0(\xi, \eta)$ which is the zero approximation. By substituting this function into integral equation (X.25), we find the first approximation, i.e., $V_1(\xi, \eta)$. Proceeding in the analogous manner, we obtain the function $V_n(\xi, \eta)$ which is the n-th approximation for the desired solution; hence in each of the approximations there are four holomorphic functions, i.e., $\psi_0(\xi)$, $\psi_0^*(\eta)$, $\psi_1(\xi)$ and $\psi_1^*(\eta)$, which can be found from the boundary conditions of the problem when conditions (X.26) are satisfied.

Application of Method of Finite Differences. As shown by experimental studies (Figures X.3 and X.4), the conditions "at infinity" (X.35) will be satisfied with an accuracy known a priori, as soon as coordinate line $\rho = \rho_1 =$ = const reaches contour L* (Figure X.2); the latter, however, will remain at some distance from the contour of the hole, not exceeding four times the greatest dimension of the hole. This can be made use of in approximating the solution of equation (X.23), and specifically, under the conditions "at infinity" (X.35), we may assume that $\rho = 1.5\text{-}2.0$ for a hole of any shape, the contour of which is given in parametric form by analytical function $\alpha + i\beta =$ $= \omega(\zeta)$ (X.11) of complex variable ζ, where $\zeta = e^{\rho+i\theta}$. For instance, by substituting by the method of finite differences the differential equation by a system of linear algebraic equations, it is necessary to add to the resultant system, in addition to the equations derived from the boundary conditions and conditions "at infinity" of the problem, six equations for functions $\hat{\phi}(\rho, \theta)$ and $\omega(\rho, \theta)$ derived from vector condition (X.28), which insures the single-valuedness of the displacement vector of the middle surface of the shell.

Numerical integration of equation (X.23) of §1 in variables ρ and θ should be performed in rectangle OABC, which is the conformal mapping of a doubly-connected region bounded by contours Γ and L onto planes (ρ, θ) (Figure X.6).

The above-described problem was solved in I. O. Bugerman's work [1], in /724 which the boundary problems for differential equation (X.23) in the case of a spherical shell weakened by elliptical and square holes (with rounded corners), are reduced to finite-difference boundary problems.

Presentation of Solution of Basic Equation for Spherical Shell. In order to analyze the stress state in shells near curvilinear holes it is necessary to have the solution of basic equation (X.42) in the form (X.47). We will consider such a solution for a spherical shell for which basic equation (X.42) has the form

$$\nabla^2(\nabla^2 - i\varkappa^2)\Phi = 0. \qquad (X.64)$$

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[1]See G. N. Savin [8], Chapter VI.
[2]The case of a square hole was analyzed by S. Mokhbaliyev [2] by this method.

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Equation (X.64) is represented as a sum of the solutions of Laplace's and Helmholtz's equations. For the j-th approximation

$$\begin{aligned}\Phi_j(r,\varphi) = (c_1^{j,0} + id_1^{j,0}) + (c_2^{j,0} + id_2^{j,0}) \ln r + \\ + (c_3^{j,0} + id_3^{j,0}) J(r\varkappa\sqrt{-i}) + (c_4^{j,0} + id_4^{j,0}) H(r\varkappa\sqrt{-i}) + \\ + \sum_{k=1}^{\infty} [(c_1^{j,k} + id_1^{j,k}) r^k + (c_2^{j,k} + id_2^{j,k}) r^{-k} + \\ + (c_3^{j,k} + id_3^{j,k}) J_k(r\varkappa\sqrt{-i}) + \\ + (c_4^{j,k} + id_4^{j,k}) H_k(r\varkappa\sqrt{-i})] \begin{matrix}\cos \\ \sin\end{matrix} k\varphi,\end{aligned} \tag{X.65}$$

where $J_k(r\varkappa\sqrt{-i}) = \mathrm{ber}_k r\varkappa + i\,\mathrm{bei}_k r\varkappa$ is Bessel's function of the first kind of the k-th order; $H_k^{(1)}(r\varkappa\sqrt{-1}) = \mathrm{her}_k r\varkappa + i\mathrm{hei}_k r\varkappa$ is Hankel's function of the first kind of the k-th order; $\mathrm{ber}_k r\varkappa$, $\mathrm{bei}_k r\varkappa$, $\mathrm{her}_k r\varkappa$ and $\mathrm{hei}_k r\varkappa$ is Thompson's function; we will omit the index 0 for the null function. The asymptotic representations of the Bessel and Hankel functions are

$$\begin{aligned}J_k(r\varkappa\sqrt{-i}) &\approx \sqrt{\frac{2}{\pi\varkappa r\sqrt{-1}}} \cos\left(r\varkappa\sqrt{-i} - \frac{\pi}{4}k - \frac{\pi}{4}\right), \\ H_k^{(1)}(r\varkappa\sqrt{-i} &\approx \sqrt{\frac{2}{\pi\varkappa r\sqrt{-i}}} e^{i\left(r\varkappa\sqrt{-i} - \frac{\pi}{4}k - \frac{\pi}{4}\right)}.\end{aligned} \tag{X.66}$$

From asymptotic representations (X.66) and the solution in the form (X.65), we see that in order to satisfy the conditions "at infinity" (X.35) it is necessary to assume

$$c_1^{j,0} = d_1^{j,0} = c_2^{j,0} = c_3^{j,0} = d_3^{j,0} = c_1^{j,k} = d_1^{j,k} = c_3^{j,k} = d_3^{j,k} = 0. \tag{X.67}$$

The solution can be used in such form to analyze the stress state in a spherical shell weakened by any curvilinear hole, the contour of which has no angular points. We will notice that the solution in the form (X.65) can be found by /725
regrouping the terms from the complete solution, presented in the monograph of I. N. Vekua [2]. From (X.44) when

$$\frac{1}{R_r} = \frac{1}{R_\theta} = \frac{1}{R}, \quad \frac{1}{R_{r\theta}} = 0 \tag{X.68}$$

we obtain the equation system for the determination of the variables u and v:

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$$\frac{\partial u^{(j)}}{\partial r} = -\frac{r_0}{R}\operatorname{Re}\Phi_j(r,\varphi) + \frac{1}{Ehnr_0}\left[\nabla^2 - (1+\nu)\frac{\partial^2}{\partial r^2}\right]\operatorname{Im}\Phi_j(r,\varphi);$$

$$\frac{1}{r}\cdot\frac{\partial v^{(j)}}{\partial\varphi} + \frac{u^{(j)}}{r} = -\frac{r_0}{R}\operatorname{Re}\Phi_j(r,\varphi) + \frac{1}{Ehnr_0}\left[(1+\nu)\frac{\partial^2}{\partial r^2} - \nu\nabla^2\right]\operatorname{Im}\Phi_j(r,\varphi);$$

$$\frac{1}{r}\cdot\frac{\partial u^{(j)}}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{v^{(j)}}{r}\right) = -2\frac{1+\nu}{Ehnr_0}\cdot\frac{\partial^2}{\partial r\partial\varphi}\cdot\frac{\operatorname{Im}\Phi_j(r,\varphi)}{r}. \qquad (X.69)$$

Into the right hand sides of (X.69) it is necessary to substitute $\Phi(r, \phi)$ from (X.65), considering (X.67). The system obtained can be integrated. Omitting the intermediate calculations and recalling (X.67), we present the values of the variables:

$$u^{(j)}(r,\varphi) = -\frac{r_0}{R}(1+\nu)\left[\frac{d_2^{j,0}}{\varkappa^2 r} + rc_4^{j,0}(\operatorname{her}\varkappa r - \right.$$

$$\left. - \operatorname{hei}''\varkappa r) - rd_4^{j,0}(\operatorname{hei}\varkappa r + \operatorname{her}''\varkappa r)\right] +$$

$$+ \frac{r_0}{R}\sum_{k=1}^{\infty}\left\{\frac{c_2^{j,k}}{k-1}r^{-k+1} + k\frac{1+\nu}{\varkappa^2}r^{-k+1}d_2^{j,k} - \right.$$

$$\left. - \frac{1+\nu}{\varkappa}(c_4^{j,k}\operatorname{hei}_k'r\varkappa + d_4^{j,k}\operatorname{her}_k'r\varkappa)\right\}\begin{matrix}\cos\\ \sin\end{matrix} k\varphi; \qquad (X.70)$$

$$v^{(j)}(r,\varphi) = \frac{r_0}{R}\sum_{k=1}^{\infty}\left\{-c_2^{j,k}\frac{r^{-k+1}}{k-1} + k\frac{1+\nu}{\varkappa^2}d_2^{j,k}r^{-k-1} + \right.$$

$$\left. + \frac{1+\nu}{r\varkappa^2}k(c_4^{j,k}\operatorname{hei}_k r\varkappa + d_4^{j,k}\operatorname{her}_k\varkappa r)\right\}\begin{matrix}\sin\\ (-\cos)\end{matrix} k\varphi.$$

If system (X.69) is integrated for k = 1, then a logarithmic term appears in $c_2^{j,1}$, therefore, in order to satisfy the conditions "at infinity" (X.35), it is necessary to assume

$$c_2^{j,1} = 0. \qquad (X.71)$$

Thus, conditions (X.70) and (X.71) insure that the conditions "at infinity" are satisfied.

We will discuss now the question of selecting the law of distribution of /726
shear force for a curvilinear hole in a spherical shell, loaded by uniform internal pressure, where the hole is covered by a roof of special construction, which transmits only the action of the shear force. We will examine holes for which

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$$f(\zeta) = \frac{1}{\zeta^N}. \tag{X.72}$$

The function $f^*(\theta)$ in the first condition of (X.33), generally speaking, can be an arbitrary integrated function, but for the above-stated curvilinear holes it should be taken in a form such that when $\varepsilon \to 0$ the distribution of shear force can be found for a round hole. The function $f^*(\theta)$ in the work of G. N. Saving and G. A. Van Fo Fy [1] for an elliptical hole was selected as the outcome of analysis of stress distribution on the contour of an elliptical plate. The shear force is represented in the form of two components, the first of which corresponds to shear force on the contour of an inscribed circle, and the second of which corresponds to an addition which approaches zero when the contour approaches the circle. This addition is assumed to be proportional to the relative difference between the areas of the ellipse and inscribed circle, and is multiplied by a trigonometric function, which insures the maximum shear force on the ends of the small axis, i.e., at the points closest to the center of the hole, and the minimum shear force on the ends of the large axis, i.e., at the points farthest from the center of the hole.

We will generalize these considerations for the case of an arbitrary hole whose contour is given by the function $\omega(\zeta) = \zeta + \varepsilon \frac{1}{\zeta^N}$ for $\rho = 1$ ($\zeta = \rho e^{i\theta}$). We will assume that $\varepsilon > 0$; then $r_1 = r_0(1 - \varepsilon)$ is the radius of the inscribed circle. The points of the contour closest to the center of the hole are found for $\theta = \pi \frac{1 + 2l}{N + 1}$ (l is a whole number), and for the shear force we will obtain the following distribution law:

$$\tilde{Q}_\rho|_r = -p\frac{D}{S}\left[1 - \frac{D - D_\theta}{D}\cos(N+1)\theta\right], \tag{X.73}$$

where $D_\theta = \pi r_0^2(1 - \varepsilon)^2$ is the area of the inscribed circle; $D = \pi r_0^2(1 - N\varepsilon^2)$ is the area of the hole; $S = 2\pi r_0(1 + \varepsilon^2 N^2/4 + \ldots)$ is the length of the arc of contour of the hole; all these values are calculated with an accuracy up to ε^2.

Finally, for shear force:

$$\tilde{Q}_\rho|_r = -\frac{pr_0}{2}\left\{1 - \varepsilon^2 N\frac{N-1}{4} - [2\varepsilon - (N+1)\varepsilon^2]\cos(N+1)\varphi + \ldots\right\}. \tag{X.74}$$

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Representation of Solutions of Basic Equation for Small Holes in a Round Cylindrical Shell. If, instead of the function $\hat{\phi}$, we introduce $-\hat{\phi}$, and for function Φ the earlier representation ($\Phi = w + in\phi$) remains valid, then equation (X.42) for a round cylindrical shell acquires the following form: /727

$$\nabla^2\nabla^2\Phi(x, y) + 8i\beta^2 \frac{\partial^2\Phi(x, y)}{\partial x^2} = 0, \tag{X.75}$$

where $\beta = \frac{r_0}{\sqrt{Rh}} \frac{\sqrt[4]{3(1-\nu^2)}}{2}$; R is the radius of the middle surface of the shell; the Ox axis is directed along the generatrix, and the Oy axis is directed along the guide. This change in definitions is made so that our definitions will coincide with those used in numerous works concerning the stress state near small holes in cylindrical shells. Then in relations (X.43) and (X.44) the sign in front of Im Φ should be changed to the opposite. From (X.44), considering this comment, we obtain an equation system for the determination of the variables:

$$\begin{aligned}
&\frac{\partial u}{\partial r} = -r_0 \frac{\sin^2\varphi}{R} \operatorname{Re}\Phi(r, \varphi) - \frac{1}{Ehnr_0}\left[\nabla^2 - (1+\nu)\frac{\partial^2}{\partial r^2}\right] \operatorname{Im}\Phi(r, \varphi);\\
&\frac{1}{r}\cdot\frac{\partial v}{\partial \varphi} + \frac{u}{r} = -r_0 \frac{\cos^2\varphi}{R} \operatorname{Re}\Phi(r, \varphi) - \frac{1}{Ehnr_0}\left[(1+\nu)\frac{\partial^2}{\partial r^2} - \right.\\
&\qquad\qquad \left. - \nu\nabla^2\right] \operatorname{Im}\Phi(r, \varphi);\\
&\frac{1}{r}\cdot\frac{\partial u}{\partial \varphi} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = r_0 \frac{\sin 2\varphi}{R} \operatorname{Re}\Phi(r, \varphi) + 2\frac{1+\nu}{Ehnr_0}\cdot\frac{\partial^2}{\partial r\partial\varphi}\cdot\frac{\operatorname{Im}\Phi(r, \varphi)}{r}.
\end{aligned} \tag{X.76}$$

For a round cylindrical shell

$$\frac{1}{R_r} = \frac{\sin^2\varphi}{R}; \quad \frac{1}{R_\varphi} = \frac{\cos^2\varphi}{R}; \quad \frac{1}{R_{r\varphi}} = \frac{1}{2}\cdot\frac{\sin 2\varphi}{R}. \tag{X.77}$$

Following the example of A. I. Lur'ye [1], the solution of equation (X.75) will be represented in the form of the sum

$$\Phi = \Phi_1 + \Phi_2, \tag{X.78}$$

where

$$\begin{aligned}
&\nabla^2\Phi_1 \pm 2(1-i)\beta\frac{\partial\Phi_1}{\partial x} = 0,\\
&\nabla^2\Phi_2 - 2(1-i)\beta\frac{\partial\Phi_2}{\partial x} = 0.
\end{aligned} \tag{X.79}$$

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We will introduce new (unknown) functions:

$$\Phi_1 = e^{-(1-i)\beta x}\psi_1(x, y); \qquad \Phi_2 = e^{(1-i)\beta x}\psi_2(x, y). \tag{X.80}$$

By substituting (X.80) into (X.79) we find that functions ψ_1 and ψ_2 satisfy the equation:

$$\nabla^2\psi + 2i\beta^2\psi = 0. \tag{X.81}$$

The solution of Hemholtz's equation (X.81) will be represented in the polar coordinate system (r, ϕ): /728

$$\psi = \sum_{k=0}^{\infty}[(a_k + ib_k)J_k(\beta r\sqrt{2i}) + (c_k + id_k)H_k^{(1)}(\beta r\sqrt{2i})]\begin{matrix}\cos\\ \sin\end{matrix} k\varphi, \tag{X.82}$$

where $J_k(\beta r\sqrt{2i}) = \mathrm{ber}_k(\beta r\sqrt{2}) - i\,\mathrm{bei}_k(\beta r\sqrt{2})$ is Bessel's function of the first kind of k-th order; $H_k^{(1)}(\beta r\sqrt{2i}) = \mathrm{her}_k(\beta r\sqrt{2}) - i\,\mathrm{hei}_k(\beta r\sqrt{2})$ is Hankel's function of the first kind of k-th order. We will write out the principal terms of the asymptotic representations of the Bessel and Hankel functions:

$$\begin{aligned} J_k(\beta r\sqrt{2i}) &\approx \sqrt{\frac{2}{\pi\beta r\sqrt{2i}}}\cos\left(r\beta\sqrt{2i} - \frac{\pi}{4}k - \frac{\pi}{4}\right);\\ H_k(\beta r\sqrt{2i}) &\approx \sqrt{\frac{2}{\pi\beta r\sqrt{2i}}}\,e^{i\left(r\beta\sqrt{2i} - \frac{\pi}{4}k - \frac{\pi}{4}\right)}. \end{aligned} \tag{X.83}$$

From the conditions "at infinity" (X.35) and asymptotic representations (X.83) we see that

$$a_k = b_k = 0. \tag{X.84}$$

By substituting (X.82) into (X.80) and (X.78), recalling (X.84), we obtain a more general solution of equation (X.75) in the form

$$\Phi = \sum_{k=0}^{\infty}[(A_k + iB_k)e^{-(1-i)\beta x} + (C_k + iD_k)e^{(1-i)\beta x}]H_k(r\beta\sqrt{2i})\begin{matrix}\cos\\ \sin\end{matrix} k\varphi. \tag{X.85}$$

From the asymptotic representation of Hankel's function (X.83) and the solution

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in the form (X.85), we see that along the generatrix (when $\phi = 0$) the function Φ is damped such that

$$\Phi \sim \frac{\text{const}}{\sqrt{r}}. \tag{X.86}$$

Consequently, this function satisfies conditions "at infinity" (X.35). We will note that the solution in form (X.85) is the solution with undivided variables.

A. I. Lur'ye [1, 2] proposed an approximation method for the solution of the problem of stress concentration near a "round" hole; this method represents the approximate representation of the solution of equation (X.75) in the form (X.85) with an accuracy up to β^2, i.e., for holes of small dimensions. We will present the solution obtained by A. I. Lur'ye's method, following the example of I. M. Pirogov [20], where this solution is represented in the form of a Fourier series and in a more complete form. First we will make a few transformations of the solution of (X.85). This solution can be represented in the form the sum of the following solutions: /729

$$\begin{aligned}\Phi_n &= [K_1(\beta x) - 2iK_3(\beta x)]\, H_n(\beta r \sqrt{2i}) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi,\\ \Phi_n &= [K_2(\beta x) - 2iK_4(\beta x)]\, H_n(\beta r \sqrt{2i}) \begin{matrix}\cos\\ \sin\end{matrix} n\varphi,\end{aligned} \tag{X.87}$$

where $K_1(\beta x)$, $K_2(\beta x)$, $K_3(\beta x)$ and $K_4(\beta x)$ is A. N. Krylov's function which is the combination $e^{-(1-i)\beta x}$ and $e^{(1-i)\beta x}$. We expand A. N. Krylov's and Hankel's functions in the vicinity of the origin of the coordinate system into series. In order to simplify the following study we will analyze separately the stress states corresponding to each of solutions (X.87).

Let us consider the solution of equation (X.75) in the form

$$\begin{aligned}\Phi_n &= [K_1(\beta x) - 2iK_3(\beta x)]\, H_n(\beta r \sqrt{2i}) \cos n\varphi \quad (n = 0, 2, 4, \ldots),\\ \Phi_n &= [K_2(\beta x) - 2iK_4(\beta x)]\, H_n(\beta r \sqrt{2i})\,(1+i) \cos n\varphi \quad (n = 1, 3, 5, \ldots).\end{aligned} \tag{X.88}$$

We will find the complex function of stresses which correspond to (X.88) in the form

$$\Phi = \sum_{n=0}^{\infty} (a_n + ib_n)\, \Phi_n, \tag{X.89}$$

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where, following A. I. Lur'ye's example, we will represent the arbitrary constants in the form

$$\left.\begin{aligned} a_n &= A_n\beta^n + B_n\beta^{n+2} + \ldots \\ b_n &= C_n\beta^n + D_n\beta^{n+2} + \ldots \end{aligned}\right\} \quad (n = 0, 2, 4, \ldots),$$
$$\left.\begin{aligned} a_n &= A_n\beta^{n-1} + B_n\beta^{n+1} + \ldots \\ b_n &= C_n\beta^{n-1} + D_n\beta^{n+1} + \ldots \end{aligned}\right\} \quad (n = 1, 3, 5, \ldots). \qquad (X.90)$$

By substituting (X.90) and (X.88), as well as A. N. Krylov's and Hankel's functions, expanded in the vicinity of the origin of the coordinate system, into (X.89), we obtain the solution represented in the form of a series with respect to small parameter β. Combining the coefficients for identical degrees of β and introducing the new definitions of the constants, we obtain, with an accuracy up to β^2, this solution in the form

$$\begin{aligned} \operatorname{Im}\Phi = {} & \frac{2}{\pi}A_0(\ln r + \gamma') + \frac{1}{\pi}A_2\cos 2\varphi + \sum_{n=2,4,\ldots}^{\infty}\left[\frac{n-2}{2\pi r^{n-2}}A_n + \right. \\ & \left. + \frac{1}{\pi r^n}(C_n + A_{n+2})\right]\cos n\varphi + \frac{1}{\pi}\beta^2\left\{2B_0(\ln r + \gamma') + (2C_0 + \right. \\ & + B_2)r^2(\ln r + \gamma') - \frac{1}{4}(2\pi A_0 + 4C_0 + \pi A_2 + B_2)r^2 + \left[(C_2 + \right. \\ & \left. + B_2)r^2(\ln r + \gamma') - \frac{1}{4}\left(\pi A_0 + \pi A_2 + \frac{2}{3}B_2\right)r^2 + F_2 + \right. \\ & \left. + \frac{F_2}{r^2}\right]\cos 2\varphi + \frac{1}{12}B_2 r^2\cos 4\varphi + \sum_{k=4,6,8,\ldots}^{\infty}\left[\frac{n-4}{24r^{n-6}}B_{n-2} + \right. \\ & \left. + \frac{D_{n-2}}{4r^{n-4}} + \frac{E_n}{r^{n-2}} + \frac{F_n}{r^n}\right]\cos n\varphi\Big\}; \qquad (X.91) \end{aligned}$$

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$$\begin{aligned} \operatorname{Re}\Phi = {} & -\frac{2}{\pi}C_0(\ln r + \gamma') - \frac{1}{\pi}B_2\cos 2\varphi - \sum_{n=2,4,6,\ldots}^{\infty}\left[\frac{n-2}{2\pi r^{n-2}}B_n + \right. \\ & \left. + \frac{1}{\pi r^n}(D_n + B_{n+2})\right]\cos n\varphi + \frac{1}{\pi}\beta^2\left\{-2D_0(\ln r + \gamma') + (2A_0 + \right. \\ & + A_2)r^2(\ln r + \gamma') + \frac{1}{4}(2\pi C_0 - 4A_0 + \pi B_2 - A_2)r^2 + \left[(A_0 + A_2)r^2(\ln r + \right. \\ & \left. + \gamma') + \frac{1}{4}\left(\pi C_0 + \pi B_2 - \frac{2}{3}A_2\right)r^2 + H_2 + \frac{K^2}{r^2}\right]\cos 2\varphi + \frac{1}{12}A_2 r^2\cos 4\varphi + \\ & \left. + \sum_{n=4,6,8,\ldots}^{\infty}\left[\frac{n-4}{24r^{n-6}}A_{n-2} + \frac{C_{n-2}}{4r^{n-4}} + \frac{H_n}{r^{n-2}} + \frac{K_n}{r^n}\right]\cos n\varphi\right\}, \end{aligned}$$

where $\gamma' = \frac{\beta\ln\gamma}{\sqrt{2}}$; γ are Euler's constants.

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This solution and the solution for the other stress states were obtained by I. M. Pirogov [20] by A. I. Lur'ye's method [1]:

$$\mathrm{Im}\Phi = \frac{A_1 + A_3}{2\pi r}\cos\varphi + \sum_{n=3,5,7,\ldots}^{\infty}\left[\frac{n-2}{2\pi r^{n-2}}A_n + \frac{1}{\pi r^n}(C_n + A_{n+2})\right]\cos n\varphi +$$
$$+ \frac{1}{\pi}\beta^2\left\{\left[2C_1 r(\ln r + \gamma') - \frac{1}{2}(\pi A_1 - C_1 + \frac{1}{6}B_3)r + \frac{1}{r}F_1\right]\cos\varphi +\right.$$
$$+\left[\frac{1}{8}(4C_1 - B_3)r + \frac{1}{r}E_3 + \frac{F_3}{r^3}\right]\cos 3\varphi +$$
$$\left. + \sum_{n=5,7,9,\ldots}^{\infty}\left[\frac{n-4}{24r^{n-6}}B_{n-2} + \frac{D_{n-2}}{4r^{n-4}} + \frac{E_n}{r^{n-2}} + \frac{F_n}{r^n}\right]\cos n\varphi\right\} + \ldots; \qquad (X.92)$$
$$\mathrm{Re}\Phi = -\frac{1}{2\pi r}(4C_1 + B_3)\cos\varphi - \sum_{n=3,5,7,\ldots}^{\infty}\left[\frac{n-2}{2\pi r^{n-2}}B_n + \frac{D_n + B_{n+2}}{\pi r^n}\right]\cos n\varphi +$$
$$+ \frac{1}{\pi}\beta^2\left\{\left[2A_1 r(\ln r + \gamma') + \frac{1}{2}(\pi C_1 + A_1 - \frac{A_3}{6})r + \frac{K_1}{r}\right]\cos\varphi +\right.$$
$$+\left[\frac{1}{8}(4A_1 + A_3)r + \frac{H_3}{r} + \frac{K_3}{r^3}\right]\cos 3\varphi + \sum_{n=5,7,9,\ldots}^{\infty}\left[\frac{n-4}{24r^{n-6}}A_{n-2} + \right. \qquad /731$$
$$\left.\left. + \frac{C_{n-2}}{4r^{n-4}} + \frac{H_n}{r^{n-2}} + \frac{K_n}{r^n}\right]\cos n\varphi\right\} + \ldots .$$

The solutions of (X.91) and (X.92) are represented in the form of a Fourier series with respect to the cosines; similarly, we may construct the solutions in the form of series with respect to sines. These solutions will differ from the solutions of (X.91) and (X.92) only in that numbers for n = 1, 2 (the terms corresponding to n = 0 are missing) which, in the case at hand, have the form

$$\mathrm{Im}\Phi = \frac{1}{2\pi r}(4A_1 + A_3)\sin\varphi + \left[\frac{A_2}{\pi} + \frac{C_2 + A_4}{\pi r^2}\right]\sin 2\varphi +$$
$$+ \frac{1}{\pi}\beta^2\left\{\left[2C_1 r(\ln r + \gamma') - \frac{1}{2}\left(\pi A_1 + C_1 + \frac{B_3}{3}\right)r + \frac{F_1}{r}\right]\sin\varphi +\right.$$
$$\left. + \left[B_2 r^2(\ln r - \gamma') - \left(\frac{\pi}{4}A_2 + \frac{B_2}{3}\right)r^2 + E_2 + \frac{1}{r^2}F_2\right]\sin 2\varphi\right\};$$
$$\mathrm{Re}\Phi = -\frac{1}{2\pi r}(4C_1 + B_3)\sin\varphi - \left[\frac{B_2}{\pi} + \frac{D_2 + B_1}{\pi r^2}\right]\sin 2\varphi + \qquad (X.93)$$
$$+ \frac{1}{\pi}\beta^2\left\{\left[2A_1 r(\ln r + \gamma') + \frac{1}{2}\left(\pi C_1 - A_1 - \frac{A_3}{3}\right)r + \frac{K_1}{r}\right]\sin\varphi +\right.$$
$$\left. + \left[A_2 r^2(\ln r + \gamma') + \left(\frac{\pi}{4}B_2 - \frac{A_2}{3}\right)r^2 + H_2 + \frac{K_2}{r^2}\right]\sin 2\varphi\right\}.$$

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The other coefficients for the trigonometric functions remain unchanged.

We will present the solution of equation (X.75) in the j-th approximation, which can be used for analysis of the stress state near a curvilinear free hole having two axes of symmetry, coinciding with the generatrix and the guide, in the basic stress state, symmetrical with respect to the axes:

$$\begin{aligned}
\operatorname{Im}\Phi_j(r,\varphi) &= \frac{2}{\pi}A_0^{(j)}(\ln r+\gamma')+\frac{\cos 2\varphi}{\pi}\left(A_2^{(j)}+\frac{B_2^{(j)}}{r^2}\right)+\\
&+\sum_{k=4,6,\ldots}^{\infty}\left(\frac{k-2}{2r^{k-2}}A_k^{(j)}+\frac{B_k^{(j)}}{r^k}\right)\frac{\cos k\varphi}{\pi}+\frac{1}{\pi}\beta^2\left\{2B_0^{(j)}(\ln r+\gamma')-\right.\\
&-\frac{\pi}{4}(2A_0^{(j)}+A_2^{(j)})r^2+\left[-\frac{\pi}{4}(A_0^{(j)}+A_2^{(j)})r^2+E_2^{(j)}+\right.\\
&\left.\left.+\frac{F_2^{(j)}}{r^2}\right]\cos 2\varphi+\sum_{k=4,6,\ldots}^{\infty}\left(\frac{E_k^{(j)}}{r^{k-2}}+\frac{F_k^{(j)}}{r^k}\right)\cos k\varphi\right\};\\
\operatorname{Re}\Phi_j(r,\varphi) &= \frac{1}{\pi}\beta^2\left\{-2D_0^{(j)}(\ln r+\gamma')+(2A_1^{(j)}+\right. \qquad \text{(X.94)}\\
&+A_2^{(j)})r^2(\ln r+\gamma')-\frac{1}{4}(4A_0^{(j)}+A_2^{(j)})r^2+[(A_0^{(j)}+\\
&+A_2^{(j)})r^2(\ln r+\gamma')-\frac{A_2^{(j)}}{6}r^2+H_2^{(j)}+\frac{K_2}{r^2}]\cos 2\varphi+\\
&+\frac{A_2^{(j)}r^2}{12}\cos 4\varphi+\sum_{k=4,6,\ldots}^{\infty}\left[\frac{(k-4)A_{k-2}^{(j)}}{24r^{k-6}}+\frac{B_{k-2}^{(j)}-A_k^{(j)}}{4r^{k-4}}+\right.\\
&\left.\left.+\frac{H_k^{(j)}}{r^{k-2}}+\frac{K_k^{(j)}}{r^k}\right]\cos k\varphi\right\}.
\end{aligned}$$

/732

The function Φ (X.94) can be obtained from (X.88)-(X.90) after the corresponding substitution of the constants.

§3. Spherical Shell with Curvilinear Hole

Let us consider the stress state in a spherical shell weakened by a curvilinear hole and under the effect of a uniform internal pressure of intensity p. The basic stress state will be assumed to be momentless. The components of the basic stress state will be represented in the form

$$\begin{aligned}
&T_r^0=p_0h;\quad T_\varphi^0=p_0h;\quad S_{r\varphi}^0=0;\quad G_r^0=0;\\
&G_\varphi^0=0;\quad \tilde{Q}_r^0=0;\quad w^0=0;\quad v^0=0; \qquad \text{(X.95)}\\
&u^0=p_0\frac{1-\nu}{E}r_0r;\qquad p_0=\frac{pR}{2h}.
\end{aligned}$$

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Round Hole. Consider the case where a hole is covered by a roof which transmits only the action of the shear force; then the boundary conditions become

$$T_r|_{r=1} = -p_0 h; \qquad S_{r\varphi}|_{r=1} = 0;$$
$$G_r|_{r=1} = 0; \qquad \tilde{Q}_r|_{r=1} = -\frac{pr_0}{2}. \tag{X.96}$$

The problem is axisymmetrical, and therefore the solution can be taken in the form (X.65) for j = k = 0. By substituting (X.43) and (X.65) into boundary conditions (X.96), we determine the constants in (X.65) for j = k = 0: /733

$$d_2^{0,0} = 0;$$
$$c_4^{0,0} = -\frac{pr_0^4}{2D\varkappa^3} \cdot \frac{\varkappa\,\mathrm{hei}''\varkappa + \nu\,\mathrm{hei}'\varkappa}{\varkappa(\mathrm{her}'\varkappa\,\mathrm{her}''\varkappa + \mathrm{hei}'\varkappa\,\mathrm{hei}''\varkappa) + \nu(\mathrm{her}'^2\varkappa + \mathrm{hei}'^2\varkappa)};$$
$$d_4^{0,0} = -\frac{pr_0^4}{2D\varkappa^3} \cdot \frac{\varkappa\,\mathrm{her}''\varkappa + \nu\,\mathrm{her}'\varkappa}{\varkappa(\mathrm{her}'\varkappa\,\mathrm{her}''\varkappa + \mathrm{hei}'\varkappa\,\mathrm{hei}''\varkappa) + \nu(\mathrm{her}'^2\varkappa + \mathrm{hei}'^2\varkappa)}. \tag{X.97}$$

By substituting (X.97) into (X.43) and (X.65), we obtain the values of the components of the stress state on the contour of the hole[1]:

$$T_\varphi^* = 2p_0 h - p_0 h\varkappa \frac{\varkappa(\mathrm{hei}''\varkappa\,\mathrm{her}\varkappa - \mathrm{hei}\,\varkappa\,\mathrm{her}''\varkappa) + \nu(\mathrm{hei}'\varkappa\,\mathrm{her}\,\varkappa - \mathrm{her}'\varkappa\,\mathrm{hei}\,\varkappa)}{\varkappa(\mathrm{her}'\varkappa\,\mathrm{her}''\varkappa + \mathrm{hei}'\varkappa\,\mathrm{hei}''\varkappa) + \nu(\mathrm{her}'^2\varkappa + \mathrm{hei}'^2\varkappa)};$$
$$G_\varphi^* = p_0 h \frac{\varkappa h}{\sqrt{12(1-\nu^2)}} \cdot \frac{(1-\nu^2)(\mathrm{her}\,\varkappa\,\mathrm{her}'\varkappa + \mathrm{hei}\,\varkappa\,\mathrm{hei}'\varkappa)}{\varkappa(\mathrm{her}'\varkappa\,\mathrm{her}''\varkappa + \mathrm{hei}'\varkappa\,\mathrm{hei}''\varkappa) + \nu(\mathrm{her}'^2\varkappa + \mathrm{hei}'^2\varkappa)}. \tag{X.98}$$

This problem was first solved by Yu. A. Shevlyakov [5, 6], who used somewhat different symbols:

$$\chi_1 = -\mathrm{her}\,\varkappa + \frac{1-\nu}{\varkappa}\mathrm{hei}'\varkappa;$$
$$\chi_2 = -\mathrm{hei}\,\varkappa - \frac{1-\nu}{\varkappa}\mathrm{her}'\varkappa; \tag{X.99}$$
$$\Psi = \chi_2\,\mathrm{her}'\varkappa - \chi_1\mathrm{hei}'\varkappa.$$

The value $T^*_{\phi}|_{r=1}$ can be represented in the form

[1] If we assume R = ∞ in (X.98), we obtain the values T_ϕ^* in a flat plate in the case of three-dimensional uniform tension.

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$$T_{\varphi}^{*}|_{r=1} = 2p_0h\left[1 + k0{,}93\ \frac{r_0}{\sqrt{Rh}}\right]. \qquad (X.100)$$

For the shell R = 200 cm; r_0 = 10 cm; h = 0.2 cm; ν = 0.3; k = 1.65.

Thus, the concentration coefficient $k = T_{\phi}^{*}|_{p_0h}$ on the contour of the hole in the given shell is 2.65 times greater than k for a plate.

We present the values of concentration coefficient k for certain values of $\varkappa = \frac{r_0}{\sqrt{Rh}}\sqrt[4]{12(1-\nu^2)}$ as determined by Yu. A. Shevlyakov [5, 6]:

$\varkappa$	1,35	1,20	1,10	1,00	0,45	0,20	0,10
k	0,90	0,85	0,75	0,70	0,40	0,15	0,08

Elliptical Hole. For this hole /734

$$r_0 = \frac{a+b}{2}; \qquad \varepsilon = \frac{a-b}{a+b};$$
$$f(\zeta) = \frac{1}{\zeta}; \qquad N = 1. \qquad (X.101)$$

We will assume that the hole is covered by a roof of special construction, which transmits only the action of the shear force; then boundary conditions (X.74), with an accuracy up to ε^2, have the following form

$$T_0 = -p_0h; \qquad S_0 = 0; \qquad G_0 = 0;$$
$$\tilde{Q}_0 = -\frac{pr_0}{2}[1 - 2(\varepsilon - \varepsilon^2)\cos 2\theta]. \qquad (X.102)$$

By expanding (X.103) into a series with respect to ε, we obtain the right hand sides of the boundary conditions for the series approximations. The solution will be found in the form (X.65). The solution of this problem was found by the "boundary form perturbations" method by G. N. Savin and A. N. Guz' [2]. This problem was also analyzed by G. N. Savin and G. A. Van Fo Fy [1], where the solution was represented through Mathieu's function, although the latter was used for relatively small values of eccentricity.

The solution in the zero approximation coincides with that for a round hole (X.96)-(X.98), and therefore we will proceed to the determination of the functions $\Phi_j(r, \phi)$, making use of the results of §2.

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To determine the constants $c_2^{1,2}; d_2^{1,2}; c_4^{1,2}; d_4^{1,2}; d_2^{2,0}; c_4^{2,0}; d_4^{2,0}; d_2^{2,2}; c_2^{2,2}; c_4^{2,2}; d_4^{2,2}; c_2^{2,4}; d_2^{2,4}; c_4^{2,4}$ and $d_4^{2,4}$, we obtain a system of algebraic equations which, in view of their cumbersomeness, we will not introduce here. The other constants in (X.65) are equal to zero for j = 1 and j = 2. To determine $T_\theta^{(1)}$ and $T_\theta^{(2)}$ on the contour of the hole, we obtain

$$T_\theta^{(1)} = \frac{1}{nr_0^2}\Big[6d_2^{1,2} + \varkappa c_4^{1,2}\,\mathrm{hei}_2^{\cdot}\varkappa + \varkappa^2 d_4^{1,2}\,\mathrm{her}_2^{\cdot}\varkappa + \\ + p_0\frac{\varkappa^4 R}{E\Psi}(\chi_1\,\mathrm{hei}'''\varkappa - \chi_2\,\mathrm{her}'''\varkappa)\Big]\cos 2\theta\,; \tag{X.103}$$

$$T_\theta^{(2)} = \frac{1}{nr_0^2}\Big(-d_2^{2,0} + c_4^{2,0}\varkappa^2\mathrm{hei}''\varkappa + d_4^{2,0}\varkappa^2\mathrm{her}''\varkappa + \\ + 6d_2^{1,2} + c_4^{1,2}\Big(\frac{\varkappa^3}{2}\,\mathrm{hei}_2'''\varkappa + \varkappa^2\mathrm{hei}_2^{\cdot}\varkappa - 4\varkappa\,\mathrm{hei}_2'\varkappa + \\ + 4\,\mathrm{hei}_2\varkappa\Big) + d_4^{1,2}\Big(\frac{\varkappa^3}{2}\,\mathrm{her}_2'''\varkappa + \varkappa^2\mathrm{her}_2^{\cdot}\varkappa - 4\varkappa\,\mathrm{her}_2'\varkappa - \\ - 4\mathrm{her}_2\varkappa) + p_0\frac{\varkappa^2 R}{4E\Psi}\Big[\chi_1(-8\varkappa\,\mathrm{hei}''\varkappa + \varkappa^3\mathrm{hei}'''\varkappa + \\ + \varkappa^2\mathrm{hei}'''\varkappa + 8\,\mathrm{hei}'\varkappa) - \chi_2(-8\varkappa\,\mathrm{her}''\varkappa + \varkappa^3\mathrm{her}^{IV}\varkappa + \\ + \varkappa^2\mathrm{her}'''\varkappa + 8\mathrm{her}'\varkappa)\Big]\Big\} + \frac{1}{nr_0^2}\Big\{6d_2^{2,2} + \varkappa^2 c_4^{2,2}\mathrm{hei}_2^{\cdot}\varkappa + \\ + \varkappa^2 d_4^{2,2}\mathrm{her}_2^{\cdot}\varkappa\Big\}\cos 2\theta + \frac{1}{nr_0^2}\Big\{20d_2^{2,4} + c_4^{2,4}\varkappa\,\mathrm{hei}_4^{\cdot}\varkappa + \\ + d_4^{2,4}\varkappa^2\mathrm{her}_4^{\cdot}\varkappa - 30d_2^{1,2} + c_4^{1,2}\Big(\frac{\varkappa^3}{2}\,\mathrm{hei}_2'''\varkappa - \varkappa^2\mathrm{hei}_2^{\cdot}\varkappa + \\ + 4\varkappa\,\mathrm{hei}_2'\varkappa - 4\,\mathrm{hei}_2\varkappa\Big) + p_0\frac{\varkappa^2 R}{4E\Psi}\Big[\chi_1(8\varkappa\,\mathrm{hei}''\varkappa + \varkappa^3\mathrm{hei}'''\varkappa - \\ - \varkappa^2\mathrm{hei}'''\varkappa - 8\mathrm{hei}'\varkappa) - \chi_2(8\varkappa\,\mathrm{her}''\varkappa + \varkappa^3\mathrm{her}^{IV}\varkappa - \\ - \varkappa^2\mathrm{her}'''\varkappa - 8\,\mathrm{her}'\varkappa)\Big]\Big\}\cos 4\theta. \tag{X.104}$$

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The values of force T_θ^* on the contour of the hole are defined with an accuracy up to ε^2:

$$T_\theta^* = T_\varphi^* + \frac{a-b}{a+b}T_\theta^{(1)} + \Big(\frac{a-b}{a+b}\Big)^2 T_\theta^{(2)}, \tag{X.105}$$

where T_ϕ^* is given by formula (X.98); $T_\theta^{(1)}$ -- (X.103) and $T_\theta^{(2)}$ -- (X.104).

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As we see, formula (X.105) makes it possible, by changing the ratio a/b, to calculate the values T^*_θ for a fixed $r_0 = (a + b)/2$ for any family of elliptical holes.

As an example we will consider the value of concentration coefficient $k = T^*_\theta/p_0 h$ for a shell with the parameters R = 200 cm; (a + b)/2 = 10 cm; h = = 0.2 cm and $\nu = 0.3$. In this case

$$k = \frac{T^*_\theta}{p_0 h} = 5{,}30 + \frac{a-b}{a+b}\,19{,}44\cos 2\theta + \\ + \left(\frac{a-b}{a+b}\right)^2 (16{,}93 - 2{,}49\cos 2\theta + 10{,}96\cos 4\theta). \qquad (X.106)$$

The approximate solution for a plate under three-dimensional uniform tension by forces of intensity $p_0 h$ was also found by this method.

The values of k calculated by formula (X.105) are listed in Table X.1 for a plate and shell when $\theta = 0$ (at the end of the large semiaxis). As we see, the maximum value of k (when $\theta = 0$) in the plate, obtained in the second approximation, even for a/b = 1.5, differs by 1.5% from the value of k obtained from the precise solution.

The formula for T^*_θ on the contour of a hole in a plate with consideration of the second approximation has the form

$$T^*_\theta = 2p_0 h\,[1 + 2\varepsilon\cos 2\theta + 2\varepsilon^2\cos 4\theta + \ldots], \qquad (X.107)$$

while the formula for T^*_θ on the contour of the hole, obtained from the precise solution is

$$T^*_\theta = 2p_0 h\,\frac{1-\varepsilon^2}{1 - 2\varepsilon\cos 2\theta + \varepsilon^2}. \qquad (X.108)$$

If T^*_θ (X.108) is expanded into a series with respect to ε, then we obtain the formula for T^*_θ (X.107) with an accuracy up to ε^2. /736

For square and equilateral triangle holes with rounded corners, we will have the analogous situation in the plate, i.e., the values of k from the precise solution, expanded into a series with respect to ε, will coincide with an accuracy up to ε^2 with the value of k found from the approximate solution with consideration of the second approximation.

Table X.1 gives the representation of the rate of convergence of the solution found for a shell weakened by an elliptical hole. Even for a/b = 1.5, the rate of convergence of the solution found should be regarded as satisfactory,

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since the maximum value of k in the first approximation is 73% greater than that in the zero approximation, while k in the second approximation is 11% greater than in the first approximation.

Figure X.8 illustrates the distribution of T_θ^*/p_0h found from (X.106) through cross sections $\theta = 0$ (curve I); $\theta = \pi/2$ (curve II) for a round hole, i.e., when a/b = 1 (l is dimensionless, related to r_0, the distance from the contour of the hole), while Figure X.9 illustrates the distribution[1] of $k = T_\theta^*/p_0h$ on the contour of an elliptical hole when R = 200 cm; r_0 = 10 cm; h = 0.2 cm; ν = 0.3; a/b = 1.5.

TABLE X.1

	Approximation	a/b					
		1,00	1,10	1,20	1.30	1,40	1,50
In plate	Precise solution	2,00	2,21	2,44	2,62	2,80	3,00
	Zero	2,00	2,00	2,00	2,00	2,00	2,00
	First	2,00	2,19	2,39	2,52	2,66	2,80
	Second	2,00	2,20	2,43	2,59	2,76	2,96
In shell	Zero	5,30	5,30	5,30	5,30	5,30	5,30
	First	5,30	6,27	7,05	7,53	8,23	9,19
	Second	5,30	6,31	7,25	8,03	8,94	10,20

Tr. Note: Commas indicate decimal points.

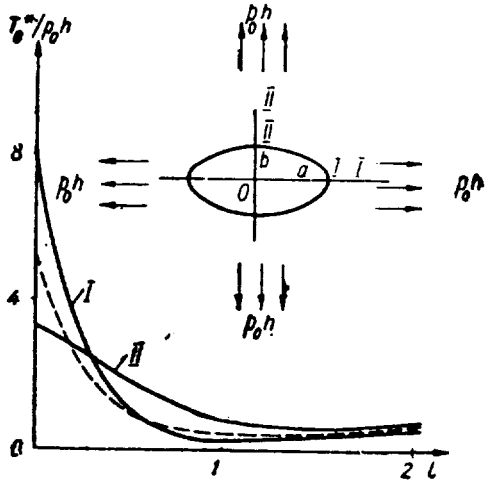


Figure X.8.

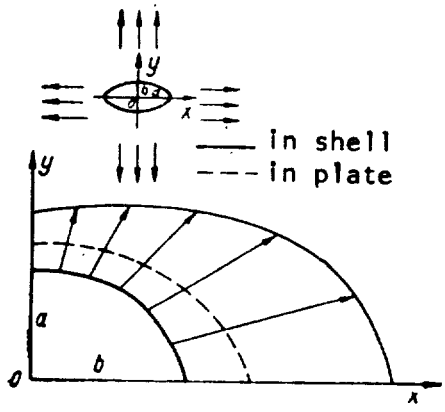


Figure X.9.

[1]The value of k in Figures X.9, X.11, X.13, X.20-X.29 is set forth (in some scale) with respect to the vector radius with its sign indicated (+ or -).

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Square Hole with Rounded Corners. We will examine the stress state in a spherical shell weakened by a square hole with rounded corners and loaded by a uniform internal pressure of intensity p. We will assume that the hole is covered with a roof which transmits the pressure imparted to it only in the form of a shear force, and the basic stress state we will assume to be momentless.

The solution of this problem was found by A. N. Guz' [10]. For a square hole with rounded corners we have

$$\varepsilon = \pm\frac{1}{9}; \quad f(\zeta) = \frac{1}{\zeta^3}. \tag{X.109}$$

In accordance with (X.109) the boundary conditions, recalling (X.74), can be written in the form /737

$$T_\varrho = -p_0 h; \quad S_{\varrho\theta} = 0;$$
$$G_\varrho = 0; \quad \widetilde{Q}_\varrho = -\frac{pr_0}{1}\left[1 - \frac{3}{2}\varepsilon^2 - 2(\varepsilon - 2\varepsilon^2)\cos 4\gamma\right]. \tag{X.110}$$

By expanding (X.110) into series with respect to ε, we obtain the boundary conditions for any approximation. The solution in the zero approximation coincides with the solution for a round hole (X.96)-(X.98). Let us turn to the determination of the function $\Phi_j(r, \phi)$ for j = 1 and j = 2, which we will write in the form (X.65). To determine the constants $c_2^{1,4}; d_2^{1,4}; c_4^{1,4}; d_4^{1,4}; d_2^{2,0}; c_4^{2,0}; d_4^{2,0}; c_2^{2,4}; d_2^{2,4}; c_4^{2,4}; d_4^{2,4}; c_2^{2,6}; d_2^{2,8}; c_4^{2,8}$ and $d_4^{2,8}$, algebraic equation systems have been derived, which we will not present here because of their awkwardness.

The other constants in (X.65) are equal to zero when j = 1 and j = 2. For the determination of $T_\theta^{(1)}$ and $T_\theta^{(2)}$ on the contour of the hole, we obtain the formulas

$$T_\theta^{(1)} = \frac{1}{nr_0^2}\left[20d_2^{1,4} + \varkappa^2 c_4^{1,4}\mathrm{hei}_4'\varkappa + \varkappa^2 d_4^{1,4}\mathrm{her}_4'\varkappa + \right.$$
$$\left. + p_0\frac{\varkappa^4 R}{E\Psi}(\chi_1\mathrm{hei}_4'''\varkappa - \chi_2\mathrm{her}_4'''\varkappa)\right]\cos 4\theta, \ldots \tag{X.111}$$
$$T_\theta^{(2)} = \frac{1}{nr_0^2}\left\{-d_2^{2,0} + c_4^{2,0}\varkappa^2\mathrm{hei}_4''\varkappa + d_4^{2,0}\varkappa^2\mathrm{her}_4''\varkappa + 60d_2^{1,4} + \right.$$
$$\left. + c_4^{1,4}\left(\frac{\varkappa^2}{2}\mathrm{hei}_4'''\varkappa + 2\varkappa^2\mathrm{hei}_4'\varkappa - 16\varkappa\,\mathrm{hei}_4'\varkappa + 16\,\mathrm{hei}_4\varkappa\right) + \right.$$

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$$+ d_4^{1,4}\left(\frac{\varkappa^3}{2}\,\mathrm{her}_4'''\varkappa + 2\varkappa^2\mathrm{her}_4''\varkappa - 16\varkappa\,\mathrm{her}_4'\varkappa + 16\,\mathrm{her}_4\varkappa\right) +$$

$$+ p_0\frac{\varkappa^2 R}{4E\Psi}\left[\chi_1(-32\varkappa\,\mathrm{hei}''\varkappa + \varkappa^3\mathrm{hei}^{IV}\varkappa + \varkappa^2\mathrm{hei}'''\varkappa + \right.$$

$$+ 32\,\mathrm{hei}'\varkappa) - \chi_2(-32\varkappa\,\mathrm{her}''\varkappa + \varkappa^3\mathrm{her}^{IV}\varkappa + \varkappa^2\mathrm{her}'''\varkappa +$$

$$\left.+ 32\,\mathrm{her}'\varkappa)\right]\Big\} + \frac{1}{nr_0^2}\left\{20d_2^{2,4} + \varkappa^2(c_4^{2,4}\mathrm{hei}_4''\varkappa + \right.$$

$$\left.+ d_4^{2,4}\mathrm{her}_4''\varkappa)\right\}\cos 4\theta + \frac{1}{nr_0^2}\left\{72d_2^{2,8} + c_4^{2,8}\varkappa^2\mathrm{hei}_8''\varkappa + \right.$$

$$+ d_4^{2,8}\varkappa^2\mathrm{her}_8''\varkappa - 180d_2^{1,4} + c_4^{1,4}\left[\frac{\varkappa^3}{2}\,\mathrm{hei}_4'''\varkappa - 2\varkappa^2\mathrm{hei}_4''\varkappa + \right.$$

$$\left.+ 16(\varkappa\,\mathrm{hei}_4'\varkappa - \mathrm{hei}_4\varkappa)\right] + d_4^{1,4}\left[\frac{\varkappa^3}{2}\,\mathrm{her}_4'''\varkappa - \right.$$

$$\left.- 2\varkappa^2\mathrm{her}_4''\varkappa + 16(\varkappa\,\mathrm{her}_4'\varkappa - \mathrm{her}_4\varkappa)\right] +$$

$$+ p_0\frac{\varkappa^2 R}{4E\Psi}\left\{\chi_1(32\varkappa\,\mathrm{hei}''\varkappa + \varkappa^3\mathrm{hei}^{IV}\varkappa - \right.$$

$$- \varkappa^2\mathrm{hei}'''\varkappa - 32\,\mathrm{hei}'\varkappa) - \chi_2(32\varkappa\,\mathrm{her}''\varkappa +$$

$$\left.+ \varkappa^3\mathrm{her}^{IV}\varkappa - \varkappa^2\mathrm{her}'''\varkappa - 32\,\mathrm{her}'\varkappa)\right\}\Big]\Big\}\cos 8\theta\,. \qquad (X.112)$$

The values of T_θ^* on the contour of the hole are determined with an accuracy up to ε^2 by the formula

$$T_\theta^* = T_\varphi^* + \varepsilon T_\theta^{(1)} + \varepsilon^2 T_\theta^{(2)} + \ldots \qquad (X.113)$$

where T_ϕ^* is given by formula (X.98), $T_\theta^{(1)}$ -- (X.111), and $T_\theta^{(2)}$ -- (X.112).

As an example, let us consider the value of the concentration coefficient $k = T_\theta^*/p_0 h$ on the contour of a hole as calculated by formula (X.113), for a shell with the parameters R = 200 cm; r_0 = 10 cm; h = 0.2 cm; ν = 0.3; ε = 1/9.

With consideration of the second approximation

$$k = 5{,}85 + 3{,}22\cos 4\theta + 1{,}01\cos 8\theta. \qquad (X.114)$$

The values k_{max} for a plate and shell are presented in the table (for θ = = 0).

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TABLE X.2

κ	Approximation			Precise solution
	Zero	First	Second	
In plate for θ=0	2,00	3,33	3,77	4,00
In shell for θ=0	5,30	8,53	10,08	—

Tr. Note: Commas indicate decimal points.

Table X.2 gives a representation of the rate of convergence of the solution obtained. As we see, the value of k in the second approximation for a plate differs from that found from the precise solution by 5.7%. The maximum value of k in a shell in the first approximation is greater than the zero approximation by 61%, while the value of k in the second approximation is 18% greater than k in the first approximation. The value of k_{max} in a shell for a square is 92% greater than that for a round hole, and in a plate, 89% greater by the approximate solution and 180% greater by the precise solution.

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Figure X.10 shows the distribution of T^*_θ/p_0h for the given example through cross sections θ = 0 (curve I); θ = π/4 (curve II), and for a round hole (ζ is dimensionless, related to r_0, the distance from the contour of the hole), while Figure X.11 shows the distribution of T^*_θ/p_0h found by formula (X.114) on the contour of a hole for ε = 1/9R = 200 cm; h = 0.2 cm; r_0 = 10 cm; ν = 0.3.

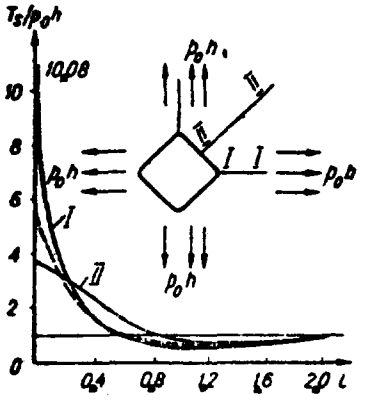


Figure X.10.

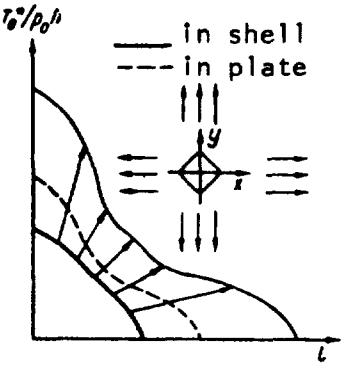


Figure X.11.

Equilateral Triangle Hole with Rounded Corners. We will examine the stress state of a spherical shell weakened by an equilateral triangle hole with rounded corners and loaded by a uniform internal pressure[1] of intensity p. We will assume that the hole is covered by a roof of special construction, which transmits the pressure imparted to it only in the form of a shear force, and the basic stress state in the shell we will assume to be momentless.

[1]See A. N. Guz' [3].

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For an equilateral triangle hole with rounded corners we have

$$\varepsilon = \pm \frac{1}{4}; \quad f(\zeta) = \frac{1}{\zeta^2}; \quad N = 2. \tag{X.115}$$

In accordance with (X.115) the boundary conditions, recalling (X.74), can be written in the form /740

$$\begin{gathered} T_0 = -p_0 h; \; S_{0\theta} = 0; \; G_0 = 0; \\ \widetilde{Q}_0 = -\frac{pr_0}{2}\left[1 - \frac{1}{2}\varepsilon^2 - (2\varepsilon - 3\varepsilon^2)\cos 3\theta\right]. \end{gathered} \tag{X.116}$$

By expanding the boundary conditions (X.116) into series with respect to ε we obtain the boundary conditions for series approximations.

The solution in the zero approximation coincides with that for a round hole (X.96)-(X.98), and therefore we will proceed to the determination of the function $\Phi(r, \phi)$ for $j = 1$ and $j = 2$, which we will taken in the form (X.63). By substituting (X.65) in the boundary conditions for the first approximation, we obtain a system of algebraic equations for the determination of the constants $c_2^{1,3}; d_2^{1,3}; c_4^{1,3}; d_4^{1,3}; d_2^{2,0}; c_4^{2,0}; d_4^{2,0}; c_2^{2,3}; d_2^{2,3}; c_4^{2,3}; c_2^{2,6}; d_2^{2,6}; c_4^{2,6}$ and $d_4^{2,6}$, which, in view of their awkwardness, we will not present here.

The other constants (X.65), for $j = 1$ and $j = 2$, are equal to zero for an equilateral triangle hole with rounded corners. For the determination of $T_\theta^{(1)}$ and $T_\theta^{(2)}$ on the contour of the hole the following expressions are obtained:

$$T_\theta^{(1)} = \frac{1}{nr_0^2}\Big\{12d_2^{1,3} + \varkappa^2 c_4^{1,3}\mathrm{her}_3''\varkappa + a_4^{1,3}\varkappa^2\mathrm{her}_3''\varkappa + \\ + p_0\frac{\varkappa^4 R}{E\Psi}(\chi_1\mathrm{hei}'''\varkappa + \chi_2\mathrm{her}'''\varkappa)\Big\}\cos 2\theta; \tag{X.117}$$

$$T_\theta^{(2)} = \frac{1}{nr_0^2}\Big\{-d_2^{2,0} + c_4^{2,0}\varkappa^2\mathrm{hei}''\varkappa + d_4^{2,0}\varkappa^2\mathrm{her}''\varkappa + \\ + 24d_2^{1,3} + c_4^{1,3}\Big(\frac{\varkappa^3}{2}\mathrm{hei}_3'''\varkappa + 3\frac{\varkappa^2}{2}\mathrm{hei}_3''\varkappa - 9\varkappa\,\mathrm{hei}_3'\varkappa + \\ + 9\mathrm{hei}_3\varkappa) + d_4^{1,3}\Big(\frac{\varkappa^3}{2}\mathrm{her}_3'''\varkappa + 3\frac{\varkappa^2}{2}\mathrm{her}_3''\varkappa - 9\varkappa\,\mathrm{her}_3'\varkappa + \\ + 9\,\mathrm{her}_3\varkappa\Big) + p_0\frac{\varkappa^2 R}{4E\Psi}\Big[\chi_1(18\varkappa\,\mathrm{hei}''\varkappa + \varkappa^3\mathrm{hei}^{IV}\varkappa + \\ + \varkappa^2\mathrm{hei}'''\varkappa + 18\,\mathrm{hei}'\varkappa) - \chi_2(-18\varkappa\,\mathrm{her}''\varkappa + \varkappa^3\mathrm{her}^{IV}\varkappa +$$

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$$+ \varkappa^2 \mathrm{her}'''\varkappa + 18\,\mathrm{her}'\varkappa)\Big]\Big\} + \frac{1}{nr_0^2}\Big\{12d_2^{2,3} + c_4^{2,3}\varkappa^2\mathrm{hei}_3''\varkappa +$$

$$+ d_4^{2,3}\varkappa^2\mathrm{her}_3''\varkappa\Big\}\cos 3\theta + \frac{1}{nr_0^2}\Big\{42d_2^{2,6} +$$

$$+ c_4^{2,6}\varkappa^2\mathrm{hei}_6''\varkappa + d_4^{2,6}\varkappa^2\mathrm{her}_6''\varkappa - 84d_2^{1,3} + c_4^{1,3}\Big(\frac{\varkappa^3}{2}\mathrm{hei}_3'''\varkappa -$$

$$- 3\frac{\varkappa^2}{2}\mathrm{hei}_3''\varkappa + 9\varkappa\,\mathrm{hei}_3'\varkappa - 9\,\mathrm{hei}_3\varkappa\Big) + d_4^{1,3}\Big(\frac{\varkappa^3}{2}\mathrm{her}_3'''\varkappa - \quad /741$$

$$- 3\frac{\varkappa^2}{2}\mathrm{her}_3''\varkappa + 9\varkappa\,\mathrm{her}_3'\varkappa - 9\,\mathrm{her}_3\varkappa\Big) + p_0\frac{\varkappa^2}{HE\Psi}\Big[\chi_1(18\varkappa\,\mathrm{hei}''\varkappa +$$

$$+ \varkappa^3\mathrm{hei}^{IV}\varkappa - \varkappa^2\mathrm{hei}'''\varkappa - 18\,\mathrm{hei}'\varkappa) - \chi_2(18\varkappa\,\mathrm{her}''\varkappa +$$

$$+ \varkappa^3\mathrm{her}^{IV}\varkappa - \varkappa^2\mathrm{her}'''\varkappa - 18\,\mathrm{her}'\varkappa)\Big]\Big\}\cos 6\theta. \tag{X.118}$$

The value T_θ^* on the contour of the hole is given by the formula

$$T_\theta^* = T_\varphi^* + \varepsilon T_\theta^{(1)} + \varepsilon^2 T_\theta^{(2)} + \varepsilon^3 T_\theta^{(3)} + \ldots, \tag{X.119}$$

where T_φ^* is determined by formula (X.98), $T_\theta^{(1)}$ by (X.117), and $T_\theta^{(2)}$ by (X.118).

As an example we will consider the values of the concentration coefficient of forces $k = T_\theta^*/p_0 h$ on the contour of the hole as calculated by formula (X.119) for a shell with the parameters R = 200 cm; r_0 = 10 cm; h = 0.2 cm; ν = 0.3 cm.

In this case, with an accuracy up to ε^2,

$$k = 6{,}50 + 6{,}37\cos 3\theta + 2{,}48\cos 6\theta. \tag{X.120}$$

The values of k_{max} are presented in Table X.3 for a plate in the case of multilateral uniform tension by forces of intensity $p_0 h$ and for a shell (for $\theta = 0$). Table X.3 gives a representation of the rate of convergence of the solution obtained. The value $(k_{max})_{\theta=0}$ in the second approximation for a plate differs from $(k_{max})_{\theta=0}$ obtained from the precise solution by 16.7%. The value $(k_{max})_{\theta=0}$ in a shell in the first approximation is 122.1% greater than the zero approximation, and 30.4% greater in the second approximation than in the first. The value k_{max} in a shell for an equilateral triangle hole with rounded corners (when $\varepsilon = 1/4$), found with an accuracy up to ε^2, is 189.6% greater than k_{max} for

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a round hole[1]. The values of k_{max} in a plate for an equilateral triangle hole (with rounded corners) is 200% greater for a round hole in comparison with the precise solution, and 150% greater in comparison with the approximate solution (with an accuracy up to ε^2).

TABLE X.3

k_{max}	Approximation			Precise solution
	Zero	First	Second	
In plate for $\theta=0$	2,00	4,00	5,00	6,00
In shell for $\theta=0$	5,30	11,77	15,35	—

Tr. Note: Commas indicate decimal points.

The distribution of T^*_θ/p_0h is given in Figure X.12 for the given example through cross sections $\theta = 0$ (curve I); $\theta = \pi/4$ (curve II) and for a round hole (ζ is dimensionless, related to r_0, the distance from the contour of the hole), and Figure X.13 shows the distribution of T^*_θ/p_0h as calculated by formula (X.120) on the contour of the hole when R = 200 cm; h = 0.2 cm; r_0 = 10 cm; ν = 0.3; ε = 1/4.

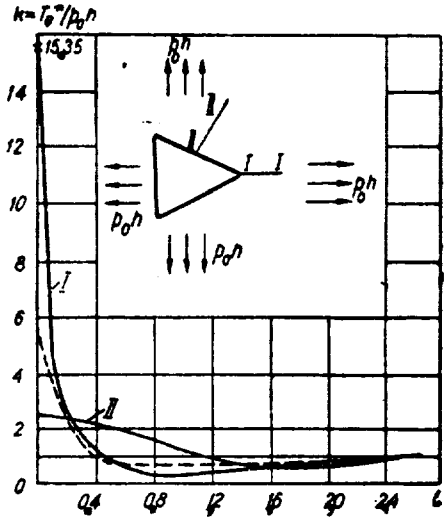


Figure X.12.

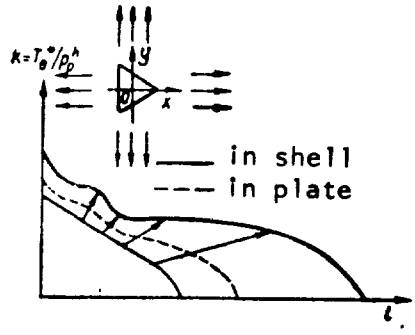


Figure X.13.

Effect of Anisotropy of Material. /742

We will analyze the stress state in a spherical orthotropic shell weakened by a round hole and under load by uniform internal pressure of intensity p. The shell has three surfaces of elastic symmetry, which coincide with the coordinate surfaces of the spherical coordinate system, the origin of which is located at the center of the shell. We will assume that the hole is covered by a roof of special construction, which transmits the pressure imparted to it in the form of a shear force, and we will further assume that the basic stress

[1]The value r_0 was assumed to be identical for the round and for the examined (equilateral) triangle holes.

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state is momentless. Moreover, we will assume that the contour of the hole coincides with one of the coordinate lines of the above-specified spherical coordinate system[1].

To analyze the additional stress state caused by the hole, we will relate the middle surface to polar coordinate system (r, ϕ), the origin of which is placed at the center of the hole.

Analysis leads to the solution of the equation

$$\nabla^2(\nabla^2 - i)\Phi = \frac{E_\varphi - E_r}{E_r}\cdot\frac{1}{\alpha^2}\left(\nabla^2 - 2\frac{1}{\alpha}\cdot\frac{d}{d\alpha}\right)\Phi \qquad (X.121)$$

under boundary conditions

$$T_r = -p_0 h;\ G_r = 0;\ \tilde{Q}_r = -\frac{pr_0}{2}, \qquad (X.122)$$

where $\Phi = w + in\hat{\phi}$; r_0 is the radius of the hole; $p_0 = \frac{pR}{2h}$;

$$n = \sqrt[4]{\frac{12(1-\nu_r\nu_\varphi)}{E_r E_\varphi h^4}};$$

$$k = \sqrt[4]{\frac{R^2h^2}{12(1-\nu_r\nu_\varphi)}\cdot\frac{E_r}{E_\varphi}}; \qquad (X.123)$$

/743

$r = k\alpha$; α is a variable; E_ϕ, E_r and ν_r, ν_ϕ are Young's modulus and Poisson's ratio, respectively.

The stress state components are found by the formulas

$$T_r = \frac{1}{nk^2\alpha}\cdot\frac{d}{d\alpha}\operatorname{Im}\Phi;\quad T_\varphi = \frac{1}{nk^2}\cdot\frac{d^2}{d\alpha^2}\operatorname{Im}\Phi;$$

$$G_r = -\frac{1}{k^2}\cdot\frac{E_r h^3}{12(1-\nu_r\nu_\varphi)}\left(\frac{d^2}{d\alpha^2} + \frac{\nu_\varphi}{\alpha}\cdot\frac{d}{d\alpha}\right)\operatorname{Re}\Phi; \qquad (X.124)$$

$$\tilde{Q}_r = -\frac{1}{k^3}\cdot\frac{E_r h^3}{12(1-\nu_r\nu_\varphi)}\left(\frac{d}{d\alpha}\nabla^2 - \frac{E_\varphi - E_r}{E_r}\cdot\frac{1}{\alpha^2}\cdot\frac{d}{d\alpha}\right)\operatorname{Re}\Phi.$$

We will introduce the parameter $\varepsilon = (E_\phi - E_r)/E_r$ and assume that $\varepsilon < 1$. If $\varepsilon > 1$ for the given material, then equation (X.121) and relation (X.124) are

[1]A. N. Guz' [1] considered this problem in such a statement.

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readily reconstructed and the parameter $\varepsilon' = (E_r - E_\phi)/E_\phi$ for which, under the condition $(E_\phi - E_r)/E_r > 1$, we obtain $(E_r - E_\phi)/E_\phi < 1$, is also readily isolated. Thus, for any relationship between Young's moduli, we can carry out expansion with respect to one of two parameters ε or ε'.

Let us consider the solution for the case $\varepsilon < 1$; for the case $\varepsilon' < 1$, however, the results obtained remain in force if E_r is substituted by E_ϕ in (X.123) and (X.124). We will represent the function Φ in the form

$$\Phi(\alpha, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \Phi_j(\alpha). \tag{X.125}$$

By substituting (X.125) into (X.123) and (X.124), recalling (X.125), we obtain a series of boundary problems for the equation

$$\nabla^2(\nabla^2 - i)\Phi_j = \frac{1}{\alpha^2}\left(\nabla^2 - \frac{2}{\alpha}\frac{d}{d\alpha}\right)\Phi_{j-1} \qquad (j = 1, 2, 3, \ldots) \tag{X.126}$$

under such boundary conditions, i.e., when $\alpha = \alpha_0$: /744

$$\begin{gathered}\frac{1}{\alpha}\cdot\frac{d}{d\alpha}\operatorname{Im}\Phi_j = -p_0 h k^2 n \delta_j^0; \\ \left(\frac{d^2}{d\alpha^2} + \frac{\nu_\varphi}{\alpha}\cdot\frac{d}{d\alpha}\right)\operatorname{Re}\Phi_j = 0; \\ \frac{d}{d\alpha}\nabla^2 \operatorname{Re}\Phi_j = \frac{1}{\alpha^2}\cdot\frac{d}{d\alpha}\operatorname{Re}\Phi_{j-1} + \frac{k^2 p r_0}{2E_r h^3}\,12\,(1 - \nu_r\nu_\varphi)\,\delta_j^0,\end{gathered} \tag{X.127}$$

where $\alpha_0 = kr_0$; $\delta_j^k = \begin{cases} 1 \text{ for } k = j, \\ 0 \text{ for } k \neq j; \end{cases}$ $\Phi_k(\alpha) \equiv 0$ for $k < 0$.

The solution of equation (X.126) in the j-th approximation, which satisfies the conditions "at infinity" (X.35), are represented in the form

$$\Phi_j(\alpha) = iC_j \ln\alpha + (A_j + iB_j) H_0^{(1)}(\alpha\sqrt{-i}) + \hat{\Phi}_j(\alpha), \tag{X.128}$$

where $H_0^{(1)}(\alpha\sqrt{-i}) = \operatorname{her}\alpha + i\operatorname{hei}\alpha$ is Hankel's function of the zero order and of the first kind; $\hat{\Phi}_j(\alpha)$ is the partial solution of equation (X.126).

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In the zero approximation we will find the function $\Phi_0(\alpha)$ (X.128) from boundary conditions (X.127) for j = 0 ($\hat{\Phi}_0 \equiv 0$):

$$\Phi_0 = D\frac{\chi_{01} - i\chi_{02}}{\Psi_0} H_0^{(1)}(\alpha\sqrt{-i}), \tag{X.129}$$

where

$$D = \frac{pr_0k^3}{2E_rh^3}12(1-\nu_r\nu_\varphi);$$

$$\chi_{01} = -\operatorname{her}\alpha_0 + \frac{1-\nu_\varphi}{\alpha_0}\operatorname{hei}'\alpha_0;$$

$$\chi_{02} = -\operatorname{hei}\alpha_0 - \frac{1-\nu_\varphi}{\alpha_0}\operatorname{her}'\alpha_0;$$

$$\Psi_0 = \chi_{02}\operatorname{her}'\alpha_0 - \chi_{01}\operatorname{hei}'\alpha_0.$$

When $E_r = E_\phi$ the function Φ_0 (X.129) coincides with the solution for an isotropic shell in the form (X.96)-(X.98).

For the determination of the function Φ_1 from (X.126), we obtain the equation

$$\nabla^2(\nabla^2 - i)\Phi_1 = D\frac{\chi_{01} - i\chi_{02}}{\Psi_0}\cdot\frac{1}{\alpha}\left(\nabla^2 - \frac{2}{\alpha}\cdot\frac{d}{d\alpha}\right)H_0^{(1)}(\alpha\sqrt{-i}), \tag{X.130}$$

the solution of which, in accordance with (X.128), can be represented in the form

$$\Phi_1(\alpha) = iC_1\ln\alpha + (A_1 + iB_1)H_0^{(1)}(\alpha\sqrt{-i}) - \\ - \frac{D}{2}\cdot\frac{\chi_{01} - i\chi_{02}}{\Psi_0}\int\frac{H_0^{(1)}(\alpha\sqrt{-i})}{\alpha}d\alpha. \tag{X.131}$$

By determining the constants C_1, A_1 and B_1 from boundary conditions (X.127), when j = 1, we obtain for $\Phi_1(\alpha)$ /745

$$\Phi_1(\alpha) = -\frac{D}{2}\cdot\frac{\chi_{01} - i\chi_{02}}{\Psi_0}\int\frac{H_0^{(1)}(\alpha\sqrt{-i})}{\alpha}d\alpha + \\ + \left[\frac{\frac{\operatorname{her}'\alpha_0}{\alpha_0}\left(\chi_{12} - \frac{1-\nu_\varphi}{\alpha_0}\chi_{11} + \chi_{01}\Psi_1\right)}{\Psi_0^2} - i\frac{\chi_{02}\Psi_1 + \frac{\operatorname{hei}'\alpha_0}{\alpha_0}\left(\chi_{12} - \frac{1-\nu_\varphi}{\alpha_0}\chi_{11}\right)}{\Psi_0^2}\right]\times \\ \times\frac{D}{2}H_0^{(1)}(\alpha\sqrt{-i}), \tag{X.132}$$

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where

$$\chi_{11} = \chi_{01} \operatorname{her} \alpha_0 + \chi_{02} \operatorname{hei} \alpha_0, \tag{X.133}$$

$$\chi_{12} = \chi_{01} \operatorname{her}' \alpha_0 + \chi_{02} \operatorname{hei}' \alpha_0;$$

$$\Psi_1 = -\chi_{01} \operatorname{hei} \alpha_0 + \chi_{02} \operatorname{her} \alpha_0. \tag{X.134}$$

By using (X.129) and (X.132), we can determine the values T^*_ϕ on the contour of the hole with an accuracy up to ε in the form[1]

$$T^*_\varphi = 2p_0h\left(1 + \frac{\alpha_0}{2}\cdot\frac{\chi_{11}}{\Psi_0}\right) + \frac{E_\varphi - E_r}{E_r}\cdot\frac{p_0h}{2}\left[1 + 2\frac{\Psi_1\chi_{11}}{\Psi_0^2} - \left(\frac{\operatorname{her}\alpha_0 \operatorname{her}'\alpha_0 + \operatorname{hei}\alpha_0 \operatorname{hei}'\alpha_0}{\Psi_0}\right)^2\right]. \tag{X.135}$$

By way of example let us examine the values $k = T^*_\phi/p_0h$ calculated by formula (X.135) on the contour of a hole for various ratios between E_r and E_ϕ, ν_r and ν_ϕ for a spherical shell with a round hole, the parameters of which are $R = 25$ cm; $r_0 = 2$ cm; $h = 0.4$ cm. The results of calculations are presented in Table X.4. The accurate solution of equation (X.121) is given in the work of V. Z. Karnaukhov [1], although it is expressed through untabulated functions.

TABLE X.4

k	E_φ/E_r							
	1,50	1,00	0,91	0,83	0,77	0,71	0,67	0,50
In shell for ν_r=0,3	3,17	2,90	2,85	2,78	2,76	2,71	2,68	2,54
In shell for ν_r=0,15	3,10	2,86	2,80	2,76	2,73	2,68	2,66	2,38
In plate	2,22	2,00	1,95	1,91	1,88	1,85	1,81	1,71

Tr. Note: Commas indicate decimal points.

Round Hole Whose Edge Is Reinforced by Thin Elastic Ring. We will examine the stress state of a spherical shell weakened by a round hole and loaded by a uniform internal pressure of intensity p. The edge of the hole is reinforced by a thin elastic ring, which we will regard as a material thread that resists /746

[1]The errors in formula (X.135) which occur in the corresponding formulas in the work of A. N. Guz' [1], concerning which the author generously commented, have been corrected.

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tension, bending and twisting; one of the axes of inertia of the cross section of the reinforcing ring lies in the middle area of the shell. We will assume that the hole is covered by a roof of special construction, which transmits the action of pressure imparted to it to the ring in the form of a shear force. The basic stress state will be regarded as momentless. This problem is examined in the works of V. N. Buyvol, S. A. Goloborod'ko and K. I. Shnerenko [1]; A. N. Guz' and G. N. Savin [1]. This problem was examined in the latter work as the zero approximation for the solution of problems of the stress state in shells near a curvilinear hole, the edge of which is reinforced by a thin elastic ring. Here we will follow the example of A. N. Guz' and G. N. Savin [1]. The boundary conditions for this problem can be obtained from (X.59) for j = 0, assuming

$$\begin{gathered} T_r^{0(0)} = p_0 h; \qquad G_r^{0(0)} = 0; \qquad \tilde{Q}_r^{0(0)} = 0; \\ u^{0(0)} = p_0 \frac{1-\nu}{E} r_0 r; \qquad v^{0(0)} = 0; \qquad w^{0(0)} = 0; \\ T_r^{(0)(0)} = 0; \quad G_r^{(0)(0)} = 0; \quad \tilde{Q}_r^{(0)(0)} = -\frac{pr_0}{2}. \end{gathered} \tag{X.136}$$

The function $\Phi(r, \phi)$ is taken in the form (X.65), considering (X.67) and (X.72); the variables are taken in the form (X.70). As we see from the boundary conditions, the problem is axisymmetric, and therefore, in (X.65) and (X.70) remain terms only for k = 0. By substituting (X.65) and (X.70) into the boundary conditions (X.59) for j = 0 and considering (X.67), (X.72) and (X.136), we determine the constants

$$\begin{gathered} d_2^{0,0} = \frac{pRE_1 F r_0 \sqrt{12(1-\nu^2)}}{Eh^2 [Ehr_0 + E_1 F(1+\nu)]}; \\ c_4^{0,0} = -\frac{pr_0^4}{2D\varkappa^3} \cdot \frac{r_0 \varkappa D \operatorname{hei}'' \varkappa + (Dr_0 \nu - A) \operatorname{hei}' \varkappa}{r_0 \varkappa D (\operatorname{her}' \varkappa \operatorname{her}'' \varkappa + \operatorname{hei}' \varkappa \operatorname{hei}'' \varkappa) + (Dr_0 \nu - A)(\operatorname{her}'^2 \varkappa + \operatorname{hei}'^2 \varkappa)}; \\ d_4^{0,0} = \frac{r_0 \varkappa D \operatorname{her}'' \varkappa + (Dr_0 \nu - A) \operatorname{her}' \varkappa}{r_0 \varkappa D \operatorname{hei}'' \varkappa + (Dr_0 \nu - A) \operatorname{hei}' \varkappa} c_4^{0,0}. \end{gathered} \tag{X.137}$$

By substituting (X.65) in (X.43) and considering (X.127), we obtain the stress state components on the contour of the hole

$$\begin{gathered} T_r^* = p_0 h \frac{2E_1 F}{Ehr_0 + E_1 F(1+\nu)}; \qquad T_\varphi^* = 2p_0 h \frac{Ehr_0 + \nu E_1 F}{Ehr_0 + E_1 F(1+\nu)} - \\ - \frac{p_0 h \varkappa}{M} [r_0 \varkappa D (\operatorname{hei}'' \varkappa \operatorname{her} \varkappa - \operatorname{hei} \varkappa \operatorname{her}'' \varkappa) + (Dr_0 \nu - A)(\operatorname{hei}' \varkappa \operatorname{her} \varkappa - \operatorname{her}' \varkappa \operatorname{hei} \varkappa)]; \\ G_r^* = -\frac{\varkappa h A}{M \sqrt{12(1-\nu^2)}} (\operatorname{hei}' \varkappa \operatorname{her}'' \varkappa - \operatorname{her}' \varkappa \operatorname{hei}'' \varkappa); \end{gathered} \tag{X.138}$$

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$$G_{\varphi}^{*} = \frac{p_0 h^2 \varkappa}{M\sqrt{12(1-\nu^2)}}\,(Dr_0 - Dr_0\nu^2 + A\nu)(\mathrm{her}\,\varkappa\,\mathrm{her}'\,\varkappa + \mathrm{hei}\,\varkappa\,\mathrm{hei}'\,\varkappa),$$

where

$$M = r_0\varkappa D\,(\mathrm{her}'\,\varkappa\,\mathrm{her}''\,\varkappa + \mathrm{hei}''\,\varkappa\,\mathrm{hei}'\,\varkappa) + (Dr_0\nu - A)(\mathrm{her}'^{2}\varkappa + \mathrm{hei}'^{2}\varkappa).$$

We will notice that T_r^* (X.138) for a shell in the case of a round hole does not depend on the radius R of the sphere and coincides with T_r^* for a round hole in a plate under multilateral tension by forces of intensity p_0h.

If we assume in (X.138) that $R = \infty$, we obtain the values of the stress state components for a flat plate; if we assume $E_1 = 0$ or $E_1 = \infty$ (E_1 enters the expressions for rigidities A, B and C linearlly), we obtain the values of the components that characterize the stress state on the contour not reinforced or reinforced, respectively, by an absolutely rigid ring, of a round hole in a spherical shell.

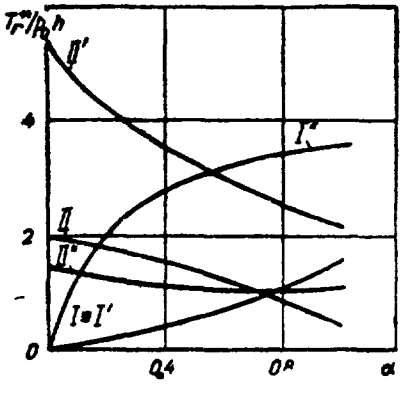


Figure X.14.

The calculations of T_r^* for a spherical shell with the parameters R = = 200 cm; h = 0.2 cm; r_0 = 10 cm; ν = = 0.3 and ν_1 = 0.3 (for a ring) and for a reinforcing ring of square cross section, the length of whose side is $0.1r_0$, are made in accordance with formula (X.138).

Figure X.14 shows the dependences of T_r^*/p_0h found from (X.138), -- curve I for a plate and shell (it coincides in accordance with (X.138)); T_ϕ^*/p_0h (X.138) -- curve II for a plate; curve II' -- for a shell; $6G_r^*/p_0h^2$ (X.138) -- curve I''; $6G_\phi^*/p_0h^2$ (X.138) -- curve II'' as a function of the parameter $\alpha = E_1/(E_1 + E)$. We see from the above-mentioned graphs that change in rigidity of the reinforcing ring can alter considerably the stress state near the hole.

In accordance with (X.43) the components characterizing the stress state in a shell near a hole can be represented in the form

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$$T_r^* = p_0 h + \frac{1}{nr_0^2}\left[\frac{d_2^{0,0}}{r^2} + \frac{\varkappa}{r}\left(c_4^{0,0}\,\mathrm{hei}'\,\varkappa r + d_4^{0,0}\,\mathrm{her}'\,\varkappa r\right)\right];$$

$$T_\varphi^* = p_0 h + \frac{1}{nr_0^2}\left[-\frac{d_2^{0,0}}{r^2} + c_4^{0,0}\varkappa^2\,\mathrm{hei}''\,\varkappa r + d_4^{0,0}\varkappa^2\,\mathrm{her}''\,\varkappa r\right];$$

$$G_r^* = -D\frac{\varkappa^2}{r_0^2}\{c_4^{0,0}[(1-\nu)\,\mathrm{her}''\,\varkappa r - \nu\,\mathrm{hei}\,\varkappa r] - d_4^{0,0}[(1-\nu)\,\mathrm{hei}''\,\varkappa r + \nu\,\mathrm{her}\,\varkappa r]\}; \qquad (X.139)$$

$$G_\varphi^* = D\frac{\varkappa^2}{r_0^2}\{c_4^{0,0}[\mathrm{hei}\,\varkappa r + (1-\nu)\,\mathrm{her}''\,\varkappa r] + d_4^{0,0}[\mathrm{her}\,\varkappa r - (1-\nu)\,\mathrm{hei}''\,\varkappa r]\}.$$

If we assume in (X.139) that r = 1 and substitute the values $d_2^{0,0}$; $c_4^{0,0}$ and $d_4^{0,0}$ from (X.137), we obtain the formula (X.138). /748

Elliptical Hole Whose Edge Is Reinforced by Elastic Ring. We will consider the stress state in a spherical shell weakened by an elliptical hole and loaded by a uniform internal pressure of intensity p. The edge of the hole is reinforced by a thin elastic ring, which we will regard as a material fiber that resists tension, bending and twisting, whereupon one of the axes of inertia of the cross section of the reinforcing ring lies in the middle surface of the shell. We will assume that the hole is covered by a roof of special construction, which transmits pressure imparted to it to the reinforcing ring in the form of a shear force, and that the basic stress state is momentless[1].

The law of distribution of the shear force is taken in the form (X.102).

The boundary conditions are found from (X.58) in the form

$$T_\varrho^0 = p_0 h; \quad T_\theta^0 = p_0 h; \quad S_{\varrho\theta}^0 = 0;$$

$$G_\varrho^0 = 0; \quad G_\theta^0 = 0; \quad \tilde{Q}_\varrho^0 = 0;$$

$$\tilde{Q}_\theta^0 = 0; \quad u^0 = p\frac{1-\nu}{E} r_0 r; \qquad (X.140)$$

$$v^0 = 0; \quad w^0 = 0; \quad T_\varrho^{(0)} = 0; \quad S_{\varrho\theta}^{(0)} = 0; \quad G_\varrho^{(0)} = 0;$$

$$Q_\varrho^{(0)} = -\frac{pr_0}{2}[1 - 2(\varepsilon - \varepsilon^2)\cos 2\theta].$$

By expanding (X.140) into series with respect to ε and substituting into (X.59), we obtain the boundary conditions for series approximations.

[1] This problem was examined by A. N. Guz' and G. N. Savin [1].

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The zero approximation coincides with the solution for a round hole (X.136) and (X.139), and therefore we will proceed to the determination of the first approximation. The function $\Phi_j(r, \phi)$ is taken in the form (X.65), considering (X.67) and (X.72), and the variables in the form (X.70) for $j = 1$. By substituting (X.65) and (X.70) into boundary conditions (X.59) for $j = 1$ and considering (X.67), (X.72) and (X.140), we obtain a system of algebraic equations for the determination of the constants $c_2^{1,2}$; $d_2^{1,2}$; $c_4^{1,2}$ and $d_4^{1,2}$, which, due to their awkwardness, will not be presented here. The other constants in (X.65) and (X.70) for $j = 1$ for the examined case are equal to zero.

For small values of eccentricity $\varepsilon = (a - b)/(a + b)$, we will confine ourselves to the zero and first approximations. Thus we have formulas for the stress state components in a shell with an accuracy up to ε in the form /749

$$T_\varrho^* = p_0 h + \frac{1}{nr_0^2}\left(\frac{d_2^{0,0}}{\varrho^2} + \varkappa\frac{c_4^{0,0}\mathrm{hei}'\varkappa\varrho + d_4^{0,0}\mathrm{her}'\varkappa\varrho}{\varrho}\right) +$$

$$+ \varepsilon\frac{\varkappa^2}{nr_0^2}\left[-6\frac{d_2^{1,2}}{\varkappa^2\varrho^4} + c_4^{1,2}(\mathrm{her}_2\varkappa\varrho - 4\mathrm{hei}_2^{\cdot}\varkappa\varrho) -\right.$$

$$- d^{1,2}(\mathrm{hei}_2\varkappa\varrho + 4\mathrm{her}''\varkappa\varrho) - \frac{2}{\varkappa^2}\cdot\frac{d_2^{0,0}}{\varrho^4} +$$

$$\left. + c_4^{0,0}\left(\frac{\mathrm{hei}''\varkappa\varrho}{\varrho^2} + \frac{\mathrm{hei}'\varkappa\varrho}{\varkappa\varrho^3}\right) + d_4^{0,0}\left(\frac{\mathrm{her}''\varkappa\varrho}{\varrho^2} - \frac{\mathrm{her}'\varkappa\varrho}{\varkappa\varrho^3}\right)\right]\cos 2\vartheta;$$

$$T_\vartheta^* = p_0 h + \frac{1}{nr_0^2}\left(-\frac{d_2^{0,0}}{\varrho^2} + c_4^{0,0}\varkappa^2\mathrm{hei}^{\cdot}\varkappa\varrho + d_4^{0,0}\varkappa^2\mathrm{her}^{\cdot}\varkappa\varrho\right) +$$

$$+ \varepsilon\frac{\varkappa^2}{nr_0^2}\left[6\frac{d_2^{1,2}}{\varkappa^2\varrho^4} + c_4^{1,2}\mathrm{hei}_2^{\cdot}\varkappa\varrho +\right. \qquad \text{(X.141)}$$

$$\left. + d_4^{1,2}\mathrm{her}_2^{\cdot}\varkappa\varrho + 2\frac{d_2^{0,0}}{\varkappa^2\varrho^4} + c_4^{0,0}\varkappa\frac{\mathrm{hei}'''\varkappa\varrho}{\varrho} + d_4^{0,0}\varkappa\frac{\mathrm{her}'''\varkappa\varrho}{\varrho}\right]\cos 2\vartheta;$$

$$G_\varrho^* = -D\frac{\varkappa^2}{r_0^2}\{c_4^{0,0}[(1-\nu)\mathrm{her}^{\cdot}\varkappa\varrho - \nu\,\mathrm{hei}\,\varkappa\varrho] - d_4^{0,0}[(1-\nu)\mathrm{hei}^{\cdot}\varkappa\varrho +$$

$$+ \nu\,\mathrm{her}\,\varkappa\varrho]\} - \varepsilon D\frac{\varkappa^2}{r_0^2}\left\{6c_2^{1,2}\frac{1-\nu}{\varkappa^2\varrho^4} + c_4^{1,2}[(1-\nu)\mathrm{her}_2^{\cdot}\varkappa\varrho - \nu\,\mathrm{hei}_2\varkappa\varrho] -\right.$$

$$- d_4^{1,2}[(1-\nu)\mathrm{hei}_2^{\cdot}\varkappa\varrho + \nu\,\mathrm{her}_2\varkappa\varrho] +$$

$$+ \frac{\varkappa}{\varrho}c_4^{0,0}[(1-\nu)\mathrm{her}'''\varkappa\varrho - \nu\,\mathrm{hei}'\varkappa\varrho] -$$

$$\left. - d_4^{0,0}\frac{\varkappa}{\varrho}[(1-\nu)\mathrm{hei}'''\varkappa\varrho + \nu\,\mathrm{her}'\varkappa\varrho]\right\}\cos 2\vartheta;$$

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$$G_\theta^* = D\frac{\varkappa^2}{r_0^2}\{c_4^{0,0}[\mathrm{hei}\,\varkappa\varrho + (1-\nu)\,\mathrm{ner}''\,\varkappa\varrho] +$$

$$+ d_4^{0,0}[\mathrm{her}\,\varkappa\varrho - (1-\nu)\,\mathrm{hei}''\,\varkappa\varrho]\} + eD\frac{\varkappa^2}{r_0^2}\left\{c_2^{1,2}\frac{1-\nu}{\varkappa^2\varrho^4} +\right.$$

$$+ c_4^{1,2}[(1-\nu)\,\mathrm{her}_2^*\,\varkappa\varrho + \mathrm{hei}_2\,\varkappa\varrho] + d_4^{1,2}[\mathrm{her}_2\,\varkappa\varrho - (1-\nu)\,\mathrm{hei}_2^*\,\varkappa\varrho] +$$

$$\left. + \frac{\varkappa}{\varrho}c_4^{0,0}[\mathrm{hei}'\,\varkappa\varrho + (1-\nu)\,\mathrm{her}'''\,\varkappa\varrho] + \frac{\varkappa}{\varrho}d_4^{0,0}[\mathrm{her}'\,\varkappa\varrho - (1-\nu)\mathrm{hei}'''\varkappa\varrho]\right\}\cos 2\vartheta.$$

In order to find the values of stresses on the contour of the hole we must assume in (X.141) that $\rho = 1$.

/750

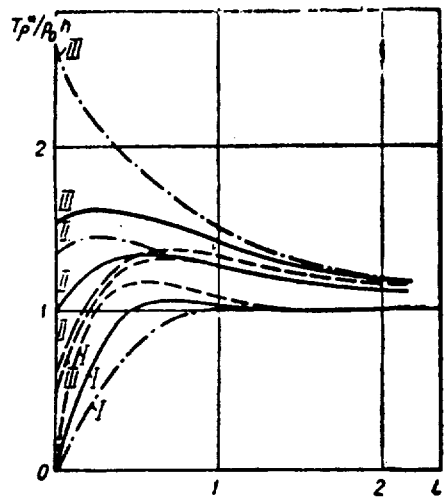


Figure X.15.

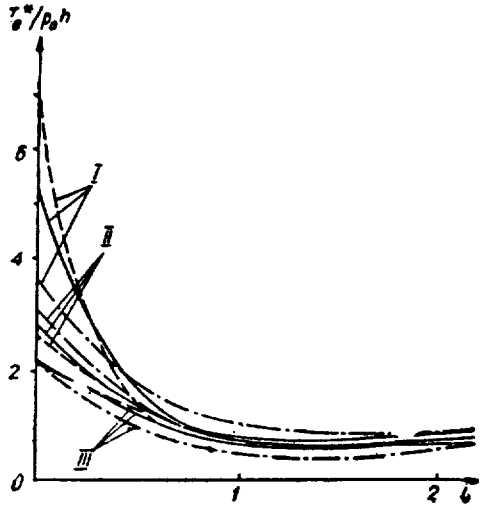


Figure X.16.

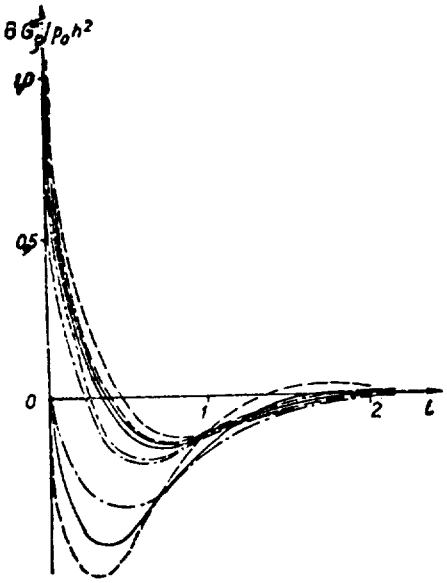


Figure X.17.

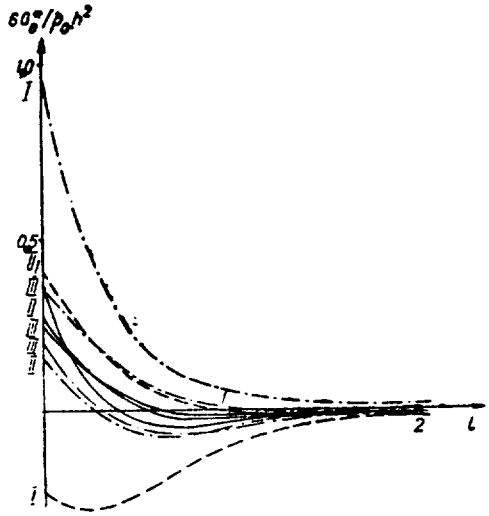


Figure X.18.

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By way of example we have determined the coefficients T_ρ^*; T_θ^*; G_ρ^* through /751
cross sections $\theta = 0$ and $\theta = \pi/2$ of a round hole for $E_1 = 0$; $E_1/E = 2.632$; $E_1 = \infty$ for a shell and a ring with the parameters $R = 200$ cm; $r_0 = 10$ cm; $h = 0.2$ cm; $\nu = 0.3$; $\nu_1 = 0.3$ (for a ring); $a/b = 1.2$. A ring of square cross section with a side of length $0.1\ r_0$ was used. The dependences T_ρ^*/p_0h; T_θ^*/p_0h; $6G_\rho^*/p_0h^2$ and $6G_\theta^*/p_0h^2$ are shown graphically in Figures X.15-X.18, respectively, as calculated by formulas (X.141), as functions of the dimensionless parameter ζ (distance from the contour of the hole), related to $r_0 = (a + b)/2$. The solid lines in these figures correspond to a round hole, the broken lines, through cross section $\theta = 0$ near the elliptical hole, and the dot-dash lines, through cross section $\theta = \pi/2$ near an elliptical hole, whereupon curves I correspond to the value $E_1 = \infty$; curves II, to the value $E_1/E = 2.632$; curves III, to the value $E_1 = 0$.

We see from these graphs that even a hole of small ellipticity ($a/b = 1.2$) has a considerable effect on the stress distribution around it. At a distance of $1.5\ r_0$-$2.0\ r_0$ from the hole the distribution of forces and moments around a curvilinear hole in a shell is very similar to the stress state near a round hole, and by measure of distance approaches the basic stress state.

A change in rigidity of the reinforcing ring has a strong effect on the stress distribution along the contour of the hole. When the reinforcing ring has a low rigidity the basic effect on stress concentration is manifested by membrane stresses; as the rigidity of the reinforcing ring increases the effect of bending stresses from G_ρ^* on the stress concentration increases, although the concentration coefficient of bending stresses from G_ρ^*, even in the case of an absolutely rigid ring, is only one half that of the concentration of stresses from T_ρ^* and G_θ^*.

As the rigidity of the reinforcing ring increases, the concentration of forces and moments increases at the end of the small semiaxis and diminishes on the end of the large semiaxis.

Similarly, stress concentration in a spherical shell near reinforced holes of different form can also be analyzed, particularly when the hole is a square, equilateral triangle, right polygon with rounded corners, etc, although the calculation for these forms of holes is quite complicated.

§4. Round Cylindrical Shell Weakened by a Small Curvilinear Hole

We will discuss the stress state in a round cylindrical shell near round, elliptical, square and equilateral triangle holes with rounded corners for various cases of reinforcement of the holes and for various external loads. The holes will be regarded as small. By small we mean such holes for which is valid the inequality

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$$\frac{r_0}{\sqrt{Rh}}\frac{\sqrt[4]{3(1-\nu^2)}}{2} < 1, \tag{X.142}$$

where r_0 is a value characterizing the absolute dimensions of the hole; in the case of a round hole this is the radius of the hole, in the case of an elliptical hole, this is the half sum of the principal semiaxes, etc; R is the radius of the middle surface of the cylindrical shell; h is the thickness of the shell. The Ox axis will be directed along the generatrix and the Oy axis along the guide.

Round Hole. The stress state near a small round hole in a cylindrical shell under uniform internal pressure and under uniaxial tension is analyzed in the works of A. I. Lur'ye[1] [1, 2]; in the work of Yu. A. Shevlyakov and F. S. Zigel' [1] for torsion. We will need these results as the zero approximation for curvilinear holes, since the zero approximation coincides with the solution for a round hole under the corresponding load. Numerous interesting problems for a cylindrical shell weakened by a small round hole under various boundary conditions and various loads are examined in the works of I. M. Pirogov [1-24]. It is pointed out in these works that in the case of small nonreinforced round holes under the momentless basic stress state, the maximum bending stresses are much less than membrane stresses. There is no basis to expect deviations for curvilinear holes from the above conclusion, and therefore, for the latter, we will analyze only the stress concentration from forces T_θ and determine only $\mathrm{Im}\Phi_j$. $\mathrm{Re}\Phi_j$ includes a sufficient number of undefined constants to insure that boundary conditions for G_ρ and $\tilde{Q}_\rho$ are satisfied. /752

We will give the values of $T_\theta^{(0)} + T_\theta^{0(0)}$ on the contour of a round hole under various loads; $T_\theta^{0(j)}$ is the coefficient for ϕ^j in the expansion T_θ^0.

Uniaxial tension along the generatrix by forces of intensity ph

$$T_\theta^{(0)} + T_\theta^{0(0)} = ph\left[1 - 2\cos 2\theta + \frac{\pi\beta^2}{2}\cos 2\theta\right]. \tag{X.143}$$

Uniform internal pressure of intensity p

$$T_\theta^{(0)} + T_\theta^{0(0)} = qh\left[\frac{3}{4} + \cos 2\theta + \pi\beta^2\left(1 + \frac{5}{4}\cos 2\theta\right)\right], \tag{X.144}$$

where

$$q = \frac{pR}{2}.$$

An error appears in this work, which was discovered by N. P. Fleyshman [1].

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Torsion by forces of intensity τh

$$T_\theta^{(0)} + T_\theta^{0(0)} = -4\tau h\left[1 + \frac{1}{2}\pi\beta^2\right]\sin 2\theta. \qquad (X.145)$$

Effect of Reinforcing Ring on Stress Concentration Near Round Hole. Assuming that the basic stress state is momentless with components

$$T_x^0 = ph; \quad T_y^0 = qh; \quad S_{xy}^0 = 0, \qquad (X.146)$$

and regarding the reinforcing ring as the material fiber that resists stretching, bending and twisting, i.e., taking the boundary conditions in the form (X.58), /753
N. P. Fleyshman [1] analyzed the stress state of a cylindrical shell weakened by a small round hole. Here we will not discuss the cumbersome intermediate calculations, but introduce only the final results of the numerical example which he analyzed.

We will consider the case where the shell is loaded by a uniform internal pressure of intensity p_0, and the hole is covered by a roof of special construction, which transmits to the contour of the hole only the action of the shear force. In this case $2p = q = p_0R/h$. The numerical example is examined for the following parameters: $r_0/\sqrt{Rh} = 0.5$; $r_0 = 15h$; $h_1 = 3h$; $h_1 = 2b$; $b = 0.1\,r_0$; $\nu = 0.3$, where h_1 and b are the height and width of the ring; E and E_1 are Young's moduli for the materials of the shell and ring, respectively; $E_1/E = \delta/(1 - \nu^2)$; $\kappa = \delta/(\delta + 20)$. The values of the total stresses on the contour of the hole for the external surface of the shell are given in Figure X.19 as functions of κ: $\left(\frac{1}{q}\sigma_\varphi\right)_{\varphi=0}$ -- curve 1; $\left(\frac{1}{q}\sigma_r\right)_{\varphi=\pi/2}$ -- curve 2 and $\left(\frac{1}{q}\sigma_r\right)_{\varphi=0}$ -- curve 3. The curves show that the rigidity of the reinforcing ring has a considerable effect on the stress concentration.

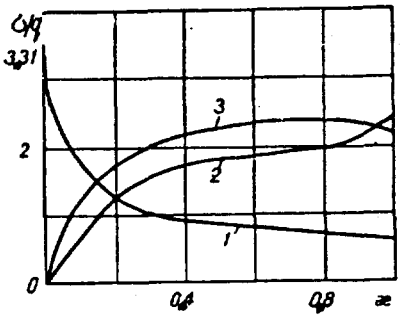


Figure X.19.

Elliptical Hole. Uniform Internal Pressure. We will consider the stress state of a round cylindrical shell weakened by a small elliptical hole and loaded by uniform internal pressure of intensity p. We will assume that the hole is covered by a roof of special construction, which transmits the action only of the shear force. In such a statement the problem is solved in the work of A. N. Guz' [8], which we will reconstruct here. For an elliptical hole

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$$r_0 = \frac{a+b}{2}; \qquad \varepsilon = \frac{a-b}{a+b}; \qquad f(\zeta) = \frac{1}{\zeta}, \tag{X.147}$$

where a and b are the semiaxes of the ellipse.

We will assume that the basic stress state is momentless; then the components of the basic stress state, expanded into series with respect to $\varepsilon = (a - b)/(a + b)$, acquire the form

$$\begin{aligned} T_Q^0 &= \frac{qh}{4}[3 - \cos 2\theta + \varepsilon(1 - \cos 4\theta) + \varepsilon^2(\cos 2\theta - \cos 6\theta)];\\ T_\theta^0 &= \frac{qh}{4}[3 + \cos 2\theta + \varepsilon(\cos 4\theta - 1) + \varepsilon^2(\cos 6\theta - \cos 2\theta)];\\ S_{Q\theta}^0 &= \frac{qh}{4}[\sin 2\theta + \varepsilon \sin 4\theta + \varepsilon^2(\sin 6\theta - \sin 2\theta)]. \end{aligned} \tag{X.148}$$

Boundary conditions /754

$$\begin{aligned} T_Q + T_Q^{0)} &= 0; \qquad S_{Q\theta} + S_{Q\theta}^0 = 0;\\ G_Q &= 0; \qquad \oint_\gamma \tilde{Q}_Q ds = -Dp, \end{aligned} \tag{X.149}$$

where D is the area of the hole; γ is the contour of the hole (see Figure X.5), must be satisfied on the contour of the hole. From (X.148) and (X.149), equating the coefficients for identical degrees of ε, we obtain the boundary conditions for series approximations.

The solution of basic equation (X.75) in the j-th approximation will be found in the form (X.94). We will determine only the concentration of forces, i.e., in the solutions of the form (X.94) we will take only[1] $\mathrm{Im}\Phi$.

Recalling the discussion presented in §2, we obtain on the contour of the hole[2]

$$T_\theta^{(1)} + T_\theta^{0(1)} = qh[(3\cos 2\theta + \cos 4\theta) + \frac{\pi\beta^2}{4}(5 + 16\cos 2\theta + 5\cos 4\theta)]; \tag{X.150}$$

$$\begin{aligned} T_\theta^{(2)} + T_\theta^{0(2)} = qh\Big[&\Big(3\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos 6\theta\Big) + \\ &+ \pi\beta^2\Big(1 + \frac{15}{4}\cos 2\theta + \frac{13}{2}\cos 4\theta + \frac{5}{4}\cos 6\theta\Big)\Big]. \end{aligned} \tag{X.151}$$

[1] $\mathrm{Re}\Phi$ includes a sufficient quantity of constants to insure that boundary conditions are satisfied for G_ρ and Q_ρ.

[2] The zero approximation coincides with the solution for a round hole (X.144).

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The values T^*_θ on the contour of the hole with an accuracy up to ε^2, considering (X.144), (X.150) and (X.151), are found by formula

$$T^*_\theta = T^{(0)}_\theta + T^{0(0)}_\theta + \varepsilon(T^{(1)}_\theta + T^{0(1)}_\theta) + \varepsilon^2(T^{(2)}_\theta + T^{0(2)}_\theta) + \varepsilon^3(\ldots) + \ldots. \qquad (X.152)$$

Formula (X.152) is conveniently represented in the form

$$T^*_\theta = T^{\text{shell}}_\theta + T^{\text{add}}_\theta, \qquad (X.153)$$

where $T^{(pl)}_\theta$ are the values of forces in a flat plate under the corresponding load, while $T^{(add)}_\theta$ is an addition which occurs in the shell as a consequence of distortion of its surface.

The values of $(T^{(add)}_\theta/qh)_{\theta=0}$ in the numerators and $(T^{(add)}_\theta/qh)_{\theta=\pi/2}$ in the denominators are presented in Table X.5 for various ratios between R; h; r_0 = = (a + b)/2; a/b for $\nu = 0.3$, when semiaxis a of the ellipse is directed along the generatrix and b is directed along the guide.

TABLE X.5.

$r_0/\sqrt{Rh}$	a/b						
	1,300	1,200	1,100	1,000	$\frac{1}{1,100}$	$\frac{1}{1,200}$	$\frac{1}{1,300}$
0,500	$\frac{+1,072}{-0,135}$	$\frac{+0,927}{-0,121}$	$\frac{+0,831}{-0,103}$	$\frac{+0,729}{-0,081}$	$\frac{+0,636}{-0,057}$	$\frac{+0,554}{-0,030}$	$\frac{+0,524}{-0,004}$
0,400	$\frac{+0,686}{-0,086}$	$\frac{+0,593}{-0,077}$	$\frac{+0,531}{-0,065}$	$\frac{+0,466}{-0,052}$	$\frac{+0,401}{-0,036}$	$\frac{+0,354}{-0,019}$	$\frac{+0,335}{-0,002}$
0,300	$\frac{+0,386}{-0,049}$	$\frac{+0,334}{-0,043}$	$\frac{+0,299}{-0,037}$	$\frac{+0,262}{-0,029}$	$\frac{+0,229}{-0,020}$	$\frac{+0,199}{-0,011}$	$\frac{+0,188}{-0,002}$
0,200	$\frac{+0,179}{-0,022}$	$\frac{+0,154}{-0,020}$	$\frac{+0,155}{-0,017}$	$\frac{+0,121}{-0,014}$	$\frac{+0,106}{-0,009}$	$\frac{+0,092}{-0,005}$	$\frac{+0,087}{-0,001}$
0,100	$\frac{+0,043}{-0,005}$	$\frac{+0,037}{-0,005}$	$\frac{+0,033}{-0,004}$	$\frac{+0,029}{-0,003}$	$\frac{+0,025}{+0,002}$	$\frac{+0,022}{-0,001}$	$\frac{+0,021}{-0,000}$

Tr. Note: Commas indicate decimal points.

We will write out the formula for the concentration coefficients $k = T^*_\theta/qh$ /755 obtained from (X.152) for $\theta = 0$ and $\theta = \pi/2$:

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$$(k)_{\theta=0}=\frac{5}{2}+4\frac{a-b}{a+b}+4\left(\frac{a-b}{a+b}\right)^2+\frac{\pi\beta^2}{4}\left[9+26\frac{a-b}{a+b}+50\left(\frac{a-b}{a+b}\right)^2\right]+\ldots\ldots \quad (X.154)$$

$$(k)_{\theta=\pi/2}=\frac{1}{2}-2\frac{a-b}{a+b}+2\left(\frac{a-b}{a+b}\right)^2+\frac{\pi\beta^2}{4}\left[-1-6\frac{a-b}{a+b}+10\left(\frac{a-b}{a+b}\right)^2\right]. \quad (X.155)$$

We see from (X.154) and (X.155) that $(k)_{\theta=0}$ and $(k)_{\theta=\pi/2}$ depends greatly on the ratio a/b, while the concentration coefficient $(k_\theta)_{\theta=0}$, moreover, depends strongly on radius R of the shell.

The values of T^*_θ/qh on the contour of the hole in both a shell and a plate are presented in Table X.6 for a/b = 1.3; ν = 0.3; $r_0/\sqrt{Rh}$ = 0.5 for various values of θ.

TABLE X.6

T^*_θ	θ°						
	0	15	30	45	60	75	90
$T^{*\,plate}_\theta/qh$	+3,10	+2,82	+2,11	+1,57	+0,70	+0,37	+0,28
$T^{*\,shell}_\theta/qh$	+4,17	+3,76	+2,73	+1,85	+0,99	+0,27	+0,15

Tr. Note: Commas indicate decimal points.

The values of T*/qh on the contour of a hole in both a shell and in a plate are presented in Table X.7 for b/a = 1.3; ν = 0.3; $r_0/\sqrt{Rh}$ = 0.5 for various values of θ.

TABLE X.7

T^*_θ	θ°						
	0	15	30	45	60	75	90
$T^{*\,plate}_\theta/qh$	+2,06	+2,02	+1,85	+1,57	+1,22	+0,89	+0,80
$T^{*\,shell}_\theta/qh$	+2,58	+2,51	+2,10	+1,85	+1,38	+0,94	+0,76

Tr. Note: Commas indicate decimal points.

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We see from Tables X.6 and X.7 that the curvature of the shell has a considerable effect on the concentration of forces when $\theta = 0$. Thus, when $a/b = 1.3$ concentration coefficient k (X.154) in the shell is 34% higher than in a plate, but only 25% higher for $b/a = 1.3$. /756

In order to make a judgment concerning the accuracy of the solution obtained, we will compare the values $(k^{(pl)})_{\theta=0}$ and $(k^{(pl)})_{\theta=\pi/2}$ (concentration coefficients of force in the plate), obtained from the approximate solution with an accuracy up to ε^2 and from the precise solution.

From (X.154) and (X.155) for $R = \infty$ we obtain the approximate value from the approximate solution

$$(k^{\text{plate}})_{\theta=0} = \frac{5}{2} + 4\frac{a-b}{a+b} + 4\left(\frac{a-b}{a+b}\right)^2; \tag{X.156}$$

$$(k^{\text{plate}})_{\theta=\pi/2} = \frac{1}{2} - 2\frac{a-b}{a+b} + 2\left(\frac{a-b}{a+b}\right)^2. \tag{X.157}$$

We will show the values of these coefficients as found from the precise solution[1] of the problem:

$$(k^{\text{plate}})_{\theta=0} = \frac{5-2\varepsilon-3\varepsilon^2}{2-4\varepsilon+2\varepsilon^2}; \tag{X.158}$$

$$(k^{\text{plate}})_{\theta=\pi/2} = \frac{1-2\varepsilon-3\varepsilon^2}{2+4\varepsilon+2\varepsilon^2} \quad \left(\varepsilon = \frac{a-b}{a+b}\right). \tag{X.159}$$

We see from (X.156)-(X.159) that the expansions of $(k^{(pl)})_{\theta=0}$ and $(k^{(pl)})_{\theta=\pi/2}$ obtained from the precise solution of (X.158) and (X.159) coincide with (X.156) and (X.157) with an accuracy up to ε^2.

The results of calculations for $a/b = 1.3$ and $a/b = 1.3$ $\left(\frac{a+b}{2\sqrt{Rh}} = 0{,}5\right)$ are presented in Table X.8.

TABLE X.8

k	Approximation						Precise solution	
	Zero		First		Second			
	a/b=1,3	a/b=1/1,3	a/b=1,3	a/b=1/1,3	a/b=1,3	a/b=1/1,3	a/b=1,3	a/b=1/1,3
In plate for $\theta=0$	2,500	2,500	3,020	1,980	3,088	2,048	3,097	2,039
In plate for $\theta=\pi/2$	0,500	0,500	0,240	0,760	0,274	0,794	0,269	0,798

Tr. Note: Commas indicate decimal points.

[1]See S. G. Lekhnitskiy [1] or G. N. Savin [8].

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We see from Table X.8 that the concentration coefficients $k^{(p1)}$ for a/b = = 1.3 and b/a = 1.3 at the ends of the small and large semiaxes of an elliptical hole, as found from the approximate solution of (X.156) and (X.157), differ from those obtained from the precise solution (X.158) and (X.159) by no more than 2%. Figure X.20 gives the distribution of T^*_θ/qh on the contour of the hole for a/b = 1.3 and X.21 for b/a = 1.3; $\frac{a+b}{2\sqrt{Rh}} = 0{,}5$

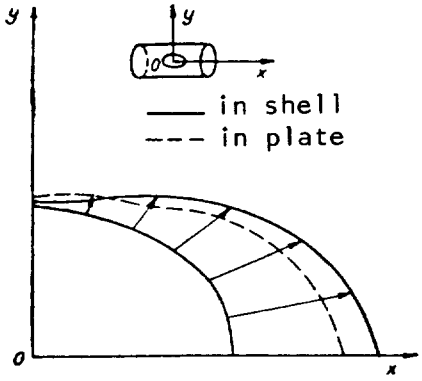


Figure X.20.

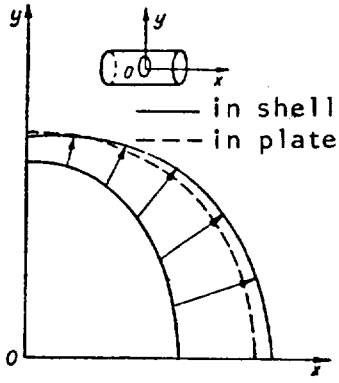


Figure X.21.

Uniaxial Tension. We will examine the stress state of a round cylindrical /757
shell weakened by a small elliptical hole[1] and subjected to tension along the generatrix by forces of intensity ph. We will assume that the hole is not reinforced.

The basic stress state is momentless with the following components

$$T_x^0 = ph; \quad T_y^0 = 0; \quad S_{xy}^0 = 0. \tag{X.160}$$

We will introduce formulas for the components of the basic stress state in the coordinate system (ρ, θ), expanding them into power series with respect to ε:

$$\begin{aligned} T_\rho^0 &= \frac{ph}{2}[1 + \cos 2\theta + \varepsilon(\cos 4\theta - 1) + \varepsilon^2(\cos 6\theta - \cos 2\theta) + \ldots]; \\ T_\theta^0 &= \frac{ph}{2}[1 - \cos 2\theta + \varepsilon(1 - \cos 4\theta) + \varepsilon^2(\cos 2\theta - \cos 6\theta) + \ldots]; \\ S_{\rho\theta}^0 &= -\frac{ph}{2}[\sin 2\theta + \varepsilon \sin 4\theta + \varepsilon^2(\sin 6\theta - \sin 2\theta) + \ldots]; \end{aligned} \tag{X.161}$$

The following boundary conditions must be satisfied on the contour of a non- /758
reinforced hole:

[1]See A. N. Guz' [6].

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$$T_\varrho + T_\varrho^0 = 0; \qquad S_{\varrho\theta} + S_{\varrho\theta}^0 = 0;$$
$$G_\varrho = 0; \qquad \tilde{Q}_\varrho = 0. \tag{X.162}$$

From (X.161) and (X.162) we obtain the boundary conditions for series approximations.

We will determine only the concentration of stresses from forces T_θ, T_ρ and $S_{\rho\theta}$, since in the case of a round nonreinforced hole the moments are negligible in absolute value. The same should also be expected for the case of an elliptical hole.

The solution of basic equation (X.75) will be found in the form (X.94).

The solution in the zero approximation coincides with solution (X.143) for a round hole.

Recalling the results of §2, we obtain for $T_\theta^{(j)} + T_\theta^{0(j)}$ (j = 1, 2) on the contour of the hole

$$T_\theta^{(1)} + T_\theta^{0(1)} = ph\left[2\operatorname{ccs} 2\theta - \frac{3}{2}\cos 4\theta - \frac{1}{2} - \frac{\pi\beta^2}{2}(1 + \cos 4\theta)\right]; \tag{X.163}$$

$$T_\theta^{(2)} + T_\theta^{0(2)} = ph\left[2\cos 4\theta - \frac{1}{2}\cos 6\theta - \frac{1}{2}\cos 2\theta - \frac{\pi\beta^2}{2}(3\cos 2\theta + \cos 6\theta)\right]. \tag{X.164}$$

The values of T_θ^* on the contour of the hole, recalling (X.143), (X.163) and (X.164), are found with an accuracy up to ε^2 by the formula

$$T_\theta^* = T_\theta^{(0)} + T_\theta^{0(0)} + \varepsilon(T_\theta^{(1)} + T_\theta^{0(1)}) + \varepsilon^2(T_\theta^{(2)} + T_\theta^{0(2)}) + \ldots. \tag{X.165}$$

The values of T_θ^* on the contour of the hole, recalling (X.143), (X.163) and (X.164) are found for $\nu = 0.3$ by the formula

$$T_\theta^* = ph\left\{\left[1 - 2\cos 2\theta + 2\varepsilon(\cos 2\theta - \cos 4\theta) + 2\varepsilon^2\left(\cos 4\theta - \frac{1}{2}\cos 2\theta - \right.\right.\right.$$
$$\left.\left.\left. - \frac{1}{2}\cos 6\theta\right)\right] - 0{,}649\frac{\varepsilon^2}{4Rh}\left[\cos 2\theta + \varepsilon(1 + \cos 4\theta) + \varepsilon^2(3\cos 2\theta + \cos 6\theta)\right]\right\} \tag{X.166}$$
$$\left(\varepsilon = \frac{a-b}{a+b}\right).$$

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Formula (X.166) can be represented in the form

$$T_\theta^* = T_\theta^{plate} + T_\theta^{add}, \qquad (X.167)$$

where $T_\theta^{(pl)}$ are the values of forces in a flat plate under the corresponding load, $T_\theta^{(add)}$ is an additive, which occurs in the shell as a consequence of distortion of its surface.

The values of $(T_\theta^{(add)}/ph)_{\theta=0}$ in the numerator and $(T_\theta^{(add)}/ph)_{\theta=\pi/2}$ in the /759 are presented in Table X.9 for $\nu = 0.3$ (on the contour of the hole) for various values of $(a + b)/2$; R; h and a/b.

TABLE X.9

$\frac{r_0}{\sqrt{Rh}}$	a/b						
	1,4	1,2	1,1	1,0	$\frac{1,0}{1,1}$	$\frac{1,0}{1,2}$	$\frac{1,0}{1,4}$
0,500	$\frac{-0,234}{0,126}$	$\frac{-0,197}{0,138}$	$\frac{-0,179}{0,148}$	$\frac{-0,162}{0,162}$	$\frac{-0,148}{0,179}$	$\frac{-0,138}{0,197}$	$\frac{-0,126}{0,234}$
0,400	$\frac{-0,149}{0,080}$	$\frac{-0,126}{0,088}$	$\frac{-0,114}{0,094}$	$\frac{-0,104}{0,104}$	$\frac{-0,094}{0,114}$	$\frac{-0,088}{0,126}$	$\frac{-0,080}{0,149}$
0,300	$\frac{-0,084}{0,045}$	$\frac{-0,070}{0,049}$	$\frac{-0,064}{0,053}$	$\frac{-0,058}{0,058}$	$\frac{-0,053}{0,064}$	$\frac{-0,049}{0,070}$	$\frac{-0,045}{0,084}$
0,200	$\frac{-0,037}{0,020}$	$\frac{-0,031}{0,022}$	$\frac{-0,028}{0,023}$	$\frac{-0,025}{0,025}$	$\frac{-0,023}{0,028}$	$\frac{-0,022}{0,031}$	$\frac{-0,020}{0,037}$

Tr. Note: Commas indicate decimal points.

We will notice that the semiaxis a is directed along the generatrix and semiaxis b, along the guide of the middle surface of the cylindrical shell.

From (X.166) it is easy to find for concentration coefficient $k = T_\theta^*/ph$, for $\theta = 0$ and $\theta = \pi/2$, the formulas

$$(k)_{\theta=0} = -1 - 0{,}649\,\frac{(a+b)^2}{4Rh}\,[1 + 2\varepsilon + 4\varepsilon^2]; \qquad (X.168)$$

$$(k)_{\theta=\pi/2} = 3 - 4\varepsilon + 4\varepsilon^2 + 0{,}649\,\frac{(a+b)^2}{4Rh}\,[1 - 2\varepsilon + 4\varepsilon^2] \qquad (X.169)$$

$$\left(\varepsilon = \frac{a-b}{a+b}\right).$$

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We see from (X.168) and (X.169) that $(k)_{\theta=0}$ and $(k)_{\theta=\pi/2}$ depend greatly on the ratio a/b, while $(k)_{\theta=0}$ also depends strongly on R. When $\theta = 0$ we obtain a qualitative difference in the distribution of forces in the shell in comparison with the plate. Thus, in the plate $(k_\theta^{(pl)})_{\theta=0} = -1$ and does not depend on the ratio a/b, while in the shell $(k)_{\theta=0}$ (X.168) depends considerably on the ratio a/b.

The values of T_θ^*/ph on the contour of the hole for $(a + b)/2\sqrt{Rh} = 0.5$; $\nu = 0.3$; $a/b = 1.4$ are presented in Table X.10 for various points of the contour of the hole.

TABLE X.10

T_θ^*	θ°						
	0	15	30	45	60	75	90
$T_\theta^{*\,shell}/ph$	—1,00	—0,62	0,28	1,28	2,00	2,36	2,44
$T_\theta^{*\,plate}/ph$	—1,23	—0,81	0,19	1,28	2,07	2,45	2,57

Tr. Note: Commas indicate decimal points.

We see from Table X.10 that $(k^{(sh)})_{\theta=0}$ is 23% greater than the value of $(k^{(pl)})_{\theta=0}$, while $(k^{(sh)})_{\theta=\pi/2}$ is only 5% greater than $(k^{(pl)})_{\theta=\pi/2}$. We will notice that in the case at hand $k_{max}^{(sh)} = (k^{(sh)}_{\theta=\pi/2}$. In the zero approximation $k_{max}^{(sh)} = 3.16$, in the first, $k_{max}^{(sh)} = 2.43$, while in the second, $k_{max}^{(sh)} = 2.57$, i.e., the first approximation differs by 23% from the zero, while the second approximation differs by 5% from the first. We will also note that $k_{\theta=0}^{(pl)}$ and $k_{\theta=\pi/2}^{(pl)}$ obtained by the approximate solution with an accuracy up to ε^2 coincide with the expansions $(k^{(pl)})_{\theta=0}$ and $(k^{(pl)})_{\theta=\pi/2}$ obtained from the precise solution. /760

The distribution of T_θ^*/ph found from (X.166) on the contour of an elliptical hole is given in Figure X.22 for $a/b = 1.4$ and in Figure X.23 for $a/b = 1.4$ and $(a + b)/2\sqrt{Rh} = 0.5$.

Square Hole with Rounded Corners. Uniform internal pressure. We will examine the stress state of a round cylindrical shell weakened by a small square hole with rounded corners and loaded by uniform internal pressure of

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intensity p. We will assume that the hole is covered by a roof of special construction, which transmits only the action of the shear force[1].

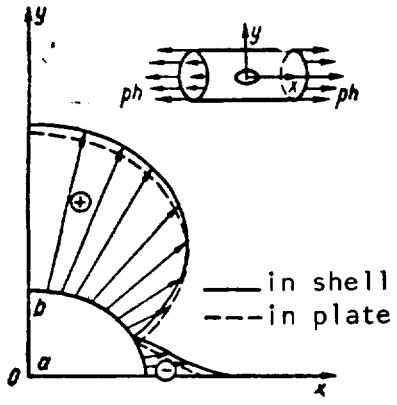


Figure X.22.

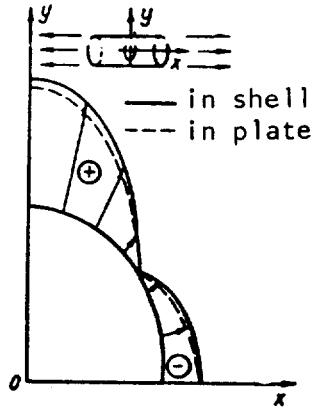


Figure X.23.

For a square hole with rounded corners

$$\varepsilon = \pm \frac{1}{9}; \qquad f(\zeta) = \frac{1}{\zeta^3}; \tag{X.170}$$

when $\varepsilon = +1/9$ the diagonal of the square is directed along the generatrix, while for $\varepsilon = -1/9$, it is directed at angle $\pi/4$ to the generatrix.

We will assume that the basic stress state is momentless; then /761

$$T_x^0 = \frac{1}{2} gh; \qquad T_y^0 = qh;$$
$$S_{xy}^0 = 0; \qquad q = p\frac{R}{h}. \tag{X.171}$$

By representing the components of the basic stress state in coordinate system (ρ, θ), and expanding them into series with respect to ε, preserving the terms with ε^2, we find

$$T_\rho^0 = \frac{qh}{4}[3 - \cos 2\vartheta - 3\varepsilon(\cos 6\vartheta - \cos 2\vartheta) - 9\varepsilon^2(\cos 10\vartheta - \cos 2\vartheta) + \ldots];$$

$$T_\theta^0 = \frac{qh}{4}[3 + \cos 2\vartheta + 3\varepsilon(\cos 6\vartheta - \cos 2\vartheta) + 9\varepsilon^2(\cos 10\vartheta - \cos 2\vartheta) + \ldots];$$

[1]See A. N. Guz' and S. A. Goloborod'ko [1].

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$$S^0_{\varrho\vartheta} = \frac{qh}{4}[\sin 2\vartheta + 3\varepsilon(\sin 6\vartheta + \sin 2\vartheta) + 9\varepsilon^2(\sin 10\vartheta - \sin 2\vartheta) + \ldots]$$

$$\left(\varepsilon = \pm\frac{1}{9}\right). \tag{X.172}$$

The following boundary conditions must be satisfied on the contour of the hole:

$$T_\varrho + T^0_\varrho = 0; \qquad S_{\varrho\vartheta} + S^0_{\varrho\vartheta} = 0;$$
$$G_\varrho = 0; \qquad \oint_\gamma Q_\varrho ds = -Dp, \tag{X.173}$$

where D is the area of the region bounded by contour γ (see Figure X.5) of the hole.

From (X.172) and (X.173), equating the coefficients for identical degrees of ε, we obtain the boundary conditions for series approximations. The solution of the basic equation (X.75) will be found in the form (X.94).

Since we are interested only in the concentration of forces near the hole, it is sufficient to analyze in functions Φ Im Φ. The function ReΦ includes as many constants as are required to satisfy the boundary conditions for Q_ρ and G_ρ.

The zero approximation coincides with the solution for a round hole (X.44).

On the basis of the results of §2, we obtain for $T^{(j)}_\theta + T^{0(j)}_\theta$ (j = 1, 2) on the contour of the hole

$$T^{(1)}_\vartheta + T^{0(1)}_\vartheta = qh\{\cos 2\vartheta + 9\cos 4\vartheta + 3\cos 6\vartheta + \frac{\pi\beta^2}{4}[1 + 7\cos 2\vartheta + 24\cos 4\vartheta + 15\cos 6\vartheta]\}; \tag{X.174}$$

/762

$$T^{(2)}_\vartheta + T^{0(2)}_\vartheta = qh\{\cos 2\vartheta + 3\cos 6\vartheta + 27\cos 8\vartheta + 9\cos 10\vartheta + \frac{\pi\beta^2}{4}(10 + 4{,}5\cos 2\vartheta + 6\cos 4\vartheta + 21\cos 6\vartheta + 72\cos 8\vartheta + 45\cos 10\vartheta)\}. \tag{X.175}$$

It is easy to find from (X.144), (X.174) and (X.175) the values T^*_θ on the contour of the hole with an accuracy up to ε^2

$$T^*_\vartheta = T^{(0)}_\vartheta + T^{0(0)}_\vartheta + \varepsilon(T^{(1)}_\vartheta + T^{0(1)}_\vartheta) + \varepsilon^2(T^{(2)}_\vartheta + T^{0(2)}_\vartheta). \tag{X.176}$$

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Formula (X.176) for $\nu = 0.3$ and $\varepsilon = 1/9$, i.e., in the case where the diagonal of the square is directed along the generatrix of a cylindrical shell, will acquire the form

$$T_\theta^* = qh\,[1{,}5 + 1{,}125\cos 2\theta + \cos 4\theta + 0{,}37\cos 6\theta + 0{,}335\cos 8\theta + \\ + 0{,}11\cos 10\theta + 0{,}325\frac{r_0^2}{Rh}(4{,}26 + 6\cos 2\theta + 3{,}41\cos 4\theta + \\ + 1{,}93\cos 6\theta + 0{,}89\cos 8\theta + 0{,}56\cos 10\theta)]. \qquad (X.177)$$

When $\varepsilon = -1/9$, i.e., when the diagonal of the square is directed at angle $\pi/4$ to the generatrix,

$$T_\theta^* = qh\,[1{,}5 + 0{,}9\cos 2\theta - \cos 4\theta - 0{,}295\cos 6\theta + 0{,}335\cos 8\theta + \\ + 0{,}11\cos 10\theta + 0{,}325\frac{r_0^2}{Rh}(4{,}01 + 4{,}44\cos 2\theta - 2{,}59\cos 4\theta - \\ - 1{,}41\cos 6\theta + 0{,}89\cos 8\theta + 0{,}56\cos 10\theta)]. \qquad (X.178)$$

We will present the values of $k = T_\theta^*/qh$ at certain points of the contour of the hole, i.e., for certain values of the parameter θ:

for $\varepsilon = 1/9$

$$(k)_{\theta=0} = 4{,}44 + 5{,}54\frac{r_0^2}{Rh};$$
$$(k)_{\theta=\frac{\pi}{2}} = 1{,}26 + 0{,}07\frac{r_0^2}{Rh}; \qquad (X.179)$$

for $\varepsilon = -1/9$

$$(k)_{\theta=\frac{\pi}{4}} = 2{,}83 + 2{,}43\frac{r_0^2}{Rh}. \qquad (X.180)$$

The values of k for a plate[1] and for a shell are presented in Table X.11 /763
for $r_0/\sqrt{Rh} = 0.6$; $\varepsilon = 1/9$ and $\nu = 0.3$ in the zero, first, and second approximations for various values of θ.

[1]See S. G. Lekhnitskiy [1] or G. N. Savin [8].

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TABLE X.11

k	Approximation			Precise solution
	Zero	First	Second	
In plate for $\theta=0$ . . .	2,50	3,94	4,44	4,68
In shell for $\theta=0$ . . .	3,55	5,61	6,43	—
In plate for $\theta=\pi/2$ . .	0,50	1,05	1,23	1,30
In shell for $\theta=\pi/2$ . .	0,38	0,97	1,24	—

Tr. Note: Commas indicate decimal points.

We see from (X.179) and (X.180) that the curvature of the shell has its greatest effect on the concentration coefficient when $\theta = 0$ and $\theta = \pi/4$, but when $\theta = \pi/2$, it has almost no effect.

From the data presented in Table X.11 we may conclude that $k^{(pl)}$ for $\theta = 0$ and $\theta = \pi/2$ calculated by the approximate solution differs by 4-5% from $k^{(pl)}$ at the very same points as calculated by the precise solution. When $\theta = 0$, both in the plate and in the shell, $(k)_{\theta=0} = k_{max}$. In the first approximation $k_{max}^{(sh)}$ is 58% greater than $k_{max}^{(sh)}$ in the zero approximation, and in the second approximation $k_{max}^{(sh)}$ is 12% greater than $k_{max}^{(sh)}$ in the first approximation. It should be noted that $k_{max}^{(sh)}$ for a square hole is 77% greater than $k_{max}^{(sh)}$ for a round hole.

The values of k calculated by formulas (X.177) and (X.178) are presented in Table X.12 for various values $r_0/\sqrt{Rh}$ when $\nu = 0.3$ for $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$.

TABLE X.12

ε; k	$r_0/\sqrt{Rh}$						
	0,6	0,5	0,4	0,3	0,2	0,1	0,0
$\varepsilon=1/9$; $(k^{(o6)})_{\theta=0}$	6,43	5,83	5,33	4,99	4,66	4,49	4,44
$\varepsilon=1/9$; $(k^{(o6)})_{\theta=\pi/2}$	1,24	1,24	1,23	1,23	1,23	1,23	1,23
$\varepsilon=1/9$; $(k^{(o6)})_{\theta=\pi/4}$	1,00	0,97	0,93	0,88	0,86	0,84	0,84
$\varepsilon=-1/9$; $(k^{(o6)})_{\theta=\pi/4}$	3,71	3,45	3,22	3,05	2,93	2,86	0,83

Tr. Note: Commas indicate decimal points.

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The curves that characterize T^*_θ/qh, calculated by formulas (X.177) and (X.178), on the contour of the hole are presented in Figure X.24 for $\varepsilon = 1/9$ and X.25 for $\varepsilon = -1/9$ when $r_0/\sqrt{Rh} = 0.5$.

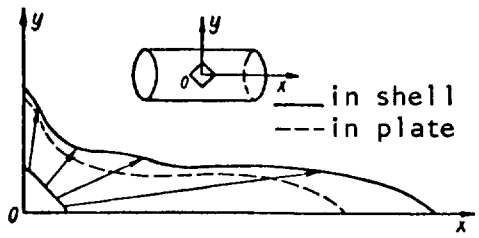


Figure X.24.

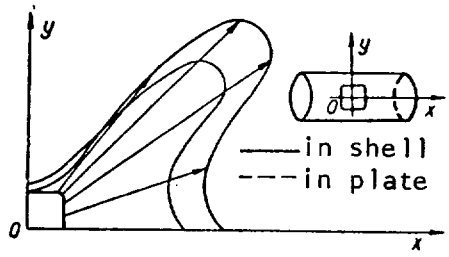


Figure X.25.

Uniaxial Tension. Let us examine the stress state of a round cylindrical shell weakened by a small square hole with rounded corners and under tension (along the generatrices) by forces of intensity ph. We will assume the hole to be nonreinforced[1]. /764

The basic stress state is momentless with the components

$$T_x^0 = ph; \quad T_y^0 = 0; \quad S_{xy}^0 = 0. \tag{X.181}$$

Components (X.181) in coordinate system (ρ, θ) is represented in the form of series with respect to ε:

$$\begin{aligned} T_\rho^0 &= \frac{ph}{2}[1 + \cos 2\theta + 3\varepsilon(\cos 6\theta - \cos 2\theta) + 9\varepsilon^2(\cos 10\theta - \cos 2\theta) + \ldots]; \\ T_\theta^0 &= \frac{ph}{2}[1 - \cos 2\theta - 3\varepsilon(\cos 6\theta - \cos 2\theta) - 9\varepsilon^2(\cos 10\theta - \cos 2\theta) + \ldots]; \\ S_{\rho\theta}^0 &= -\frac{ph}{2}[\sin 2\theta + 3\varepsilon(\sin 6\theta + \sin 2\theta) + 9\varepsilon^2(\sin 10\theta - \sin 2\theta) + \ldots]. \end{aligned} \tag{X.182}$$

On the contour of a nonreinforced hole the following boundary conditions must be satisfied:

$$\begin{aligned} T_\rho + T_\rho^0 &= 0; \qquad S_{\rho\theta} + S_{\rho\theta}^0 = 0; \\ G_\rho &= 0; \qquad \tilde{Q}_\rho = 0. \end{aligned} \tag{X.183}$$

[1]See G. N. Savin and A. N. Guz' [1].

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From (X.182) and (X.183) we obtain the boundary conditions for series approximations.

We will determine only the stress concentration from forces T_θ, T_ρ and $S_{\rho\theta}$, since in this case the values of the moments on the contour of the hole will be negligible.

The solution of basic equation (X.75) will be taken in the form (X.94).

In the zero approximation the desired solution coincides with the solution for a round hole (X.143).

On the basis of the results of §2 we will find the expressions /765

$$T_\theta^{(1)} + T_\theta^{0(1)} = ph\left\{-2\cos 2\theta + 6\cos 4\theta - 6\cos 6\theta - \right.$$
$$\left. - \frac{\pi\beta^2}{2}[1 + 3\cos 2\theta + 3\cos 6\theta]\right\}; \qquad (X.184)$$

$$T_\theta^{(2)} + T_\theta^{0(2)} = ph\left\{-2\cos 2\theta - 6\cos 6\theta + 18\cos 18\theta - \right.$$
$$\left. - 18\cos 10\theta + \frac{\pi\beta^2}{2}[4 - 4\cos 2\theta - 12\cos 4\theta - 18\cos 6\theta - 18\cos 10\theta]\right\}. \qquad (X.185)$$

Forces T_θ^* on the contour of the hole, as follows from (X.143), (X.184) and (X.185), are determined with an accuracy up to ε^2 by the formula

$$T_\theta^* = T_\theta^{(0)} + T^{0(1)} + \varepsilon(T_\theta^{(1)} + T_\theta^{0(1)}) + \varepsilon^2(T^{(2)} + T^{0(2)}). \qquad (X.186)$$

The values T_θ^* (X.186) for $\nu = 0.3$, found with an accuracy up to ε^2, will have the following form:

for $\varepsilon = 1/9$

$$T_\theta^* = ph[1 - 2{,}24\cos 2\theta + 0{,}66\cos 4\theta + 0{,}73\cos 6\theta +$$
$$+ 0{,}22\cos 8\theta - 0{,}22\cos 10\theta - 0{,}65\frac{r_0^2}{Rh}(0{,}08 +$$
$$+ 1{,}34\cos 2\theta + 0{,}06\cos 4\theta - 0{,}44\cos 6\theta + 0{,}11\cos 10\theta]; \qquad (X.187)$$

$$T_\theta^* = ph\left[1 - 1{,}8\cos 2\theta - 0{,}66\cos 4\theta + 0{,}59\cos 6\theta + \right.$$
$$+ 0{,}22\cos 8\theta - 0{,}22\cos 10\theta + 0{,}65\frac{r_0^2}{Rh}(0{,}12 -$$
$$\left. - 0{,}7\cos 2\theta - 0{,}06\cos 4\theta + 0{,}22\cos 6\theta - 0{,}11\cos 10\theta)\right]. \qquad (X.188)$$

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We present the values of concentration coefficient $k = T^*_\theta/ph$, found from (X.187) and (X.188), on the contour of the hole for certain values of the parameter θ:

for $\varepsilon = 1/9$

$$(k)_{\theta=0} = -1{,}31 - 1{,}02\pi\beta^2,$$
$$(k)_{\theta=\frac{\pi}{2}} = 5{,}07 + 0{,}87\pi\beta^2; \qquad \text{(X.189)}$$

for $\varepsilon = -1/9$

$$(k)_{\theta=\frac{\pi}{4}} = 1{,}88 + 0{,}09\pi\beta^2. \qquad \text{(X.190)}$$

The values of k for a plate and for a shell, obtained in the zero, first and second approximations for $\theta = \pi/2$; $r_0/\sqrt{Rh} = 0.5$; $\varepsilon = 1/9$ and $\nu = 0.3$ are presented in Table X.13. /766

TABLE X.13

k	Approximation			Precise solution
	Zero	First	Second	
In plate for $\theta=\pi/2$ . .	3,00	4,55	5,09	5,38
In shell for $\theta=\pi/2$ . .	3,16	4,80	5,37	—

Tr. Note: Commas indicate decimal points.

The distribution of T^*_θ/ph (X.187) and (X.188) on the contour of the hole is shown in Figure X.26 for $\varepsilon = 1/9$ and in X.27 for $\varepsilon = -1/9$ when $r_0/\sqrt{Rh} = 0.6$.

Torsion. Let us examine the stress state of a round cylindrical plate weakened by a square, nonreinforced hole with rounded corners. The shell is twisted by moments applied to its ends[1]. The basic stress state will be assumed as momentless with the components

$$T_x^0 = 0; \quad T_y^0 = 0; \quad S_{xy}^0 = \tau h. \qquad \text{(X.191)}$$

[1]See A. N. Guz' [9].

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We will represent the components of the basic stress state in coordinate system (ρ, θ), expanding them into power series with respect to ε and retaining the terms with ε^2:

$$T_\varrho^0 = \tau h\,[\sin 2\vartheta + 3\varepsilon(\sin 6\vartheta + \sin 2\vartheta) + 9\varepsilon^2(\sin 10\vartheta - \sin 2\vartheta) + \ldots];$$
$$T_\vartheta^0 = -T_\varrho^0; \qquad (X.192)$$
$$S_{\varrho\vartheta}^0 = \tau h\,[\cos 2\vartheta + 3\varepsilon(\cos 6\vartheta - \cos 2\vartheta) + 9\varepsilon^2(\cos 10\vartheta - \cos 2\vartheta) + \ldots].$$

On the contour of the hole, free of the effect of external forces, the following boundary conditions must be satisfied:

$$T_\varrho + T_\varrho^0 = 0; \quad S_{\varrho\vartheta}^0 + S_{\varrho\vartheta} = 0;$$
$$G_\varrho = 0; \quad \tilde{Q}_\varrho = 0. \qquad (X.193)$$

By equating to zero the coefficients for identical degrees of ε, we obtain from (X.192) and (X.193) the boundary conditions for the zero, first and second approximations.

The solution of the basic equation (X.75) will be taken in the form (X.91) and (X.94).

We will notice that here, as in the preceding problems, we are interested only in the concentration of forces, and therefore we will determine only the function $\mathrm{Im}\Phi_j$. The magnitudes of the moments G_θ on the contour of the hole in the case of nonreinforced holes are negligible. /767

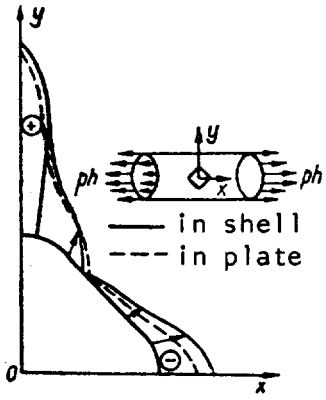


Figure X.26.

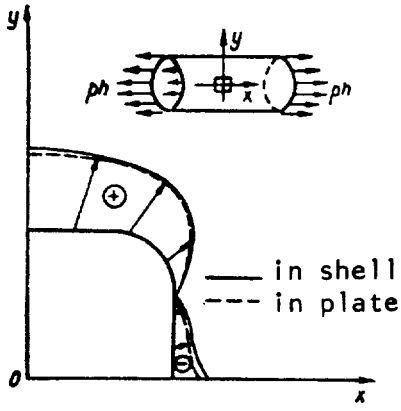


Figure X.27.

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The zero approximation coincides with the solution for a round hole (X.145).

Proceeding similarly as indicated in the preceding cases, we obtain on the contour of the hole, on the basis of the discussion in §2,

$$T_\theta^{(1)} + T_\theta^{0(1)} = 4\tau h\,[\sin 2\theta - 3\sin 6\theta + \pi\beta^2(\sin 2\theta - 1{,}5\sin 6\theta)]; \tag{X.194}$$

$$T_\theta^{(2)} + T_\theta^{0(2)} = -\tau h\,[6\sin 2\theta - 12\sin 6\theta + 38\sin 10\theta + \\ + \pi\beta^2(7\sin 2\theta - 12\sin 6\theta + 19\sin 10\theta)]. \tag{X.195}$$

Forces T_θ^* on the contour of the hole are determined with an accuracy up to ε^2 with consideration of (X.145), (X.194) and (X.195) by formula (X.186).

We will present the values of T_θ^* (X.186) (for the case of torsion of the shell) on the contour of the hole with an accuracy up to ε^2:

for $\varepsilon = 1/9$

$$T_\theta^* = -\tau h\Big[3{,}63\sin 2\theta + 1{,}18\sin 6\theta + 0{,}47\sin 10\theta + \\ + 0{,}78\sqrt{3(1-\nu^2)}\,\frac{r_0^2}{Rh}\,(1{,}65\sin 2\theta + 0{,}51\sin 6\theta + 0{,}23\sin 10\theta)\Big]; \tag{X.196}$$

for $\varepsilon = -1/9$

$$T_\theta^* = -\tau h\Big[4{,}51\sin 2\theta - 1{,}48\sin 6\theta + 0{,}47\sin 10\theta + \\ + 0{,}78\sqrt{3(1-\nu^2)}\,\frac{r_0^2}{Rh}\,(2{,}53\sin 2\theta - 0{,}81\sin 6\theta + 0{,}23\sin 10\theta)\Big]. \tag{X.197}$$

As we see from (X.196) and (X.197), the maximum concentration coefficient $k = T_\theta^*/\tau h$ on the contour of the hole is achieved when $\theta = \pi/4$: /768

for $\varepsilon = 1/9$

$$(k)_{\theta=\pi/4} = -2{,}98 - 1{,}07\sqrt{3(1-\nu^2)}\,\frac{r_0^2}{Rh}; \tag{X.198}$$

for $\varepsilon = -1/9$

$$(k)_{\theta=\pi/4} = -6{,}48 - 2{,}97\sqrt{3(1-\nu^2)}\,\frac{r_0}{Rh}. \tag{X.199}$$

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The values of $(k)_{\theta=\pi/4}$ (X.198) are presented in Table X.14 for $\varepsilon = -1/9$, $\nu = 0.3$ and for various values $r_0/\sqrt{Rh}$. When $r_0/\sqrt{Rh}$ we obtain the values of concentration coefficient k for a plate.

TABLE X.14

Concentration coefficient	$r_0/\sqrt{Rh}$						
	0,0	0,10	0,20	0,30	0,40	0,50	0,60
	In plate	In shell					
$(k)_{\theta=\pi/4}$	−6,46	−6,51	−6,64	−6,87	−7,20	−7,61	−8,11
$\left(\frac{k^{shell}}{k^{plate}}\right)_{\theta=\pi/4}$	1,00	1,01	1,03	1,06	1,11	1,18	1,25

Tr. Note: Commas indicate decimal points.

The values of $(k)_{\theta=\pi/4}$ (X.199) are presented in Table X.15 for $\varepsilon = -1/9$, $\nu = 0.3$, $r_0/\sqrt{Rh} = 0.6$, both in a plate and in a shell, in the zero, first and second approximations, and the value of $(k)_{\theta=\pi/4}$, obtained from the precise solution, is also presented for a plate. It follows from Table X.15 that the value $(k^{(pl)})_{\theta=\pi/4} = -6.46$, obtained in the second approximation, differs by 4.3% from its precise value (6.75), while $(k^{(pl)})_{\theta=0}$ and $(k^{(pl)})_{\theta=\pi/2}$, as the calculations show, coincide with the precise value. The value of $(k^{(sh)})_{\theta=\pi/4}$ for a shell in the first approximation is greater by 46% than in the zero approximation, while in the second approximation it is 12% greater than in the first approximation. The magnitude of $(k^{(sh)})_{\theta=\pi/4}$ for a square hole differs by 65% from the same value for a round hole.

TABLE X.15

k при $\theta=\pi/4$	Approximation			Precise solution
	Zero	First	Second	
In plate	−4,00	−5,77	−6,46	−6,75
In shell	−4,92	−7,20	−8,11	—

Tr. Note: Commas indicate decimal points.

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The distribution of k found from (X.196) and (X.197) on the contour of a square hole is shown in Figure X.28 for $\varepsilon = 1/9$ and in Figure X.29 for $\varepsilon = -1/9$, for shell and plate when $\nu = 0.3$ and $r_0/\sqrt{Rh} = 0.6$.

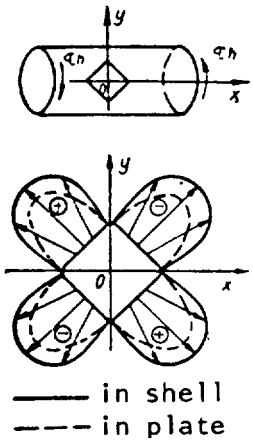


Figure X.28.

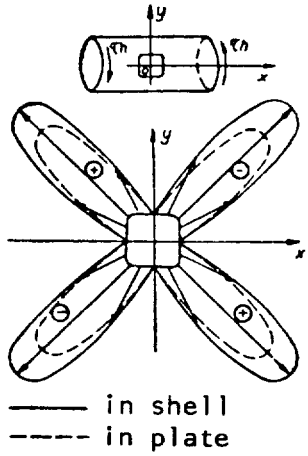


Figure X.29.

Torsion of Cylindrical Shell Weakened by Equilateral Triangle Hole. We will consider the stress state of a cylindrical shell subjected to torsion by moments applied to its ends. The shell is weakened by a nonreinforced equilateral triangle hole with rounded corners[1]. For such a hole /769

$$\varepsilon = \pm\frac{1}{4}; \qquad f(\zeta) = \frac{1}{\zeta^3}. \tag{X.200}$$

The holes are distributed in the shell as shown in Figure X.30, a for $\varepsilon = 1/4$ and Figure X.30, b for $\varepsilon = -1/4$.

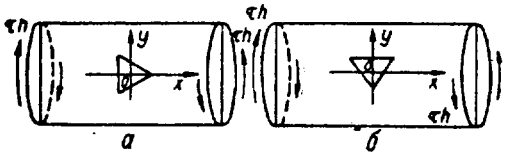


Figure X.30.

The basic stress state will be regarded as momentless with components

$$T_x^0 = 0; \qquad T_y^0 = 0; \qquad S_{xy}^0 = \tau h. \tag{X.201}$$

We will write out the components of the basic stress state in coordinate system (ρ, θ) in the form of power series with respect to ε, retaining in these expansions only the terms with ε^2. The expansions of components (X.201) on the contour of the hole are /770

[1]See A. N. Guz' [13].

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$$T_0^0 = \tau h [\sin 2\theta + 2\varepsilon (\sin 5\theta + \sin \theta) + 4\varepsilon^2 (\sin 8\theta - \sin 2\theta) + \ldots];$$

$$T_\theta^0 = -T_0^0;$$

$$S_{0\theta}^0 = \tau h [\cos 2\theta - 2\varepsilon (\cos \theta - \cos 5\theta) - 4\varepsilon^2 (\cos 2\theta - \cos 8\theta) + \ldots].$$

TABLE X.16

θ	Approximation		
	Zero	First	Second
0	0,00 0,00	0,00 0,00	0,00 0,00
$\frac{\pi}{4}$	−4,52 −4,52	−2,92 −6,12	−2,92 −6,12
$\frac{\pi}{3}$	−3,89 −3,89	−1,95 −5,83	−2,84 −6,72
$\frac{\pi}{2}$	0,00 0,00	−2,26 2,26	−2,26 2,26
$\frac{2\pi}{3}$	3,89 3,89	5,83 1,95	6,72 2,84
$\frac{3\pi}{4}$	4,52 4,52	6,12 2,92	6,12 2,92
π	0,00 0,00	0,00 0,00	0,00 0,00
$\frac{5\pi}{4}$	−4,52 −4,52	−6,12 −2,92	−6,12 −2,92
$\frac{4\pi}{3}$	3,89 3,89	1,95 5,83	1,04 4,94
$\frac{3\pi}{2}$	0,00 0,00	−2,26 2,26	−2,26 2,26
$\frac{5\pi}{3}$	3,89 3,89	1,95 5,83	2,84 6,72
$\frac{7\pi}{4}$	4,52 4,52	2,92 6,12	2,03 5,53

Tr. Note: Commas indicate decimal points.

On the contour of the hole upon which external forces do not act, boundary conditions (X.193) must be satisfied. From (X.193) and (X.202), equating to zero the coefficients for identical degrees of ε, we obtain the boundary conditions for the zero, first and second approximations.

The solution of the basic equation (X.75) will be taken in the form (X.91)-(X.94).

The zero approximation coincides with the solution for a round hole (X.145).

On the basis of the results of §2, we obtain, on the contour of the hole,

$$T_\theta^{(1)} + T_\theta^{0(1)} = -8\tau h \left(1 + \frac{1}{2}\pi\beta^2\right) \sin 5\theta; \quad \text{(X.203)}$$

$$T_\theta^{(2)} + T_\theta^{0(2)} = -16\tau h \left(1 + \frac{1}{8}\pi\beta^2\right) \sin 8\theta. \quad \text{(X.204)}$$

The values of T_θ^* on the contour of the hole are determined with an accuracy up to ε^2, with consideration of (X.145), (X.203) and (X.204), by formula (X.186). We will present these values on the contour of the hole (with an accuracy up to ε^2):

for $\varepsilon = 1/4$ (see Figure X.30, a)

$$T_\theta^* = -4\tau h [\sin 2\theta + 0{,}5 \sin 5\theta + 0{,}25 \sin 8\theta + 0{,}5\pi\beta^2 (\sin 2\theta + 0{,}5 \sin 5\theta + 0{,}0625 \sin 8\theta)]; \quad \text{(X.205)}$$

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for $\varepsilon = -1/4$ (see Figure X.30, b)

$$T_\theta^* = -4\tau h [\sin 2\theta - 0{,}5 \sin 5\theta + 0{,}25 \sin 8\theta + \\ + 0{,}5\pi\beta^2 (\sin 2\theta - 0{,}5 \sin 5\theta + 0{,}0625 \sin 8\theta)]. \qquad (X.206)$$

The values of $k = T_\theta^*/\tau h$ at certain points of the contour of the hole, calculated by formulas (X.205) and (X.206), are presented in Table X.16. The values of k for $\varepsilon = 1/4$ are given in the numerators, and in the denominators for $\varepsilon = -1/4$, for $\nu = 0.3$ and $r_0/\sqrt{Rh}$. As we see, k_{max} is achieved when $\theta = 2\pi/3$ in the case where $\varepsilon = 1/4$ and for $\theta = \pi/3$ in the case where $\varepsilon = -1/4$. At these values of θ, the magnitude of $k^{(sh)}$ is 12% greater than $k^{(pl)}$. At the points $\theta = 0$; $\theta = \pi/4$; $\theta = \pi/2$; $\theta = 3\pi/4$; $\theta = \pi$; $\theta = 5\pi/4$ and $\theta = 3\pi/2$ the second approximation coincides with the first; at other points the convergence is not so good. Thus, $k_{max}^{(sh)}$ in the first approximation is 50% greater than the zero approximation, and 15% greater in the second approximation than in the first. /771

§5. Conclusions

In this chapter we have examined the problems of stress concentration near a single free or reinforced hole in spherical and cylindrical shells. The solutions given herein do not exhaust the family of problems being investigated at the present time, and these solutions are presented in extremely brief form. Therefore, in order to afford the reader with the possibility of becoming more familiar with works of which we have spoken, and with works which we could not mention, a rather detailed list of literature has been compiled. This bibliography also includes works that pertain to the problem of stress concentration near holes in shells which, due to the limitedness of the monograph, could not be discussed in the given chapter.

Important results are now being obtained from the analysis of stress concentration near holes in shells of positive, zero and negative gauss curvatures. These results can be classified as follows: 1) analysis of stress state in shells near large holes; 2) application of variation methods to the problem of stress concentration; 3) analysis of stress state near a hole in an elasto-plastic statement; 4) analysis of geometrically nonlinear problems, i.e., of stress state in shells near holes during supercritical deformation; 5) analysis of the effect of physical nonlinearity of the material of the shell on the stress concentration near holes; 6) analysis of multiply-connected regions, i.e., in the case of both a finite and infinite number of holes, investigation of their interaction, effect of the edge of the shell, rigidity of reinforcing rings, etc; 7) periodic problems for shells weakened by holes, etc.

A special monograph will be devoted to the discussion of the above mentioned results of analysis and to more detailed analysis of the results of the given chapter.

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CHAPTER XI. DYNAMIC PROBLEMS OF STRESS DISTRIBUTION NEAR HOLES

Abstract. This chapter is devoted to the investigation of the stressed state in the vicinity of a hole in the case of dynamic problems. Problems are considered on distribution of the transient cylindrical elastic waves, generated on a circular hole in a thin infinite plate. The plate material was assumed to be both isotropic and anisotropic and non-uniform of special type. The stresses at the hole contour and wave front are determined.

The preceding chapters have been devoted to the problem of stress concentration near holes under static, i.e., extremely slow loading. In practice, however, cases are often encountered where the character of external loading is such that it is not possible to discard the inertial terms in the equilibrium equations. This is related to the fact that the applied forces change in time quite rapidly, while the interval of time of observation is less than that during which the body experiences the state of static equilibrium. In these case we will have propagation within the body (continuous medium) of perturbations, or waves moving at certain velocities. /780

Some of the commonest problems related to the analysis of elastic[1] wave motions in regions weakened by holes will be considered below.

§1. Statement of Problem

In the case of the plane problem of elasticity theory it is convenient to introduce two functions $\phi(x, y)$ and $\vec{\psi} = \vec{k}\psi(x, y)$, where $\vec{k}$ is the unit vector of the normal to plane xOy, related to displacement vector $\vec{u}$ by the relation

$$\vec{u} = \operatorname{grad}\varphi + \operatorname{rot}\vec{\psi}. \tag{XI.1}$$

Then, from the known Lame equations[2], we find that the functions $\phi(x, y, t)$ and $\psi(x, y, t)$ should satisfy the following two wave equations:

$$\nabla^2\varphi - \frac{1}{c_1^2}\frac{\partial^2\varphi}{dt^2} = 0,$$

---

[1]For more detailed accounts of nonlinear, elastoviscous and plastic waves, the reader is referred to the reports of H. Kolsky [1]; Kh. A. Rakhmatulin and G. S. Shapiro [1, 2]; V. Ol'shaka, Z. Mruz and P. Pezhina [1] and the monographs of Kh. A. Rakhmatulin and Yu. M. Dem'yanova [1].

[2]The mass forces are omitted in these equations. The motion originating from the mass forces are described by A. Lyav [1], pp. 317-319.

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$$\nabla^2\psi - \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} = 0; \qquad \text{(XI.2)}$$

where ∇^2 is the Laplace operator.

Equations (XI.2) describe two types of elastic waves that occur in an unbounded homogeneous medium. The function $\phi(x, y, t)$ corresponds to the vortex-free wave ($\vec{u}_1 = \text{grad}\ \phi$), i.e., to such as for which /781

$$\text{rot}\,\vec{u}_1 = 0,$$

while three dimensional expansion is

$$\Delta = \text{div}\,\vec{u}_1 = \nabla^2\varphi.$$

The function $\psi(x, y, t)$ corresponds to equivoluminal wave ($\vec{u}_2 = \text{rot}\ \vec{k}\psi$). For this type of waves

$$\text{div}\,\vec{u}_2 = 0, \quad \vec{\Omega} = \text{rot}\,\vec{u}_2 = -\vec{k}\nabla^2\psi.$$

Here $\vec{\Omega}$ is the vector of instantaneous rotation[1].

In equations (XI.2) c_1 is the velocity of the vortex-free wave and c_2 is the velocity of the equivoluminal wave.

In the case of plane deformation

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{g}}, \qquad c_2 = \sqrt{\frac{\mu}{g}}, \qquad \text{(XI.3)}$$

and in the case of the (generalized) plane stress state

$$c_1 = \sqrt{\frac{\lambda' + 2\mu}{g}} = \sqrt{\frac{E}{(1-\nu^2)g}}, \qquad c_2 = \sqrt{\frac{\mu}{g}} = \sqrt{\frac{G}{g}}, \qquad \lambda' = \frac{2\lambda\mu}{\lambda + 2\mu}. \qquad \text{(XI.4)}$$

Here λ, μ are Lame's constants; g is the density of the medium.

[1]Also often used are the terms "longitudinal wave" and "transverse wave," since in the first case particles of a continuous medium accomplish motion in the direction of propagation of the elastic wave, and in the second, perpendicular to it. In the following discussion we will use the terms "compression or tension wave" and "displacement wave."

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If the external load changes in time harmonically, i.e., if the functions that describe the wave field are of the form

$$\varphi = \varphi(M)\, e^{-i\omega t}, \qquad \psi = \psi(M)\, e^{-i\omega t},$$

where ω is the rotary frequency, then for $\phi(M)$, $\psi(M)$, which depend on the spatial coordinates of the point M, we obtain from (XI.2) the Helmholtz equations

$$\nabla^2\varphi + \frac{\omega^2}{c_1^2}\varphi = 0, \qquad \nabla^2\psi + \frac{\omega^2}{c_2^2}\psi = 0. \tag{XI.5}$$

If the solutions of equations (XI.5) are multiplied by $e^{-i\omega t}$ and the real part in the latter is separated, we obtain the solution of equations (XI.2) for the case of steady wave motions.

In the case of the unsteady process the region occupied by the propagating perturbation is bounded by a closed cylindrical surface (curve) in the plane xOy, which moves in the direction of its normal at a velocity of c_1 or c_2. During motion on it, certain conditions of a kinematic and dynamic character[1] must be satisfied. It follows from the conditions of continuity that the displacement vector on this surface should be equal to zero. /782

If an elastic wave of one type with a free boundary is encountered, then there will be reflected waves of both types[2].

If the boundary is the line (surface) of separation of two media with different elastic properties, elastic waves of both types will also occur in the second medium.

For diverging waves, both reflected and refracted, certain conditions that preclude the possibility of waves arriving from infinity must be satisfied. For unsteady motions this is a condition of damping of perturbations at infinity. For steady motions, however, they consist of the following.

We will represent the displacement vector $\vec{u}$ in the form of the sum of its potential $\vec{u}_S$ and solenoidal $\vec{u}_R$ parts:

$$\vec{u} = \mathrm{Re}\left[(\vec{u}_R + \vec{u}_S)\, e^{-i\omega t}\right].$$

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[1]See A. Lyav [1], pp. 308-310.
[2]See H. Kolsky [2].

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Whereas the component of the vectors $\vec{u}_R$, $\vec{u}_S$ for $r \to \infty$, where $\vec{r}$ is the vector radius, is constructed from the origin of the coordinate system, the following conditions must be satisfied[1]:

$$u_R = O\left(\frac{1}{\sqrt{r}}\right), \quad \frac{\partial U_R}{\partial r} - i\frac{\omega}{c_1}u_R = O\left(\frac{1}{\sqrt{r}}\right),$$
$$u_S = O\left(\frac{1}{\sqrt{r}}\right), \quad \frac{\partial u_S}{\partial r} - i\frac{\omega}{c_2}u_S = O\left(\frac{1}{\sqrt{r}}\right), \qquad \text{(XI.6)}$$

which will be referred to below as the "emission conditions." Here [g,p.78] is a function which, for $r \to +\infty$, has the value $O\left(\frac{1}{\sqrt{r}}\right) \leqslant M\frac{1}{\sqrt{r}}$, $M = \text{const}$, and $O\left(\frac{1}{\sqrt{r}}\right)$ is a function for which the ratio $O\left(\frac{1}{\sqrt{r}}\right):\frac{1}{\sqrt{r}}$ approaches zero uniformly in relation to the direction of vector radius $\vec{r}$.

The propagation of waves in the middle surface of thin plates is described satisfactorily by the equations of the generalized plane stress state in the case of long waves, the length of which is greater than the thickness of the plate. In terms of frequency this signifies that the theory of the generalized plane stress state produces good results for frequencies that are much lower /783 than the basic frequency of vibrations through thickness. The frequencies of the first types of vibrations through thickness are determined by the formulas

$$\omega = \frac{\pi}{2h}\sqrt{\frac{\lambda + 2\mu}{g}}, \quad \omega = \frac{\pi}{h}\sqrt{\frac{\mu}{g}},$$

where h is the thickness of the plate; g is the density of the material of the plate.

§2. Round Hole. Axisymmetric Problems

If to the contour of a round hole of radius a, located in an infinite elastic plane, a uniform pressure or displacement, changing in time, is applied, a cylindrical elastic wave of compression or displacement, respectively, will occur in this plane. In the given case it is convenient to solve the problem in displacements, since due to axial symmetry, one of the components of the displacement vector will be equal to zero.

Let us examine the case of dynamic pressure.

In polar coordinates r, θ, the center of which coincides with the center of the hole, we obtain the equation for radial displacement u:

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[1]See V. D. Kupradze [2].

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$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{1}{c_1^2}\cdot\frac{\partial^2 u}{\partial t^2}. \qquad \text{(XI.7)}$$

In the case of steady vibrations the general solution of equation (XI.7) has the form

$$u(r, t) = \left[AH_1^{(1)}\left(\frac{\omega}{c_1}r\right) + BH_1^{(2)}\left(\frac{\omega}{c_1}r\right)\right]e^{-i\omega t},$$

where $H_1^{(1)}$, $H_1^{(2)}$ are Hankel's functions of kinds I and II; A, B are undefined constants.

From emission conditions (XI.6) it follows that B = 0. By satisfying harmonic condition

$$\sigma_r|_{r=a} = -\sigma_0 e^{-i\omega t}, \qquad \text{(XI.8)}$$

we obtain

$$u(r, t) = \mathrm{Re}\left\{-\sigma_0 \frac{H_1^{(1)}\left(\frac{\omega}{c_1}r\right)e^{-i\omega t}}{\frac{\nu-1}{a}H_1^{(1)}\left(\frac{\omega}{c_1}a\right) + \frac{\omega}{c_1}H_0^{(1)}\left(\frac{\omega}{c_1}a\right)}\right\}. \qquad \text{(XI.9)}$$

For an elastic wave of different configuration we may use the superposition of the solutions of (XI.9), taking the desired solution in the form of a Fourier series or integral. Also effective is the Laplace transform[1]. If /784 equation (XI.7) is multiplied by e^{-pt} (Rep > 0) and the result is integrated with respect to time from zero to infinity, we obtain in the region of the Laplace transformations, for zero initial conditions, the following equation[2]:

$$\frac{d^2U}{dr^2} + \frac{1}{r}\cdot\frac{dU}{dr} - \left(\frac{p}{c_1^2} + \frac{1}{r^2}\right)U = 0. \qquad \text{(XI.10)}$$

Here U(r, p) is the transformation u(r, t), with the form

$$U(r, p) = \int_0^\infty e^{-pt}u(r, t)\,dt \div\!\!> u(r, t). \qquad \text{(XI.11)}$$

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[1]See M. A. Lavrent'yev and B. V. Shabat [1].
[2]See A. Kromm [1, 2]; I. Miklowitz [1].

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If the material of the plate is cylindrically anisotropic, such that the axis of anisotropy passes through the center of the hole, the propagation of the cylindrical compression wave in the space of the transformations is described by the equation[1]

$$\frac{d^2U}{dr^2}+\frac{1}{r}\cdot\frac{dU}{dr}-\left(\frac{k}{r^2}U+\frac{p^2}{c_1^2}\right)U=0, \tag{XI.12}$$

where

$$c_1^2=\frac{E_1}{(1-\nu_r\nu_\theta)g},\quad k=\frac{E_2}{E_1}=\frac{\nu_r}{\nu_\theta}.$$

If, on the other hand, the material is continuously heterogeneous, such that Young's moduli are gradual functions of the radial coordinate r, i.e.,

$$E_r=E_1r^m,\quad E_\theta=E_2r^m,\quad \nu_r\nu_\theta=\text{const}, \tag{XI.13}$$

then, for the transformation of displacement U we have the following equation[2]:

$$\frac{d^2U}{dr^2}+\frac{m+1}{r}\cdot\frac{dU}{dr}+\frac{m\nu_\theta-k}{r^2}U=\frac{g(1-\nu_r\nu_\theta)}{E_1}r^{-m}p^2U. \tag{XI.14}$$

Equations (XI.10), (XI.12) and (XI.14) reduce to Bessel's equation. Let us examine equation (XI.14), a particular case of which are equations (XI.10) and (XI.12). Introducing the dimensionless values

$$\bar r=\frac{r}{a},\quad \bar t=\frac{c_0t}{a},\quad c_0^2=\frac{E_1a^m}{(1-\nu_r\nu_\theta)g},\quad \bar u=\frac{E_1a^{m-1}}{(1-\nu_r\nu_\theta)\sigma_0}u,$$

$$\bar\sigma_r=\frac{\sigma_r}{\sigma_0},\quad \bar\sigma_\theta=\frac{\sigma_\theta}{\sigma_0},$$

we write, in the space of transformations, the general solution of equation (XI.14): /785

$$\bar U(\bar r,p)=\bar r^{-\frac{m}{2}}\left[A(p)K_n\left(\frac{p}{1-\frac{m}{2}}\bar r^{1-\frac{m}{2}}\right)+B(p)I_n\left(\frac{p}{1-\frac{m}{2}}\bar r^{1-\frac{m}{2}}\right)\right], \tag{XI.15}$$

$$n=\frac{1}{2-m}\sqrt{m^2+4(k-m\nu_\theta)}.$$

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[1]See G. Eason [1].
[2]See V. D. Kubenko [1, 2].

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In the solution of (XI.15) K_n is McDonald's function; I_n is a modified Bessel function[1].

When m = 2 the solution of equation (XI.14) becomes

$$\bar{U}(\bar{r}, p) = A_1(p)\bar{r}^{d_1} + B_1(p)\bar{r}^{d_2},$$
$$d_{1,2} = -1 \mp \sqrt{p^2 + k + 1 - 2\nu_\theta}. \tag{XI.16}$$

The boundary conditions consist in the assignment of $\bar{\sigma}_r$ on the contour of the hole and in the damping condition of perturbations at infinity, i.e.,

$$\bar{\sigma}_r = \begin{cases} -H(\bar{t}), & \bar{r} = 1; \\ 0, & \bar{r} \to \infty. \end{cases} \tag{XI.17}$$

Here H(t) is Heaviside's function

$$H(\bar{t}) = \begin{cases} 0, & \bar{t} < 0; \\ 1, & t \geqslant 0. \end{cases}$$

In the space of transformations conditions (XI.17) will acquire the form

$$\int_0^\infty e^{-p\bar{t}}\bar{\sigma}_r(\bar{r}, \bar{t})\,d\bar{t} = \begin{cases} -\dfrac{1}{p}, & \bar{r} = 1; \\ 0, & \bar{r} \to \infty. \end{cases} \tag{XI.18}$$

In the case m < 2 the solution of equation (XI.14) under conditions (XI.18) becomes[2]

$$\bar{U}(\bar{r}, p) = \bar{r}^{\alpha-1}\frac{\frac{1}{p^2}K_n\left(\frac{p}{\alpha}\bar{r}^\alpha\right)}{\frac{1-\alpha+n\alpha-\nu_\theta}{p}K_n\left(\frac{p}{\alpha}\right) + K_{n-1}\left(\frac{p}{\alpha}\right)}, \tag{XI.19}$$

where

$$\alpha = 1 - \frac{m}{2}.$$

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[1]See M. A. Lavrent'yev and B. V. Shabat [1].
[2]From the second condition (XI.18) it follows that B = 0.

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Transition to the space of the originals can be accomplished by various means[1]. For instance, by representing the function K_n through Laplace's integral and using the theorem of convolution of originals, we find[2] that the displacement, velocity, and direction satisfy Voltaire's integral equation of kind I: /786

$$\int_0^{\bar{t}} s(\bar{r}, \tau) K(\bar{t}-\tau)\, d\tau = R_s(\bar{r}, \bar{t}). \tag{XI.20}$$

Here $s(\bar{r}, \tau)$ denotes the functions $\bar{u}$, $\bar{u} = \partial u/\partial t$, $\overline{\sigma_r}$, $\overline{\sigma_\theta}$. The kernel K will be the same for all equations (XI.20):

$$K(x) = \frac{1-\alpha+\alpha n-\nu_\theta}{2n}\left[(\alpha x+1+\sqrt{(\alpha x+1)^2-1})^n - \right.$$
$$\left. -(\alpha x+1-\sqrt{(\alpha x+1)^2-1})^n\right] +$$
$$+\frac{(\alpha x+1+\sqrt{(\alpha x+1)^2-1})^{n-1}+(\alpha x+1-\sqrt{(\alpha x+1)^2-1})^{n-1}}{2\sqrt{(\alpha x+1)^2-1}}.$$

The right hand side of (XI.20) has the form

$$\begin{aligned} R_{\bar{u}} &= z(\bar{z}, \bar{t})\bar{r}^{\alpha-1}, \\ R_{\bar{\sigma}_r} &= (\alpha+\nu_\theta-1)\, z(\bar{r}, \bar{t})\bar{r}^{-\alpha} + z'(\bar{r}, \bar{t})\bar{r}^{1-\alpha}, \\ R_{\bar{\sigma}_\theta} &= (\alpha\nu_\theta+k-\nu_\theta)\, z(\bar{r}, \bar{t})\bar{r}^{-\alpha} + \nu_\theta z'(\bar{r}, \bar{t})\bar{r}^{1-\alpha}, \\ R_{\dot{\bar{u}}} &= \dot{z}(\bar{r}, \bar{t})\bar{r}^{\alpha-1}, \end{aligned} \tag{XI.21}$$

where

$$z(\bar{r}, \bar{t}) = H\left(\bar{t}-\frac{\bar{r}^\alpha-1}{\alpha}\right)\frac{\bar{r}^\alpha}{2\alpha n(n^2-1)}\left\{\left(\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}+\sqrt{\frac{(\alpha\bar{t}+1)^2}{\bar{r}^{2\alpha}}-1}\right)^{n-1}\times\right.$$
$$\times\left[(n-1)\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}\left(\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}+\sqrt{\frac{(\alpha\bar{t}+1)^2}{\bar{r}^{2\alpha}}-1}\right)-2n\right]-$$
$$-\left(\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}-\sqrt{\frac{(\alpha\bar{t}+1)^2}{\bar{r}^{2\alpha}}-1}\right)^{n-1}\left[(n-1)\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}\times\right.$$
$$\left.\left.\times\left(\frac{\alpha\bar{t}+1}{\bar{r}^\alpha}-\sqrt{\frac{(\alpha\bar{t}+1)^2}{\bar{r}^{2\alpha}}-1}\right)-2n\right]\right\};\quad z'=\frac{\partial z}{\partial\bar{r}}; \tag{XI.22}$$

[1]See H. L. Selberg [1]; I. Miklowitz [1].
[2]See A. Kromm [1]; V. D. Kubenko [1].

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for n = 1

$$z(\bar{r}, \bar{t}) = H\left(\bar{t} - \frac{\bar{r}^{\alpha} - 1}{\alpha}\right)\left[\frac{\alpha\bar{t} + 1}{2\bar{r}^{\alpha}}\sqrt{(\alpha\bar{t} + 1)^2 - \bar{r}^{2\alpha}} - \right.$$
$$\left. - \frac{\bar{r}^{\alpha}}{2\alpha}\ln\frac{\alpha\bar{t} + 1 + \sqrt{(\alpha\bar{t} + 1)^2 - \bar{r}^{2\alpha}}}{\bar{r}^{\alpha}}\right]. \qquad \text{(XI.23)}$$

During interval of time /787

$$0 < t < \frac{\bar{r}^{\alpha} - 1}{\alpha}$$

the right hand sides of (XI.20), on the basis of (XI.22), (XI.23), are equal to zero.

On the wave front we have

$$\bar{t} = \frac{\bar{r}^{\alpha} - 1}{\alpha}.$$

Hence the velocity of the wave front is

$$c = \frac{d\bar{r}}{d\bar{t}} = \bar{r}^{1-\alpha} = \bar{r}^{\frac{m}{2}}, \qquad \text{(XI.24)}$$

or, in the initial symbols:

$$c = \sqrt{\frac{E_1 r^m}{(1 - \nu_r\nu_\theta)\,g}}.$$

By substituting into integral equations (XI.20) $t = \frac{r^{\alpha} - 1}{\alpha} + \varepsilon$, where ε approaches zero, we obtain the following relations on the wave surface:

$$\bar{u} = 0,\ \bar{\sigma}_r = -\bar{r}^{\frac{m}{4} - \frac{1}{2}},\ \bar{\sigma}_\theta = -\nu_\theta \bar{r}^{\frac{m}{4} - \frac{1}{2}},\ \dot{\bar{u}} = r^{-\frac{1}{2} - \frac{1}{4}m} \qquad \text{(XI.25)}$$

From (XI.25) it follows that the discontinuity of stresses on the wave front depends on nonhomogeneity index m, and not on the anisotropy of the material.

Equations (XI.20) can be solved numerically by reducing them to a system of algebraic equations. The stresses $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ as solved by this method in the

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work of A. Kromm [1] are shown in Figures XI.1-XI.3 as functions of time and the radial coordinate for a homogeneous isotropic medium, i.e., for m = 0. For these calculations it was assumed that $\nu_r = \nu_\theta = 0.25$.

If the order of the cylindrical functions in (XI.15) is equal to a whole number plus one-half, it is possible to make the transition to the space of originals by way of contour integration using the remainders theorem[1]. Then the expressions for displacement and stresses can be found in closed form. Some of the results obtained by this method are illustrated in Figures XI.4-XI.6.

When m = 2 we have the solution of (XI.16). If boundary conditions (XI.18) are satisfied, we obtain

$$\bar{U}(\bar{r}, p) = \frac{e^{-\ln\bar{r}\sqrt{p^2+h^2}}}{\bar{r}p(b+\sqrt{p^2+h^2})}, \tag{XI.26}$$

where

$$b = 1 - \nu_\theta, \quad h^2 = k + 1 - 2\nu_\theta.$$

Converting to the original, we obtain[2] /789

$$\bar{u}(\bar{r}, \bar{t}) = H(\bar{t} - \ln\bar{r})\frac{1}{\bar{r}}\left[\frac{1}{b} - \frac{1}{b}\bar{r}^b e^{-b\bar{t}} - \bar{r}^b \int_{\ln\bar{r}}^{\bar{t}} d\tau \int_0^{h\sqrt{\tau^2-\ln^2\bar{r}}} e^{-\frac{b}{h}\sqrt{h^2\tau^2-z^2}} J_1(z)dz\right];$$

$$\bar{\sigma}_r(\bar{r}, \bar{t}) = H(\bar{t} - \ln\bar{r})\left[-1 + h\ln\bar{r}\int_{\ln\bar{r}}^{\bar{t}} \frac{J_1(h\sqrt{\tau^2-\ln^2\bar{r}})}{\sqrt{\tau^2-\ln^2\bar{r}}} d\tau\right];$$

$$\bar{\sigma}_\theta(\bar{r}, \bar{t}) = H(\bar{t} - \ln\bar{r})\left[\frac{b-1+k}{b} + \frac{(b-1)^2-k}{b}\bar{r}^b e^{-b\bar{t}} + \right. \tag{XI.27}$$

$$+ (b^2 - 2b - k + 1)\bar{r}^b \int_{\ln\bar{r}}^{\bar{t}} d\tau \int_0^{h\sqrt{\tau^2-\ln^2\bar{r}}} e^{-\frac{b}{h}\sqrt{h^2\tau^2-z^2}} J_1(z)\,dz -$$

$$\left. - h(b-1)\ln\bar{r}\int_{\ln\bar{r}}^{\bar{t}} \frac{J_1(h\sqrt{\tau^2-\ln^2\bar{r}})}{\sqrt{\tau^2-\ln^2\bar{r}}} d\tau.\right]$$

[1]See G. Eason [1]; V. D. Kubenko [1].
[2]See V. D. Kubenko [2].

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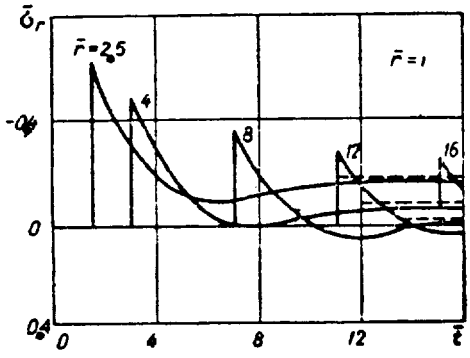


Figure XI.1.

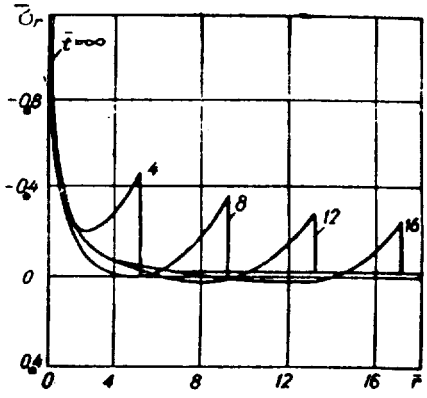


Figure XI.2.

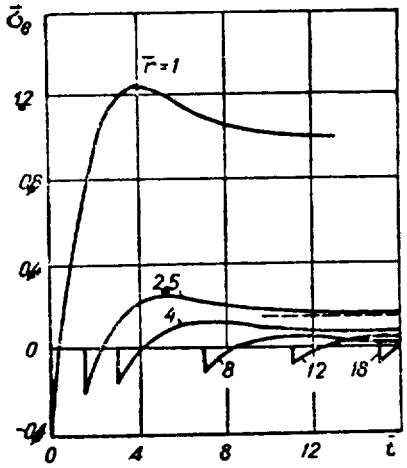


Figure XI.3.

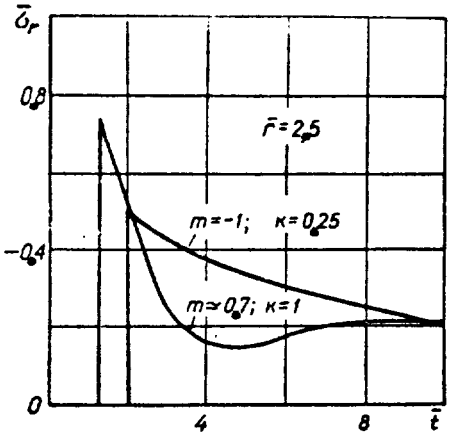


Figure XI.4.

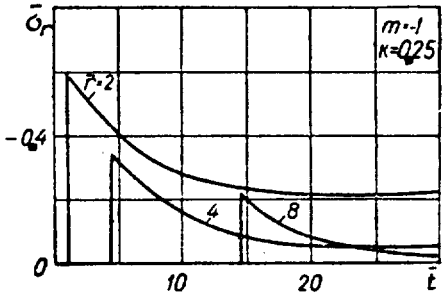


Figure XI.5.

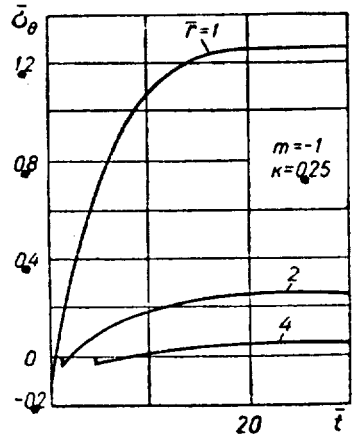


Figure XI.6.

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The wave front propagates at the velocity $c = \bar{r}$, whereupon, on the front,

$$\bar{u} = 0, \quad \dot{\bar{u}} = \bar{r}^{-1}, \quad \bar{\sigma}_r = -1, \quad \bar{\sigma}_\theta = -\nu_\theta. \tag{XI.28}$$

Hence the discontinuity of stresses on the wave front remains constant during the propagation of the wave. The change of $\bar{\sigma}_r$ with time is shown in Figure XI.7. For the calculations it was assumed that $\nu_\theta = 1/4$, $\nu_r = 1/14$.

Finally, in the case $m > 2$ the wave reaches an infinitely distant point beyond which the terminal time interval is

$$\bar{t}_\infty = -\frac{1}{1-\frac{m}{2}}.$$

V. D. Kubenko [2] demonstrated that this circumstance necessitates the use of a condition other than the condition of damping of perturbations[1], specifically, to assume the absence of displacement at infinity, since in accordance with (XI.13) for $r \to \infty$ the material (at infinite points) becomes rigid. Then the solution found for these new conditions will converge upon the static solution with time[2].

In the case where the density of the medium also changes by a gradual law as a function of r, the solution of the problem does not differ in principle from the above.

If the dependence of the boundary condition on time is defined not by Heaviside's function, but by some function P(t), then in this case we may use Duhamel's formula for the determination of the solution. If through $f_H(\bar{r}, \bar{t})$ we denote the solution derived from Heaviside's function, then for the function P(t) we will have /790

$$f_P(\bar{r}, \bar{t}) = P(0) f_H(\bar{r}, \bar{t}) + \int_0^{\bar{t}} \frac{dP(\tau)}{d\tau} f_H(\bar{t} - \tau, \bar{r})\, d\tau. \tag{XI.29}$$

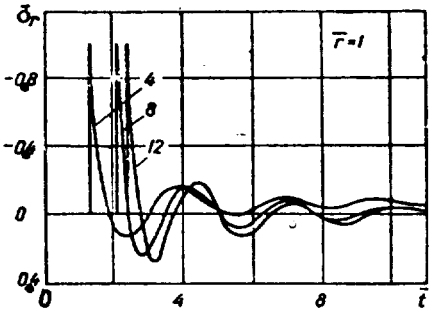


Figure XI.7.

In like manner we may solve problems in the case where a suddenly applied

[1]The second condition in (XI.18).
[2]See S. G. Lekhnitskiy [1].

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uniform force of displacement acts upon the contour of a round hole or a radial or circular velocity is imparted to the points of the contour[1]. E. Sternberg and I. C. Charkraborty [1] examined the problem of the propagation of a cylindrical wave of displacement in an elastic medium whose displacement modulus is a gradual function of the radial coordinate.

§3. Round Hole. Incident Plane Harmonic Wave

Of great importance in application is the problem which is the dynamic analog of the problem of stress concentration near a hole in the case of the biaxial basic stress state. In the dynamic case the latter is accomplished by means of a plane elastic wave[2], incident upon the hole, which is located at the origin of the coordinate system. The incident wave generates reflected waves, and the total wave field determines the stress state in an elastic plane.

Y. H. Pao [1], Y. H. Pao, C. C. Mow [1], C. C. Mow, L. J. Mente [1] examine dynamic problems of stress concentration near a round free hole or near a soldered rigid inclusion (Figure XI.8) in a thin infinite plate, within which a plane wave of compression or displacement (steady vibrations) propagates. The generalized plane stress state is assumed, i.e., the case of a thin plate is examined.

Reflected waves of both compression and displacement occur on the contour of the hole. Their potentials ϕ and ψ satisfy Helmholtz equations (XI.5), the general solution of which, in polar coordinates (r, θ), can be represented in the form /791

$$\varphi^*(r,\theta)=\sum_{m=0}^{\infty}\left(A_m\cos m\theta+B_m\sin m\theta\right)H_m^{(1)}\left(\frac{\omega}{c_1}r\right),$$
$$\psi^*(r,\theta)=\sum_{m=0}^{\infty}\left(C_m\cos m\theta+D_m\sin m\theta\right)H_m^{(1)}\left(\frac{\omega}{c_2}r\right). \qquad (XI.30)$$

Here A_m, B_m, C_m, D_m are undefined (constant) coefficients; $\alpha = \omega/c_1$, $\beta = \omega/c_2$ are wave numbers; $H_m^{(1)} = J_m + iN_m$ is Hankel's function of kind I, of the m-th order; ω is rotary frequency. Hankel's function of kind II is discarded in formulas (XI.30), as it does not satisfy emission conditions (XI.6). The asterisk in the ϕ and ψ functions in (XI.30) indicates that these are the potentials of reflected waves.

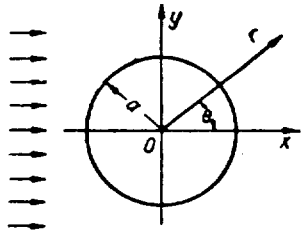


Figure XI.8.

[1]See A. Kromm [2]; I. N. Godier, W. E. Johsman [1].
[2]See F. M. Mors, G. Feshbakh [1].

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Infinte Compression Wave. Free Hole. A plane harmonic compression wave propagating in the direction of increasing values of x is defined by the following potentials:

$$\varphi^{I} = \Phi_0 e^{i(\alpha x - \omega t)},$$
$$\psi^{I} = 0. \tag{XI.31}$$

The superscript I in the ϕ and ψ functions (XI.31) indicates that these functions characterize the basic stress state

$$\sigma_x^I = -\beta^2 \mu \Phi_0 e^{i(\alpha x - \omega t)},$$
$$\sigma_y^I = -\left(\frac{\beta^2}{\alpha^2} - 2\right)\mu\alpha^2 \Phi_0 e^{i(\alpha x - \omega t)}, \tag{XI.32}$$
$$\tau_{xy}^I = 0.$$

A plane wave in polar coordinates is represented[1] in the form

$$e^{i(\alpha x - \omega t)} = \sum_{m=0}^{\infty} \epsilon_m i^m J_m(\alpha r) \cos m\theta e^{-i\omega t}, \tag{XI.33}$$

where J_m is the Bessel function;

$$\epsilon_m = \begin{cases} 1, & m = 0, \\ 2, & m > 1. \end{cases}$$

In the case of a free round hole of radius a the boundary conditions on the contour of the hole are /792

$$(\sigma_r^I + \sigma_r^*)_{r=a} = 0,$$
$$(\tau_{r\theta}^I + \tau_{r\theta}^*)_{r=a} = 0. \tag{XI.34}$$

From (XI.30), (XI.32)-(XI.34) we obtain albegraic equation systems for the determination of coefficients A_m, B_m, C_m, D_m. Consequently the expressions for stresses and displacements are obtained in the form of Fourier-Bessel series. For instance, the stress σ_θ on the contour of the hole is

[1]See F. M. Mors, G. Feshbakh [1].

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$$\sigma_\theta|_{r=a} = -2\left(\frac{\beta^2}{\alpha^2} - 1\right)\mu\alpha^2\Phi_0 \sum_{m=0}^{\infty} \epsilon_m i^m S_m \cos m\theta e^{-i\omega t}$$

$$S_m = i\frac{2}{\pi}\left[\alpha a H_{m-1}(\alpha a) - H_m(\alpha a)\right] q_m(\beta a)^{-1}, \qquad \text{(XI.35)}$$

$$q_m(\beta a)^{-1} = \frac{(m^3 - 1)\beta a H_{m-1}(\beta a) - \left(m^3 - m + \frac{\beta^2 a^2}{2}\right) H_m(\beta a)}{\left(m^3 - m + \frac{\beta^2 a^2}{2}\right)\beta a H_{m-1}(\beta a) - \left(m^2 + m - \frac{\beta^2 a^2}{4}\right)\beta^2 a^2 H_m(\beta a)}$$

If in formulas (XI.35) α approaches zero and we use an asymptotic Hankel's function for small values of the argument, then we obtain at the limit a known static solution of the corresponding problem.

For all waves except "very long" and "very short" waves (the convergence of series depreciates as the frequency increases), the stresses are determined by summing series (XI.35). Consequently we obtain for σ_θ on the contour of the hole the expression

$$\sigma_\theta|_{r=a} = \mathrm{Re}\{(R + iI)e^{-i\omega t}\} = \sqrt{R^2 + I^2}\,\mathrm{Re}\,e^{-i(\omega t - \delta)},$$

$$\delta = \arctan\frac{I}{R}. \qquad \text{(XI.36)}$$

During the total period of vibration (0, T = $2\pi/\omega$) the real part of R represents the stress at the moment t = 0; during this moment the stresses in the incident wave achieve their maximum for $\theta = \pi/2$. The imaginary part I gives the stress at the moment t = T/4; the stresses are equal to zero in the incident wave at this moment when $\theta = \pi/2$. The absolute value of $\sqrt{R^2 + I^2}$ gives the maximum value of stress σ_θ.

It follows from (XI.32) that the maximum stresses in the incident wave are

$$\sigma_x^1 = \sigma^1 = -\frac{\beta^2}{\alpha^2}P; \quad \sigma_y^1 = -\left(\frac{\beta^2}{\alpha^2} - 2\right)P; \quad P = \mu\alpha^2\Phi_0.$$

The relation $\left(\frac{\sigma_\theta}{\sigma^1}\right)_{r=a} = \bar{\sigma}_\theta$ can be regarded as the dynamic stress concentration coefficient.

The concentration coefficient on the contour of a round hole is shown in Figure XI.9 as a function of frequency ($\alpha = \omega/c_1$) when $\theta = \pi/2$ for an incident compression wave for various Poisson's ratios. /793

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If an additional field is superimposed on stress field (XI.32) such that the total wave produces the basic stress state of the form

$$\sigma_x^I = -P\left[\frac{\beta^2}{\alpha^2}e^{i\alpha x} - \left(1 - 2\frac{\alpha^2}{\beta^2}\right)\left(\frac{\beta^2}{\alpha^2} - 2\right)e^{i\alpha y}\right]e^{-i\omega t},$$
$$\sigma_y^I = -P\left[\left(\frac{\beta^2}{\alpha^2} - 2\right)e^{i\alpha x} - \left(1 - \frac{2\alpha^2}{\beta^2}\right)\frac{\beta^2}{\alpha^2}e^{i\alpha y}\right]e^{-i\omega t}, \qquad \text{(XI.37)}$$

then, from (XI.37) for $\alpha \to 0$, we will have the uniaxial stress state

$$\sigma_{x\,\text{ст}}^I = -4\left(1 - \frac{\alpha^2}{\beta^2}\right)P, \quad \sigma_{y\,\text{ст}}^I = 0.$$

The concentration coefficient as a function of frequency is represented in this case in Figure XI.10.

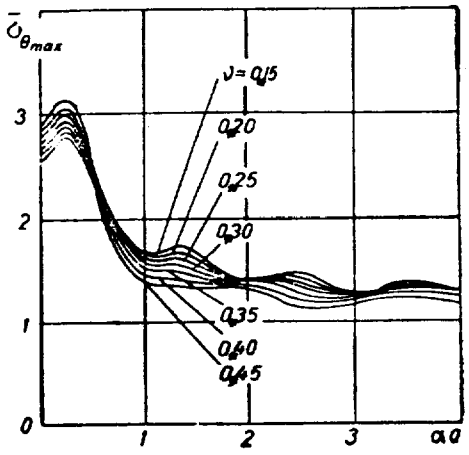


Figure X.9.

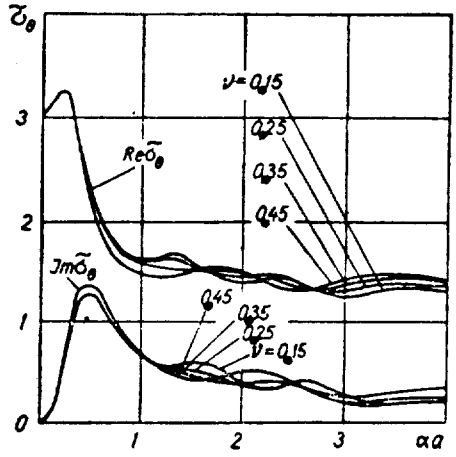


Figure XI.10.

The distribution of σ_θ along the contour of a free round hole for various values of αa in basic stress state (XI.32) is illustrated in Figure XI.11 (ν = = 0.35).

These graphs show that the stress state depends considerably on the ratio between the wavelength and the dimension of the hole and on Poisson's ratio. In a certain range of wave numbers the concentration coefficient is about 10% greater than in the static case.

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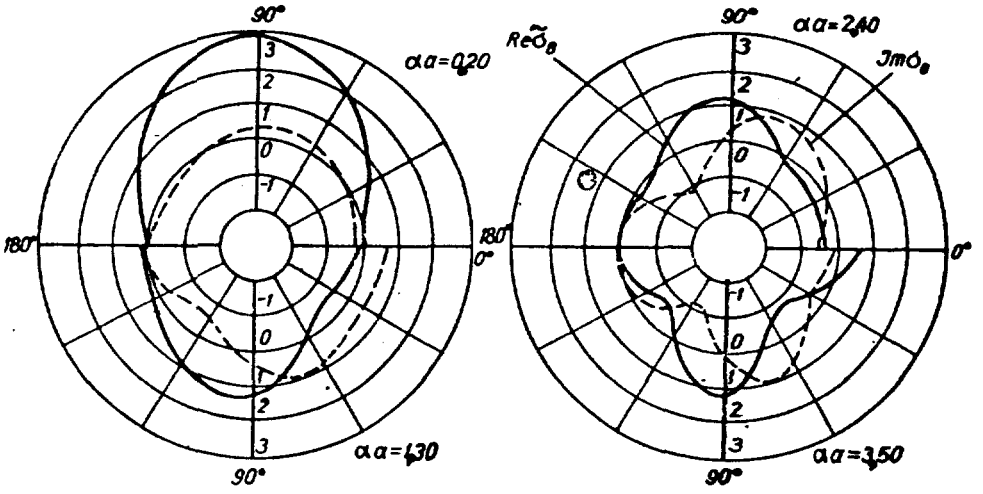


Figure XI.11.

Incident Compression Wave. Rigid Inclusion. The dynamic pattern of stress distribution in a plate with a rigid soldered circular nucleus during the action of plane harmonic compression wave (XI.31) is examined by Y. H. Pao, C. C. Mow [1]. /794

In static problems, as we know, the boundary conditions consist in the fact that the displacements along the soldered joint between the plate and rigid inclusion are equal to zero. In dynamics this condition is valid only when the inclusion is fixed in space by the corresponding external forces or possesses infinite density. If, on the other hand, these two factors do not obtain, the inclusion, when acted upon by the incident wave, will be displaced in space -- it will oscillate as a rigid body. In this case the kinematic boundary conditions will be such[1] that the displacements caused by the incident and reflected waves will produce a linear displacement and rotation of the inclusion, which moves as a rigid body.

Fixed Rigid Inclusion. For stresses on the boundary r = a of the seam we obtain the following formulas:

$$\sigma_r|_{r=a} = \frac{2}{\pi}\mu\Phi_0\beta^2\sum_{m=0}^{\infty}\varepsilon_m i^{m-1}\frac{\beta a H_m'(\beta a)}{\Delta_m}\cos m\theta e^{-i\omega t},$$

$$\tau_{r\theta}|_{r=a} = \frac{2}{\pi}\mu\Phi_0\beta^2\sum_{m=0}^{\infty}\varepsilon_m i^{m-1}\frac{mH_m(\beta a)}{\Delta_m}\sin m\theta e^{-i\omega t},$$

[1]See H. Lamb [1], K. Sezawa [1].

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$$\sigma_\theta|_{r=a} = \left(1 - 2\frac{\alpha^2}{\beta^2}\right)\sigma_r|_{r=a},$$

$$\Delta_m = \alpha\beta a^2 H_m'(\alpha a) H_m'(\beta a) - m^2 H_m(\alpha a) H_m(\beta a), \qquad (XI.38)$$

$$H_m' = \frac{dH_m}{dr}, \qquad /795$$

since $0 < \nu < 1/2$ and $\alpha^2/\beta^2 = (1 - \nu)/2$, $\sigma_\theta/_{r=a}$ is always less than $\sigma_r/_{r=a}$.

Nonfixed Rigid Inclusion. If u and v are radial and tangential components of the displacement vector, then the boundary conditions will acquire the form

$$u|_{r=a} = u^I + u^* = U\cos\theta,$$
$$v|_{r=a} = v^I + v^* = -U\sin\theta, \qquad (XI.39)$$

where U is displacement of the nucleus in the direction of propagation of the wave, and is found from the motion equation for r = a

$$\pi a^2 g_1 \ddot{U} = \int_0^{2\pi} (\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\, a\,d\theta, \qquad (XI.40)$$

where g_1 is the density of the material of the nucleus.

Since the desired stresses are represented in the form of series with respect to sin mθ, cos mθ, by substituting them into (XI.40), using the orthogonality of the trigonometric functions, we obtain

$$U = \eta\frac{1}{a}[2i\Phi_0 J_1(\alpha a) + A_1 H_1(\alpha a) + B_1 H_1(\beta a)],$$
$$\eta = \frac{g}{g_1}. \qquad (XI.41)$$

Here g is the density of the medium.

Consequently we have for displacements and stresses on the boundary, the following expressions:

$$u|_{r=a} = \frac{4\Phi_0}{a\pi\delta_1}\eta[2H_1(\beta a) - \beta a H_0(\beta a)]\cos\theta,$$

$$v|_{r=a} = -\frac{4\Phi_0}{a\pi\delta_1}\eta[2H_1(\beta a) - \beta a H_0(\beta a)]\sin\theta,$$

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$$\sigma_r|_{r=a} = \frac{2\mu\Phi_0\beta^2}{\pi}\left\{\frac{i}{\alpha a H_1(\alpha a)} - 2\left[(1+\eta)H_1(\beta a) - \right.\right.$$

$$\left.\left. - \beta a H_0(\beta a)\right]\frac{1}{\delta_1}\cos\theta + 2\sum_{m=2}^{\infty} i^{m-1}\frac{\beta a H_m'(\beta a)}{\Delta_m}\cos m\theta\right\},$$

$$\tau_{r\theta}|_{r=a} = \frac{2\mu\Phi_0\beta^2}{\pi}\left[2(1-\eta)H_1(\beta a)\frac{1}{\delta_1}\sin\theta + 2\sum_{m=2}^{\infty} i^{m-1}\frac{mH_m(\beta a)}{\Delta_m}\sin m\theta\right], \quad \text{(XI.42)}$$

$$\sigma_\theta|_{r=a} = \left(1 - 2\frac{\alpha^2}{\beta^2}\right)\sigma_r|_{r=a}.$$

Figure XI.12 shows the distribution of stress σ_r on the boundary r = a for /797 $\eta = 0$, $\nu = 0.25$, $\alpha a = 0.10$, determined by formulas (XI.38), and Figure XI.13 shows the same as found by formulas (XI.42) for $\eta = 0.5$, $\alpha a = 0.10$ and $\alpha a = 2.0$, $\nu = 0.25$.

The change of the principal stress σ_1, calculated by formula

$$\sigma_{1,2} = \frac{1}{2}(\sigma_r + \sigma_\theta) \pm \sqrt{\frac{1}{4}(\sigma_r - \sigma_\theta)^2 + \tau_{r\theta}^2},$$

is shown in Figures XI.14 and XI.15 as a function of αa at the points $\theta = 0$ and $\theta = \pi$ for the case $\eta = 0$ and $\eta = 0.5$.

We see in Figure XI.14 that in the case of an inclusion that is fixed in space, as the wave number approaches zero at the points $\theta = 0$ and $\theta = \pi$, the stresses σ_r become infinitely large.

If the force applied to the boundary r = a and directed along the Ox axis is calculated on the basis of (XI.38), we obtain

$$X = \int_0^{2\pi}(\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\,ad\theta = \quad \text{(XI.43)}$$

$$= 4\mu\Phi_0\beta^2\left[\beta a H_1'(\beta a) - H_1(\beta a)\right]\frac{1}{\Delta_1}.$$

As αa approaches zero we will have the following asymptotics:

$$\frac{\beta a H_1'(\beta a)}{\Delta_1} \to \frac{1}{\left(1+\frac{c_1^2}{c_2^2}\right)\alpha a}, \quad \frac{H_1(\beta a)}{\Delta_1} \to -\frac{1}{\left(1+\frac{c_1^2}{c_2^2}\right)\alpha a}.$$

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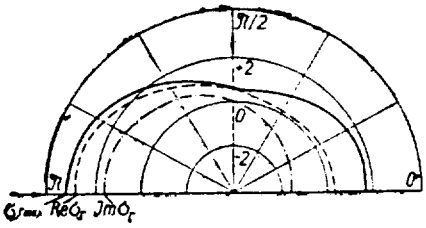


Figure XI.12

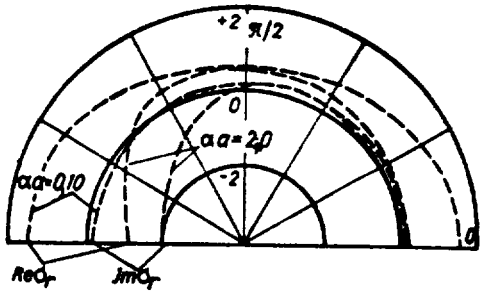


Figure XI.13.

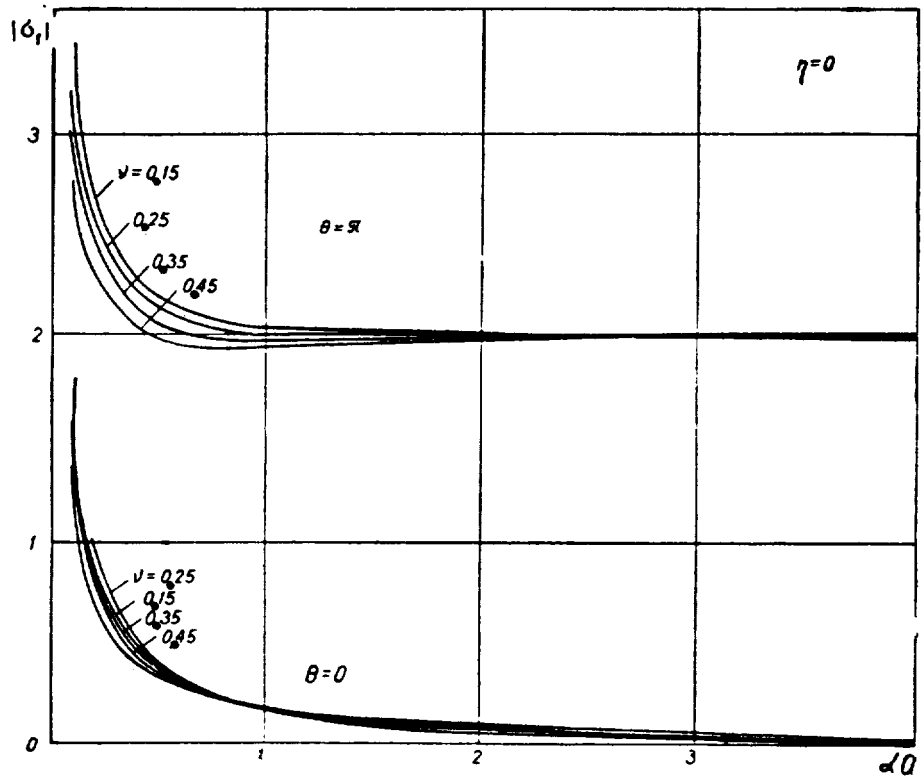


Figure XI.14.

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These expressions show that when the values of αa are small the force that keeps the inclusion fixed has to be great. In the case of standing waves this does not occur[1] since we are not posed with the question of whether or not the inclusion moves. Standing waves are realized in the following manner.

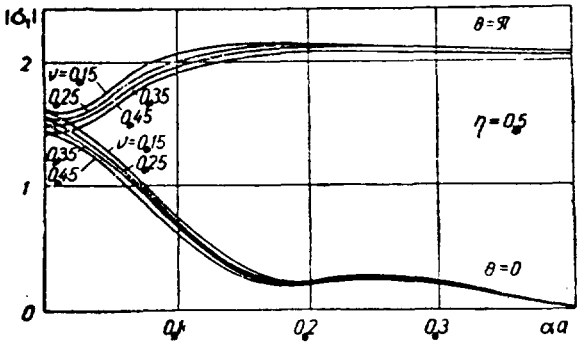


Figure XI.15.

If on wave field (XI.31) we superimpose a wave with the same amplitude and frequency, but which moves in the opposite direction, we obtain the displacement potentials

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$$\varphi^{I} = 2\Phi_0 \cos \alpha x e^{-i\omega t}, \qquad \psi^{I} = 0. \tag{XI.44}$$

These are standing waves. In the given case, only the terms that correspond to even m remain in the expansion of the plane wave (XI.33). The stresses and displacements are given by formulas (XI.35), (XI.38) and (XI.42), in which the terms with even m are omitted.

Incident Displacement Wave. Free Hole. A plane harmonic displacement wave moving in the direction of the x axis is given by wave potentials

$$\varphi^{I} = 0, \qquad \psi^{I} = \Psi_0 e^{i(\beta x - \omega t)}. \tag{XI.45}$$

In the case of a round hole, the stress and deformation states are determined in the same manner as for an incident compression wave. As a result we obtain for the stress σ_θ on the contour of the hole

$$\sigma_\theta|_{r=a} = -\frac{8}{\pi}\left(1 - \frac{\alpha^2}{\beta^2}\right)\beta^2 \mu \Psi_0 \sum_{m=1}^{\infty} i^m S_m \sin m\theta e^{-i\omega t},$$

$$S_m = im\left(m^2 - 1 - \frac{1}{2}\beta^2 a^2\right) H_m(\alpha a)\, \Delta_m^{-1},$$

$$\Delta_m = \beta^2\alpha^2\left(m^2 + m - \frac{\beta^2 a^2}{4}\right) H_m(\alpha a) H_m(\beta a) + \tag{XI.46}$$

$$+ \alpha\beta a^2 (m^2 - 1)\, H_{m-1}(\alpha a)\, H_{m-1}(\beta a) + \left(m - m^3 - \frac{\beta^2 a^2}{2}\right)\beta a H_m(\alpha a) \times$$

$$\times H_{m-1}(\beta a) + \alpha a H_{m-1}(\alpha a)\, H_m(\beta a).$$

[1] Due to symmetry with respect to the y axis.

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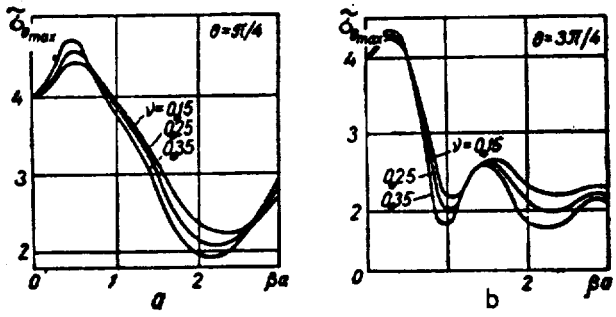


Figure XI.16.

The values $\sigma_\theta|_{r=a} = \frac{\sigma_\theta^{max}}{\mu\beta^2\Psi_0} = \tilde{\sigma}_\theta^{max}$ /799

are shown in Figure XI.16 as functions of the parameter βa for θ = π/4 (Figure XI.16,a) and θ = 3π/4 (Figure XI.16,b), for various values of ν.

Incident Displacement Wave. Rigid Inclusion. In the case of a rigid soldered nucleus (inclusion) of arbitrary density, the resultant forces and torque acting on the contour of the soldered joint r = a upon the inclusion from the direction of the incident and reflected waves are determined by the formulas

$$\begin{aligned} X &= \int_0^{2\pi} (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta)\, a d\theta, \\ Y &= \int_0^{2\pi} (\sigma_r \sin\theta + \tau_{r\theta} \cos\theta)\, a d\theta, \\ M_{cr} &= \int_0^{2\pi} \tau_{r\theta} a^2 d\theta. \end{aligned} \tag{XI.47}$$

From Newton's second law

$$\begin{aligned} \pi a^2 g_1 \ddot{U}_x &= X, \\ \pi a^2 g_1 \ddot{U}_y &= Y, \\ \frac{\pi a^4 g_1}{2} \ddot{\vartheta} &= M_{cr} \end{aligned} \tag{XI.48}$$

(where ϑ is rotation of the nucleus) and from relations

$$\begin{aligned} u &= u_x \cos\theta + u_y \sin\theta, \\ v &= -u_x \sin\theta + u_y \cos\theta + a\vartheta \end{aligned} \tag{XI.49}$$

we obtain the conditions for the determination of the constants in the general solution of (XI.30). The series obtained, due to its awkwardness, will not be written out here[1]. Figure XI.17 illustrates the change of $\overline{\tau}_{r\theta}$ on the boundary r = a of the seam for θ = π/2 and θ = π as a function of frequency for various $\eta = g/g_1$ when ν = 0.25.

[1]See C. C. Mow, L. I. Mente [1].

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Nonstationary Asymmetric Problems. Round Hole. Ya. M. Mindlin [1] constructed an integral of the wave equation that is a generalization of the Lamb-d'Alembert formula for the case of the asymmetric problem:

$$\Phi = \int_0^\infty \left[A_n^{(1)}\left(t - \frac{r}{c}\operatorname{ch}\xi\right) + A_n^{(2)}\left(t + \frac{r}{c}\operatorname{ch}\xi\right)\right] \operatorname{ch} n\xi d\xi e^{in\theta}, \qquad \text{(XI.50)}$$

where n is a whole positive number or zero, and the function $\Phi(r, \theta)$ satisfies the wave equation

$$\nabla^2\Phi - \frac{1}{c^2}\cdot\frac{\partial^2\Phi}{\partial t^2} = 0. \qquad \text{(XI.51)}$$

The functions $A_n^{(1)}$, $A_n^{(2)}$ obey the conditions /800

$$\lim_{\mu\to-\infty} \mu^{n+\varepsilon} A_n^{(1)}(\mu) = 0, \qquad \lim_{\mu\to\infty} \mu^{n+\varepsilon} A_n^{(2)}(\mu) = 0,$$

where ε is any positive number. Ya. A. Mindlin [2] applies the solution in the form (XI.50) to the problem of plane elasticity theory, where the displacements on the boundary of a round hole are given and represented in the form of a Fourier series with respect to an angular coordinate. When the initial and boundary conditions are satisifed, integral Voltaire equations[1] of kind I are obtained for functions A_n.

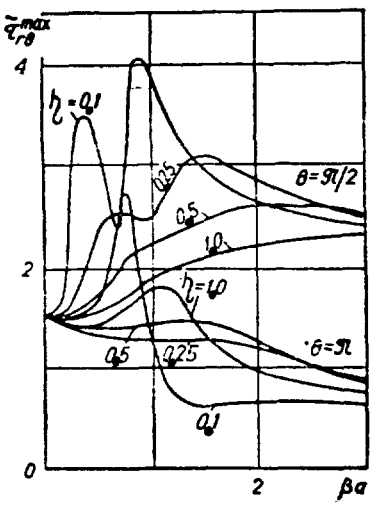


Figure XI.17.

In certain works the plane problem with a nonstationary external action for a round hole (cavity) is examined with the aid of integral transformations. Briefly, the essence of the solution method is as follows.

In the space of the Laplace transformations equations (XI.51) acquire the form

$$\nabla^2\Phi - \frac{p^2}{c_1^2}\Phi = 0, \qquad \nabla^2\Psi - \frac{p^2}{c_2^2}\Psi = 0, \qquad \text{(XI.52)}$$

[1]M. M. Sidlyar [1] examines the analogous problem in stresses by this method.

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where

$$\Phi(r, \theta, p) = \int_0^\infty e^{-pt}\varphi(r, \theta, t)\, dt,$$

$$\Psi(r, \theta, p) = \int_0^\infty e^{-pt}\psi(r, \theta, t)\, dt.$$

Equations (XI.52) in polar coordinates have a solution, damping at infinity, of the following form:

$$\Phi = \sum_{n=0}^{\infty} K_n\left(\frac{p}{c_1} r\right)(A_n \cos n\theta + B_n \sin n\theta).$$
$$\Psi = \sum_{n=0}^{\infty} K_n\left(\frac{p}{c_2} r\right)(C_n \cos n\theta + D_n \sin n\theta), \qquad \text{(XI.53)}$$

where K_n is McDonald's function.

Equations (XI.52) should be satisfied both by the potentials of the waves reflected on the hole and by the potentials that describe the basic stress state, i.e., the state caused by the load applied. /801

The basic stress state is represented in the form Fourier series

$$\sigma_r = \sum_{n=0}^{\infty} [a_n(r, t)\cos n\theta + b_n(r, t)\sin n\theta],$$
$$\tau_{r\theta} = \sum_{n=0}^{\infty} [c_n(r, t)\cos n\theta + d_n(r, t)\sin n\theta]. \qquad \text{(XI.54)}$$

From the boundary conditions, subjected to Laplace transformation, we find A_n, B_n, C_n, D_n as functions of p, after which we must convert to the space of the originals. As we know, the function f(t) is defined through its transformation F(p) using the operator:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\, dp \qquad (\gamma > 0).$$

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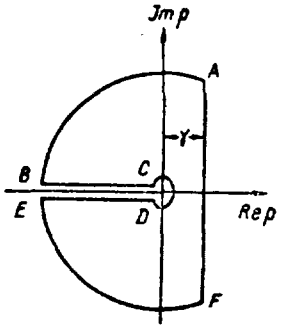


Figure XI.18.

In the problems[1] at hand we have a branching point of the function under the integral at the origin of the coordinate system and of the semi-axis in the left hand half plane, since the stress state, as time passes, should approach the static state asymptotically. Therefore, as the contour of integration we may use the contour represented in Figure XI.18. As the radius of arcs AB and EF increases the functions under the integral approach zero uniformly with respect to arg p. Then the value of the integral for straight line AF is equal to its values calculated for the sides of section BC and ED, to the sum of remainders of the integral function at its poles, and to the integral for the infinitely small vicinity at the origin of the coordinate system.

M. L. Baron and A. T. Mattews [1, 2] analyzed the stress and deformation states in the vicinity of a circular plane[2], where a plane stress wave of a stepped form moves within the medium (in the points of the medium which reach the wave front the stresses increase suddenly, and then remain constant). The expansion of the basic stress state on the contour of the hole into a Fourier series shows that the harmonics n = 0 and n = 2 in formulas (XI.54) continue to be nonzero values from the moment the wave front passes the cavity. The calculations made by the above-cited authors show that during certain moments of time (after the wave has passed the cavity) the stress concentration exceeds the static value by 9-11%, and then approaches the static value as time passes.

§4. Curvilinear Hole with Rather Smooth Contour /802

The problems examined in the preceding sections of this chapter pertain only to a round hole. They are all related to the separation of variables, which in boundary problems of elasticity theory is done only in Cartesian, cylindrical (polar) and spherical coordinate systems. Therefore, in order to solve dynamic problems for holes whose shapes are not circular, it is necessary to find different methods. One such method[3] is the conversion to a system of integral equations for the determination of displacement potentials. However, the determination of the solution to its completion by this method, i.e., the determination of sufficiently accurate numerical results, necessitates the solution of voluminous algebraic equation systems[4]. Simultaneously, for a row of holes that are "close" (in a given sense) to round it is possible to arrive at a more or less compact approximate solution that yields to qualitative analysis. Such a solution for a steady process was found by the "boundary form perturbations" method by V. D. Kubenko [3-6].

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[1]See M. M. Sidlyar [2].
[2]I.e., in the case of plane deformation.
[3]See V. D. Kupradze [2].
[4]See R. P. Banaugh, W. Goldsmith [1].

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Solution Method. We will outline briefly the idea of this method in application to dynamic problems[1].

We introduce the dimensionless values

$$\bar{r} = \frac{r}{a_0}, \qquad \bar{\theta} = \theta, \qquad \bar{t} = \frac{c_2 t}{a_0},$$
$$\bar{\omega} = \frac{a_0\omega}{c_2}, \qquad \vec{u} = \frac{E}{2(1+\nu)a_0Q_0}\vec{u}, \tag{XI.55}$$
$$\{\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\tau}_{r\theta}\} = \frac{1}{Q_0}\{\sigma_r, \sigma_\theta, \tau_{r\theta}\},$$

where r, θ are polar coordinates; $\vec{u}(u, v)$ is the displacement vector; a_0 is a linear dimension (radius of round hole, close to the examined contour); Q_0 is the intensity of the stress state.

In the new symbols, the bar above which we will omit, we find that the displacement potentials satisfy Helmholtz equations

$$(\nabla^2 + \omega^2\xi^2)\varphi = 0, \tag{XI.56}$$
$$(\nabla^2 + \omega^2)\psi = 0,$$

where

$$\xi^2 = \frac{1-\nu}{2}.$$

The stresses in polar coordinates are found through the functions ϕ and ψ from formulas /803

$$\sigma_r = -\nu\omega^2\varphi + 2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial\psi}{\partial\theta}\right),$$
$$\sigma_\theta = -\omega^2\varphi - 2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial\psi}{\partial\theta}\right), \tag{XI.57}$$
$$\tau_{r\theta} = -\omega^2\psi + 2\left(\frac{1}{r}\cdot\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{1}{r}\cdot\frac{\partial\varphi}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2}\right).$$

Consider an infinite plane z weakened by a hole with contour Γ, whose center of gravity coincides with the origin of the coordinate system (Figure XI.19). Let the function

[1]The application of this solution method to static problems is described in Chapters VI and X.

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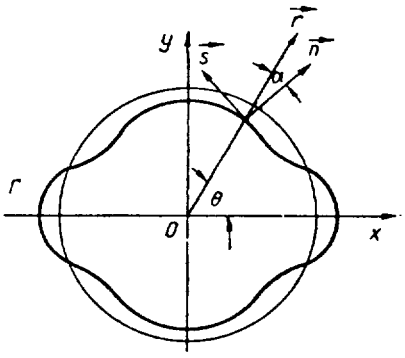


Figure XI.19.

$$z = \omega_1(\zeta) = \zeta + \varepsilon f(\zeta), \qquad z = re^{i\theta},$$
$$\zeta = \varrho e^{i\vartheta} \qquad (|\varepsilon| \ll 1), \tag{XI.58}$$

map conformally the infinite plane ξ with a hole in the form a unit circle onto infinite plane z with hole Γ. The choice of the function $f(\xi)$ in (XI.58) enables us to find a hole of rather arbitrary form without angular points.

From (XI.58) we obtain the following relations[1]:

$$r = \sqrt{\zeta\bar{\zeta} + \varepsilon[\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)] + \varepsilon^2 f(\zeta)\overline{f(\zeta)}},$$
$$\theta = \arctan\frac{\varrho\sin\vartheta + \varepsilon\,\mathrm{I_m} f(\zeta)}{\varrho\cos\vartheta + \varepsilon\,\mathrm{Re}\,f(\zeta)}, \qquad e^{i\alpha} = \frac{\overline{\zeta\omega_1(\zeta)}\,\omega_1'(\zeta)}{|\zeta|\,|\omega_1(\zeta)|\,|\omega_1'(\zeta)|}, \tag{XI.59}$$

where α is the angle between the radial direction in plane z and the direction of the external normal to the curves into which the circles constructed from the origin of the coordinate system in plane ζ are transformed by the function $\omega_1(\zeta)$ (XI.58).

The boundary conditions in the case of the first basic problem consist in assigning the stresses on Γ

$$\sigma_n|_\Gamma = h(\vartheta, \varepsilon)\,e^{-i\omega t},$$
$$\tau_{ns}|_\Gamma = g(\vartheta, \varepsilon)\,e^{-i\omega t}. \tag{XI.60}$$

As we know, the general solution of equations (XI.56) (with consideration of the emission conditions) in polar coordinates r, θ has the form /804

$$\varphi(r, \theta) = \sum_{m=0}^{\infty} (A_m \cos m\,\theta + B_m \sin m\theta)\,H_m(\omega\xi r),$$
$$\psi(r, \theta) = \sum_{m=0}^{\infty} (C_m \cos m\theta + D_m \sin m\theta)\,H_m(\omega r). \tag{XI.61}$$

[1]The rigorous derivation of these relations is given in Chapter VI.

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Stresses σ_r, σ_θ, $\tau_{r\theta}$ (XI.57) are related to stresses σ_n, σ_s, τ_{ns} by the known transformation formulas:

$$\begin{aligned}\sigma_n &= \sigma_r \cos^2\alpha + \sigma_\theta \sin^2\alpha + 2\tau_{r\theta}\sin\alpha\cos\alpha,\\ \sigma_s &= \sigma_r \sin^2\alpha + \sigma_\theta \cos^2\alpha - 2\tau_{r\theta}\sin\alpha\cos\alpha,\\ \tau_{ns} &= (\sigma_\theta - \sigma_r)\sin\alpha\cos\alpha + \tau_{r\theta}(\cos^2\alpha - \sin^2\alpha).\end{aligned} \tag{XI.62}$$

We will represent the solution of equations (XI.56) and stress components σ_n, σ_s and τ_{ns} in the form of series with respect to parameter ε:

$$\begin{aligned}\varphi(r,\theta) &= \sum_{j=0}^{\infty}\varepsilon^j\varphi_j(r,\theta), \qquad \psi(r,\theta) = \sum_{j=0}^{\infty}\varepsilon^j\psi_j(r,\theta),\\ \sigma_n(r,\theta) &= \sum_{j=0}^{\infty}\varepsilon^j\sigma_n^{(j)}(r,\theta), \qquad \sigma_s(r,\theta) = \sum_{j=0}^{\infty}\varepsilon^j\sigma_s^{(j)}(r,\theta),\\ \tau_{ns}(r,\theta) &= \sum_{j=0}^{\infty}\varepsilon^j\tau_{ns}^{(j)}(r,\theta).\end{aligned} \tag{XI.63}$$

We expand functions h and g in (XI.60) by degrees of the small parameter ε

$$h(\vartheta,\varepsilon) = \sum_{j=0}^{\infty}\varepsilon^j h_j(\vartheta), \qquad g(\vartheta,\varepsilon) = \sum_{j=0}^{\infty}\varepsilon^j g_j(\vartheta). \tag{XI.64}$$

By substituting (XI.63), (XI.64) into (XI.56), (XI.57) and (XI.62), as well as the expansions (XI.59) with respect to ε, equating the coefficients for identical degrees of ε, we obtain equations for the j-th approximation

$$\begin{aligned}(\nabla^2 + \omega^2\xi^2)\varphi_j(r,\theta) &= 0,\\ (\nabla^2 + \omega^2)\psi_j(r,\theta) &= 0\end{aligned} \tag{XI.65}$$

and the boundary conditions

$$\sigma_n^{(j)}|_\Gamma = h_j(\vartheta), \qquad \tau_{ns}^{(j)}|_\Gamma = g_j(\vartheta). \tag{XI.66}$$

The general solution of (XI.65), in accordance with (XI.61), has the form /805

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$$\varphi_j(r,\theta)=\sum_{m=0}^{\infty}H_m(\omega\xi r)\,(A_m^{(j)}\cos m\theta+B_m^{(j)}\sin m\theta),$$
$$\psi_j(r,\theta)=\sum_{m=0}^{\infty}H_m(\omega r)\,(C_m^{(j)}\cos m\theta+D_m^{(j)}\sin m\theta). \tag{XI.67}$$

In the j-th approximation the stresses are given by the formulas

$$\sigma_n^{(j)}=e^{-i\omega t}\sum_{\substack{k=0\\p=0\\s=0\\k+p+s=j}}^{j}[M_r^{(k)}(\Phi_s)\,l_1^{(p)}+M_\theta^{(k)}(\Phi_s)\,l_2^{(p)}+M_{r\theta}^{(k)}(\Phi_s)\,l_3^{(p)}],$$
$$\sigma_s^{(j)}=e^{-i\omega t}\sum_{\substack{k=0\\p=0\\s=0\\k+p+s=j}}^{j}[M_r^{(k)}(\Phi_s)\,l_4^{(p)}+M_\theta^{(k)}(\Phi_s)\,l_5^{(p)}+M_{r\theta}^{(k)}(\Phi_s)\,l_6^{(p)}], \tag{XI.68}$$
$$\tau_{ns}^{(j)}=e^{-i\omega t}\sum_{\substack{k=0\\p=0\\s=0\\k+p+s=j}}^{j}[M_r^{(k)}(\Phi_s)\,l_7^{(p)}+M_\theta^{(k)}(\Phi_s)\,l_8^{(p)}+M_{r\theta}^{(k)}(\Phi_s)\,l_9^{(p)}],$$

where for brevity the pair of functions $\varphi_s(\varrho,\vartheta)$ and $\psi_s(\varrho,\vartheta)$, is expressed through Φ_s, i.e., $\Phi_s\sim\varphi_s(\varrho,\vartheta),\ \psi_s(\varrho,\vartheta)$.

The differential operators in (XI.68) have the form

$$M_r^{(k)}(\Phi_s)=-\nu\omega^2L_1^{(k)}(\varphi_s)+2L_2^{(k)}(\varphi_s)+2L_3^{(k)}(\psi_s),$$
$$M_\theta^{(k)}(\Phi_s)=-\omega^2L_1^{(k)}(\varphi_s)-2L_2^{(k)}(\varphi_s)-2L_3^{(k)}(\psi_s),$$
$$M_{r\theta}^{(k)}(\Phi_s)=-\omega^2L_1^{(k)}(\psi_s)-2L_2^{(k)}(\psi_s)+2L_3^{(k)}(\varphi_s), \tag{XI.69}$$
$$k=0,1,2,$$

where[1]

$$L_1^{(0)}=1;\qquad L_2^{(0)}=\frac{\partial^2}{\partial\varrho^2};\qquad L_3^{(0)}=\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho\partial\vartheta}-\frac{1}{\varrho^2}\cdot\frac{\partial}{\partial\vartheta};$$
$$L_1^{(1)}=\frac{\zeta\bar f+\bar\zeta f}{2\varrho}\cdot\frac{\partial}{\partial\varrho}+\left(\frac{f-\bar f}{2i\varrho}\cos\vartheta-\frac{f+\bar f}{2\varrho}\sin\vartheta\right)\frac{\partial}{\partial\vartheta};$$

[1]For brevity we will write f in formulas (XI.70) instead of f(ζ).

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$$L_2^{(1)} = \frac{\zeta\bar{f} + \bar{\zeta}f}{2\varrho} \cdot \frac{\partial^3}{\partial\varrho^3} + \left(\frac{f-\bar{f}}{2i\varrho}\cos\vartheta - \frac{f+\bar{f}}{2\varrho}\sin\vartheta\right)\frac{\partial^3}{\partial\varrho^2\partial\vartheta};$$

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$$L_3^{(1)} = \frac{\zeta\bar{f} + \bar{\zeta}f}{2\varrho^2} \cdot \frac{\partial^3}{\partial\varrho^2\partial\vartheta} + \frac{\zeta\bar{f} + \bar{\zeta}f}{\varrho^4} \cdot \frac{\partial}{\partial\vartheta} + \left(\frac{f-\bar{f}}{2i\varrho^2}\cos\vartheta - \frac{f+\bar{f}}{2\varrho^2}\sin\vartheta\right)\frac{\partial^3}{\partial\varrho\partial\vartheta^2} -$$
$$- \frac{\zeta\bar{f} + \bar{\zeta}f}{\varrho^3} \cdot \frac{\partial^2}{\partial\varrho\partial\vartheta} - \left(\frac{f-\bar{f}}{2i\varrho}\cos\vartheta - \frac{f+\bar{f}}{2\varrho}\sin\vartheta\right)\frac{\partial^2}{\partial\vartheta^2};$$

$$L_1^{(2)} = \frac{(\zeta\bar{f} + \bar{\zeta}f)^2}{8\varrho^2} \cdot \frac{\partial^2}{\partial\varrho^2} + \frac{\zeta\bar{f} + \bar{\zeta}f}{\varrho^2}\left(\frac{f-\bar{f}}{2i}\cos\vartheta - \frac{f+\bar{f}}{2}\sin\vartheta\right)\frac{\partial^2}{\partial\varrho\partial\vartheta} +$$
$$+ \frac{4if\bar{f} - 2i(f^2+\bar{f}^2)\cos 2\vartheta - 2(f^2-\bar{f}^2)\sin 2\vartheta}{8i\varrho^2} \cdot \frac{\partial^2}{\partial\vartheta^2} -$$
$$- \frac{(\bar{\zeta}f - \zeta\bar{f})^2}{8\varrho^3} \cdot \frac{\partial}{\partial\varrho} + \frac{(f^2+\bar{f}^2)\,i\sin 2\vartheta - (f^2-\bar{f}^2)\cos 2\vartheta}{4i\varrho^2} \cdot \frac{\partial}{\partial\vartheta};$$

$$L_2^{(2)} = \frac{(\zeta\bar{f} + \bar{\zeta}f)^2}{8\varrho^2} \cdot \frac{\partial^4}{\partial\varrho^4} + \frac{\zeta\bar{f} + \bar{\zeta}f}{\varrho^3}\left(\frac{f-\bar{f}}{2i}\cos\vartheta - \frac{f+\bar{f}}{2}\sin\vartheta\right)\frac{\partial^4}{\partial\varrho^3\partial\vartheta} +$$
$$+ \frac{4if\bar{f} - 2i(f^2+\bar{f}^2)\cos 2\vartheta - 2(f^2-\bar{f}^2)\sin 2\vartheta}{8i\varrho^2} \cdot \frac{\partial^4}{\partial\varrho^2\partial\vartheta^2} - \qquad \text{(XI.70)}$$
$$- \frac{(\bar{\zeta}f - \zeta\bar{f})^2}{8\varrho^3} \cdot \frac{\partial^3}{\partial\varrho^3} + \frac{(f^2+\bar{f}^2)\,i\sin 2\vartheta - (f^2-\bar{f}^2)\cos 2\vartheta}{4i\varrho^2} \cdot \frac{\partial^3}{\partial\varrho^2\partial\vartheta};$$

$$L_3^{(2)} = \frac{(\zeta\bar{f} + \bar{\zeta}f)^2}{8\varrho^3} \cdot \frac{\partial^4}{\partial\varrho^3\partial\vartheta} + \frac{\zeta\bar{f} + \bar{\zeta}f}{2\varrho^3}\left(\frac{f-\bar{f}}{2i}\cos\vartheta - \frac{f+\bar{f}}{2}\sin\vartheta\right)\frac{\partial^4}{\partial\varrho^2\partial\vartheta^2} +$$
$$+ \frac{4if\bar{f} - 2i(f^2+\bar{f}^2)\cos 2\vartheta - 2(f^2-\bar{f}^2)\sin 2\vartheta}{8i\varrho^3} \cdot \frac{\partial^4}{\partial\varrho\partial\vartheta^2} +$$
$$+ \frac{(f^2+\bar{f}^2)\sin 2\vartheta + i(f^2-\bar{f}^2)\cos\vartheta - 3(\zeta\bar{f} + \bar{\zeta}f)\left(\frac{f-\bar{f}}{2i\varrho}\cos\vartheta - \frac{f+\bar{f}}{2\varrho}\sin\vartheta\right)}{4\varrho^3} \cdot \frac{\partial^3}{\partial\varrho\partial\vartheta^2} -$$
$$- \frac{4if\bar{f} - 2i(f^2+\bar{f}^2)\cos 2\vartheta - 2(f^2-\bar{f}^2)\sin 2\vartheta}{8i\varrho^4} \cdot \frac{\partial^3}{\partial\vartheta^3} -$$
$$- \frac{\zeta^2\bar{f}^2 + \bar{\zeta}^2f^2 + 6\varrho^2f\bar{f}}{8\varrho^4} \cdot \frac{\partial^3}{\partial\varrho^2\partial\vartheta} + \frac{9\zeta^2\bar{f}^2 + 9\bar{\zeta}^2f^2 + 6\varrho^2f\bar{f}}{8\varrho^3} \cdot \frac{\partial^2}{\partial\varrho\partial\vartheta} +$$
$$+ \frac{-(f^2+\bar{f}^2)\sin 2\vartheta + i(f^2-\bar{f}^2)\cos 2\vartheta + (\zeta\bar{f} + \bar{\zeta}f)\left(\frac{f-\bar{f}}{i\varrho}\cos\vartheta - \frac{f+\bar{f}}{\varrho}\sin\vartheta\right)}{2\varrho^4} \cdot \frac{\partial^2}{\partial\vartheta^2} +$$
$$+ \frac{\zeta^2\bar{f}^2 + \bar{\zeta}^2f^2 + \varrho^2f\bar{f}}{\varrho^5} \cdot \frac{\partial}{\partial\vartheta}.$$

In (XI.68) we denote through $l_m^{(p)}$ the values

$$l_1^{(0)} = 1;\ l_1^{(1)} = 0;\ l_1^{(2)} = \frac{(\zeta\bar{f} - \bar{\zeta}f + \varrho^2f' - \varrho^2\bar{f}')^2}{4\varrho^4};$$
$$l_2^{(0)} = 0;\quad l_2^{(1)} = 0;\quad l_2^{(2)} = -l_1^{(2)};$$

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$$l_3^{(0)} = 0; \quad l_3^{(1)} = \frac{\zeta\bar{f} - \bar{\zeta}f + \varrho^2(f' - \bar{f}')}{i\varrho^2}; \quad l_3^{(2)} = \frac{\bar{\zeta}^2 f^2 - \zeta^2\bar{f}^2 + \varrho^4(f'^2 - \bar{f}'^2)}{2i\varrho^4};$$

$$l_4^{(0)} = 0; \quad l_4^{(1)} = 0; \quad l_4^{(2)} = -l_1^{(2)};$$

$$l_5^{(0)} = 1; \quad l_5^{(1)} = 0; \quad l_5^{(2)} = l_1^{(2)};$$

$$l_6^{(0)} = 0; \quad l_6^{(1)} = -l_3^{(1)}; \quad l_6^{(2)} = -l_3^{(2)};$$

$$l_7^{(0)} = 0; \quad l_7^{(1)} = -\frac{1}{2}l_3^{(1)}; \quad l_7^{(2)} = -\frac{1}{2}l_3^{(2)};$$

$$l_8^{(0)} = 0; \quad l_8^{(1)} = \frac{1}{2}l_3^{(1)}; \quad l_8^{(2)} = \frac{1}{2}l_3^{(2)};$$

$$l_9^{(0)} = 1; \quad l_9^{(1)} = 0; \quad l_9^{(2)} = 2l_1^{(2)}.$$

In the j-th approximation, as follows from (XI.68), only the functions $\varphi_j(\varrho, \vartheta)$ and $\psi_j(\varrho, \vartheta)$ remain unknown.

Comparing (XI.68) with (XI.57), we see that all values of the j-th order are determined through the functions $\phi_j(\rho, \vartheta)$ and $\psi_j(\rho, \vartheta)$, just as in (XI.57), if we substitute formally in them r by ρ and θ by ϑ. The functions $\phi_j(\rho, \vartheta)$ and $\psi_j(\rho, \vartheta)$ themselves are found by simple substitution of r, θ by ρ, ϑ in functions $\phi_j(r, \theta)$, $\psi_j(r, \theta)$, which are the solutions of equations (XI.65).

Thus, in each of the series approximations the problem reduces to the determination of the solution for the exterior of the hole in the form of a unit circle, which enables us to use the method of variable separation.

We will examine a few specific problems.

Oscillating Pressure Applied to Contour of Hole. We will assume that a uniform pressure, changing harmonically with time, is applied to the contour of hole Γ in an infinite elastic plate. The boundary conditions on the hole have the form

$$\sigma_n|_\Gamma = -e^{-i\omega t}, \quad \tau_{ns}|_\Gamma = 0. \tag{XI.71}$$

By expanding the right hand side of (XI.71) with respect to ε, we will have, in accordance with (XI.60) and (XI.64),

$$h_0 = -1, \; h_1 = h_2 = \ldots = h_m = \ldots = g_0 = g_1 = \ldots = g_m = \ldots = 0. \tag{XI.72}$$

Thus, the boundary conditions are:

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in the zero approximation

$$\sigma_n^{(0)}|_{Q=1} = -e^{-i\omega t}, \quad \tau_{ns}^{(0)}|_{Q=1} = 0; \tag{XI.73}$$

in the first approximation

$$\sigma_n^{(1)}|_{Q=1} = 0, \quad \tau_{ns}^{(1)}|_{Q=1} = 0; \tag{XI.74}$$

in the second approximation

$$\sigma_n^{(2)}|_{Q=1} = 0, \quad \tau_{ns}^{(2)}|_{Q=1} = 0. \tag{XI.75}$$

In boundary conditions (XI.73)-(XI.75) the left hand sides are represented by expansions (XI.68). /808

The function $f(\zeta)$ in (XI.58) is taken in a more convenient[1] form:

$$f(\zeta) = \frac{1}{\zeta^N}, \tag{XI.76}$$

where N is a whole positive number.

The function (XI.76) enables us to obtain various holes with rounded corners[2]. Thus, when N = 1 we have an elliptical hole with semiaxes a and b, i.e.,

$$\varepsilon = \pm\frac{a-b}{a+b};$$

when N = 3 and $\varepsilon = \pm 1/9$ or $\varepsilon = \pm 1/6$, a square hole with rounded corners; when N = 2 and $\varepsilon = \pm 1/4$, a triangular hole with rounded corners. The sign of parameter ε determines the orientation of the hole in relation to the x and y axes.

Proceeding to the solution of the problem we see that in the zero approximation it is sufficient to take as the solution of equations (XI.65)

[1]More complex expressions for the function $f(\zeta)$ do not, in principle, add anything new, but simply impede the mathematical calculations.

[2]G. N. Savin and N. A. Shul'gi [1] proposed for such forms of holes the method of solution of analogous dynamic problems for a material whose elastic state is characterized by an asymmetric stress tensor (Chapter VI).

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$$\varphi_0 = A_0^{(0)} H_0(\omega\xi r), \quad \psi_0 = 0. \tag{XI.77}$$

In the zero approximation the solution of the problem as stated for boundary conditions (XI.65) is the solution of the axisymmetric problem for a round hole.

In the first approximation the displacement potentials should be taken in the form

$$\begin{aligned} \varphi_1 &= A_{N+1}^{(1)} H_{N+1}(\omega\xi r)\cos(N+1)\theta, \\ \psi_1 &= D_{N+1}^{(1)} H_{N+1}(\omega r)\sin(N+1)\theta. \end{aligned} \tag{XI.78}$$

We see from (XI.78) that even in the first approximation a displacement wave appears.

Finally, the displacement potentials in the second approximation have the form

$$\begin{aligned} \varphi_2 &= A_0^{(2)} H_0(\omega\xi r) + A_{2N+2}^{(2)} H_{2N+2}(\omega\xi r)\cos 2(N+1)\theta, \\ \psi_2 &= D_{2N+2}^{(2)} H_{2N+2}(\omega r)\sin 2(N+1)\theta. \end{aligned} \tag{XI.79}$$

The boundary conditions (XI.73)-(XI.75) are used for the determination of the constants $A_0^{(0)}, A_{N+1}^{(1)}, D_{N+1}^{(1)}, A_0^{(2)}, A_{2N+2}^{(2)}$ и $D_{2N+2}^{(2)}$.

Figure XI.20 shows the stress $\sigma_n(\rho)^{\max}$ along the x axis for $\nu = 0.28$ for a round (curve 1), elliptical (curve 2) $\varepsilon = 1/5$ and square (curve 3) $\varepsilon = 1/9$ hole when $\omega = 1$. As we see, the curvilinearity of the hole has an effect on the stresses only in the vicinity of the contour. /809

Figure XI.21 shows the stress $\sigma_s^{\max}$ as a function of frequency ω on the contour of a round (curve 1), elliptical (curve 2) $\varepsilon = 1/5$ and square (curve 3) $\varepsilon = 1/9$ hole at the point of maximum concentration of the stresses. As we see, in a certain range of frequencies the stress σ_s exceeds its static value by an average of 15%.

The convergence of the series approximations for the elliptical $\varepsilon = 1/5$ and square $\varepsilon = 1/9$ holes is illustrated in Table XI.1.

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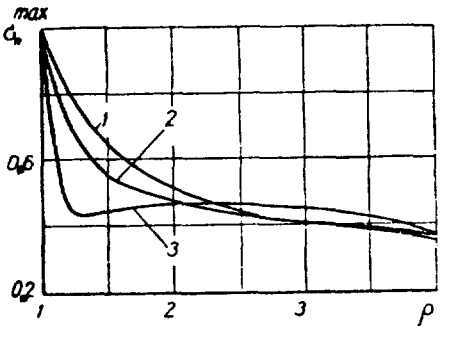


Figure XI.20.

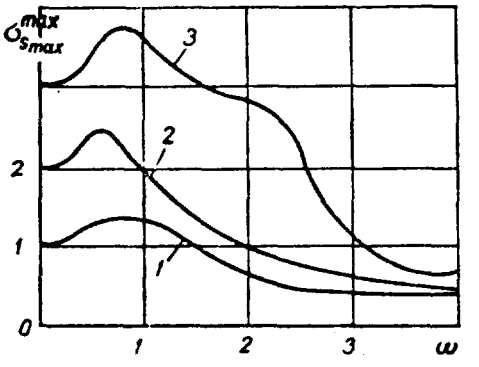


Figure XI.21.

TABLE XI.1.

	Hole	ω	Approximation: Zero	Approximation: First	Approximation: Second	Precise solution
σ_s^{max}	Elliptical	0	1,000	1,800	1,960	2,000
		0,6	1,261	2,215	2,395	—
	Square	0	1,000	2,333	2,777	3,000
		0,8	1,307	2,894	3,637	—

Tr. Note: Commas indicate decimal points.

Incident Compression Wave. Elliptical and Square Free Holes. We will give a plane harmonic wave by displacement potentials[1]

$$\varphi^I = -\frac{1}{\omega^2} e^{i\omega(\xi x - t)}, \quad \psi^I = 0, \tag{XI.80}$$

to which corresponds the basic stress state of the form /810

$$\sigma_x^I = e^{i\omega(\xi x - t)}, \quad \sigma_y^I = \nu e^{i\omega(\xi x - t)}, \quad \tau_{xy}^I = 0. \tag{XI.81}$$

The asterisk denotes the potentials of the reflected waves and the stresses corresponding to them.

The boundary conditions on the contour of a free hole have the form

$$(\sigma_n^I + \sigma_n^*)_\Gamma = 0, \quad (\tau_{ns}^I + \tau_{ns}^*)_\Gamma = 0. \tag{XI.82}$$

[1]See V. D. Kubenko [4, 5].

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The potentials ϕ^*, ψ^* are the solutions of Helmholtz equation (XI.5).

According to the solution method described above, we will represent σ_n^1, σ_s^1, τ_{ns}^1 in a form of series with respect to parameter ε. For this purpose we will use the expansion of the plane wave in polar coordinates (XI.32) and expand the functions r, θ and $e^{i\alpha}$ (XI.59) with respect to ε. We then obtain formulas for σ_n^1 and τ_{ns}^1 with consideration of the zero, first and second approximations:

$$\sigma_n^1|_\Gamma = h(\vartheta, \varepsilon) = e^{-i\omega t}\sum_{m=0}^{\infty} \in_m i^m \left\{J_m(\omega\xi)\left[\frac{1}{2}\cos m\vartheta + \frac{1}{4}\cos(m-2)\vartheta + \right.\right.$$

$$\left.+\frac{1}{4}\cos(m+2)\vartheta\right] + \varepsilon\,[k_m^{(0)}\cos(m-N-3)\vartheta + k_m^{(1)}\cos(m-N-1)\vartheta +$$

$$+ k_m^{(2)}\cos(m-N+1)\vartheta + k_m^{(3)}\cos(m+N-1)\vartheta +$$

$$+ k_m^{(4)}\cos(m+N+1)\vartheta + k_m^{(5)}\cos(m+N+3)\vartheta] +$$

$$+ \varepsilon^2[K_m^{(0)}\cos(m-2N-4)\vartheta + K_m^{(1)}\cos(m-2N-2)\vartheta +$$

$$+ K_m^{(2)}\cos(m-2N)\vartheta - K_m^{(3)}\cos(m-2)\vartheta - K_m^{(4)}\cos m\vartheta +$$

$$+ K_m^{(5)}\cos(m+2)\vartheta + K_m^{(6)}\cos(m+2N)\vartheta +$$

$$\left.+ K_m^{(7)}\cos(m+2N+2)\vartheta + K_m^{(8)}\cos(m+2N+4)\vartheta] + \ldots\right\};$$

$$\tau_{ns}^1|_\Gamma = g(\vartheta, \varepsilon) = e^{-i\omega t}\sum_{m=0}^{\infty} \in_m i^m \left\{J_m(\omega\xi)\left[\frac{1}{4}\sin(m-2)\vartheta - \right.\right.$$

$$\left.-\frac{1}{4}\sin(m+2)\vartheta\right] + \varepsilon\,[k_m^{(0)}\sin(m-N-3)\vartheta - k_m^{(1)}\sin(m-N+1)\vartheta +$$

$$+ k_m^{(3)}\sin(m+N+1)\vartheta - k_m^{(5)}\sin(m+N+3)\vartheta] +$$

/811

$$+ \varepsilon^2[K_m^{(0)}\sin(m-2N-4)\vartheta - K_m^{(2)}\sin(m-2N)\vartheta - \qquad \text{(XI.83)}$$

$$- K_m^{(3)}\sin(m-2)\vartheta - K_m^{(5)}\sin(m+2)\vartheta +$$

$$\left.+ K_m^{(6)}\sin(m+2N)\vartheta - K_m^{(8)}\sin(m+2N+4)\vartheta] + \ldots\right\};$$

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$$(\sigma_n + \sigma_s)_\Gamma = e^{-i\omega t} \sum_{m=0}^{\infty} \in_m i^m \{J_m(\omega\xi)\cos m\vartheta\,(1+\nu) +$$
$$+ \varepsilon 2\,[k_m^{(1)}\cos(m-N-1)\,\vartheta + k_m^{(4)}\cos(m+N+1)\,\vartheta] +$$
$$+ \varepsilon^2 2\,[-K_m^{(4)}\cos m\vartheta + K_m^{(1)}\cos(m-2N-2)\,\vartheta +$$
$$+ K_m^{(7)}\cos(m+2N+2)\,\vartheta] + \ldots\},$$

where

$$k_m^{(0)} = (1-\nu)\left[\frac{\omega\xi}{8}J_{m-1} + \frac{N}{4}J_m\right],$$
$$k_m^{(1)} = (1+\nu)\frac{\omega\xi}{4}J_{m-1},$$
$$k_m^{(2)} = (1-\nu)\left[\frac{\omega\xi}{8}J_{m-1} - \frac{N}{4}J_m\right],$$
$$k_m^{(3)} = (1-\nu)\left[\frac{\omega\xi}{8}J_{m-1} - \frac{m+N}{4}J_m\right],$$
$$k_m^{(4)} = (1+\nu)\left[\frac{\omega\xi}{4}J_{m-1} - \frac{m}{2}J_m\right],$$
$$k_m^{(5)} = (1-\nu)\left[\frac{\omega\xi}{8}J_{m-1} - \frac{m-N}{4}J_m\right],$$
$$K_m^{(0)} = (1-\nu)\left[\frac{m+2N-1}{16}\omega\xi J_{m-1} + \frac{8N^2-\omega^2\xi^2}{32}J_m\right],$$
$$K_m^{(1)} = (1+\nu)\left[\frac{m-1}{8}\omega\xi J_{m-1} - \frac{\omega^2\xi^2}{16}J_m\right],$$
$$K_m^{(2)} = (1-\nu)\left[\frac{m-2N-1}{16}\omega\xi J_{m-1} - \frac{\omega^2\xi^2}{32}J_m\right], \qquad \text{(XI.84)}$$
$$K_m^{(3)} = \frac{4N(m+N)+\omega^2\xi^2}{16}J_m(1-\nu),$$
$$K_m^{(4)} = (1+\nu)\frac{\omega^2\xi^2}{8}J_m,$$
$$K_m^{(5)} = \frac{4N(m-N)+\omega^2\xi^2}{16}J_m(1-\nu),$$

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$$K_m^{(6)} = \left[-\frac{m+2N+1}{16}\omega\xi J_{m-1} + \frac{4m(m+2N+1)-\omega^2\xi^2}{32}J_m\right](1-\nu),$$
$$K_m^{(7)} = (1+\nu)\left[-\frac{m+1}{8}\omega\xi J_{m-1} + \frac{4m(m+1)-\omega^2\xi^2}{16}J_m\right],$$
$$K_m^{(8)} = (1-\nu)\left[-\frac{m-2N-1}{16}\omega\xi J_{m-1} + \right.$$
$$\left. + \frac{4m(m-2N+1)+8N^2-\omega^2\xi^2}{32}J_m\right],$$
$$J_m = J_m(\omega\xi).$$

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By substituting (XI.83) into (XI.82), recalling that $\sigma_n^{*(j)}$, $\tau_{ns}^{*(j)}$ are defined by formulas (XI.68), we obtain algebraic equation systems for the determination of the coefficients $A_m^{(j)}$, $B_m^{(j)}$, $C_m^{(j)}$, $D_m^{(j)}$ $(m=0, 1, 2, \ldots;\ j=0, 1, 2, \ldots)$.

The stresses σ_n, σ_s and τ_{ns} are calculated by formulas (XI.63). Hence in each of the approximations the real and imaginary parts have been calculated, the physical sense of which was explained in §3. During the calculations series (XI.83) were terminated at the 17-th term, which insures sufficient accuracy (10^{-4}) in the frequency range under consideration. A BESM-2M computer was used for the calculations and it was assumed that $\nu = 0.28$.

Figure XI.22 shows the change of σ_s with frequency on the contour of an elliptical hole with the semiaxes a = 2 and b = 3 when $\theta = \pi/2$. The broken curve corresponds to stresses at the origin of the period, the dot-dash curve, to stresses following one-fourth of the period, and the solid curve, to maximum σ_s. As we see, in a certain frequency range σ_s^{max} exceeds its static value. In particular, when $\omega = 0.4$ the stress $\sigma^{max} = 4.106$, which is 8.2% greater than the static value.

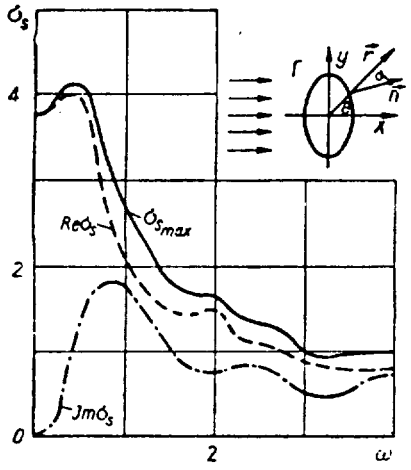


Figure XI.22.

TABLE XI.2.

	ω	Approximation			Precise solution
		Zero	First	Second	
σ_s^{max}	0	2,720	3,520	3,680	3,720
	0,4	2,944	3,914	4,106	—

TABLE XI.3.

	ω	Approximation			Precise solution
		Zero	First	Second	
σ_s^{max}	0	2,720	4,053	4,498	4,720
	0,4	2,944	4,584	5,120	—

Tr. Note: Commas indicate decimal points.

The convergence of approximations for an elliptical hole when $\varepsilon = -1/5$ at the point $\theta = \pi/2$ is shown in Table XI.2. /813

The distribution of σ_s on the contour of the hole at the beginning of the period (real part of the solution) is shown in Figure XI.23. On the basis of the graphs presented herein we may conclude that the stress state changes considerably with frequency.

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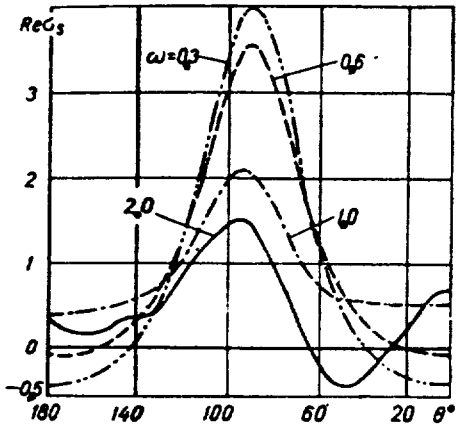


Figure XI.23.

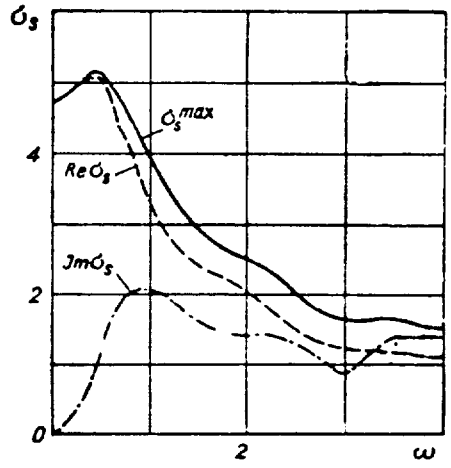


Figure XI.24.

Figure XI.24 shows (just as in Figure XI.23) σ_s as a function of ω on the contour of a square hole for $\varepsilon = 1/9$ when $\theta = \pi/2$. Here too there is a frequency range in which the stress exceeds its static value. The values of the series approximations for a square hole ($\varepsilon = 1/9$, $\theta = \pi/2$) are presented in Table XI.3.

Incident Displacement Wave. Square Free Hole. Let the basic stress state be realized in a plate with the aid of a plane harmonic displacement wave whose displacement potentials have the form

$$\varphi^I = 0, \qquad \psi^I = -\frac{1}{\omega^2} e^{-i\omega(x-t)}. \tag{XI.85}$$

To these potentials corresponds the stress state

$$\sigma_x^I = \sigma_y^I = 0, \qquad \tau_{xy}^I = e^{-i\omega(x-t)}. \tag{XI.86}$$

As in the preceding case, for boundary conditions (XI.82) we obtain a representation of the basic stress state in the form of the series /814

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$$\sigma_{n|\Gamma}^{1} = h(\vartheta, \varepsilon) = e^{-i\omega t} \sum_{m=0}^{\infty} \in_m i^m \left\{ \frac{1}{2} J_m(\omega) [\sin(m+2)\vartheta - \right.$$

$$- \sin(m-2)\vartheta] + \varepsilon 2 [- \bar{k}_m^{(0)} \sin(m-N-3)\vartheta +$$

$$+ \bar{k}_m^{(2)} \sin(m-N+1)\vartheta - \bar{k}_m^{(3)} \sin(m+N-1)\vartheta +$$

$$+ \bar{k}_m^{(5)} \sin(m+N+3)\vartheta] + \varepsilon^2 2 [- \bar{K}_m^{(0)} \sin(m-2N-4)\vartheta +$$

$$+ \bar{K}_m^{(2)} \sin(m-2N)\vartheta - \bar{K}_m^{(3)} \sin(m-2)\vartheta +$$

$$+ \bar{K}_m^{(5)} \sin(m+2)\vartheta - \bar{K}_m^{(6)} \sin(m+2N)\vartheta +$$

$$\left. + \bar{K}_m^{(8)} \sin(m+2N+4)\vartheta] + \ldots \right\}; \qquad \text{(XI.87)}$$

$$\tau_{ns}^{1}|_{\Gamma} = g(\vartheta, \varepsilon) = e^{-i\omega t} \sum_{m=0}^{\infty} \in_m i^m \left\{ \frac{1}{2} J_m(\omega) [\cos(m+2)\vartheta + \right.$$

$$+ \cos(m-2)\vartheta] + \varepsilon 2 [\bar{k}_m^{(0)} \cos(m-N-3)\vartheta + \bar{k}_m^{(2)} \cos(m-N+1)\vartheta +$$

$$+ \bar{k}_m^{(3)} \cos(m+N-1)\vartheta + \bar{k}_m^{(5)} \cos(m+N+3)\vartheta] +$$

$$+ \varepsilon^2 2 [\bar{K}_m^{(0)} \cos(m-2N-4)\vartheta + \bar{K}_m^{(2)} \cos(m-2N)\vartheta -$$

$$- \bar{K}_m^{(3)} \cos(m-2)\vartheta + \bar{K}_m^{(5)} \cos(m+2)\vartheta +$$

$$\left. + \bar{K}_m^{(6)} \cos(m+2N)\vartheta + \bar{K}_m^{(8)} \cos(m+2N+4)\vartheta] + \ldots \right\}.$$

The values $\bar{k}_m^{(s)}$, $\bar{K}_m^{(s)}$ ($s = 0, 1, \ldots, 8$; $m = 0, 1, 2, \ldots$) are given by formulas (XI.84), in which $\omega\xi$ must be replaced by ω, and the factors $1 + \nu$, $1 - \nu$ are omitted.

By substituting (XI.87) and (XI.68) into (XI.82) we obtain an equation system for the determination of arbitrary constants. The calculations were made as in the case of a compression wave. The graphs for σ_s on the contour of a square hole for $\varepsilon = -1/6$ at the points $\theta = \pi/4$ and $\theta = 3\pi/4$ are presented in Figures XI.25 and XI.26. In this example we see clearly the asymmetry in the distribution of stresses in the "shadow" and "illuminated" regions[1]. Thus, if /815 σ_s drops to 4 when $\theta = 3\pi/4$ and $\omega = 1$, then when $\theta = \pi/4$ and $\omega = 1$ it exceeds 7. In the "shadow" region the frequency range with the high concentration

[1]The "illuminated" part of the contour is distinguished from the "shadow" by the greatest diameter of the hole, perpendicular to the direction of propagation of the incident wave.

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coefficient is extended more than in the "illuminated" part. This difference is even more marked when the real or imaginary values of σ_s at these points are compared. In particular, when $\theta = \pi/4$ and $\omega = 0.4$, stresses $\sigma_s^{max} = 9.234$, i.e., 15.4% greater than the static value.

§5. Some Comments Concerning Problems for Multiply-Connected Regions

The solution of the problems of steady oscillations of an elastic body that occupies a multiply-connected region can be found in the monograph of V. D. Kupradze [2] and in the article of D. I. Sherman [1]. In V. D. Kupradze's monograph [2] all basic boundary problems of elastic vibrations are reduced to integral equations. In D. I. Sherman's work [1] only the first boundary problem is examined. The solution is obtained in the form of a system of two integral Friedholm equations of the second order.

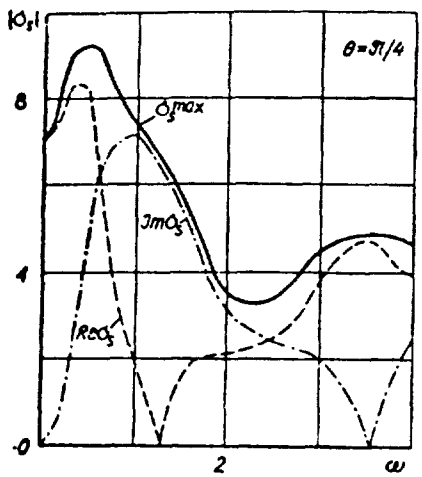


Figure XI.25.

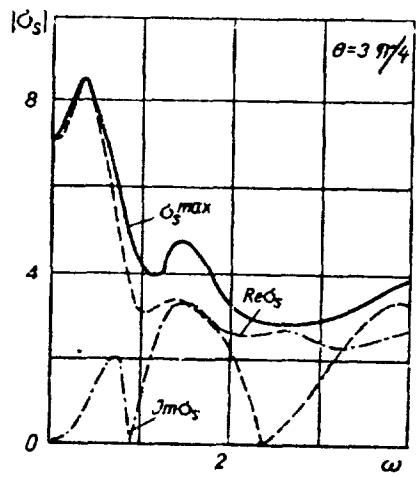


Figure XI.26.

A. N. Guz' [1, 2] gives the solution of the dynamic plane problem for an infinite multiply-connected region bounded by circular contours when the displacements of the points of the boundary are known. An infinite system of algebraic equations is obtained for the constants of the solution, represented in the form of a series. After substitution of the unknowns, the latter is transformed /817 into an infinite quasiregular system with a normal determinant.

Basic plane boundary problems of the theory of steady vibrations are solved for a finite multiply-connected region by an analogous method in the article of V. T. Golovchan [1].

When the method proposed by A. N. Guz' [1] is used the approximate solution of finite algebraic equation systems can be found by the reduction method. The computer makes it possible to find the solution that satisfies the boundary conditions with the required degree of accuracy.

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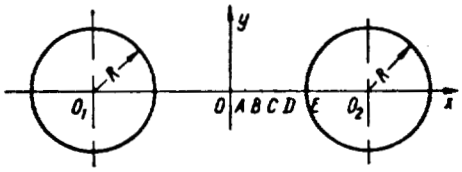


Figure XI.27.

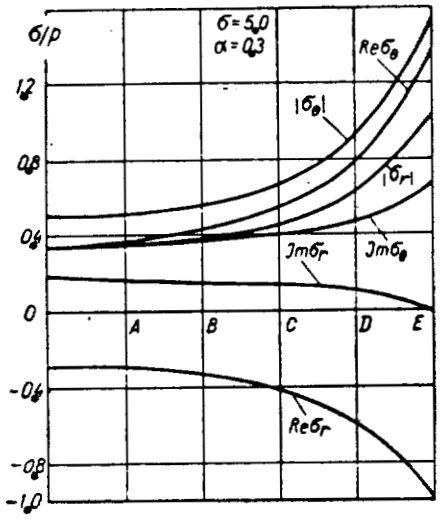


Figure XI.28.

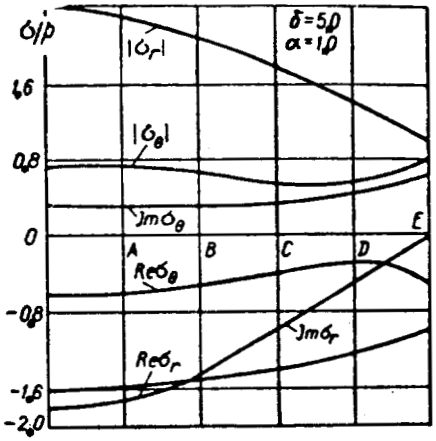


Figure XI.29.

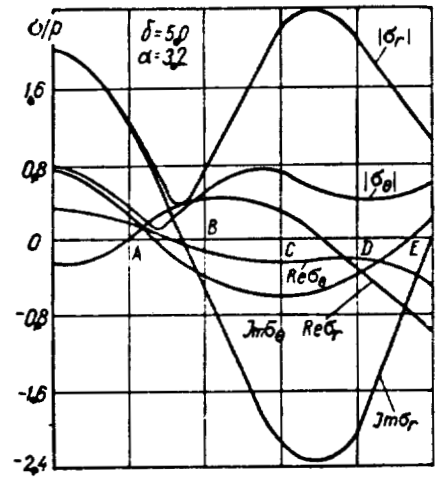


Figure XI.30.

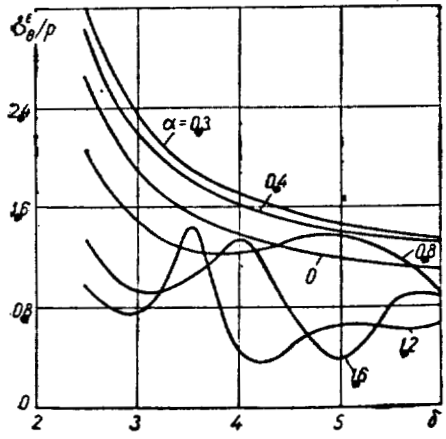


Figure XI.31.

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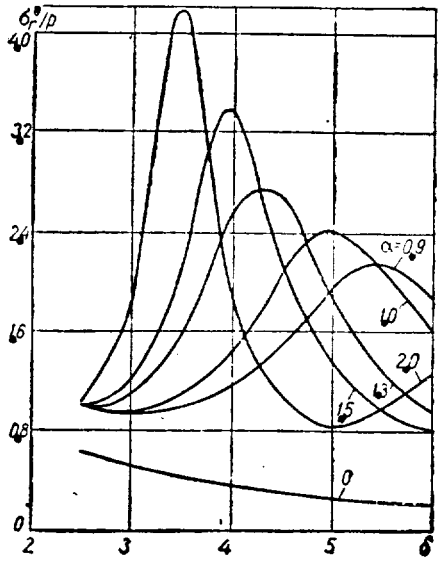


Figure XI.32.

V. T. Golovchan [2, 3] solved the problem of the dynamic concentration of stresses in an infinite plate with two identical round holes (Figure XI.27), the contours of which are stressed by harmonic pressure, by this method. Numerous graphs are presented in these articles for stresses at points on the center line of the holes as a function of the distance between the holes and the parameter $\alpha = 2\pi R/L$, where R is the radius of the holes, and L is the length of an arbitrary wave.

The results obtained show that the character of distribution of stresses in a plate for a given $\delta \left(\delta = \frac{O_1O_2}{R}\right)$ -- relative distance; (see Figure XI.27) depends greatly on the parameter α, i.e., on the length of the wave propagating in the plate. Figures XI.28, XI.30 illustrate the graphs of stresses, related by p, at the half way point between the holes for $\delta = 5$, for certain characteristic values of the parameter α. The specific values of α are close to those for which $|\sigma_\theta^E|$ (Figure XI.28) or $|\sigma_r^0|$ (Figure XI.29 and XI.30) acquire their maximum values in the range $0 \leqslant \alpha \leqslant 3.2$.

Figures XI.31 and XI.32 show the curves of change of amplitude of stresses σ_θ^E and σ_r^0 as functions of δ.

It can be concluded from these graphs that the stress state in a plate near holes during dynamic loading differs greatly, both qualitatively and quantitatively, from the static loading corresponding to the case $\alpha = 0$.

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CHAPTER XII. EFFECT OF VISCOELASTIC PROPERTIES OF MATERIAL ON STRESS DISTRIBUTION NEAR HOLES

Abstract. The effect of viscoelastic properties of the material on the stress distribution near holes is discussed. The general relations of the linear theory of the viscoelastic media and the boundary value problems are formulated. The stress concentration is studied near the circular and elliptical holes in the anisotropic viscoelastic plate made of unidirectional glass-reinforced plastic.

§1. Basic Relations of Linear Theory of Viscoelastic Media

Statement of Problem. Numerous experiments show that the time factor /820
plays a considerable part in the deformation of such materials as concrete, wood, high polymers, and even metals, particularly at high temperatures. In bodies of such materials, if after the application of a load, destruction does not occur, the steady state is achieved after a certain period of time, which, depending on the method of loading and other factors, can last from a few seconds to many years.

During a period commensurate with the service life of a machine or installation, the stress state in parts of these machines or elements of facilities will change considerably, and therefore it cannot be regarded as steady. The time factor affects the deformation process as a whole, i.e., it affects all three stages -- elastic, plastic and fracturing.

Hence it becomes necessary to use, instead of the known relations of elasticity in the form of Hooke's generalized law, which assume the instantaneous establishment within a body of completely defined and unchanging, in time, stress and deformation states, such relations and laws that will correlate the stresses and deformations with time.

The best relations for the description of the process of deformation in time are those that contain time integral operators with relaxation and consequence kernels. Such relations were first applied by V. Volterra [1], and then by Yu. N. Rabotnov [1], M. I. Rozovskiy [1], N. Kh. Arutyunyan [1], and others.

For many materials the processes of deformation during time can be described satisfactorily on the basis of linear theories of viscoelastic media, the most important of which is V. Volterra's theory of viscoelastic heredity.

The equations of V. Volterra's heredity theory of elasticity are found[1] by substituting in the relations of classical elasticity theory, the constants E, G and ν by integral operators $\overline{E}$, $\overline{G}$ and $\overline{\nu}$:

---

[1]See Yu. No Rabotnov [1].

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$$\bar{E}\varepsilon_x = \sigma_x - \bar{\nu}(\sigma_y + \sigma_z), \tag{XII.1}$$

$$\tau_{xy} = 2\bar{G}\varepsilon_{xy} \qquad (x, y, z), \tag{XII.2}$$

where

/821

$$\begin{aligned} \bar{G}f &= G_0(1 - \mathring{R}_1)f; \\ \bar{E}f &= E_0(1 - \mathring{R}_2)f; \\ \bar{\nu}f &= \nu_0(1 + \mathring{R}_3)f. \end{aligned} \tag{XII.3}$$

Integral operators $\overset{*}{R}_i$ (i = 1, 2, 3) with relaxation kernels $R_i(t, s)$ acting on some function f(x, y, z, t) of coordinates (x, y, z) and time t, have the form

$$\mathring{R}_i f = \int_0^t R_i(t, s) f(x, y, z, s)\, ds \quad (i = 1, 2, 3). \tag{XII.4}$$

In relations (XII.3), G_0 and E_0 are the instantaneous moduli of displacement and normal deformation, respectively; ν_0 is the instantaneous Poisson's ratio. The symbol (x, y, z) shows that the other relationships in (XII.1) and (XII.2) between stress components σ_x, σ_y, ..., τ_{xy} and deformation components ε_x, ε_y, ..., ε_{xy} are obtained by the circular permutation of the letters x, y and z.

The operators $\bar{G}$ and $\bar{E}$ are called the relaxation operators. The creep operators $\bar{P}_1$ and $\bar{P}_2$ are defined as the operators that are inverse to operators $\bar{G}$ and $\bar{E}$, respectively:

$$\bar{P}_1 = \frac{1}{\bar{G}} = \frac{1}{G_0(1 - \mathring{R}_1)} = \frac{1}{G_0}(1 + \mathring{P}_1), \tag{XII.5}$$

$$\bar{P}_2 = \frac{1}{\bar{E}} = \frac{1}{E_0(1 - \mathring{R}_2)} = \frac{1}{E_0}(1 + \mathring{P}_2), \tag{XII.6}$$

where

$$\frac{1}{1 - \mathring{R}_i} = 1 + \mathring{R}_i + \mathring{R}_i^2 + \mathring{R}_i^3 + \ldots + \mathring{R}_i^{n-1} + \ldots \quad (i = 1, 2). \tag{XII.7}$$

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In expansion (XII.7) the degrees of operator $\overset{*}{R}_i$ are expressed through repeated kernels of the original kernel $\overset{*}{R}_i$ as follows:

$$\overset{*}{R}_i^{n-1} f = \int_0^t R_{i;n-1}(t, s) f(s)\, ds, \tag{XII.8}$$

where

$$R_{i;n-1}(t, s) = \int_s^t R_i(t, u) R_{i;n-2}(u, s)\, du. \tag{XII.9}$$

The kernel $P_1(t, s)$ of integral operator $\overset{*}{P}_1$ and the kernel $R_1(t, s)$ of integral operator $\overset{*}{R}_1$ are found by means of twist testing specimens: the first from the creep curve, and the second from the relaxation curve. The kernel $P_1(t, s)$ is the resolvant of the kernel $R_1(t, s)$. Series (XII.7), as we know[1], converges uniformly for any finite range of change of arguments t and s for a regular or slightly singular[2] kernel. /822

In expression (XII.6) the kernels $P_2(t, s)$ and $R_2(t, s)$ of integral operators $\overset{*}{P}_3$ and $\overset{*}{R}_2$, in contrast to the above-mentioned kernels, are found from stretch tests or pure bending tests of samples: the first from the creep curve, and the second from the relaxation curve.

Operator $\bar{\nu}$ can be expressed through the above integral operators. Thus, disregarding the effect of the time factor (heredity) during three-dimensional compression, i.e., considering the three-dimensional deformation to be elastic, Yu. N. Rabotnov [2] obtained the formula

$$\bar{\nu} = \nu_0 \left(1 + \frac{1-2\nu_0}{2\nu_0} \overset{*}{R}_2\right). \tag{XII.10}$$

The operator $\bar{\nu}$ cannot be expressed through operator $\overset{*}{R}_1$. According to M. I. Rozovskiy [2], we will have

$$\bar{\nu} = \nu_0 + (1 + \nu_0) \overset{*}{H}, \tag{XII.11}$$

---

[1]See N. N. Dolinina [1].
[2]I.e., for kernels of the form (XII.12)-(XII.14).

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where the kernel $H(t, s)$ of integral operator $\overset{*}{H}$ represents the resolvant of kernel $0.5(1 - 2\nu_0)R_1(t, s)$.

The use of kernels $R_1(t, s)$ and $R_2(t, s)$ for expressing operator $\overline{\nu}$ in the general case, i.e., with consideration of the effect of the time factor on expansion, requires the calculation of the ratio of the operators[1] $\overline{E}$ and $\overline{G}$.

The time integral operators $\overline{G}$, $\overline{E}$ and $\overline{\nu}$ (XII.3) and spatial operators of differentiation and integration with respect to the coordinates, after multiplication, possess properties of commutativity. Therefore any[2] problem that considers the effect of the time factor (hereditary elasticity) can be solved as a problem of ordinary elasticity theory, and only in the final result is it necessary to substitute the elastic constants G, E and ν by operators $\overline{G}$, $\overline{E}$ and $\overline{\nu}$, defined by relations (XII.3). The basic difficulty encountered when using V. Volterra's principle is in the determination of the various functions of the operators, appearing as the result of the above-mentioned substitution of the elastic constants E, G and ν in the solution of the three-dimensional (or plane) problem for an ideal elastic body, by operators $\overline{E}$, $\overline{G}$ and $\overline{\nu}$. /823

Selection of Kernels of Integral Operators. The functions that are characteristic of a given material, which reflect the effect of unit stress acting during a unit period of time τ, on deformation at the moment of time t are used as the kernels of the integral operators. The form of these functions is established on the basis of analysis of given experimental studies of creep or relaxation. The consequence and relaxation kernels experimentally determined in this manner possess a singularity when $t - \tau = 0$.

We will present the analytical expressions of the kernels for certain materials, obtained on the basis of tests.

Through his tests on belts, G. Duffing [1] established that the consequence kernel should be taken in the form

---

[1]See M. I. Rozovskiy [2].

[2]Except for those problems in which the boundaries change with time. See, for instance, A. B. Yefimov [1] and G. A. C. Graham [1]. In particular, this change of boundaries will occur in the case of the so-called "contact problems of stress concentration" -- problems of stress concentration near reinforced holes, -- where the reinforcing ring is simply placed in (and not sealed!) a hole in an elastic plane without "tightening." For instance, when a plate with a hole reinforced by this method is subjected to uniaxial tension, the material of the elastic plate will separate from the ring in some places, and in this case the area of contact between the ring and the elastic plate will not be known beforehand; due to creep of the material it will change with time. The latter (area of contact) should be determined as the result of the solution of the problem.

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$$A(t-\tau)^{-\alpha}, \tag{XII.12}$$

where $\alpha = 4/5$; A is a constant.

As the result of the analysis of the relaxation curves of resin, A. P. Bronskiy [1] found that the desired kernel is conveniently represented in the form

$$A(t-\tau)^{\alpha}\exp[-(t-\tau)^{1-\alpha}], \quad \alpha > 0. \tag{XII.13}$$

The resolvant of kernel (XII.13) could not be found.

For the consideration of the effect of increasing change of concrete, N. Kh. Arutyunyan [1] proposed a kernel of the form

$$\frac{\partial}{\partial\tau}\left(C_{\cdot}+\frac{A}{\tau}\right)(1-e^{-\gamma(t-\tau)}), \tag{XII.14}$$

where A, C and γ are constants.

Integral operator (XII.14) is not invariant in the case of change of the origin of time measurements, and this makes it possible, using this operator, to reflect the effect of physico-chemical processes in certain "aging" materials, particularly concrete, occurring with the passing of time, on their mechanical properties.

To insure that V. Volterra's principle will be an effective means of solving problems with consideration of the effect of the time factor, Yu. N. Rabotnov [2] proposed for the construction of the relaxation operator the exponential function of fractional order

$$\mathcal{E}_{\alpha}(\beta; t-\tau) = (t-\tau)^{\alpha}\sum_{n=1}^{\infty}\frac{\beta^{n}(t-\tau)^{n(1+\alpha)}}{\Gamma[(n+1)(1+\alpha)]} \quad (0>\alpha>-1). \tag{XII.15}$$

Series (XII.15) converges for any finite t - τ. A kernel of the type (XII.15) possesses the same singularity as does kernel (XII.13), and encompasses as a partial case the kernel (XII.12). /824

Yu. N. Rabotnov [2] showed that integral operators with kernels of the type (XII.15) possess the following properties:

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$$\dot{\vartheta}_\alpha(x)\dot{\vartheta}_\alpha(y) = \frac{\dot{\vartheta}_\alpha(x) - \dot{\vartheta}_\alpha(y)}{x-y} \quad \text{for} \quad x \neq y,$$

$$\dot{\vartheta}_\alpha^2 = \frac{\partial \dot{\vartheta}_\alpha(x)}{\partial x} \quad \text{for} \quad x = y; \tag{XII.16}$$

$$\frac{1}{1-\varkappa\dot{\vartheta}_\alpha(\beta)} = 1 + \varkappa\dot{\vartheta}_\alpha(\varkappa+\beta) + \ldots; \tag{XII.17}$$

M. I. Rozovskiy [3, 5] established that

$$\frac{1}{1-\sum_{k=1}^{m}\xi_k\dot{\vartheta}_\alpha(x_k)} = 1 + \sum_{k=1}^{m} a_k\dot{\vartheta}_\alpha(\mu_k), \tag{XII.18}$$

where μ_k (k = 1, 2, 3, ..., m) are the roots of the equation

$$1 + \sum_{n=1}^{m}\frac{\xi_n}{x_n-\mu} = 0, \tag{XII.19}$$

and the coefficients a_k are determined from a system of m linear algebraic equations

$$1 + \sum_{k=1}^{m}\frac{a_k}{x_n-\mu_k} = 0 \quad (n = 1, 2, 3, \ldots, m), \tag{XII.20}$$

$$\prod_{i=1}^{n}\dot{\vartheta}_\alpha^{m_i}(x_i) = \frac{1}{\prod_{i=1}^{n}(m_i-1)!} - \frac{\partial^{m_1+m_2+\ldots+m_n-n}}{\partial x_1^{m_1-1}\partial x_2^{m_2-1}\ldots\partial x_n^{m_n-1}}\left[\sum_{i=1}^{n}\frac{\dot{\vartheta}_\alpha(x_i)}{\prod_{k=1}^{n}(x_i-x_k)}\right]. \tag{XII.21}$$

Formula (XII.21) makes it possible to express the product of various degrees of the operators through derivatives with respect to parameters x_i of the original kernel $\overset{*}{\vartheta}_\alpha(x_i)$.

After the final calculation of the desired stresses and deformations as functions of the coordinates and time, it is possible in practical calculations to use an approximation of the $\overset{*}{\vartheta}_\alpha$-operator[1] in the form /825

[1]See M. I. Rozovskiy [4], and also V. D. Annin [1].

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$$\dot{\vartheta}_\alpha(-\beta)\,1 \approx \frac{1}{\beta}(1 - e^{-\beta\gamma t^{(1+\alpha)}}). \qquad \text{(XII.22)}$$

where

$$\gamma = (1+\alpha)^{(1+\alpha)};\ -1 < \alpha < 0;\ \mathrm{Re}\,\beta > 0.$$

The $\overset{*}{\vartheta}_\alpha$ $(-\beta)$-operator can be expressed[1] during action on unity through the Mittag-Lefler function of the order $(1 + \alpha)$ in the form

$$\dot{\vartheta}_\alpha(-\beta)\,1 = \frac{1}{\beta}[1 - E_{1+\alpha}(-\beta t^{1+\alpha})]. \qquad \text{(XII.23)}$$

The numerical values[2] of the function $E_{1+\alpha}(-\xi)$ $(\xi = \beta t^{1+\alpha})$ in (XII.23) are presented below for $\alpha = -0.7$:

ξ	$E_{0,3}(-\xi)$	ξ	$E_{0,3}(-\xi)$
0	1	0,6	0,5882
0,1	0,8985	0,7	0,5504
0,2	0,8149	0,8	0,5112
0,3	0,7450	0,9	0,4922
0,4	0,6782	1,0	0,4904
0,5	0,6246		

In the limit case $\alpha = 0$ the function $\vartheta_\alpha(\beta,\ t - \tau)$ (XII.15) changes into an ordinary exponential function.

Exponential kernels of the form

$$e^{-b_k(t-\tau)} \ldots \qquad \text{(XII.24)}$$

are the simplest, but they, as is evident, do not possess a singularity when $t - \tau = 0$.

It is readily shown by simple differentiation that when certain limitations are imposed on the constants b_k and the initial conditions, integral operators with exponential kernels are equivalent to differential operators[3] of the form

[1] See M. I. Rozovskiy [2].
[2] See M. I. Rozovskiy [6].
[3] J. R. Radok [1], using relations of the form (XII.25), investigates the stress concentration near a round (both free and with a sealed elastic or absolutely rigid collar) hole under the conditions of plane deformation in an incompressible material of the Voight and Maxwell types.

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$$\sum_{k=0}^{n} a_k \frac{\partial^k \sigma}{\partial t^k} = \sum_{s=0}^{m} b_s \frac{\partial^s \varepsilon}{\partial t^s}. \qquad \text{(XII.25)}$$

Experimental Determination of Rheologic Parameters. The analysis of experimental creep (aftereffect) curves and relaxation curves can be accomplished by means of approximation (XII.22) or precise representation (XII.23). /826

Such analysis is most readily accomplished by using approximation (XII.22). In this case the curve of simple creep is operated upon in accordance with formula

$$\frac{\varepsilon(t) - \varepsilon_0}{\varepsilon_0} = a[1 - \exp(-b\gamma t^{1+a})], \qquad \text{(XII.26)}$$

where $\varepsilon(t)$ is deformation (during stretching or simple deflection) at moment of time t; ε_0 is instantaneous deformation; a and b are creep parameters.

The relaxation curves are processed by analogous formulas. If the relaxation curve is not available, then the relaxation parameters τ and λ can be calculated by formulas

$$\lambda_2 = \frac{a}{1+a} \quad \text{and} \quad \tau_2 = \left[\frac{\gamma}{(1+a)b}\right]^{\frac{1}{1-a}} \qquad \text{(XII.27)}$$

for known values of creep parameters a and b.

The creep and relaxation curves obtained from twist tests of samples are processed similarly as described above.

§2. Stress Distribution Near Hole in Homogeneous Viscoelastic Materials

Distribution of Deflection Moments Near Holes During Twisting of Thin Plates. Let a thin isotropic plate of thickness h be subjected to the action of moments of torque H = const, applied on the entire edge of the plate. If to four faces of a rectangular plate are applied distributed moments of torque H, then[1] the distribution of moments on the contour of an elliptical, square or triangular hole can be represented in the form

$$M_\theta = \varphi(\nu)\psi(\theta). \qquad \text{(XII.28)}$$

The extraction of the function $\psi(\theta)$ that characterizes the distribution of deflecting moments M_θ on the contour of the hole, in the form of a cofactor

[1]See G. N. Savin [1], Chapter VI, §4.

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denotes that the distribution M_θ on the contour of the hole will be geometrically similar for successive fixed moments of time.

Let us consider, by way of example, the dependence of M_θ on the contour of a square hole on time. To simplify further calculations in the formula[1]

$$M_\theta = \frac{48(1+\nu)}{19+5\nu} \cdot \frac{\mathcal{H} \sin 2\theta}{5+4\cos\theta}, \qquad \text{(XII.29)}$$

which gives the distribution of bending moments M_θ for t = 0 (elasto-instantaneous state), it is convenient to extract from the right hand side of (XII.29) the term that does not depend on ν: /827

$$M_\theta = \left(1 - \frac{14}{5\nu+19}\right)\psi(\theta), \qquad \text{(XII.30)}$$

where

$$\psi(\theta) = \frac{48\mathcal{H}\sin 2\theta}{25+20\cos 4\theta}.$$

By substituting in (XII.30), instead of ν, the operator

$$\bar{\nu} = \nu_0[1 + \delta\dot{\mathcal{J}}_\alpha(-\beta)], \qquad \text{(XII.31)}$$

where ν_0 is Poisson's ratio for t = 0, δ and β are rheologic characteristics, we obtain

$$M_\theta(t) = M_\theta(0)\left[1 + \frac{14\nu_0\delta}{(1+\nu_0)(19+5\nu_0)}\dot{\mathcal{J}}_\alpha(-\beta_1)\right]; \qquad \text{(XII.32)}$$

here

$$\beta_1 = \beta + \frac{5\nu_0\delta}{19+5\nu_0}.$$

Conversion of the operators is accomplished by formula (XII.17).

By integrating with consideration of formula (XII.22), we obtain

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[1]See G. N. Savin [1], formula (6.96), and also Figure 152.

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$$M_\theta(t) \approx M_\theta(0)\left\{1 + \frac{14\nu_0\delta\,[1 - \exp(-\gamma\beta_1 t^{1+\alpha})]}{(1+\nu_0)\,[19\beta + 5\nu_0(\beta+\delta)]}\right\}. \qquad \text{(XII.33)}$$

The parameters β, δ, α and ν_0 from experimental data obtained under the condition of simple relaxation of copper and aluminum samples at room temperature[1], are presented in Table XII.1.

TABLE XII.1

Material	ν_0	α	β	δ
Aluminum	0,25	—0,50	$6{,}65\cdot10^{-3}\ \text{sec}^{-0,5}$	$6{,}15\cdot10^{-3}\ \text{sec}^{-0,5}$
Copper	0,25	—0,50	$9{,}2\cdot10^{-3}\ \text{sec}^{-0,5}$	$6{,}65\cdot10^{-3}\ \text{sec}^{-0,5}$

Tr. Note: Commas indicate decimal points.

The results of calculations made by formula (XII.33) for the values of the rheologic characteristics presented in Table XII.1, show that in the steady (limit) state the bending moments for an aluminum plate are 12% higher, and for a copper plate, 10% higher. When $t \geqslant 300$ hr, the stress state in the plate near a hole can be regarded as practically steady.

For $t \to \infty$ in (XII.33), we obtain /828

$$(M_\theta)_\infty = M_\theta(0)\left\{1 + \frac{14\nu_0\delta}{(1+\nu_0)\,[19\beta + 5\nu_0(\beta+\delta)]}\right\}. \qquad \text{(XII.34)}$$

Deflection of Triangular Plate Weakened by a Round Hole. Let us examine the change in time of the stress state near a round hole during the bending of a thin plate in the form of an equilateral triangle[2]. Let uniformly distributed bending moments M be applied to the outer contour of the plate. The expression for the bending moments in operator form will be

$$M_\theta(t) = M_\theta(0)\left[1 + \bar{\nu} - \frac{2R}{C}\left(\bar{\nu} - 3 + \frac{8}{\bar{\nu}+3}\right)\cos 3\theta\right]. \qquad \text{(XII.35)}$$

Hence, recalling formula (XII.31), we find

[1]See B. M. Rovinskiy and V. G. Lyuttsau
[2]See G. N. Savin [1], Chapter VI, §5, formula (6.108).

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$$M_\theta(t) = M_\theta(0)[1 + a_1\dot{\vartheta}_\alpha(-\beta) + a_2\dot{\vartheta}_\alpha(-\beta_1)], \qquad \text{(XII.36)}$$

where

$$a_1 = \frac{1-\psi(\theta)}{M_\theta(0)}\nu_0\delta; \quad a_2 = \frac{2(5-\nu_0)\psi(\theta)}{(3+\nu_0)^2 M_\theta(0)}\nu_0\delta;$$

$$\beta_1 = \beta + \frac{\nu_0\delta}{3+\nu_0}; \quad \psi(\theta) = \frac{2R}{C}\cos 3\theta.$$

The conversion of the operators is accomplished by formula (XII.17).

It is easy to see that the geometric similarity of distribution of deflection moments M_θ (XII.36) on the contour of the hole will now be absent, since the change of M_θ with time depends on θ.

By calculating operators $\dot{\vartheta}^*_\alpha$ in (XII.36) in like manner as above for the case of the twisting of thin plates, and by performing the calculations for the very same rheologic characteristics, recalling R/C = 0.1, we find that the maximum value of M_θ, which is achieved when $\theta = 0$, is 16% higher for aluminum when $t \to \infty$, while the minimum (when $\theta = 180°$) is 20% higher when $t \to \infty$. Thus, the difference between the greatest and least values of M_θ decreases with time.

Plate with Triangular Hole. During the pure cylindrical deflection of a plate weakened by a triangular hole, the deflection moments on the contour of the hole[1] are

$$M_\theta(t) = M\left\{\psi(\theta) + \frac{2}{3+\tilde{\nu}}[1-\psi(\theta)]\right\}, \qquad \text{(XII.37)}$$

where

$$\psi(\theta) = \frac{5 + 12\cos\theta - 18\cos 3\theta}{13 - 12\cos 3\theta}.$$

Recalling expression (XII.31) and performing conversion by formula (XII.17), we find /82

$$M_\theta(t) = M_\theta(0)\left\{1 - \frac{2\delta[1-\psi(\theta)]}{(3+\nu_0)^2 M_\theta(0)}\dot{\vartheta}^*_\alpha(-\beta_1)\right\}. \qquad \text{(XII.38)}$$

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[1]See G. N. Savin [1], Chapter VI, §3, formula (6.84).

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where

$$\beta_1 = \beta + \frac{\nu_0 \delta}{3 + \nu_0}.$$

Hence, on the basis of approximation (XII.22), we obtain

$$M_\theta(t) = M_\theta(0)\left\{1 - \frac{2\delta\,[1-\psi(\theta)]\,[1-\exp(-\gamma\beta_1 t^{1+\alpha})]}{(3+\nu_0)\,[3\beta + \nu_0(\delta+\beta)]\,M_\theta(0)}\right\}. \qquad \text{(XII.39)}$$

The extremal values of M_θ/M, found on the basis of formula (XII.39) for aluminum for the rheologic parameters listed in Table XII.1 are given in Table XII.2.

TABLE XII.2

$\theta°$	$M_\theta(0)/M$	$M_\theta(\infty)/M$
0	0,23	0,15
4	0,38	0,31
120	3,90	4,23

We see from Table XII.2 that the difference between the greatest and least magnitudes of the deflection moments M_θ (XII.39) on the contour of the hole increases with time[1]. The "critical" values of angles θ_{cr} at which $M_\theta(t)$ will be[2] equal to $M_\theta(0)$ for any value of t, as seen in formula (XII.39), is found from the condition

$$1 - \psi(\theta_{cr}) = 0. \qquad \text{(XII.40)}$$

Tr. Note: Commas indicate decimal points.

Uniaxial Tension of Plate Weakened by a Round Hole with a Soldered Circular Collar. If into a circular hole of an infinite viscoelastic isotropic plate, subjected to tension "at infinity" by stresses p, a circular collar of a different (elastic) material is sealed, then the stresses on the contour of the seam for r = R, according to the principle of V. Volterra, will have the form[3]

$$\frac{2\sigma_r}{p} = 1 - \frac{R^2}{r^2} + \left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta + (1+\bar{\varkappa})\frac{R^2}{r^2}\left(\frac{1}{\bar{b}_1} + \right.$$

$$\left. + \frac{4}{\bar{b}_2}\cos 2\theta\right) - 3\frac{R^4}{r^4}\cdot\frac{1+\bar{\varkappa}}{\bar{b}_2}\cos 2\theta; \qquad \text{(XII.41)}$$

$$\frac{2\sigma_\theta}{p} = 1 + \frac{R^2}{r^2} - \left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta - (1+\bar{\varkappa})\frac{R^2}{r^2}\left(\frac{1}{\bar{b}_1} - \frac{3}{\bar{b}_2}\cdot\frac{R^4}{r^4}\cos 2\theta\right);$$

[1]Since the greatest value of M_θ (XII.39) increases with time, while the least decreases.

[2]At which M_θ in (XII.39) ceases to decrease and begins to increase.

[3]See G. N. Savin [1], Chapter V, §2, formulas (5.15) and (5.20).

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$$\frac{2\tau_{r\theta}}{p} = -\left[2\frac{R^2}{r^2} + 1 - 3\frac{R^4}{r^4} - 2\frac{R^2}{r^2}\cdot\frac{1+\bar{\varkappa}}{\bar{b}_2}\left(1 - 1{,}5\frac{R^2}{r^2}\right)\right]\sin 2\theta,$$

where

$$\bar{\varkappa} = \varkappa_0[1 - q\mathring{\vartheta}_\alpha(-\eta)],\ \bar{\varkappa}_1 = \varkappa_{10}[1 - q_1\mathring{\vartheta}_\alpha(-\eta_1)]; \tag{XII.42}$$

$$\bar{G} = G_0[1 - Q\mathring{\vartheta}_\alpha(-\eta^*)],\ \bar{G}_1 = G_{10}[1 - Q_1\mathring{\vartheta}_\alpha(-\eta_1^*)]; \tag{XII.43}$$

$$\bar{b}_1 = \frac{\bar{G}}{\bar{G}_1} - \bar{\varkappa}_1\frac{\bar{G}}{\bar{G}_1} - 2,\quad \bar{b}_2 = \bar{\varkappa} + \frac{\bar{G}}{\bar{G}_1}. \tag{XII.44}$$

The parameters q, Q, η, η^* are expressed through β and δ, the numerical values of which are given in Table XII.1. We will consider a thin plate with a sealed elastic (steel) collar, for which $\mu = G = 8.10 \cdot 9.81 \cdot 10^9$ n/m^2;

$$\varkappa = \frac{3-\nu}{1+\nu} = 2{,}08;\ \nu = 0{,}3.$$

Using formula (XII.17), we find

$$\bar{\varkappa} = \frac{3-\bar{\nu}}{1+\bar{\nu}} = \varkappa_0\left[1 - \frac{4\nu_0\delta}{(1+\nu_0)(3-\nu_0)}\mathring{\vartheta}_\alpha\left(-\beta - \frac{\nu_0\delta}{1+\nu_0}\right)\right]. \tag{XII.45}$$

Comparison of (XII.42) and (XII.43) shows that

$$q = \frac{4\nu_0\delta}{(1+\nu_0)(3-\nu_0)},\quad \eta = \beta + \frac{\nu_0\delta}{1+\nu_0}.$$

To find Q and η^*, we will use the known relation

$$\bar{G} = \frac{\bar{E}}{2(1+\bar{\nu})}, \tag{XII.46}$$

and also, assuming the three-dimensional deformation of the material of the plate to be elastic, we will use the relation

$$K_0 = \frac{1-2\nu_0}{E_0} = \frac{1-2\bar{\nu}}{\bar{E}}. \tag{XII.47}$$

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Then

$$\bar{G} = G_0\left[1 - \frac{3\nu_0\delta}{(1-2\nu_0)(1+\nu_0)}\,\dot{\vartheta}_\alpha\left(-\beta - \frac{\nu_0\delta}{1+\nu_0}\right)\right]$$

and consequently,

$$Q = \frac{3\nu_0\delta}{(1-2\nu_0)(1+\nu_0)}, \quad \eta^* = \eta.$$

Analogously,

$$\eta = \beta_1 + \frac{\nu_{10}\delta_1}{1+\nu_{10}}, \qquad \dot{\eta_1} = \eta_1,$$

$$q_1 = \frac{4\nu_{10}\delta_1}{(1+\nu_{10})(3-\nu_{10})}, \qquad Q_1 = \frac{3\nu_{10}\delta_1}{(1-2\nu_{10})(1+\nu_{10})}.$$

By substituting (XII.42) and (XII.43) into (XII.44) we obtain /831

$$\bar{b}_1 = b_{10}[1 - a_1\dot{\vartheta}_\alpha(-\eta) + a_2\dot{\vartheta}_\alpha(Q_1 - \eta_1), \tag{XII.48}$$

where

$$a_1 = \frac{G_0 Q[(\eta_1 - \eta)(1-\varkappa_{10}) + q_1\varkappa_{10}]}{G_{10}b_{10}(\eta_1 - \eta - Q_1)},$$

$$a_2 = \frac{G_0[Q_1 - \varkappa_{10}(Q_1 - q_1)](\eta_1 - \eta - Q_1 + Q)}{G_{10}b_{10}(\eta_1 - \eta - Q_1)};$$

$$b_{10} = \frac{G_0}{G_{10}} - \varkappa_{10}\frac{G_0}{G_{10}} - 2, \tag{XII.49}$$

$$\bar{b}_2 = b_{20}[1 - a_3\dot{\vartheta}_\alpha(-\eta) + a_4\dot{\vartheta}_\alpha(Q_1 - \eta_1)].$$

Here

$$a_3 = \frac{q\varkappa_0}{b_{20}} + \frac{G_0 Q(\eta_1 - \eta)}{b_{20}G_{10}(\eta_1 - \eta - Q_1)};$$

$$a_4 = \frac{G_0 Q_1(\eta_1 - \eta - Q_1 + Q)}{G_{10}b_{20}(\eta_1 - \eta - Q_1)};$$

$$b_{20} = \varkappa_0 + \frac{G_0}{G_{10}}.$$

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In order to convert the aggregate of operators (XII.28) and (XII.29) we will use formulas (XII.18)-(XII.20), according to which

$$\frac{1}{\bar{b}_1} = \frac{1}{b_{10}}[1 + A_1\dot{\vartheta}_\alpha(\mu_1) + A_2\dot{\vartheta}_\alpha(\mu_2)],$$
$$\frac{1}{\bar{b}_2} = \frac{1}{b_{20}}[1 + A_3\dot{\vartheta}_\alpha(\mu_3) + A_4\dot{\vartheta}_\alpha(\mu_4)]. \qquad \text{(XII.50)}$$

Here μ_i are the roots of quadratic equations

$$\mu_i^2 + p_1\mu_i + T_1 = 0 \quad (i = 1, 2),$$
$$\mu_i^2 + p_2\mu_i + T_2 = 0 \quad (i = 3, 4),$$

where

$$p_1 = \eta_1 + \eta + a_2 - a_1 - Q_1, \quad T_1 = (\eta_1 - Q_1)(\eta - a_1) + a_2\eta,$$
$$p_2 = \eta_1 + \eta + a_4 - a_3 - Q_1, \quad T_2 = (\eta_1 - Q_1)(\eta - a_3) + a_4\eta.$$

Coefficients A_i are found from the algebraic equation system

$$1 - \sum_{i=j}^{j+1} \frac{A_i}{r_j + \mu_i} = 0 \quad \begin{matrix}(j = 1, 3),\\ (n = 1, 2)\end{matrix} \quad r_1 = \eta, \quad r_2 = \eta_1 - Q_1.$$

Then

$$A_i = \frac{(-1)^{i+1}(\eta + \mu_i)(\eta_1 - Q_1 + \mu_i)}{\mu_1 - \mu_2} \quad (i = 1, 2),$$
$$A_i = \frac{(-1)^{i+1}(\eta + \mu_i)(\eta_1 - Q_1 + \mu_i)}{\mu_3 - \mu_4} \quad (i = 3, 4).$$

By substituting (XII.50) and (XII.42) into (XII.41) and using the formulas /832 for the multiplication of operators $\overset{*}{\vartheta}_\alpha$, we obtain

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$$\frac{2\sigma_r(t)}{p} = \frac{2\sigma_r(0)}{p} + \frac{R^2\varkappa_0 q}{r^2}\left[\frac{1}{b_{10}}\left(\sum_{i=1}^{2}\frac{A_i}{\eta+\mu_i} - 1\right) + \frac{1}{b_{20}}\left(4 - 3\frac{R^2}{r^2}\right)\times\right.$$

$$\left.\times\left(\sum_{i=3}^{4}\frac{A_i}{\eta+\mu_i} - 1\right)\cos 2\theta\right]\dot{\vartheta}_\alpha(-\eta) + \frac{R^2(1+\varkappa_0)}{r^2}\left[\sum_{i=1}^{2}N_{i1}\dot{\vartheta}_\alpha(\mu_i) + \right.$$

$$\left. + \left(4 - 3\frac{R^2}{r^2}\right)\sum_{i=3}^{4}N_{i2}\dot{\vartheta}_\alpha(\mu_i)\cos 2\theta\right];$$

$$\frac{2\sigma_\theta(t)}{p} = \frac{2\sigma_\theta(0)}{p} - \frac{R^2\varkappa_0 q}{r^2}\left[\frac{1}{b_{10}}\left(\sum_{i=1}^{2}\frac{A_i}{\eta+\mu_i} - 1\right) - \frac{3R}{b_{20}r^2}\left(\sum_{i=3}^{4}\frac{A_i}{\eta+\mu_i} - 1\right)\times\right.$$

$$\left.\times\cos 2\theta\right]\dot{\vartheta}_\alpha(-\eta) + \frac{R^2(1+\varkappa_0)}{r^2}\left[\sum_{i=1}^{2}N_{i1}\dot{\vartheta}_\alpha(\mu_i) - 3\frac{R^2}{r^2}\sum_{i=3}^{4}N_{12}\dot{\vartheta}_\alpha(\mu_i)\cos 2\theta\right];$$

$$\frac{2\tau_{r\theta}}{p} = \frac{2\tau_{r\theta}(0)}{p} + \frac{R^2}{r^2}\left(2 - 3\frac{R^2}{r^2}\right)\left[\frac{\varkappa_0 q}{b_{20}}\left(\sum_{i=3}^{4}\frac{A_i}{\eta+\mu_i} - 1\right)\dot{\vartheta}_\alpha(-\eta) + \right.$$

$$\left. + (1+\varkappa_0)\sum_{i=3}^{4}N_{i2}\dot{\vartheta}_\alpha(\mu_i)\right]\sin 2\theta.$$

On the contour of the seam, i.e., when

$$\frac{2\sigma_r(t)}{p} = \frac{2\sigma_r(0)}{p} + (1+\varkappa_0)\left[(N_0 + N_1\cos 2\theta)\dot{\vartheta}_\alpha(-\eta) + \right.$$

$$\left. + \sum_{i=1}^{2}N_{i1}\dot{\vartheta}_\alpha(\mu_i) + \cos 2\theta\sum_{i=3}^{4}N_{i2}\dot{\vartheta}_\alpha(\mu_i)\right],$$

$$\frac{2\sigma_\theta(t)}{p} = \frac{2\sigma_\theta(0)}{p} - (1+\varkappa_0)\left[(N_0 - 3N_1\cos 2\theta)\dot{\vartheta}_\alpha(-\eta) + \right. \qquad \text{(XII.51)}$$

$$\left. + \sum_{i=1}^{2}N_{i1}\dot{\vartheta}_\alpha(\mu_i) - 3\cos 2\theta\sum_{i=3}^{4}N_{i2}\dot{\vartheta}_\alpha(\mu_i)\right],$$

$$\frac{2\tau_{r\theta}(t)}{p} = \frac{2\tau_{r\theta}(0)}{p} - (1+\varkappa_0)\left[N_1\dot{\vartheta}_\alpha(-\eta) + \sum_{i=3}^{4}N_{i2}\dot{\vartheta}_\alpha(\mu_i)\right]\sin 2\theta,$$

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where

$$N_0 = \frac{\varkappa_0 q}{(1+\varkappa_0)\, b_{10}}\left(\sum_{i=1}^{2}\frac{A_i}{\eta+\mu_i}-1\right), \quad N_1 = \frac{\varkappa_0 q}{(1+\varkappa_0)\, b_{20}}\left(\sum_{i=3}^{4}\frac{A_i}{\eta+\mu_i}-1\right),$$

$$N_{ij} = A_i b_{j0}^{-1}\left[1-\frac{\varkappa_0 q}{(1+\varkappa_0)(\eta+\mu_i)}\right] \quad (i=1,2,3,4; \quad j=1,2).$$

To simplify calculations it is convenient to use a representation of $\overset{*}{\vartheta}_\alpha 1$ that will enable us to express the solution of (XII.52) through a finite number of tabulated functions:

$$\dot{\vartheta}_\alpha(-\beta) = \dot{J}_\alpha \beta_1 \beta^{-1}\left[1-\beta_1 \dot{\vartheta}_0(-\beta_1)\right] \sum_{k=0}^{m-1}\sum_{l=1}^{n} B_k a_{kl}\, \dot{J}_0^{\frac{l-1}{n}}, \qquad \text{(XII.52)}$$

where

$$\beta_1 = -(-\beta)^{n/m}, \quad B_k = \prod_{\substack{j=0 \\ j\neq k}}^{m-1}(r_{k0}-r_{j0})^{-1},$$

$$\sum_{l=1}^{n} a_{kl}\dot{J}_0^{\frac{l-1}{n}} = \prod_{l=1}^{n-1}(\dot{J}_0^{1/n}-r_{kl}) \quad (k=0,1,\ldots,m-1),$$

$$r_{kl} = |\beta^{-\frac{1}{m}}|\exp\left[\left(\frac{n\varphi_{k0}+2\pi l}{n}\right)i\right], \quad \varphi_{k0} = [\pi(2k-1)-\varphi]\, m^{-1},$$

$$\varphi = \arg\beta, \quad J_\alpha(t-\tau) = (t-\tau)^\alpha/\Gamma(1+\alpha), \quad i = \sqrt{-1}.$$

Here

$$\dot{J}_\alpha \dot{J}_0^{\frac{l-1}{n}}\left[1-\beta_1\dot{\vartheta}_0(-\beta_1)\right]1 = \frac{\exp(-\beta_1 t)}{\Gamma(s+1)}\int_0^t z^s e^{\beta_1 z}dz,$$

$$s = (m+l-n-1)\, n^{-1} \quad (l=1,2,\ldots,n). \qquad \text{(XII.53)}$$

The limit (when $t \to \infty$) value of solution (XII.51) can be found[1] for

$$\frac{a_i}{r_1} - \frac{a_{i+1}}{r_2} < 1 \quad (i=1,3) \qquad \text{(XII.54)}$$

[1]See I. I. Krush [1].

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by substituting in (XII.41) $\overline{\kappa}$, $\overline{\kappa}_1$, $\overline{b}$, $\overline{b}_1$ by their limit values for $t \to \infty$. If conditions (XII.54) are not satisfied, the solution (XII.51) will be unstable for $t \to \infty$.

By way example we will consider the effect of the time factor on stresses $\sigma_r(t)$ on the contour of the seam, i.e., when $r = R$, in an aluminum plate with a soldered copper collar. The rheologic parameters are presented in Table XII.1. The ratio of the displacement coefficient for copper G_{01} to the displacement ratio for aluminum G_0 is assumed to be 1.62. For $\alpha = -0.53$ we find from (XII.52) /834

$$\Im_\alpha(-\beta)\,1 = \frac{1}{\beta}\left\{1 - \exp(\beta^2 t)\,[1 - \Phi(\beta\sqrt{t})]\right\}, \qquad \text{(XII.55)}$$

where $\Phi(\beta\sqrt{t})$ is the probability integral.

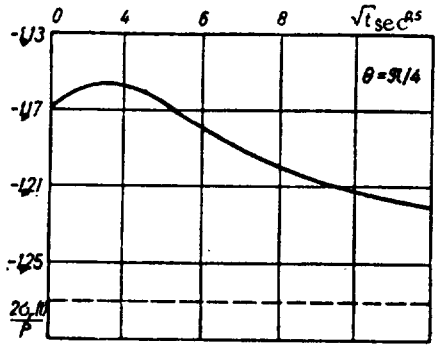


Figure XII.1.

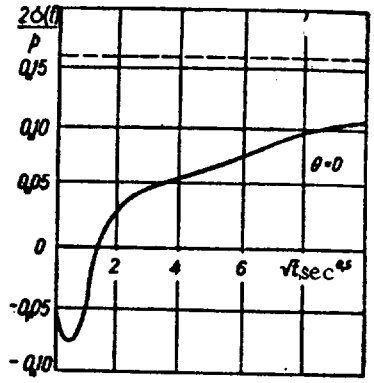


Figure XII.2.

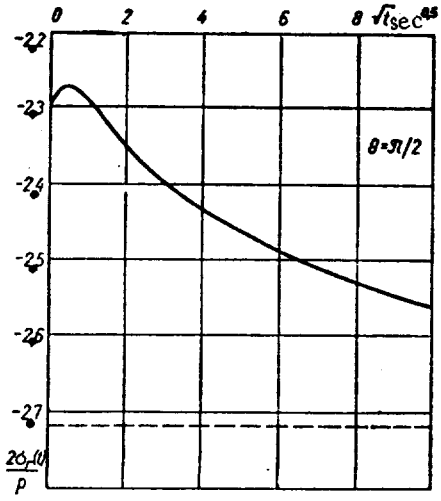


Figure XII.3.

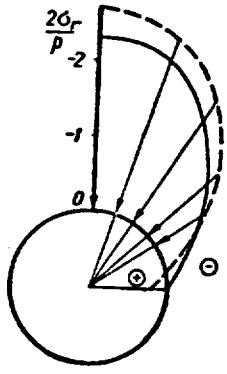


Figure XII.4.

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Figures XII.1-XII.4 show the results of calculations of $2\sigma_r(t)/p$ for $\theta = \pi/4$ (Figure XII.1); $\theta = 0$ (Figure XII.2); $\theta = \pi/2$ (Figure XII.3) and the values $2\sigma_z(0)/p$ on the contour of the hole, i.e., as functions of angle θ (Figure XII.4). The value $2\sigma_r(t)/p$ (dimensionless) is arranged in Figures XII.1-XII.3 on the ordinate axis, while the time, measured in seconds, is arranged in steps of 0.5 along the abscissa. The broken lines in Figures XII.1-XII.4 represent the limit values of $2\sigma_r(\infty)/p$. /835

The data presented in these figures show the considerable effect of the creep of the material both on the character and on the magnitude (of maximum values) of the stress state near a hole[1].

§3. Stress Distribution Near Holes in Anisotropic and Heterogeneous Materials

Stress Distribution in Structure of Heterogeneous Materials. Synthetic materials based on glass fiber and polymers, and also on other compositions, possess a wide variety of physico-mechanical properties and are being widely used as construction materials in all branches of technology.

Of particular importance are the oriented glass plastics, the structure of which consists of rectified fibers. On the basis of this "fibrous" structure it is quite possible to design strong structures with optimum (in a certain sense) mechanical properties. Among this type of construction materials are those obtained on the basis of glass fiber or glass thread (the so-called "fibrous" materials), glass tape, -- "laminar" materials.

The physico-mechanical properties of glass fibers, including the elasticity modulus, resistance to rupture, thermal expansion, etc., differ by one to two orders from those of polymer bonding agents, and therefore glass plastics represent a heterogeneous system, the properties of which depend considerably on the orientation of the glass filler (glass fiber or glass thread), the form of polymer bonding agent and the manufacturing technology.

Studies show[2] that in heterogeneous materials such as glass plastics, two types of stresses can be detected: stresses in the structure of the material, arising due to the interaction of the rigid glass filler and the comparatively soft polymer bonding agent, and medium stresses, the distribution of which is attributed to the geometry of the particular product. Therefore, in glass plastics and other heterogeneous materials the concentration of medium stresses that occur due to the presence of a hole, grooves, and hollow chamfers in a part, is accumulated into perturbations in the stress state within the material itself.

---

[1]A large number of examples applicable to the mining industry concerning the redistribution of the stress state near both reinforced and free holes, caused by the creep of rocks, is discussed in the monograph of Zh. S. Yerzhanov [1], and also in the articles of Sh. M. Aytaliyev and Zh. S. Yerzhanov [1]; V. S. Kuksina [1].

[2]See G. A. Van Fo Fy [1]; G. A. Van Fo Fy, G. N. Savin [1, 2].

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Experimental and theoretical studies[1] of the internal field of stresses in a linearly oriented glass plastic indicate the existence of a stress concentration near the fibers. The concentration coefficient of the structural stresses in glass plastics depends on the properties and volumetric content of the filler and bonding agent, mutual arrangement of the fibers and the manufacturing technology. /836

Studies of the internal stress field in linearly oriented glass plastics with a hexagonal structure[2] indicate that the local stresses on the contour of the fibers exceeds the medium stresses.

We will consider briefly[3] the problem of the distribution of macroscopic characteristics and stress concentration near fibers in the structure of linearly oriented glass plastic, the base of which consists of spun rectified identical fibers, arranged in nodes of right triangular net, the space between which is filled with a homogeneous polymer bonding agent.

At room temperature the glass fiber is elastic, and its properties obey Hooke's law all the way to rupture.

Thermal reactive polymers, used in the manufacture of glass plastics, are viscoelastic materials, the growth of deformations within which occurs with lag. The properties of such polymers can be described satisfactorily by the theory of elasto-hereditary media, discussed briefly in §1.

In the case at hand, the fibers form a double periodic structure, and therefore the solution of the problem is found with the aid of the theory of elliptical functions of the complex variable $x_2 + ix_3$, where the Ox_1 axis coincides in direction with the orientation of the fibers.

If an element of the volume is located far from the edge, the stress and deformation states are analyzed on the basis of expansions of the Kolosov-Muskhelishvili complex potentials:

$$\Phi_s(z,t) = \Phi_0(z) + c_0(t) + \sum_{k=0}^{\infty} c_{2k+2}(t)\frac{\lambda^{2k+2}\zeta_1^{(2k+1)}(z)}{(2k+1)!};$$

$$\Psi_s(z,t) = \Psi_0(z) + d_0(t) + \sum_{k=0}^{\infty} d_{2k+2}(t)\frac{\lambda^{2k+2}\zeta_1^{(2k+1)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} c_{2k+2}(t)\frac{\lambda^{2k+2}\zeta_2^{(2k+2)}(z)}{(2k+1)!}; \quad \text{(XII.56)}$$

[1]See G. A. Van Fo Fy and G. N. Savin [2]; G. A. Van Fo Fy [4].
[2]With the hexagonal structure, the fibers are arranged in the glass plastic in nodes of the proper net.
[3]For a more detailed description see G. A. Van Fo Fy [5].

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$$\Phi_a(z,t) = \sum_{n=0}^{\infty} a_{2n}(t) z^{2n};$$

$$\Psi_a(z,t) = \sum_{n=0}^{\infty} b_{2n}(t) z^{2n},$$

where the subscripts s and a denote functions pertaining to the regions occupied by the bonding agent and by the glass filler, respectively;

$$\zeta_1(z) = \frac{1}{z} + \sum_{m,n}{}' [(z-p)^{-1} + p^{-1} + zp^{-2}];$$

$$\zeta_2(z) = \sum_{m,n}{}' \bar{p} [(z-p)^{-1} + z^2 p^{-3} + zp^{-2}]; \qquad \text{(XII.57)}$$

$p = m\omega_1 + n\omega_2$, $\omega_2 = \omega_1 e^{i\alpha}$; $\alpha = \pi/3$; ω_1, ω_2 - are the periods of the net.

The unknown parameters $a_{2n}(t)$, $b_{2n}(t)$, $c_{2n}(t)$ and $d_{2n}(t)$ are found from the conditions of equality of the stresses and displacements on the boundary $r = r_0$ of contact of the fibers with the bonding agent, or resin ($z = \tau = r_0 e^{i\theta}$).

For regular structures, due to the double periodicity of the problem, it is sufficient to satisfy the boundary conditions

$$\Phi_a(\tau,t) + \overline{\Phi_a(\tau,t)} - e^{2i\theta}[\bar{\tau}\Phi_a'(\tau,t) + \Psi_a(\tau,t)] =$$
$$= \Phi_s(\tau,t) + \overline{\Phi_s(\tau,t)} - e^{2i\theta}[\bar{\tau}\Phi_s'(\tau,t) + \Psi_s(\tau,t)];$$
$$\left(1 - \frac{G}{G_s}\right)\Phi_a(\tau,t) + \left(1 + \varkappa_a \frac{\bar{G}}{G_a}\right)\overline{\Phi_a(\tau,t)} - \qquad \text{(XII.58)}$$
$$- \left(1 - \frac{G}{G_a}\right) e^{2i\theta}[\bar{\tau}\Phi_a'(\tau,t) + \Psi_a(\tau,t)] = (\bar{\varkappa} + 1)\overline{\Phi_s(\tau,t)}$$

on one arbitrary contour (of transverse cross section) of the fiber.

The conditions on the contour of an elementary cell are reduced to the assignment of mean stresses $\langle\sigma_{ik}\rangle$ or deformations $\langle\varepsilon_{ik}\rangle$, where

$$\langle\sigma_{ik}\rangle = \frac{1}{F}\int_F \sigma_{ik}\, dF; \quad \langle\varepsilon_{ik}\rangle = \frac{1}{F}\int_F \varepsilon_{ik}\, dF; \quad F = \omega_1\omega_2 \sin\alpha. \qquad \text{(XII.59)}$$

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Through $\xi = \pi r_0^2/F$ and $\eta = 1 - \xi$ we will denote the relative volumetric content of the glass and bonding agent in the glass plastic.

The distribution of tangential stresses in the structure of a composition material for the hexagonal (densest) packing of the fibers in glass plastics during displacement in the (x_1, x_2) and (x_1, x_3) planes (Figure XII.5) is illustrated by the curves presented in Figure XII.6.

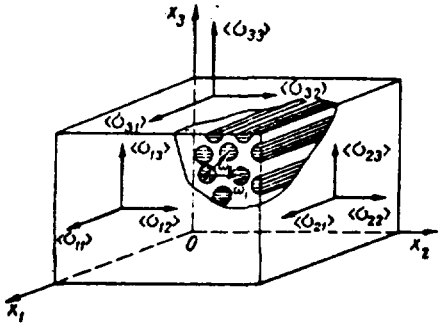


Figure XII.5.

For comparison, Figure XII.6 also shows broken curves that characterize the distribution of stresses for a small volumetric quantity of glass fiber ($\xi = 0.23$). Here, on the ordinate axis, are arranged the values of the concentration coefficients of tangential stresses $\tau_{r\theta}$ (through cross section $\theta = 0$) in the structure of the material, and on the abscissa is the relative distance between the fibers. The stresses are calculated on planes perpendicular to the Ox_2 axis. /838

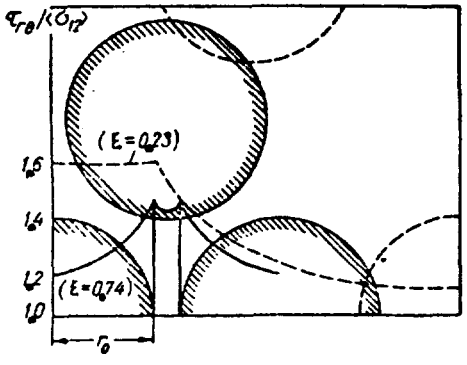


Figure XII.6.

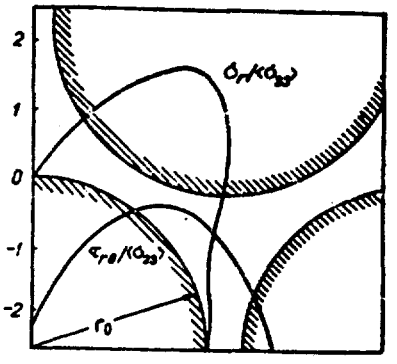


Figure XII.7.

These curves show that as the volumetric content ξ of the glass filler increases, the stress distribution in the structure of the material becomes more uniform. The maximum stress concentration coefficients are achieved at the boundary between the fibers and bonding agents.

The distribution of normal σ_r and tangential $\tau_{r\theta}$ stresses on the contour of a fiber ($\xi = 0.74$) during displacement of the glass plastic in the (x_2, x_3) plane is represented in Figure XII.7. The values of the stress concentration

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coefficient on the surfaces perpendicular to the radius are measured, as usual, from the contour line, i.e., from the circle $r = r_0$.

During the stretching of glass plastics in the direction of orientation, i.e., along the glass fibers, the distribution of normal σ_r and tangential $\tau_{r\theta}$ stresses ($\xi = 0.74$) on the contour of the fibers on the radial planes is illustrated by the curves in Figure XII.8.

The distribution of tangential $\tau_{r\theta}$ and normal σ_θ stresses on the contour of a fiber, both on the radial and on the tangential planes, during the stretching of reinforced plastics by stresses $\langle\sigma_{22}\rangle$ or $\langle\sigma_{33}\rangle$, is represented by the curves in Figure XII.9.

As we see, the concentration of stresses σ_θ on the boundary between the /839
fibers and bonding agent is slight if the adhesion of the fibers to the resin is good.

Because of the viscoelastic properties of the polymer bonding agent, the redistribution of stresses in time occurs in the composition of the material between the filler and the bonding agent.

Studies show[1] that as the bonding agent undergoes creep the redistribution of stresses in the structure of a composition material does not exceed 10% of the instantaneous value.

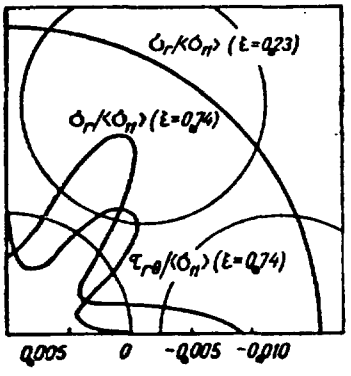


Figure XII.8.

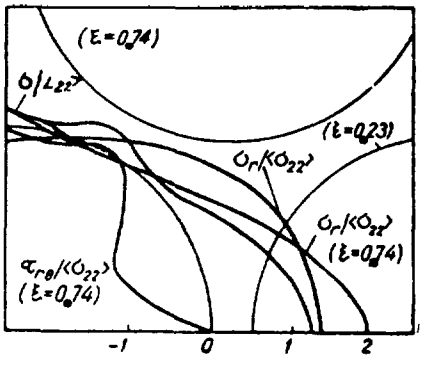


Figure XII.9.

During the constant deformation of glass plastic, no relaxation of the stresses occurs on the boundary of the fibers. It turns out here that the relaxation of the structural stresses is proportional to the relaxation of the mean stresses.

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[1]See G. A. Van Fo Fy [4].

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The true stresses σ_x, σ_y, ..., σ_{xy} in the case of hexagonal structure (Figure XII.5) are related to the mean stresses $\langle\sigma_{ik}\rangle$ (i, k = 1, 2, 3) by the relations

$$\sigma_x = k_{c1}^{(x)}\langle\sigma_{11}\rangle + k_{c2}^{(x)}\langle\sigma_{22}\rangle + k_{c3}^{(x)}\langle\sigma_{33}\rangle + k_{c4}^{(x)}\langle\sigma_{23}\rangle + \\ + k_{c5}^{(x)}\langle\sigma_{31}\rangle + k_{c6}^{(x)}\langle\sigma_{12}\rangle \qquad \text{(XII.60)}$$

(and analogously for σ_y, ..., τ_{xy}), where $k_{ci}^{(x)}$, $k_{ci}^{(y)}$, ..., $k_{ci}^{(xy)}$ (i = 1, 2, 3, ..., 6) are the structural stress concentration coefficients.

Basic Relations of Theory of Elastic Heredity for Heterogeneous Materials. The base of heterogeneous fibrous materials of the oriented glass plastic type is made up of a linearly reinforced layer consisting of rectified fibers.

If the axis (Figure XII.5) is directed along the orientation of the fibers, then the relationship between mean stresses $\langle\sigma_{ik}\rangle$ and mean deformations $\langle\varepsilon_{ik}\rangle$ for a linearly reinforced layer, the fibers in which form a hexagonal structure, has the form /840

$$\langle\varepsilon_{11}\rangle = \frac{1}{E_{11}}\langle\sigma_{11}\rangle - \frac{\nu_{21}}{E_{11}}[\langle\sigma_{22}\rangle + \langle\sigma_{33}\rangle];$$

$$\langle\varepsilon_{22}\rangle = \frac{1}{\bar{E}_{22}}\langle\sigma_{22}\rangle - \frac{\nu_{21}}{E_{11}}\langle\sigma_{11}\rangle - \frac{\bar{\nu}_{23}}{\bar{E}_{22}}\langle\sigma_{33}\rangle$$

$$\langle\varepsilon_{33}\rangle = \frac{1}{E_{22}}\langle\sigma_{33}\rangle - \frac{\nu_{21}}{E_{11}}\langle\sigma_{11}\rangle - \frac{\bar{\nu}_{23}}{\bar{E}_{22}}\langle\sigma_{22}\rangle; \qquad \text{(XII.61)}$$

$$\langle\varepsilon_{12}\rangle = \frac{1}{\bar{G}_{12}}\langle\sigma_{12}\rangle;\ \langle\varepsilon_{13}\rangle = \frac{1}{\bar{G}_{12}}\langle\sigma_{13}\rangle;\ \langle\varepsilon_{23}\rangle = \frac{2(1+\bar{\nu}_{23})}{E_{22}}\langle\sigma_{23}\rangle.$$

If the fibers form a more complex structure, then[1] the number of elastic constants increases and, in the general case, reaches 13.

The hexagonal distribution of the fibers is most likely, and therefore the relations given above will be used as the basis of the theory of oriented glass plastics.

In formulas (XII.61) the bar denotes the operator moduli; the other values in the case of viscoelastic deformations of the polymer bonding agent can be assumed as the constants of the material with an accuracy up to 5%[2].

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[1]See G. A. Van Fo Fy [2].
[2]See G. A. Van Fo Fy [3].

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The operator and ordinary moduli in relations (XII.61) can be expressed through the physico-mechanical properties of the filler and bonding agent. The approximate (with an accuracy up to 7-10%) formulas for the operators and constants of a composition material are

$$\frac{1}{G_{12}} = \frac{1}{G_{012}}\left[1 + \frac{a}{\left[\eta + (1+\xi)\frac{G_0}{G_a}\right]\eta}\,\dot{\vartheta}_a(-\omega_\infty)\right];$$

$$\frac{1}{\bar{E}_{22}} = \frac{1}{E_{22}^0}\Bigg[1 + \left(\frac{a\omega_0}{\omega_0 - b_1}\cdot\frac{1}{1+c} + \frac{a\omega_0}{\omega_0 - b_2}\cdot\frac{c}{1+c}\right)\dot{\vartheta}_a(-\omega_\infty) +$$

$$+ \frac{\omega_0 - a - b_1}{\omega_0 - b_1}\cdot\frac{b_1}{1+c}\,\dot{\vartheta}_a(-\omega + b_1) + \frac{\omega_0 - a - b_2}{\omega_0 - b_2}\cdot\frac{b_2 c}{1+c}\,\dot{\vartheta}_a(-\omega + b_2)\Bigg]; \tag{XII.62}$$

$$\frac{\bar{\nu}_{23}}{\bar{E}_{22}} \approx -\frac{1}{\bar{E}_{22}} + \frac{1}{2\bar{G}} - \frac{\xi(1-\nu_0)\,b_2\left(1 - \frac{G_0}{G_a}\right)}{(1-2\nu_0)\,\omega_0 G_0}\left[1 + \frac{a\omega_0}{\omega_0 + b_2}\,\dot{\vartheta}_a(-\omega_\infty) - \right.$$

$$\left. - b_2\frac{\omega_0 - a + b_2}{\omega_0 + b_2}\,\dot{\vartheta}_a(-\omega - b_2)\right] + \ldots,$$

where /841

$$\varkappa = 3 - 4\nu;$$

$$a = \frac{3\omega_0}{2 + 2\nu_0}; \quad b_1 = \frac{2(1-2\nu_0)\,\omega_0\,\xi}{1 + \eta + \xi\varkappa_0 + \eta(\varkappa_a - 1)\frac{G_0}{G_a}};$$

$$b_2 = \frac{2(1-2\nu_0)\,\omega_0}{\xi + \varkappa_0 + \eta_0\frac{G_0}{G_a}}; \quad c = -\frac{2\xi^2 b_2\left(1 - \frac{G_0}{G_a}\right)}{b_1\left[2 + (\varkappa_a - 1)\frac{G_0}{G_a}\right]}. \tag{XII.63}$$

Here the subscripts 0 and a denote the instantaneous elastic constants of the bonding agent and glass fibers, respectively; ω_0, ω_∞ are the parameters that characterize the rheologic properties of the polymer bonding agent.

In the case of the unsteady process of creep $\omega_\infty \to 0$. For polymers based on a maleic-epoxy composition, we may assume for t = 25°C that $\omega_0 = 0.052\ h^{1+\alpha}$; $\alpha = 0.5$; $\omega_\infty = 0.12\ h^{1+\alpha}$; $\nu_0 = 0.382$; $E_0 = 0.981 \times 0.351 \cdot 10^{10}\ n/m^2$. For alumi-numborosilicate glass fibers at the same temperature, $\nu_a = 0.2$; $E_a = 0.981 \cdot 7.0 \cdot 10^{10}\ n/m^2$.

The formula for the operator $1/\bar{E}_{22}$ can be represented conveniently in the form

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$$\frac{1}{\bar{E}_{22}} = \frac{1}{E_{22}^0}\left[1 - \sum_{k=1}^{3} A_k \dot{\vartheta}_a(-\varkappa_k)\right]. \qquad \text{(XII.64)}$$

The instantaneous elastic constants of a composition material are

$$\frac{1}{E_{22}^0} = \frac{\nu_{21}^2}{E_{11}} - \frac{1-\nu_0^2}{E_0}\left[\frac{2+(\varkappa_a-1)\frac{G_0}{G_a}}{1+\eta+\xi\varkappa_0+\eta(\varkappa_a-1)\frac{G_0}{G_a}} - 2\xi\frac{1-\frac{G_0}{G_a}}{\xi+\varkappa_0+\eta\frac{G_0}{G_a}}\right];$$

$$\nu_{12} = \nu_{21}\frac{E_{22}^0}{E_{11}};$$

$$\frac{1}{G_{12}} = \frac{\eta+(1+\xi)\frac{G_0}{G_a}}{1+\xi+\eta\frac{G_0}{G_a}}\cdot\frac{1}{G_0} + \ldots; \qquad \text{(XII.65)}$$

$$E_{11} = \xi E_a + \eta E_0 + \ldots;$$

$$\nu_{21} \approx \nu_0 - \frac{(\nu_0-\nu_a)(\varkappa_0+1)}{2+(\varkappa_0-1)\xi+\eta(\varkappa_a-1)\frac{G_0}{G_a}} + \ldots;$$

$$\nu_{32} \approx 1 - 2\nu_{21}\nu_{12} - \frac{1-\nu_0^2}{E_0}E_{22}\left[\frac{1-2\nu_0}{1-\nu_0} - 2\xi\frac{\varkappa_0-1-(\varkappa_a-1)\frac{G_0}{G_a}}{1+\eta+\xi\varkappa_0+\eta(\varkappa_a-1)\frac{G_0}{G_a}}\right].$$

In order to analyze the relaxation processes of stresses, we must have the operators /842

$$\bar{G}_{12} = G_{12}^0\left[1 - \frac{a}{\left[\eta+(1+\xi)\frac{G_0}{G_a}\right]\eta}\dot{\vartheta}_a\left(-\omega_\infty - \frac{a}{\left[\eta+(1+\xi)\frac{G_0}{G_a}\right]\eta}\right)\right],$$

$$\bar{E}_{22} = E_{22}^0\left[1 + \sum_{k=1}^{3} f_k \dot{\vartheta}_a(p_k)\right], \qquad \text{(XII.66)}$$

where p_k (k = 1, 2, 3) are the roots of the equation

$$1 + \sum_{k=1}^{3}\frac{A_k}{x_k - p} = 0, \qquad \text{(XII.67)}$$

and f_k is found from the solution of the system of three equations

$$1 + \sum_{k=1}^{3}\frac{f_k}{x_l - p_k} = 0 \quad (l = 1,2,3). \qquad \text{(XII.68)}$$

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Stress Concentration Near Round and Elliptical Holes in Plate Made of Fibrous Materials. We will examine the concentration of mean stresses $\langle\sigma_{ik}\rangle$ near round and elliptical holes in a thin linearly reinforced plate made of glass plastic based on a maleic-epoxy bonding agent under simple tension by stresses $\langle\sigma_{\alpha}\rangle$.

Let the fibers be oriented along the Ox_1 axis. We will use the Cartesian coordinate system (x_1, x_2), located in the middle surface of the plate, with the origin of the coordinates located at the center of the hole.

From the condition of equilibrium, disregarding the stresses $\langle\sigma_{33}\rangle$, $\langle\sigma_{23}\rangle$ and $\langle\sigma_{13}\rangle$,

$$\frac{\partial\langle\sigma_{11}\rangle}{\partial x_1}+\frac{\partial\langle\sigma_{12}\rangle}{\partial x_2}=0,$$
$$\frac{\partial\langle\sigma_{12}\rangle}{\partial x_1}+\frac{\partial\langle\sigma_{22}\rangle}{\partial x_2}=0 \qquad \text{(XII.69)}$$

and from the equations of continuity it follows that the stress function U should satisfy the biharmonic equation with operator coefficients:

$$\frac{1}{E_{22}}\cdot\frac{\partial^4 U}{\partial x_1^4}+\frac{1-2\nu_{21}}{E_{11}}\cdot\frac{\partial^4 U}{\partial x_1^2\partial x_2^2}+\frac{1}{E_{11}}\cdot\frac{\partial^4 U}{\partial x_2^4}=0. \qquad \text{(XII.70)}$$

The function U should be constructed such that the contour of the hole will be free of external forces, and the stresses "at infinity," defined by this function, will approach the given stress state[1]. The solution of this problem is given in 1, Chapter III with the aid of functions of complex variables $z_1 = x_1 + s_1x_2$ and $z_2 = x_1 + s_2x_2$, where /843

$$s_1s_2=-\sqrt{\frac{E_{11}}{E_{22}}};\quad s_1+s_2=i\sqrt{2\left(\sqrt{\frac{E_{11}}{E_{22}}}-\nu_{21}\right)+\frac{E_{11}}{G_{12}}}. \qquad \text{(XII.71)}$$

Using V. Volterra's principle, we can determine[2] the redistribution of stresses in time near an elliptical (and in particular, round) hole.

Let α be the angle of slope of the direction of the stresses of tensions $\langle\sigma_{\alpha}\rangle$ to the Ox_1 axis; the mean stresses in the plate are given by the formulas[3]

---

[1]See Chapter II, §1.
[2]See G. A. Van Fo Fy [1]; G. A. Van Fo Fy, G. N. Savin [2].
[3]See formula (III.23).

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$$\begin{aligned}
\langle\sigma_{11}\rangle &= \langle\sigma_\alpha\rangle \cos^2\alpha + 2\langle\sigma_\alpha\rangle \operatorname{Re}[s_1^2\varphi_0'(z_1) + s_2^2\psi_0'(z_2)];\\
\langle\sigma_{12}\rangle &= \langle\sigma_\alpha\rangle \sin\alpha\cos\alpha - 2\langle\sigma_\alpha\rangle \operatorname{Re}[s_1\varphi_0'(z_1) + s_2\psi_0'(z_2)];\\
\langle\sigma_{22}\rangle &= \langle\sigma_\alpha\rangle \sin^2\alpha + 2\langle\sigma_\alpha\rangle \operatorname{Re}[\varphi_0'(z_1) + \psi_0'(z_2)],
\end{aligned} \tag{XII.72}$$

where the functions are

$$\varphi_0(z_1) = -\frac{i(a - is_1b)}{4(s_1 - s_2)}\left[\frac{b(s_2\sin 2\alpha + 2\cos^2\alpha)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}} + \frac{ia(2s_2\sin^2\alpha + \sin 2\alpha)}{z_1 + \sqrt{z_1^2 - (a^2 + s_1^2b^2)}}\right];$$

$$\psi_0(z_2) = \frac{i(a - is_2b)}{4(s_1 - s_2)}\left[\frac{b(s_1\sin 2\alpha + 2\cos^2\alpha)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}} + \frac{ia(2s_1\sin^2\alpha + \sin 2\alpha)}{z_2 + \sqrt{z_2^2 - (a^2 + s_2^2b^2)}}\right]; \tag{XII.73}$$

here a and b are the large and small semiaxes of the ellipse, respectively.

Of greatest importance is the distribution of normal stresses $\langle\sigma_\vartheta\rangle$ acting on the planes perpendicular to the tangent to the contour of the hole, and the distribution of tangential stresses $\langle\sigma_{12}\rangle$ between the fibers.

On the contour of the hole

$$\langle\sigma_\vartheta\rangle = \langle\sigma_\alpha\rangle\left\{\frac{\sin^2\vartheta}{\sin^2\vartheta + k^2\cos^2\vartheta} + \right.$$
$$\left. + \frac{k}{\sin^2\vartheta + k^2\cos^2\vartheta}\operatorname{Re}\left[\frac{e^{i\vartheta}}{s_1 - s_2}\left(\frac{(s_1\sin\vartheta + k\cos\vartheta)^2}{\sin\vartheta - ks_1\cos\vartheta} - \frac{(s_2\sin\vartheta + k\cos\vartheta)^2}{\sin\vartheta - ks_2\cos\vartheta}\right)\right]\right\}, \tag{XII.74}$$

where $k = b/a$, $x_2 = a\cos\vartheta$; $x_3 = b\sin\vartheta$.

From (XII.74) we have:

for $\vartheta = 0$

$$\langle\sigma_\vartheta\rangle = -\langle\sigma_\alpha\rangle\sqrt{\frac{E_{22}}{E_{11}}}; \tag{XII.75}$$

for $\vartheta = \pi/2$

$$\langle\sigma_\vartheta\rangle = \langle\sigma_\alpha\rangle\left[1 + k\sqrt{2\left(\sqrt{\frac{E_{11}}{E_{22}}} - \nu_{21}\right) + \frac{E_{11}}{G_{12}}}\right]. \tag{XII.76}$$

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Figure XII.10 illustrates stresses[1] $\langle\sigma_\vartheta\rangle$ on the contour of a round hole. /844 These stresses were calculated by formula (XII.74) for a = b and $\alpha = 0$, i.e., for tension along the glass fibers[2] by stresses $\langle\sigma_\alpha\rangle$. Curves 1 and 2 characterize the distribution of these normal stresses for elastic isotropic and anisotropic materials, respectively. The redistribution of mean stresses $\langle\sigma_\vartheta\rangle$ (XII.74) during the prolonged ($t \to \infty$) effect of external stresses $\langle\sigma_\alpha\rangle$ during the stable process of creep is represented by curve 3. This curve shows clearly that stresses $\langle\sigma_\vartheta\rangle$ (XII.74) increase with time on the planes reinforced by the glass fiber; the mean stresses on the planes parallel to the orientation of the fibers, due to the development of viscoelastic deformations in the polymer bonding agent, decrease.

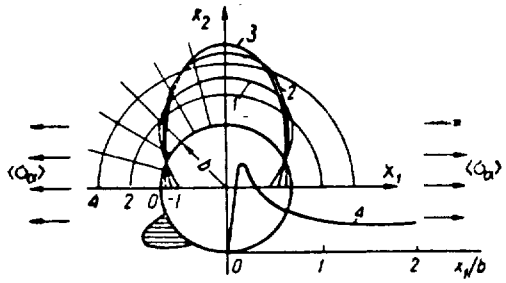


Figure XII.10.

In order to study the stress state in a plate near a hole at an arbitrary moment of time, it is convenient to use the formula

$$\sqrt{1 + a\dot{\vartheta}_\alpha(-\omega)} = 1 + \sum_{k=1}^{\infty} \frac{(2k-3)!}{(2k)!} \cdot \frac{\omega^k}{(k-1)!} \cdot \frac{\partial^{k-1}}{\partial\omega^{k-1}} \dot{\vartheta}_\alpha(-\omega). \qquad \text{(XII.77)}$$

The distribution of tangential stresses $\langle\sigma_{12}\rangle$ on plane ($x_2 = b$, x_1) is represented by curve 4 (Figure XII.10).

The true tangential stresses on the planes parallel to the fibers ($\xi = 0.74$) are

$$\sigma_{12} = k_c k_0 \langle\sigma_{12}\rangle = 0{,}81 \langle\sigma_\alpha\rangle. \qquad \text{(XII.78)}$$

The resistance of the polymer bonding agent to cutting is less than its resistance to stretching, and therefore the bearing capacity of the plate after the tangential stresses $\langle\sigma_{12}\rangle$ have reached their limit values is determined by the resistance of the unweakened part of the glass plastics to rupture. The

---

[1]More accurately, concentration coefficients k_0 of these stresses.

[2]The direction of the glass fibers is shown in Figure XII.10 (third quadrant on the left) and in Figures XII.12-XII.15 (fourth quadrant on the right).

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shaded area in Figure XII.11 is eliminated from work due to the cutting of the bonding agent along the line aa', while the remainder of the material is in a uniform stress state.

Change in the concentration coefficient k_0 of stresses $\langle\sigma_\vartheta\rangle$ (XII.74) on the contour of an elliptical hole, the large axis of which is parallel to the orientation of the fibers[1] (for a/b = 2) is represented by curve 1 in Figure XII.12. Curve 2 characterizes the redistribution of stresses $\langle\sigma_\vartheta\rangle$ on the contour of the hole during the prolonged loading of the plate. /845

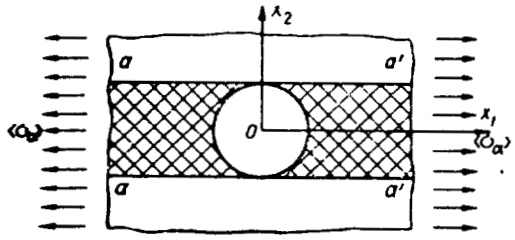


Figure XII.11.

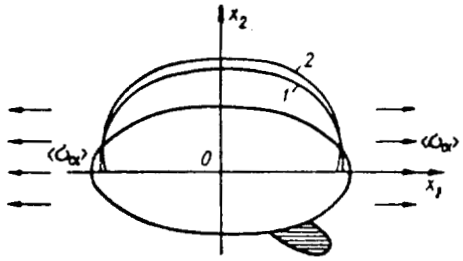


Figure XII.12.

The change in the concentration coefficient of stresses $\langle\sigma_\vartheta\rangle$ in the very same plate, but for the case where the large axis of the elliptical hole is perpendicular to the direction of tension, directed along the glass fibers, is represented in Figure XII.13. Curve 1 corresponds to the initial stress state in the plate, and curve 2, to the stress state after prolonged loading (about 500 hours), i.e., after the relaxation processes have practically concluded.

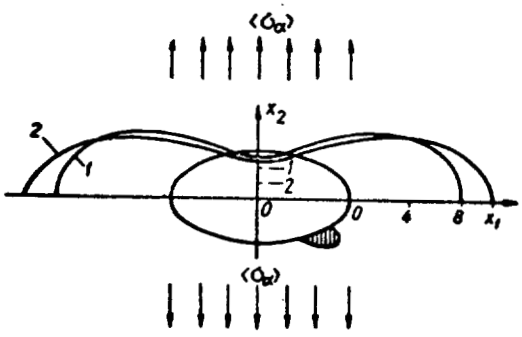


Figure XII.13.

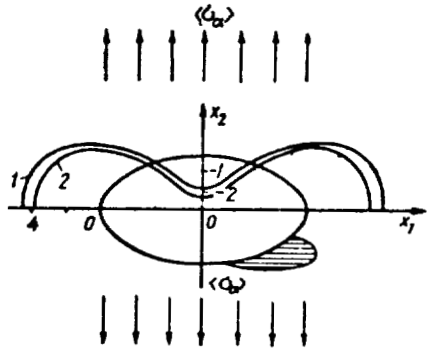


Figure XII.14.

[1]See the shading of part of the plate near the contour in the fourth quadrant.

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The change in time of the concentration coefficient of stresses $\langle\sigma_\vartheta\rangle$ in a plate reinforced along the large axis of an ellipse (when a/b = 2), for tension perpendicular to the orientation of the fibers, after prolonged loading, is shown by curve 2 in Figure XII.14. Curve 1 corresponds to the instantaneous stress state at the initial moment of time. Due to the redistribution of stresses, the maximum concentration coefficient k_0 of the stresses decreases in time. /846

The maximum concentration coefficient of the true stresses ($\xi = 0.74$) is

$$k_{max} = k_c k_0 = 1{,}9 \cdot 4{,}6 = 8{,}74. \qquad \text{(XII.79)}$$

If the plate consists of intersecting reinforced layers of glass tape[1] of equal thickness, arranged symmetrically with respect to its middle surface, then the mechanical properties are readily determined for the entire body as a whole.

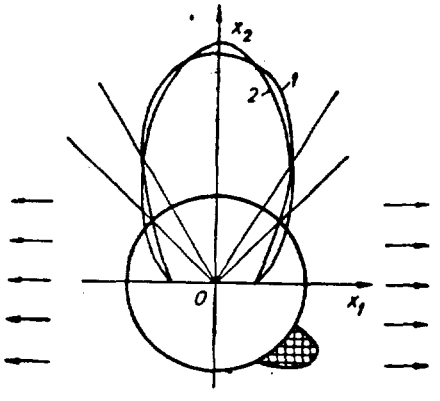


Figure XII.15.

From the conditions of deformation compatability of the layers we have

$$\overline{G}_{12}^{\bullet} = G_{12}; \qquad \overline{\nu}_{21} = \nu_{21} \frac{2\frac{\overline{E}_{22}}{E_{11}}}{1+\frac{\overline{E}_{22}}{E_{11}}};$$

$$\overline{E}_{11} = \overline{E}_{22} = \frac{E_{11}}{2} \frac{\left(1+\frac{\overline{E}_{22}}{E_{11}}\right)^2 - 4\nu_{21}^2\left(\frac{\overline{E}_{22}}{E_{12}}\right)^2}{\left(1+\frac{\overline{E}_{22}}{E_{11}}\right)\left(1-\nu_{21}^2\frac{\overline{E}_{22}}{E_{11}}\right)}. \qquad \text{(XII.80)}$$

The operator moduli of displacement $\overline{G}_{12}$ and $\overline{E}_{22}$, $\overline{E}_{11}$, ν_{21} are calculated on the basis of the formulas given above.

Let a laminar plate of uniform strength be weakened by a round hole of radius b and subjected to tension along one of its principal directions of anisotropy.

The distribution of mean stresses $\langle\sigma_\vartheta\rangle$ on the contour of the hole is given by formula (XII.74), in which it is necessary to assume

[1]See the shaded area in the fourth quadrant of the part of the plate near the contour in Figure XII.15.

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$$s_1 s_2 = -1; \qquad s_1 + s_2 = i \sqrt{2(1 - \bar{\nu}_{21}) + \frac{\bar{E}_{11}}{\bar{G}_{12}}}. \qquad \text{(XII.81)}$$

As follows from (XII.75), stresses $\langle\sigma_\vartheta\rangle$ do not change in time when $\vartheta = 0$.

The distribution of stresses $\langle\sigma\rangle$ (XII.74) on the contour of the hole at the initial moment of time (curve 1) can be traced by the change of the stress concentration coefficient k_0, shown in Figure XII.15. Curve 2 shows the change in k_0 during prolonged loading.

It is interesting to note that during unstable creep of the polymer bonding agent, the stresses on the plane $\vartheta = 0$ do not change in time, while the stresses on the plane $\vartheta = \pi/2$ increase.

Therefore the curves shown in Figures XII.10-XII.15 show that the concentration of structural stresses near rigid inclusions in composition materials of the glass plastic type, because of the considerable difference in the mechanical characteristics of the component parts, is an additional source of perturbation of the stress state. /847

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APPENDIX

EXPERIMENTAL DATA ON STRESS DISTRIBUTION NEAR HOLES

§1. General Comments

In the preceding chapters, for the theoretical solution of the problem of /849
the effect of various holes on the stress state pattern in a heterogeneous field, we made an important assumption to simplify the problem of interest. We assumed that in parts of the plate or shell that are sufficiently distant from a hole the stress state remains the same as in a plate (beam or rod) or shell without a hole.

It is obvious that the theoretical solutions derived in these chapters will characterize more accurately the stress patterns in finite plates or shells weakened by some hole as the dimensions of the holes decrease in comparison with the dimensions of the plates and shells. However, the theoretical solution of the corresponding problems for doubly-connected regions involves considerable mathematical difficulties and yields, as a rule, to an approximate solution. Therefore the experimental method is preferable in many cases. Experimental studies are also required in those cases where it is impossible to evaluate the degree of approximation of the theoretical solution found. The vast majority of experimental studies concerning stress concentration near holes in plates and shells have been conducted by the photoelasticity[1] and photoelastic coatings[2] methods. The methods of tensometry[3] and lacquer coatings[4] have also been used to a lesser degree for this purpose.

In this appendix it will not be our purpose to attempt to describe all experimental investigations concerning stress concentration near holes in plates and shells, but we will describe some of those that make it possible to establish the range of applicability of the theoretical solutions obtained above to plates and shells of finite dimensions, as well as those works which make it possible to evaluate the degree of error of the approximate solutions derived. For many cases the isoclines, isostats corresponding to them, and pattern of interference bands[5] are presented. Isostats and interference bands afford a graphic representation of the dimensions of a region to which spreads the effect of a hole on the stress state in a given plate or beam and thus

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[1]See M. M. Frokht [1, 2]; L. Kiker, E. Faylon [1]; A. M. Prigorovskiy [1]; H. T. Jessop, C. Snell, J. M. Alisson [1].
[2]See A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1].
[3]See F. J. Mehringer, W. E. Cooper [1]; M. A. Loshkarev [1].
[4]See K. Fink, Kh. Rorbakh [1].
[5]An isocline is a curve that passes through points with principal stresses of identical directions. An isostat is a curve, the tangent to which has the direction of the principal stress at the given point. An interference band has identical maximum tangential stresses at the median points.

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makes it possible to judge the range of applicability of the theoretical solutions obtained in the preceding chapters to finite regions.

The numerical values of stresses obtained theoretically yield to convenient comparison with those experimental results which were obtained under conditions analogous to the theoretical solution. Hence, it is not necessary to use as the basis of comparison the values of stresses at the regular points of holes and cutouts. Near the above-stated points the field of stresses changes very rapidly. At these points, slight changes in the curvature of the contour, which are almost impossible to perceive, are markedly reflected in the results. Therefore, for comparison of the theoretical and experimental data, stress strain diagrams of stresses through certain cross sections will be presented, as well as the values of stresses at the nonangular points of contours of holes.

§2. Stretching of Plate with Holes

/850

Circular Hole. The experimental results for a stretched beam with a central circular hole were presented in Figure II.67. We will also present here the experimental data found by the photoelasticity method on bakelite models (see M. M. Frokht [1, 2]).

The pattern of isoclines and isostats for a beam subjected to tension by uniform forces p along the Oy axis is represented in Figure A.1. The width of the plate is 3.3 times the diameter of the hole (the width of the plate is 26.4 mm, the diameter of the circular hole is 8 mm). The isocline with the parameter 5° (or 85°) passes through the points in which the principal stresses are rotated by 5° in relation to the principal directions of the uniform stress field. The greatest distance between this isocline and the center of the hole does not exceed 2.5 d (d is the diameter of the hole). Consequently, it may be assumed that at distances greater than 2.5 d from the center of the circular hole, the latter has practically no effect on the stress field. This is also supported by the pattern of isostats represented in the upper part of the figure. By comparing the isostats shown in the figure for a beam with a central circular hole, obtained experimentally, with the isostat pattern shown in Figure II.37, found theoretically for an infinite plate with a circular hole, we see that if the width of the stretched beam is

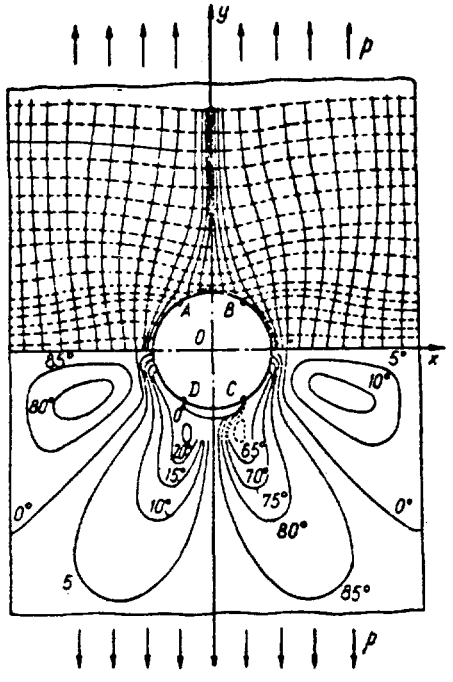


Figure A.1.

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greater than 5d (d is the diameter of the hole), then for the theoretical solution of the problem the beam can be regarded as an infinite plate.

In Figure A.2 we find the interference bands that give a representation of the distribution of maximum tangential stresses τ_{max} in the specified beam with a round hole. By comparing the pattern of interference bands with the lines of equal τ_{max} shown in Figure II.36, obtained theoretically for an unbounded plate under tension, weakened by a round hole, we see that they are quite similar. Thus, the ratio of $(\sigma_\theta)_{max}$ on the contour of the circular hole in the stretched beam to the stress $\sigma^{(\infty)} = p$ at points that are remote from the hole is equal to 3.33. In Table II.22, for $\lambda = 0.3$ and $\theta = 90°$, this ratio is equal to 3.36.

Elliptical Hole. Figure A.3 shows the isoclines and isostats for a plate under tension with an elliptical hole[1] (b/a = 2/3). The ratio of the width of the plate to the large axis of the ellipse is equal to 4.23. The ratio of the axes of the ellipse is a/b = 3/2. Tension is applied on the small axis. By comparing the lines of principal stresses (isostats) obtained experimentally with the lines of principal stresses represented in Figure II.34 we see, first, that they coincide satisfactorily, and second, that at a distance equal to 2.5 a from the origin of the coordinate system, the perturbation caused by the elliptical hole is practically indiscernible.

Square Hole. In Figure A.4, in the upper half of the figure, are the isoclines, and in the bottom half, the isostats[2] for a uniaxially stretched beam with a square hole. The sides of the square are equal to a. The rounding /852
radius of its angles is 1/6 a. The width of the beam is twice the length of the side of the square; the width of the beam is 0.473 cm. The load is 20 kg. The distribution of principal normal stresses through cross sections x = 0, y = 0, x = a (i.e., stress at the points of the vertical edge of the beam) and along the contour of the hole, where P and Q are the principal normal stresses, is represented in Figure A.5. The stresses on the middle sides, parallel to the longitudinal axis, is 1.4 times greater than the mean stress[3] through the weakened cross section. In the case of an unbounded plate, as follows from Table II.1, the stresses at these points ($\theta = 90°$) of a square hole are equal to 1.47.

Rectangular Hole. For a plate with a rectangular hole with sides a and b (a > b), compressed in two directions, the isostat pattern, for stresses at points distant from the hole (p and q = 1/4 p) is shown[4] in Figure A.6. The /854
ratio of the sides of the rectangle is a/b = 1/8, where the large axis of the rectangle forms an angle of 20° with the direction of forces p. As we see, the size of the perturbing zone, caused by the rectangular hole under consideration,

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[1]See E. Koker and L. Faylon [1], p. 457.
[2]Ibid, p. 556.
[3]See E. Koker and L. Faylon [1], p. 417.
[4]See L. G. Afendik and A. M. Yershov [1].

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does not depart from the round region (whose center is at the center of gravity of the square hole), of radius r_0 = 1.5 a.

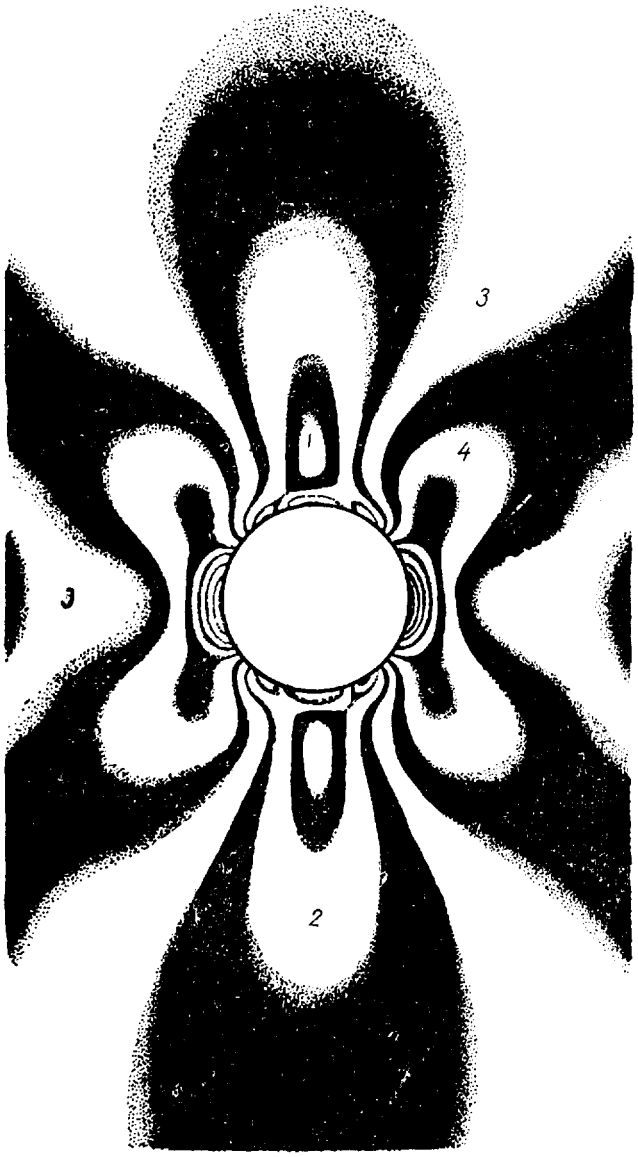


Figure A.2.

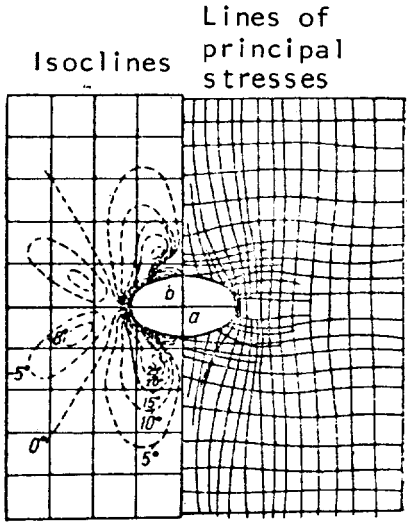


Figure A.3.

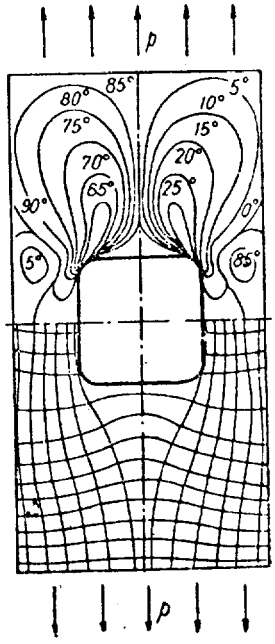


Figure A.4.

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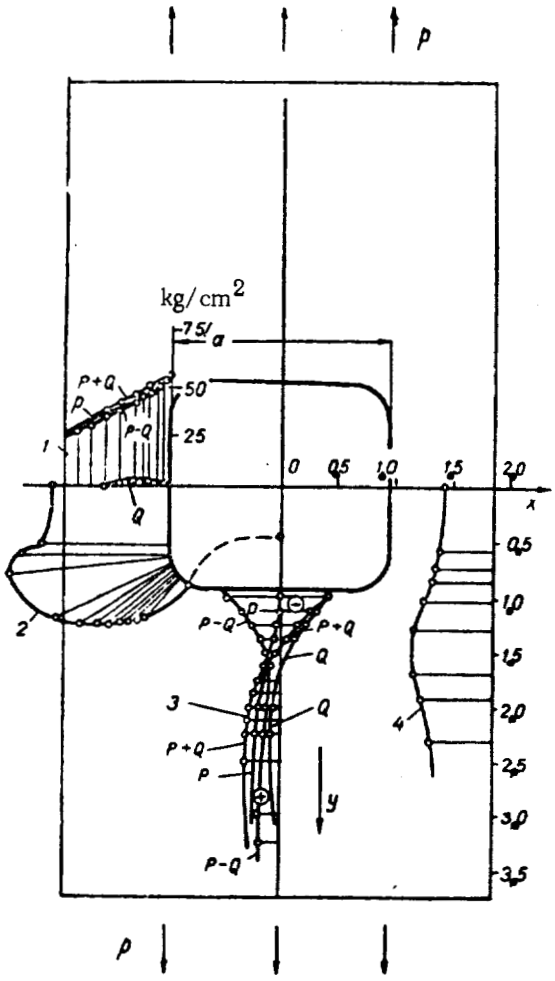


Figure A.5.

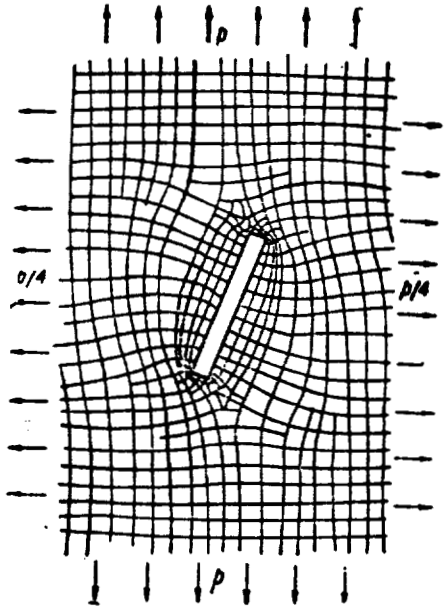


Figure A.6.

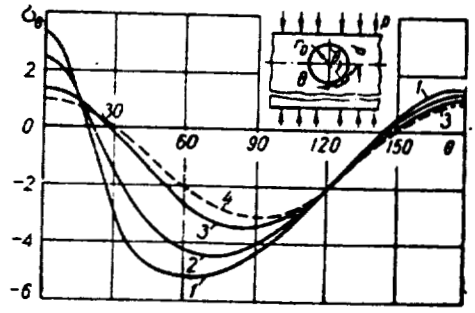
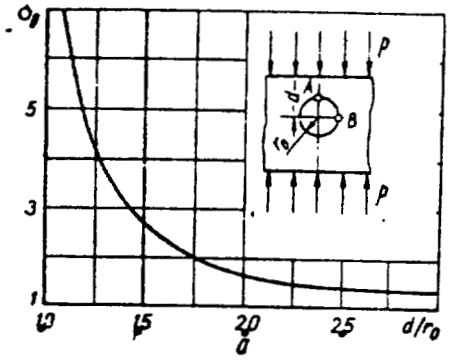
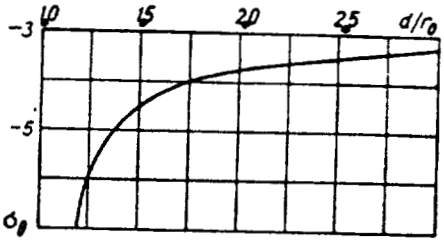


Figure A.7.



a



b

Figure A.8.

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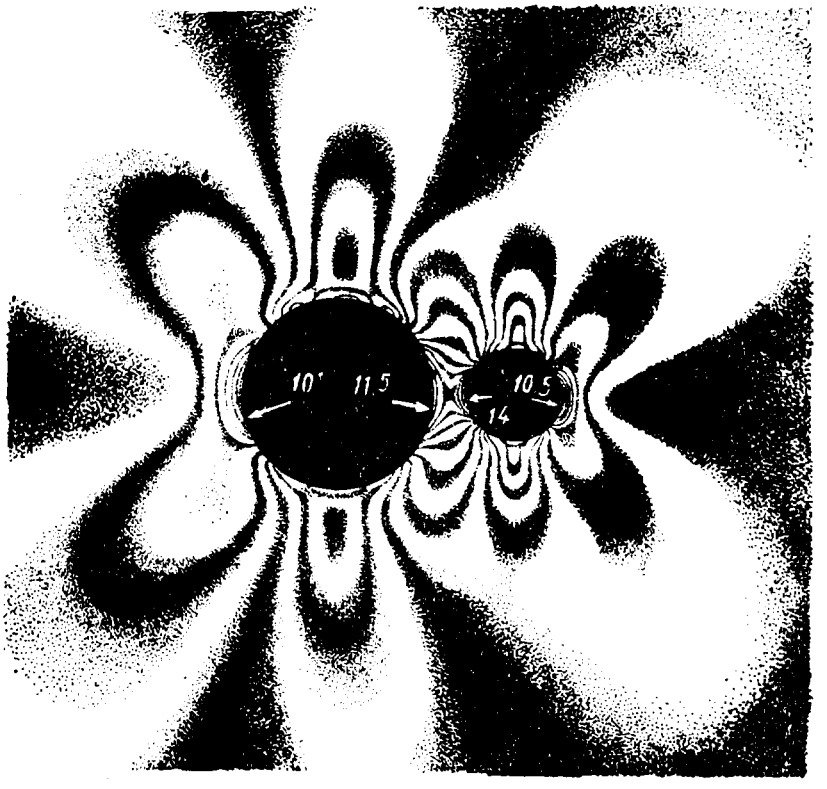


Figure A.9.

Transverse Compression of Wide Beam with a Circular Hole Near the Edge. The stress strain diagrams of stresses σ_θ along the contour of a round hole, found by R. D. Vagapov and O. I. Shishorina [1] by the photoelasticity method, where the center of the hole is located distances $d/r_0 = 1.34$ (curve 1), 1.54 (curve 2) and 2.58 (curve 3) from the upper edge of the beam, are shown in Figure A.7. Broken curve 4 corresponds to an unbounded plate, i.e., $d/r_0 = \infty$.

Figure A.8 shows stresses σ_θ at points A (Figure A.8, a) and B (Figure A.8, b), respectively, as functions of d/r_0, where d is the distance of the center of the circular hole from the upper edge of the beam, and r_0 is its radius. As we see, the perturbation zone near the given hole for $d/r_0 \geqslant 3$ differs little from the analogous zone near a circular hole for an infinite plate.

From the data presented in Figure A.2-A.8 it is possible to determine (with a known accuracy) the least distance d of the center of gravity of the hole in the form of an ellipse or right polygon (and also rectangle) with rounded corners from the edge of the beam, when it is possible to use the

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corresponding (to the given hole) solution for an infinite plate, which solution is given in Chapter II.

Two Round Holes. The stress state near two unequal round holes in the case of the tension of a plate "at infinity" by forces p = const in the direction perpendicular to their center lines, was examined by R. D. Vagapov, O. I. Shishorina and L. A. Khripina [1-3].

Figure A.9 shows the characteristic pattern of isochromes in the case of two unequal circular holes, where the radius of the larger hole is r = 14 mm and of the smaller, r = 6.97 mm, when the width of the area between the holes δ is 5.2 mm. It has been established that in the case of two unequal circular holes in a plate under tension, as shown in Figure A.9, when one of the holes is much smaller than the other, i.e., when $r_1 \gg r_2$, the greatest stress σ_θ (on the contours of these holes) will occur on the contour of the smaller hole at point A_2 (Figure A.10). As the holes become closer together, i.e., when the space δ between them decreases, unloading[1] will take place on the contour of the larger hole in the vicinity of point A_1. Figure A.10 shows the values of concentration coefficients $k_2 = \sigma_\theta/p$ at the point A_2 as functions of δ/r_1. The solid and broken curves are constructed on the basis of the approximate theoretical solution found by the authors mentioned above (which is in very good agreement with the accurate solution presented in §9, Chapter II) for $r_2/r_1 = 1$ (solid curve) and $r_2:r_1 = 1:30$ (broken curve). The different points in Figure A.10 correspond to concentration coefficients $k_2 = \sigma_\theta/p$ at point A_2, found by the photoelasticity methd for various values of δ/r_1. As we see, all experimental points lie close together near the theoretical curves for $r_2/r_1 = 1$ and $r_2:r_1 = 1:30$, which, in turn, are located very near to each other. Hence, we may use, in the broad range $1/30 \leqslant r_2/r_1 \leqslant 1$, and with a high degree of accuracy, concentration coefficients for $r_2/r_1 = 1$.

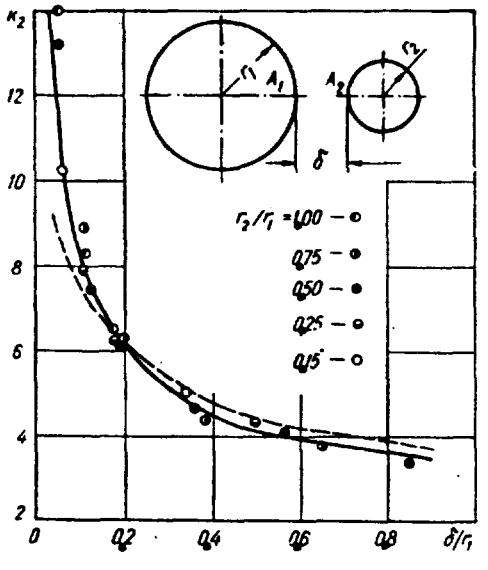


Figure A.10.

/856

[1]This may also be used for the unloading of strongly stressed areas near holes.

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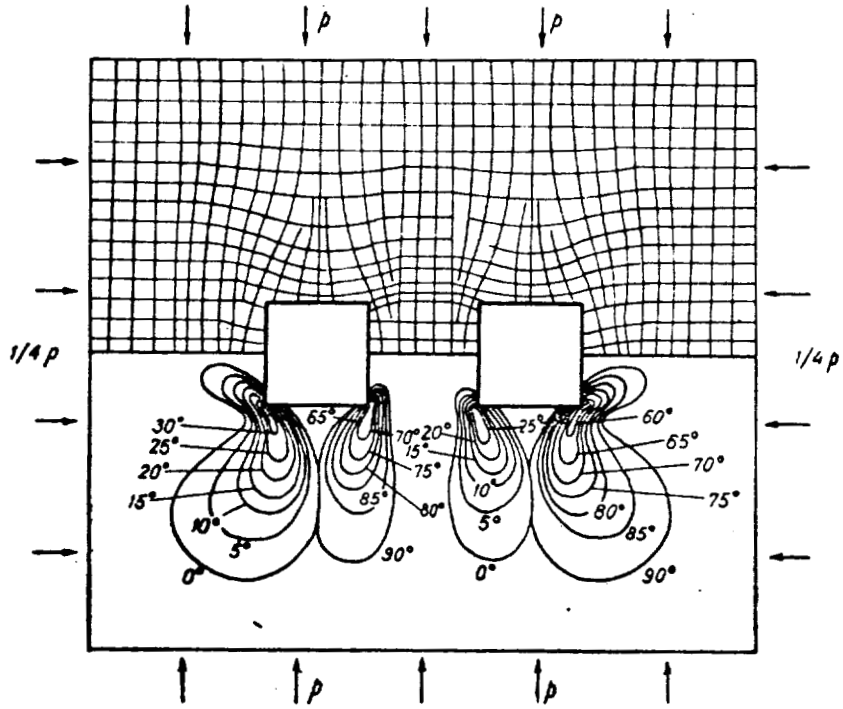


Figure A.11.

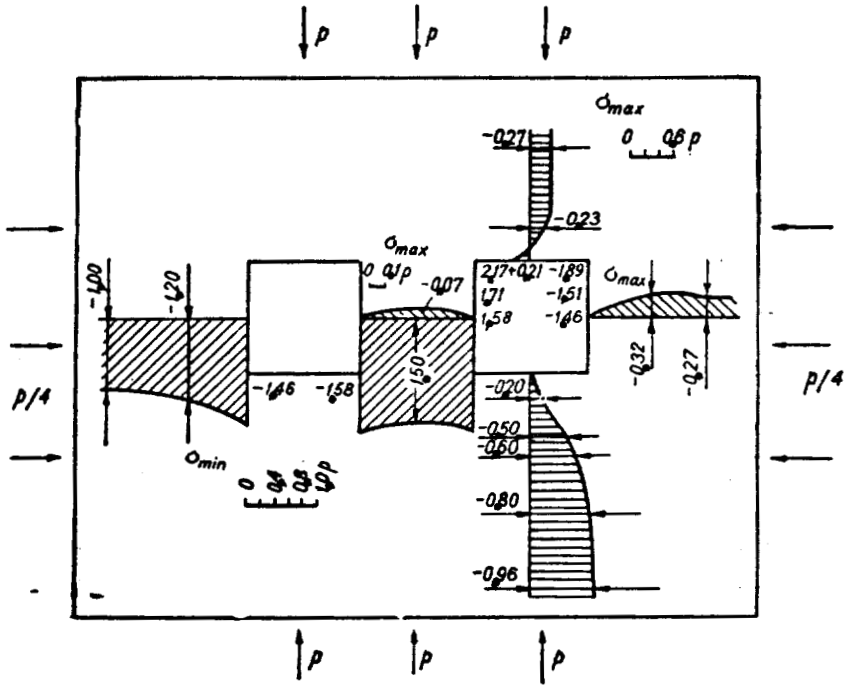


Figure A.12.

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Two Square Holes. The isoclines and isostats for a plate with two square holes, located at a distance equal to the length of the side of the square from each other, are shown in Figure A.11. The plate is compressed in two directions by forces p and q = 1/4 p. Figure A.12 shows the pattern of stress distribution through two cross sections for this case[1].

The isoclines and isostats for a plate weakened by two identical square holes and compressed in one direction by forces p are shown in Figure A.13, and the distribution of principal stresses along the axes of symmetry is shown in Figure A.14. The length of the plate is assumed to be 10 times greater than the side of the square. The distance between the square holes is equal to triple the length of the side of the square. When the distance between the centers of the squares is taken as 3 times the length of the side of the square (see Figure A.14), the effect of one square hole on the stress field near the other is comparatively slight. Thus, the stresses on the middle side, perpendicular to the direction of the compressive forces p, have the ratio 0.60/0.72. /857

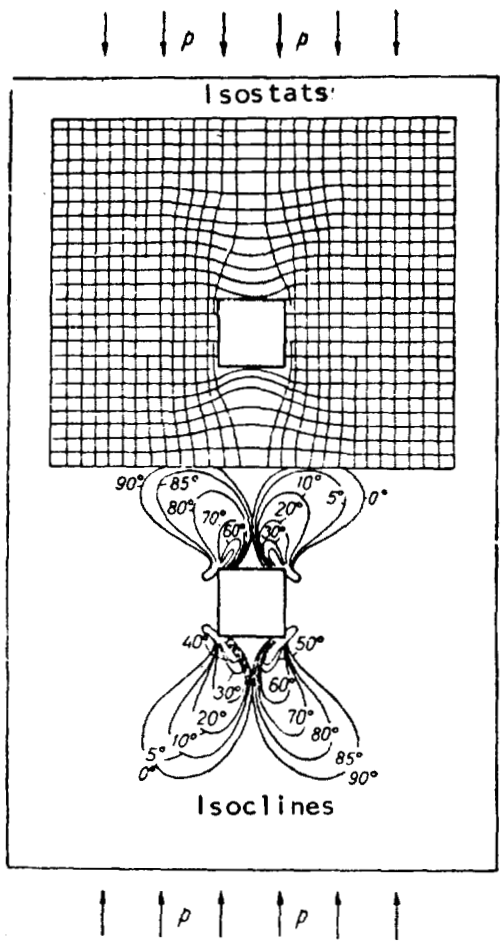


Figure A.13.

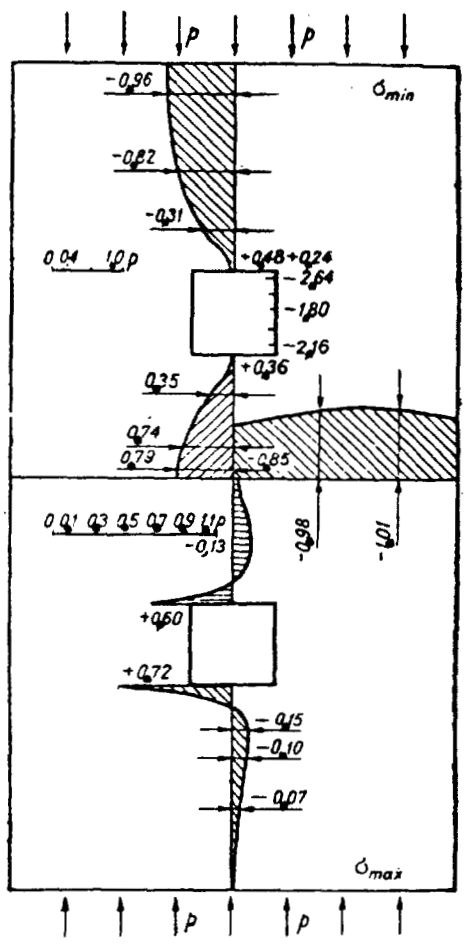


Figure A.14.

[1]See L. G. Afendik [1].

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§3. Pure Deflection of Beam with Hole

Circular Hole. Figure A.15 illustrates the diagram of an adaptation by which a beam with a round hole is subjected to pure deflection. This figure also indicates the dimensions of the beam and the position of the hole.

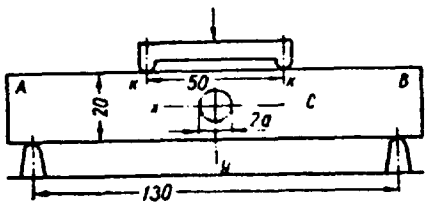


Figure A.15.

Figure A.16 shows[1] the stress-strain diagrams of stresses σ_θ on the contour of a round hole of radius a. The broken curves correspond to stresses σ_θ as determined by formula (II.105). The maximum σ_θ for a = 6 mm is 12.9, and for a = 5 mm, 10.77. The solid lines with the points correspond to the values of σ_θ determined experimentally on the contours of these holes. Here σ_θ = 13.7 for a = 6 mm and σ_θ = 10.5 for a = 5 mm. These values of σ_θ differ from those given by formula (II.105) by no more than 7%.

Figure A.17 shows the stress-strain diagrams of stresses σ_x and σ_y through cross section x = 0, where the radius of a round hole is a = 6 mm. The solid lines represent experimental data, while the broken curves correspond to theoretical data calculated by formulas (II.104), i.e., for an unbounded beam.

Figure A.18 shows the stress-strain diagrams of stresses σ_x, σ_y and τ_{xy} through cross section x = 7.5 mm, where the radius of the circular hole is a = 6 mm. The solid lines were found experimentally and the broken lines, from the theoretical solution of (II.104).

Figure A.19 represents the trajectories of principal stresses in a beam with a circular hole of radius a = 6 mm.

From the data presented in Figures A.16-A.19, we see that the diameter of a circular hole in a beam does not have to be small; it can reach up to 2/3 of its height, and here the theoretical solution of (II.104), within the limits of usual accuracy, is completely valid.

Square and Rectangular Holes. The photoelasticity method[2] was used on celluloid models to investigate the effect of square and rectangular holes on the stress state pattern in a beam (rod) under pure deflection. The height of the hole was 1/3 the height of the beam (Figure A.20).

---

[1]See Z. Tuzi [1].
[2]See G. N. Savin [1].

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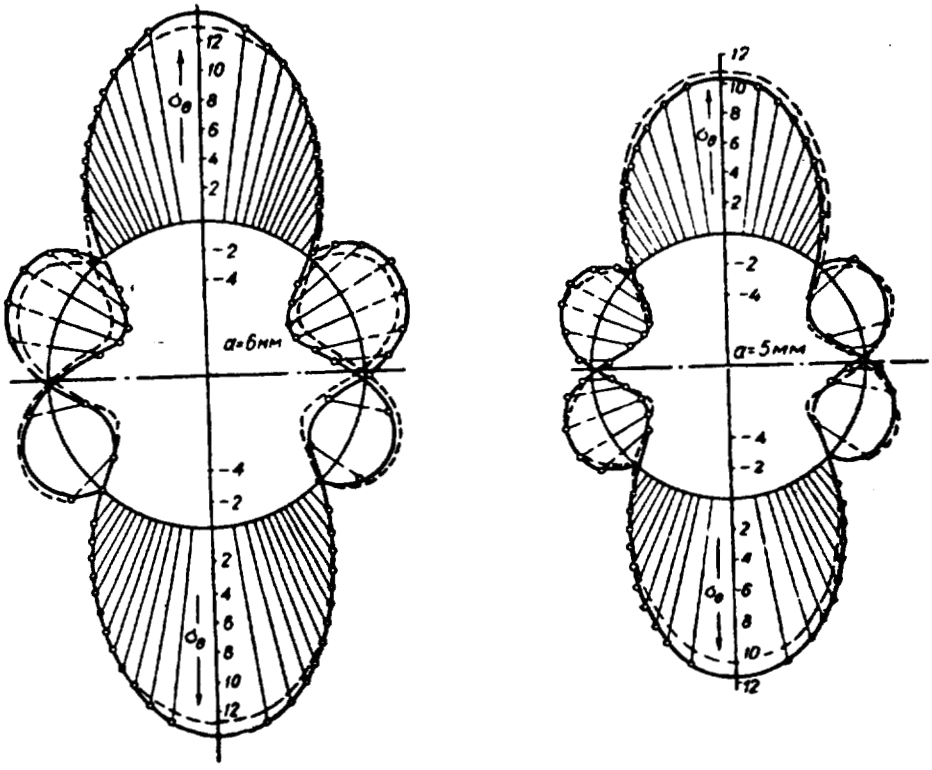


Figure A.16.

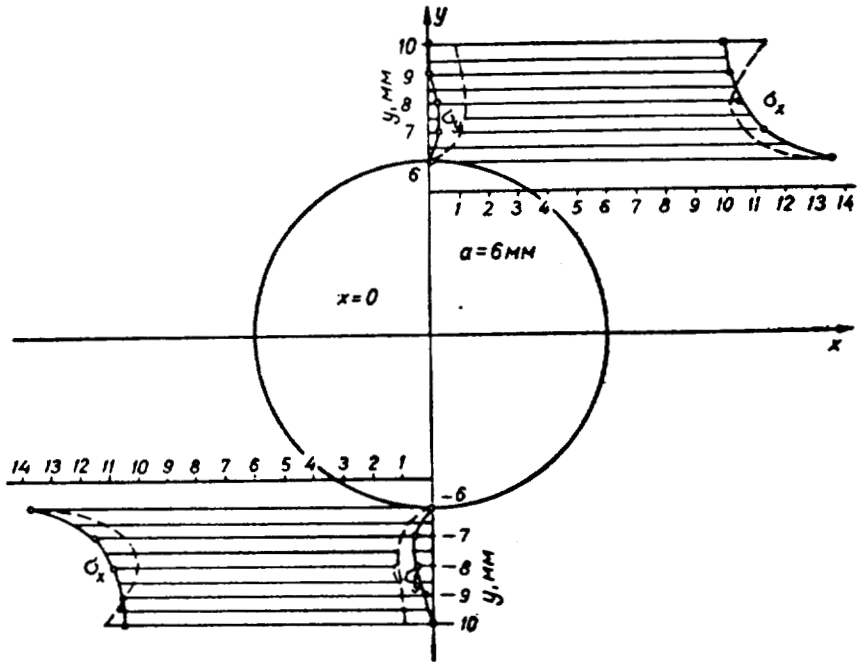


Figure A.17.

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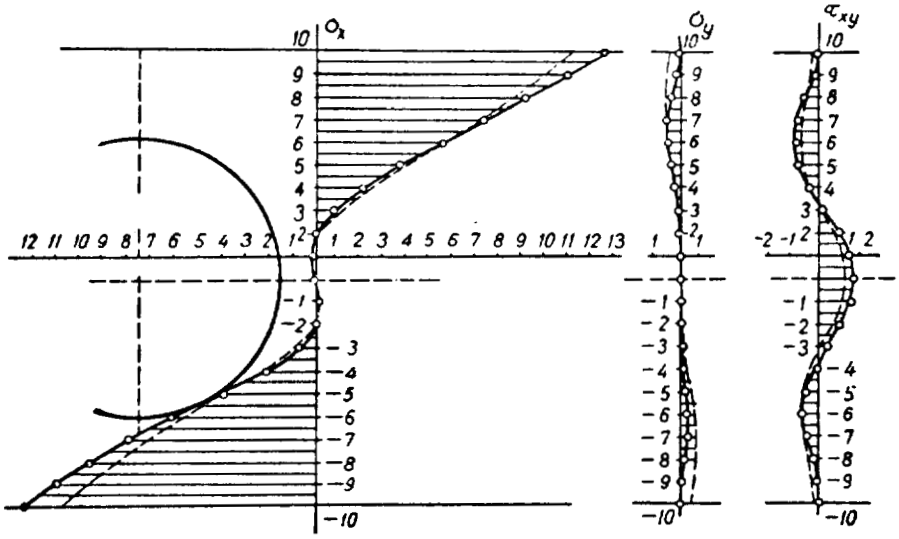


Figure A.18.

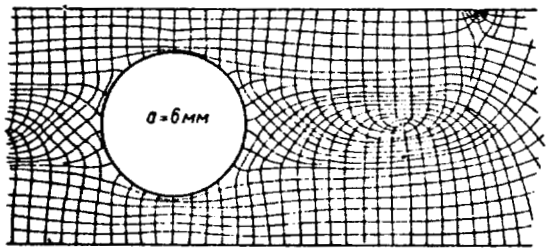


Figure A.19.

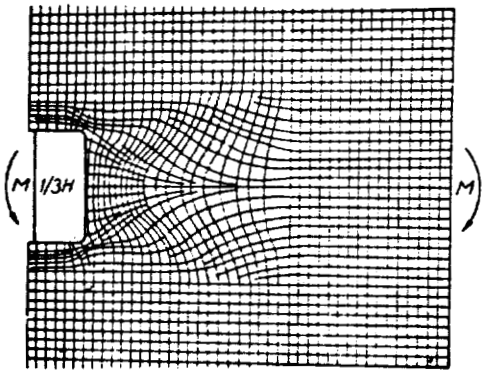


Figure A.20.

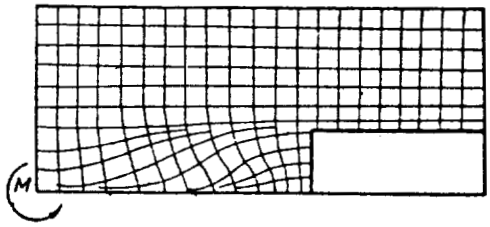


Figure A.21.

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Figure A.20 shows the field of isostats for a square hole. As we see, the perturbation caused by the square hole extends along the neutral axis of the beam to a distance equal to twice the length of the side of the square, starting from the contour of the hole, and in the perpendicular direction, to a distance of only 1/3 the length of the side of the square, also starting from the contour of the hole.

The analogous experiments were conducted for a rectangular hole with the side ratio a/b = 3 and with a height equal to 1/3 the height of the cross section of the beam. The field of isostats for this case is shown in Figure A.21.

The experimental data presented in Figures A.20 and A.21 make it possible to establish quite readily the range of applicability of the theoretical solutions in §2 and 3 of Chapter II to finite regions.

§4. Deflection of Cantilever Beam Weakened by Circular Holes /860

S. P. Shikhobalov [1] used the photoelasticity method to investigate the stress distribution in a cantilever beam weakened by one or two circular holes. The beam (rod) is bent by a concentrated force applied on the free end. The models were made of annealed mirror glass 7.5 mm in thickness. The calculated length of the beam l = 100 mm, the height h = 24 mm, the diameter of the holes d = 12 mm.

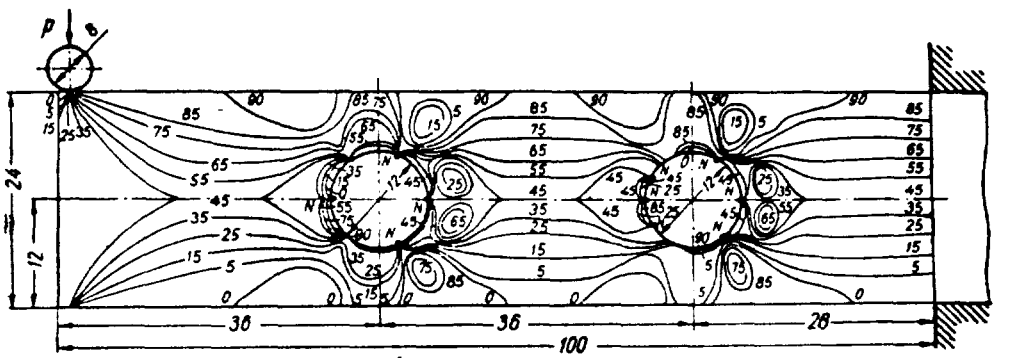


Figure A. 22.

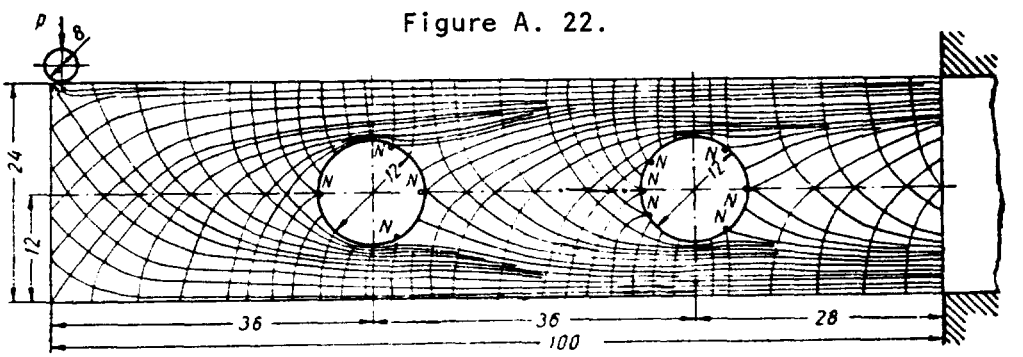


Figure A.23.

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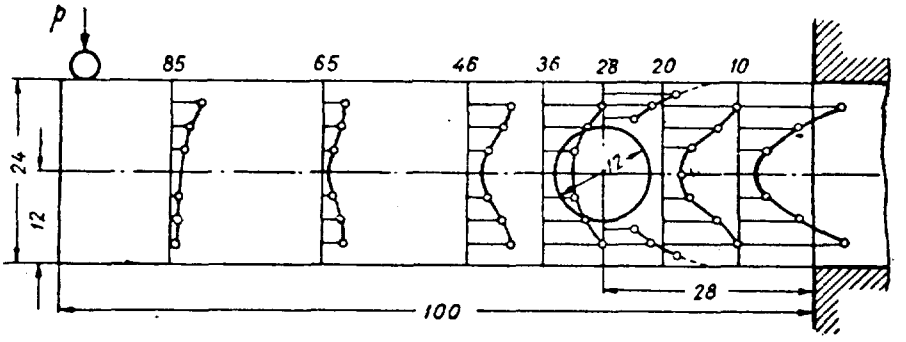


Figure A.24.

Figures A.22 and A.23 show the isoclines and isostats in a beam with two identical circular holes (d = 12 mm), located at a distance of 36 mm from each other; at this distance the mutual effect of the holes on each other is slight. /861

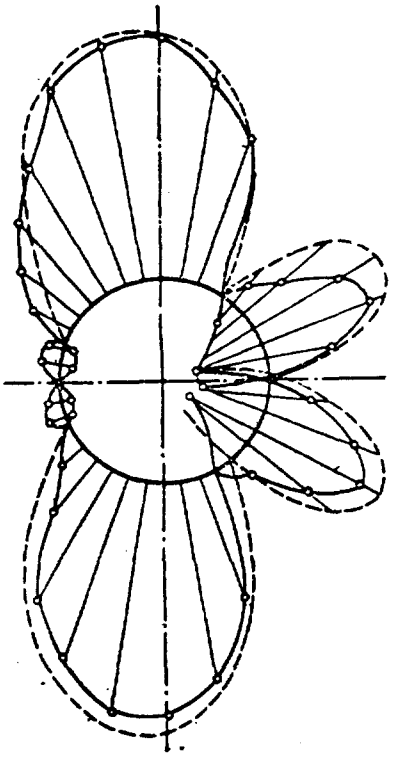


Figure A.25.

The distribution of maximum tangential stresses through certain cross sections and of stresses σ_θ on the contour of the hole for a beam with one hole is shown in Figures A.24 and A.25, where the broken lines represent the theoretical curves, while the solid lines represent experimental data.

Analysis of the isoclines shows that the effect of each hole is spread along the axis of the beam to a distance of not much more than 0.75-0.80 the diameter of the hole, starting from the contour of the hole.

From the data presented in Figures A.22-A.25 we derive the following conclusions:

1) the maximum stress σ_θ on the contour of the hole will occur at the top point of the contour, in the zone of stretch;

2) the stress concentration coefficient[1] in cantilever beams with

[1]The ratio of stresses σ_θ in the upper point of the hole to the corresponding stress σ_x in beam without a hole at the same point.

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circular holes whose centers are located on the neutral axis (for the given ratios of the dimensions) is equal to two;

3) stresses σ_x on the outer contour of the beam (rod) exceeds the corresponding stresses on the contour of the hole;

4) the most dangerous cross section is not that cross section weakened by the hole, but rather the one located at the point of pinching;

5) the principal normal stresses achieve maximum values at the point of pinching;

6) even for the ratio h/d = 2 (where h is the height of the beam and d is the diameter of the hole), the stresses on the contour of a circular hole can be calculated on the basis of formula (II.105).

§5. Uniaxial Tension of Plate Weakened by Circular Hole Whose Edge Is Reinforced by Elastic Ring

Holes are often reinforced as illustrated in Figure A.26. In the design of such rings it is assumed[1] that the entire cross section of the reinforcement, i.e., the entire height H of the ring, participates in the work of the plate. However, the stress state in the ring changes through its height. It is clear that the greater the height H of the ring in comparison with thickness h of the plate, the greater will be the difference between the stress state in its extreme cross sections and the stress state in the cross section lying in the median plane of the plate.

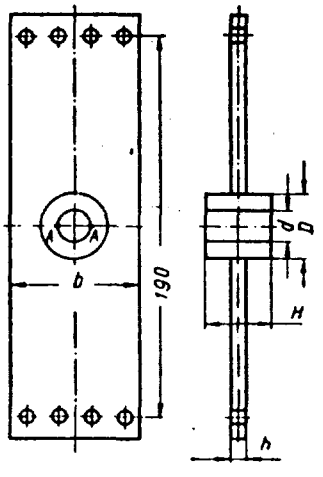


Figure A.26.

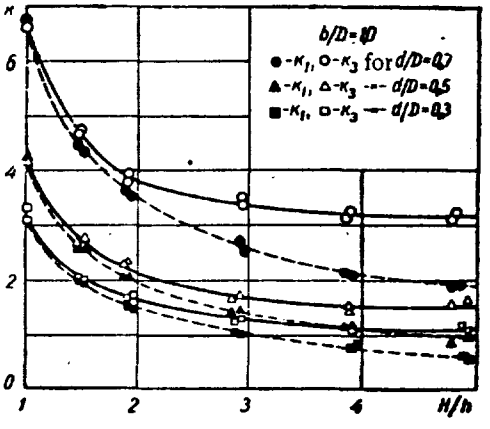


Figure A.27.

[1]See S. P. Timoshenko [1].

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The works of M. Seika, M. Ishii, I. Kurutsu [1]; M. Seika, M. Ishii [1], and also M. Reiner, H. Lerchental [1] all pertain to the experimental investi- /862
gation of the given problem by the photoelasticity method.

The stress concentration coefficients $k_1 = \sigma_1/\sigma_0$ and $k_3 = \sigma_3/\sigma_0$ at point A (see Figure A.26) are presented in Figure A.27 for b/D = 1 as functions of H/h, which changes in the range $1 \leqslant H/h \leqslant 5$, for three ratios d/D. Here σ_0 is the nominal stress in an unweakened plate; σ_1 is the maximum (average through height H) stress in the ring; σ_3 is the average stress through the thickness h of the plate.

The stresses σ_0, σ_1 and σ_3 are determined by illumination with polarized light in the direction perpendicular to the median plane of a "frozen" plate with a reinforced hole. Illumination at point A (Figure A.26) through height H gives the stress σ_1, and through thickness h, i.e., after the removal of the protruding part of the ring (removal of the part H - h of the ring), the stress /863
σ_3, which, in comparison with the same stress for a plate without a ring, found by the same method of illumination of a "frozen" plate with a nonreinforced hole of the same dimension, will be much less, since part of the strength of the ring is "spent on itself." The curves for k_1 and k_3 are illustrated in Figures A.27-A.29 for b/D, equal to 1.0, 1.5, 2.0 and 4.0, respectively. The dot-dash curve in Figure A.28 represents C. Curney's data [1], calculated theoretically in the assumption that the stresses are propagated uniformly through the thickness of the ring. The top two curves in Figure A.29, solid for k_3 and broken for k_1, correspond to b/D = 1.5, and the other three pairs of curves, to b/D = 2.0.

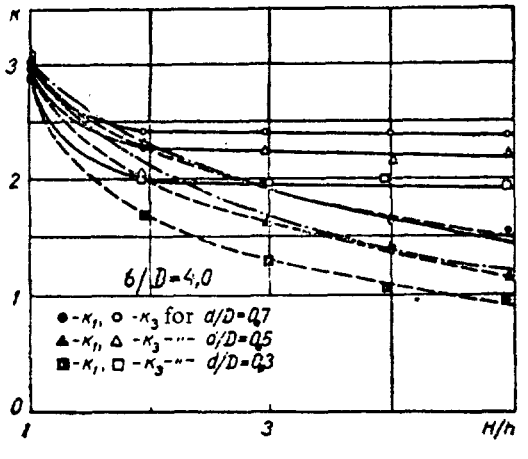


Figure A.28.

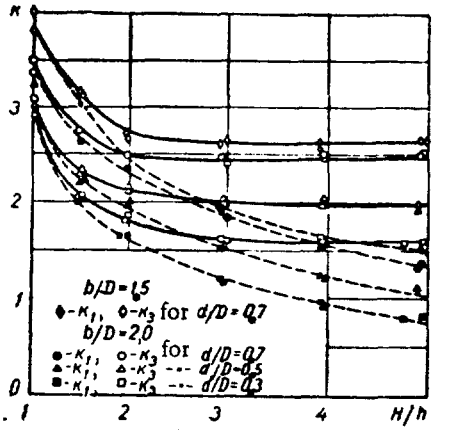


Figure A.29.

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In Figures A.27-A.29, the points enclosed by circles, triangles and squares, lying on the ordinate axis, i.e., corresponding to H/h = 1, represent data found by the approximate formula of R. B. Heywood [1], which was found by simplifying the precise solution, but which is inconvenient in applications of Howland's formula, discussed in §6, Chapter II. We see from the figures that these points coincide satisfactorily with experimental data.

The curves in Figure A.27 show that the ratio d/D has a considerable effect on the magnitude of k_3. Thus, when H/h = 1.5, the concentration coefficient $k_3 = 2.0$ for d/D = 0.3 and $k_3 = 4.5$ for d/D = 0.7. By comparing the curves presented in Figures A.28 and A.29, for the very same values of d/D, it is easy to obtain the representation of the effect of the width b of the plate on the concentration coefficient k_3.

As we see from Figures A.27-A.29, for each value of the ratio d/D, the curves for k_3, starting with some value H/h, become nearly parallel to the abscissa. This indicates that, starting with certain values of H/h, a further increase in the height H of the ring is, for all practical purposes, useless, since only a slight decrease in the magnitude of concentration coefficient k_3 results. Thus, for b/D = 1 (Figure A.27), starting with H/h = 3, all curves for k_3 become nearly parallel to the abscissa; for b/D = 1.5 and 2.0 (Figure A.29) this parallelism occurs at H/h = 2.5, and for b/D = 4.0 (Figure A.28), at H/h = 2.0. On the basis of these experimental data it is easy to show the effective height of the reinforcing ring. We will notice that the curves for k_3 for b/D = 4.0 coincide, for all practical purposes, with the same curves construc- for a "infinite" plate[1].

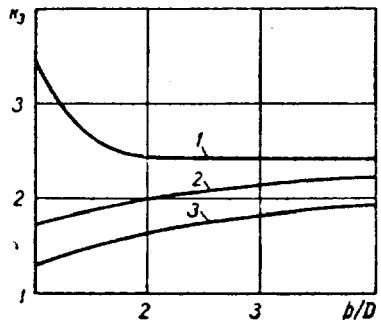


Figure A.30.

The graphs for k_3 are given in Figure A.30 as functions of b/D for the above values of d/D, equal to 0.7 (curve 1), 0.5 (curve 2) and 0.3 (curve 3) when H/h = 3.0.

§6. Rectangular Holes in Gravitational Field

In many problems of technology, particularly mining, the forces that cause stress concentration near holes are those of the gravity γ = const of the material.

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[1]See M. Seika, M. Ishii, I. Kurutsu [1].

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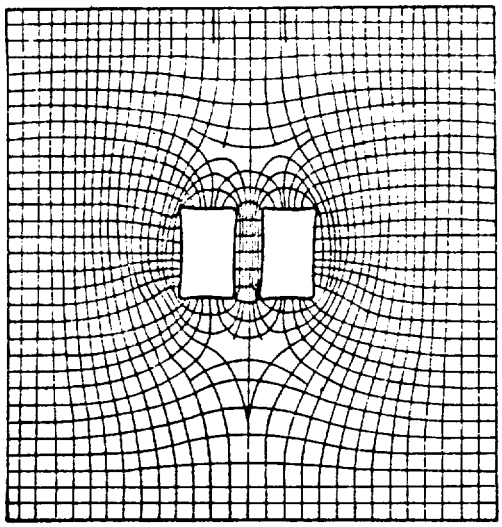


Figure A.31.

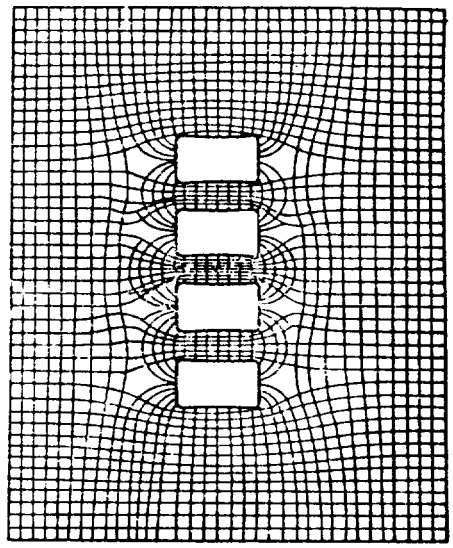


Figure A.32.

The patterns of the isostats for two and four rectangular holes, taken from the work of P. M. Tsimbarevich [1], are shown in Figures A.31 and A.32, respectively. He analyzed the stress distribution near the above-described holes in a massive plate using an optically active material (igdantine), the elastic constants of which are $E = 1045\ g/cm^2$, $\nu = 0.34$ and $\gamma = 1.175\ g/cm^3$. These figures show clearly the size of the perturbation zone near the holes, i.e., the effect of the holes on the stress distribution in the plane field under the effect of the forces of gravity.

§7. Effect of Gauss Curvature on Size of Perturbation Zone Near Hole /864

Experimental studies of the stress state near rather smooth holes (contours of which have no angular points) show that the perturbation stress zone near a hole in a shell has a local character[1], occupying a small (encompassing the hole) section of the shell.

The patterns of the perturbation zones near a "circular" hole[2] in a circular tore, which was located in the zones of positive (Figure A.36, I), null (Figure A.36, II), and negative (Figure A.36, III) gauss curvature, respectively, are illustrated in Figures A.33-A.35. The tore is subjected to internal hydrostatic pressure $p = 0.45$ atm. The dimensions of the tore are as follows: external diameter 850 mm, cross sectional diameter 200 mm, thickness of wall (shell) 2 mm. The diameter (outer) of the circular cylinder, by means of which the "circular" hole is cut on the surface of the tore, is 32 mm. This

---

[1]See G. N. Savin [3].

[2]Found at the intersection of the tore and circular cylinder, the axis of which is directed along the normal to the surface of the tore at its corresponding point.

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hole was stopped up with a thin resin film measuring 110 × 110 mm with a thickness of 0.6 mm.

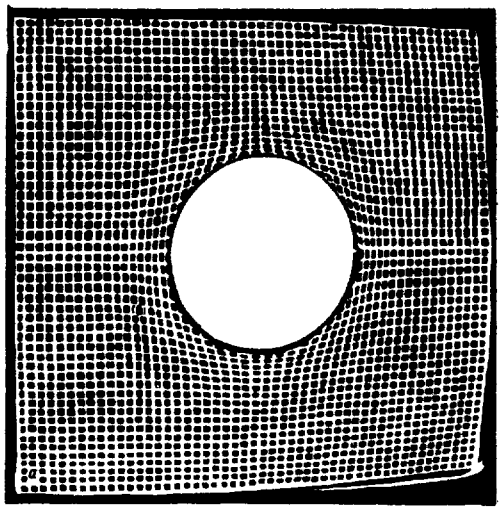


Figure A.33.

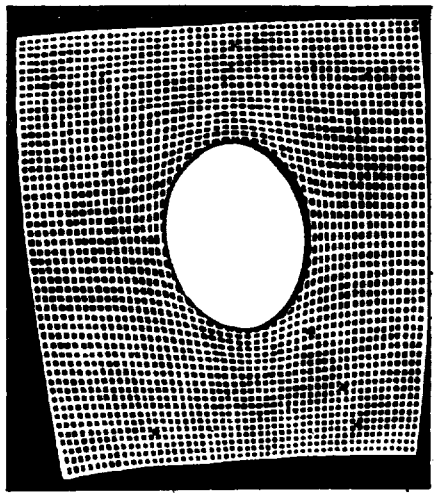


Figure A.34.

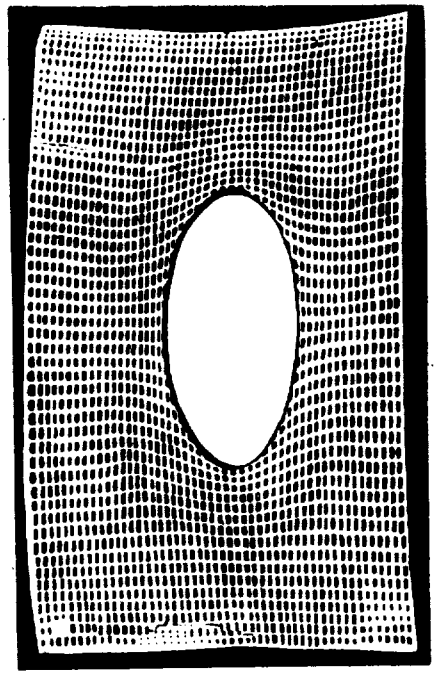


Figure A.35.

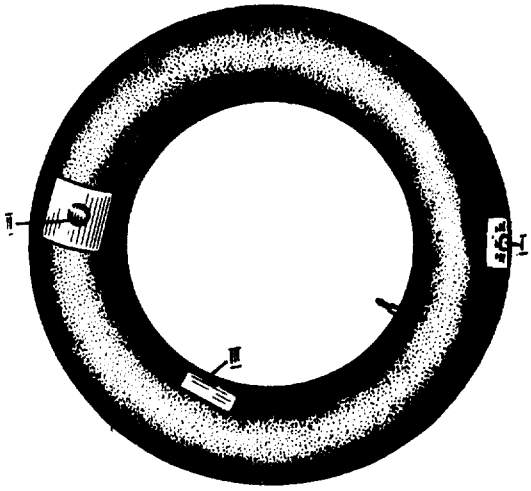


Figure A.36.

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Thus, from the above figures we see that the perturbation zone near a hole has a local character in all cases, although the dimensions of these zones depend on the sign of the gauss curvature of the shell. If, therefore, the maximum dimension of the perturbation zone for case I (Figure A.36) is assumed to be unity, then it will of the order of 1.5 for case II and 2.0 for case III.

A shell in the form of a tore makes it possible to observe simultaneously the perturbation zones near a given[1] (prior to deformation) hole located in zones with different gauss curvature.

§8. Cylindrical Shell Weakened by a Single Hole

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Circular Hole. Axial Compression. Experimental studies of stress concentration near circular holes in cylindrical (round) shells, conducted by the photoelasticity method[2] on models of shells made of optically active material, as well as on models of shells made of metal by the photoelastic coatings method[3], show that for small holes, for which the inequality

$$\omega = \frac{r_0^2}{Rh} < 1,$$

is valid, experimental data agree satisfactorily with A. I. Lur'ye's solution [1]. Here r_0 is the radius of the hole in the shell; R is the radius of the median surface of the shell; h is the thickness of the shell.

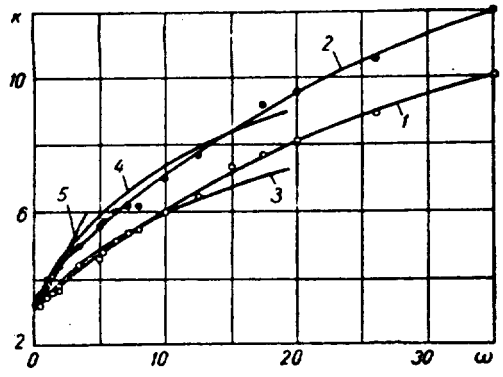


Figure A.37.

The stress concentration coefficients are $k = \sigma_\theta/\sigma_0$ (σ_θ are the maximum stresses on the contour of the hole and σ_0 are normal stresses in the shell in the zone of the uniform stress state) acquire their maximum values at points A (see Figure A.50) during the stretching of the shell along the generatrices of the cylinder by uniformly distributed forces $p = \sigma_0 h$.

The dependence of the maximum stress concentration coefficient k on the magnitude of parameter ω is represented in Figure A.37 by curves 1 and 2,

---

[1] I.e., near a hole located at the intersection of the same surfaces: of the tore and of the (circular) cylinder.

[2] See N. A. Fledrova [1]; D. S. Houghton [1]; D. S. Houghton, A. Rothwell [1]; H. T. Jessop, C. Snell, J. M. Allison.

[3] See A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1].

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constructed on the basis of experimental points for the median and internal surfaces, respectively. As we see, these curves do not coincide, as should be expected, since instead of the tangential components of forces in the zone of stress concentration around the hole in the shell, there are moments that produce stresses of deflection, depending on the parameter ω of the shell. Curves 3 and 4 are constructed on the basis of the solutions obtained by A. K. Privarnikov, V. N. Chekhov, Yu. A. Shevlyakov[1] and J. G. Lekkerkerker [1], obtained for large holes. Curve 5 corresponds to A. I. Lur'ye's solution [1] for small circular holes. As we see, the theoretical solutions for large holes agree quite well with the experimental data up to values of the parameter $\omega = 12$.

It follows from the above mentioned work of A. Ya. Aleksandrov, et al [1] that the perturbation zone around the hole, even when $\omega = 12.5$, propagates from the edge of the hole to a distance not exceeding 3.5 the radius of the hole r_0. According to N. A. Flerova's data [1], this zone of perturbation is even smaller and propagates from the edge of the hole to a distance not exceeding one radius r_0 of the hole.

The results of experimental studies mentioned above show that the perturbation zone, even around a rather large hole, has a local character. Therefore, for the theoretical solution of the problem, the hole, in many cases, can be assumed to be "small," and the region, infinite.

Elliptical Hole. Axial Compression. The pattern of the stress state around nonreinforced elliptical holes in circular cylindrical shells[2] under axial compression (Figure A.38), for the case of four shells whose parameters are presented in Table A.1, was examined by Yu. I. Vologzhaninov and S. G. Shokot'ko [1] by the photoelasticity method using the "deformation freezing" method.

TABLE A.1

Parameters of shell and hole	Number of shell			
	1	2	3	4
R	38,0	50,5	50,5	50,5
h	1,5	2,25	2,0	2,0
H	100,0	130,0	120,0	120,0
a	3,0	6,1	4.0	6,0
b	4,5	4,6	6,0	4,0
h/R	0,039	0,044	0,039	0,039
$r_0/\sqrt{Rh}$	0,5	0,5	0,5	0,5

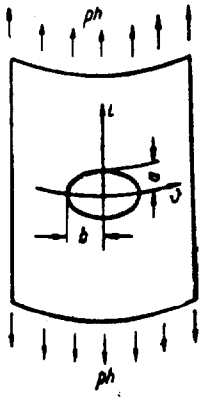


Figure A.38.

Tr. Note: Commas indicate decimal points.

[1]See A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1].
[2]See also D. S. Houghton [1].

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The following parameters are given in Table A.1: R, the radius of the mean surface of the shell; h, the thickness of the shell; H, height of the shell; a and b, semiaxes of the elliptical hole (see Figure A.38) (all linear dimensions are given in mm). /867

In the cases under examination, the perturbation zone around an elliptical hole occupied a small region, extending along the generatrices of the cylinder (Figure A.38). The distance to which the perturbation propagated did not exceed two large axes of the ellipse from the edge of the hole. Here the perturbation from the holes extended along the generatrices (a > b) was propagated to a lesser distance.

Figure A.39 shows the distribution of stress concentration coefficients, equal to the ratio of contour forces T_s to the forces of the basic stress state T_1. The solid lines are constructed on the basis of experimental data, and the broken, in accordance with formula (X.165).

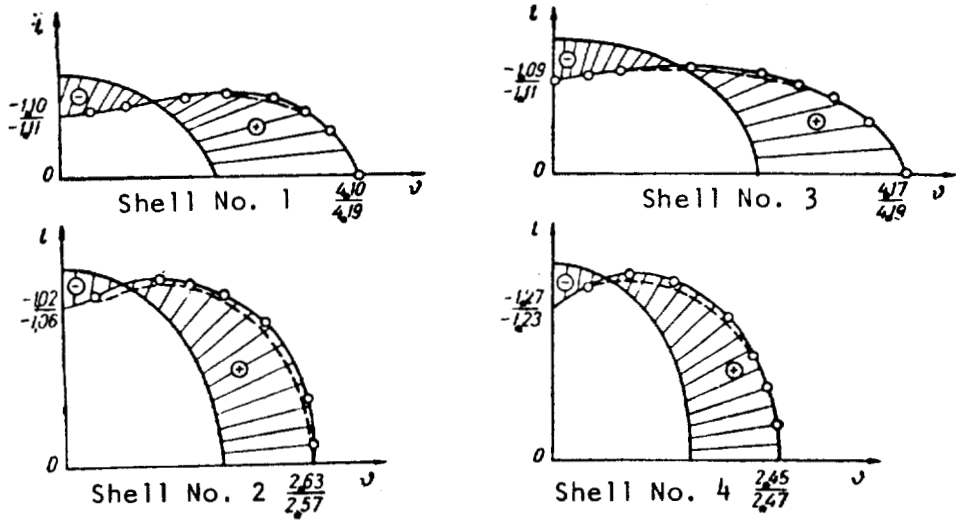


Figure A.39.

For the case under examination, formula (X.165) has the form

$$k^{th} = \frac{T_s}{T_t} = \frac{1}{2}\{(2 - \varepsilon\pi\beta^2) + [4\varepsilon - (4 + \pi\beta^2) - \varepsilon^2(2 + 3\pi\beta^2)]\cos 2\vartheta + \\ + [4\varepsilon^2 - (4 + \pi\beta^2)\varepsilon]\cos 4\vartheta - (2 + \pi\beta^2)\varepsilon^2\cos 6\vartheta\}, \quad (A.1)$$

where

$$\varepsilon = \frac{a-b}{a+b}; \qquad \beta = \frac{\sqrt[4]{3(1-\nu^2)}}{2}\cdot\frac{r_0}{\sqrt{Rh}}.$$

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In comparing the experimental results with the theoretical, given by formula (A.1), it is necessary to take into account the relationship between angle ϕ of the plane system of coordinates, the origin of which coincides with the geometrical center (of gravity) of the hole, and the parameter ϑ. Angle ϕ is read off from the l axis (see Figure A.38):

$$\phi = \arctan \frac{1-\varepsilon}{1+\varepsilon} \tan \vartheta.$$

For the data presented in the fourth column of Table A.1, formula (A.1), for $\nu = 0.5$ (Poisson's ratio for "freezing") acquires the form

$$k^{th} = 0{,}97 - 1{,}805 \cos 2\vartheta - 0{,}349 \cos 4\vartheta - 0{,}046 \cos 6\vartheta. \qquad (A.2)$$

The values of concentration coefficients k at the points of intersection of the contour of the hole with the l axis ($\vartheta = 0$) (see Figure A.38) and the ϑ axis ($\vartheta = \pi/2$) are presented in Table A.2. The values k^{exp} and k^{th} were calculated on the basis of formula (A.2). For comparison, the value k^{pl}, the concentration coefficient of forces T_s, found by formula (II.63) for $\sigma_x^{(\infty)} = 0$, $\sigma_y^{(\infty)} = -p$ is also presented.

TABLE A.2

Concentration Coefficients	ϑ	Number of shell 1	2	3	4
$k^{(exp)}$	0	1,10	1,02	1,09	1,27
	$\pi/2$	4,10	2,63	4,17	2,45
$k^{(th)}$	0	1,11	1,06	1,11	1,23
	$\pi/2$	4,19	2,57	4,19	2,47
$k^{(pl)}$	0	1,00	1,00	1,00	1,00
	$\pi/2$	4,00	2,51	4,00	2,33

Tr. Note: Commas indicate decimal points.

The data presented in Figure A.39 and Table A.2 show that the maximum divergence of the experimental data with the corresponding approximate theoretical solution in Chapter X, does not exceed 3-5%. /868

For a cylindrical shell, on the basis of (theoretical) studies of several problems[1], it is concluded that in determining the maximum stress concentration coefficients in a cylindrical shell near small holes (the contours of which are free of external forces) of arbitrary form, but not possessing angular points when the cylindrical shell is under tension or compression along the generatrix, it is possible, with an accuracy of 7-10%, to use the corresponding solutions of the plane problem, described in Chapter II.

Square Hole with Rounded Corners. Axial Compression. A round hole of radius r_0 was first cut in a shell, and the hole was then gradually transformed

[1]See G. N. Savin, A. N. Guz' [1], and also G. N. Savin [2].

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from experiment to experiment into a square hole with angles, rounded with a given radius[1].

Figure A.40 represents the curves for the value k as a function of r/r_0 (r is the radius of rounding of the corners) for two parameters $\omega_0 = 5.08$ and $\omega_0 = 8.0$ when $\omega_0 = r_0^2/Rh$. The solid curves correspond to the value k on the inside surface, and the broken curves, in the median surface of the shell. As we see, first, as the radius of rounding of the corner decreases, the value k decreases also, reaching its minimum at $r/r_0 = 0.45$-0.50. A further reduction /869 in the radius of round of the corners results in an increase in the value of k. The maximum stresses σ_θ on the contour of the hole are concentrated at the points of contact of the rectilinear parts of the contour of the hole with the rounding curves of the corners on the sides of the square, parallel to the axis of stretching.

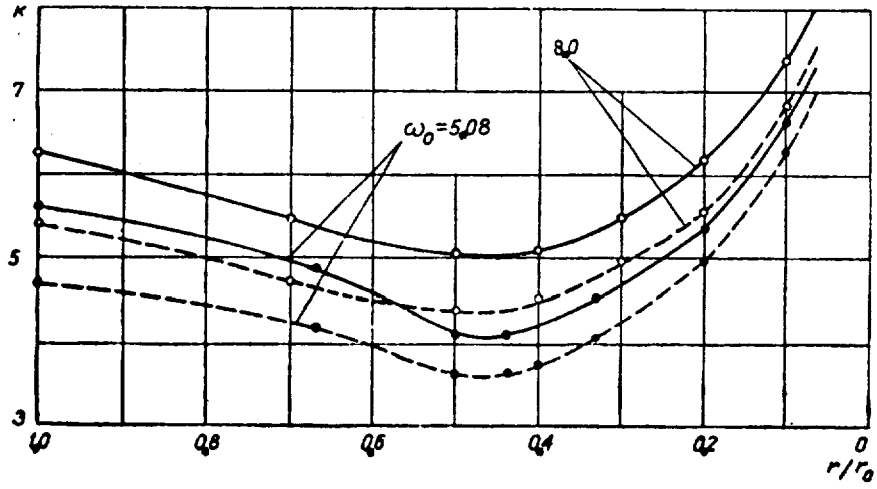


Figure A.40.

The law of change of the value of k along the contour of the hole for $\omega_0 = 8.0$ and $r/r_0 = 0.5$ and for the case where the shell is under tension along the x axis is shown in Figure A.41. The broken curve corresponds to the value of k for the mean surface, and the solid, for the internal surface of the shell. The postive values of k correspond to the stretching stresses and are arranged from the outer side of the contour of the hole.

Round Hole. Twisting of Cylindrical Shell. Curves 1 and 2 in Figure A.42 were constructed on the basis of experimental[2] data and correspond to stresses in the median and outer surfaces of the shell. Curve 5 was constructed from

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[1]See A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1].
[2]See A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1].

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the solution of Yu. A. Shevlyakov, F. S. Zigel [1] for the mean surface of the shell. Curves 3 and 4 were constructed on the basis of J. G. Lekkerkerker's data [1], and curve 6, corresponding to stresses on the inner surface of the shell, on the basis of D. Withum's data [1]. J. G. Lekkerkerker's solution [1] for parameters $\omega_0 \leqslant 10$ demonstrates the best agreement with the experimental results.

The distribution of stresses σ_θ on the contour of a circular hole for four parameters ω, equal to 1.0, 7.1, 15.0 and 20.0, are presented in Figure A.43, where the solid lines illustrate the distribution of stresses in the median surface of the shell, and the broken curves represent the bending stresses on the inner surface of the shell.

Tension and Torsion of Cylindrical Shell with Reinforced Holes. The curves for the value k_{max} as a function of the magnitude of reinforcement $v = H/h$ for points lying on the inner contour of a hole in the median surface are presented in Figure A.44. These curves were constructed experimentally with the aid of strips of optically active material. These strips were glued onto the end near the hole at the level of the median surface. Curves 1 and 2 correspond to a circular hole for $\omega = 4.0$ and $r_0/r = 0.6$. Curves 3 and 4 correspond to a rectangular hole with parameters $ab/Rh = 13.3$; $a/b = 1.5$; $c/b = 0.48$ and $r'_0/b = 0.20$. Curves 5 and 6 correspond to a square hole with parameters $a^2/Rh = 8.85$; $c/a = 0.48$ and $r'_0/a = 0.12$. Here the solid lines /871 correspond to tension and the broken lines to torsion. As we see[1], as the parameter $v = H/h$ increases, the concentration of maximum stresses on the contour of the hole first decreases sharply[2], and then, starting approximately with $v = 5.0$, this decrease slows down. Therefore, for each form of hole it is possible to select the optimal magnitude of height H of the ring.

Stresses around Circular Hole in Cylindrical Shell during the Elastoplastic Stage of Deformation. The stress concentration around a circular hole, free of external forces, in duraluminum and steel[3] shells under tension by forces $p = \sigma_0/h$ along the generatrices, was examined by A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, A. S. Rakin [1] by the photoelastic coatings method.

The experimental studies show that as the loading parameter $\lambda = \sigma_0/\sigma_{0,2}$ increases, the stress concentration coefficients k in points of the median surface of the shell decrease sharply. Here σ_0 is the nominal principal stress at points of the shell that are rather remote from the hole, i.e., in the zone

---

[1]The analogous picture is seen also for the reinforcing ring of a hole in the case of a flat plate (see Figures A.26-A.29).
[2]In Figures A.44 and A.45, for greater clarity, the scales are somewhat displaced in relation to each other along the abscissa.
[3]The stress diagram of the given steel did not possess an area of flow.

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of the uniform stress state of the shell; $\sigma_{0,2}$ is the conditional yield point of the material of the shell.

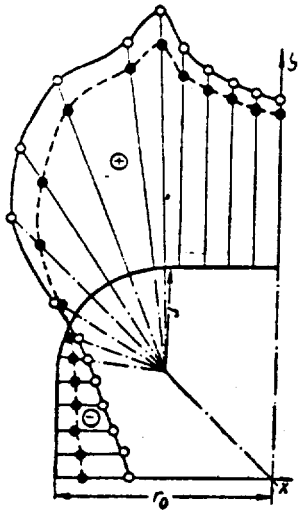


Figure A.41.

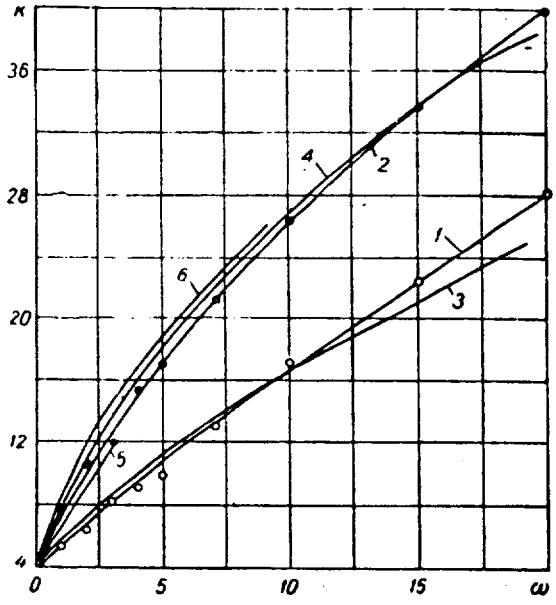


Figure A.42.

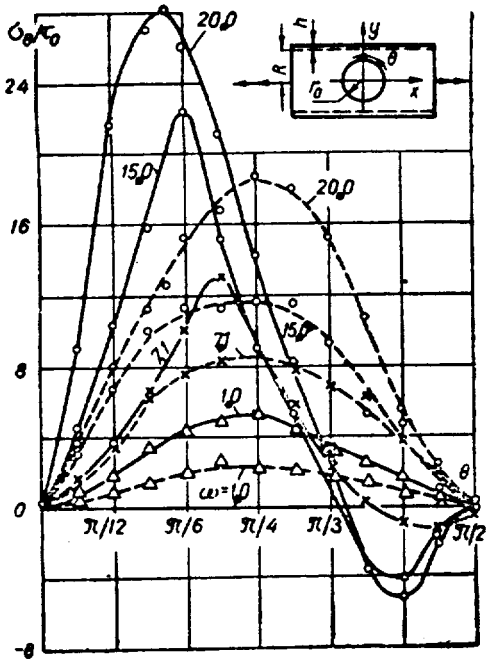


Figure A.43.

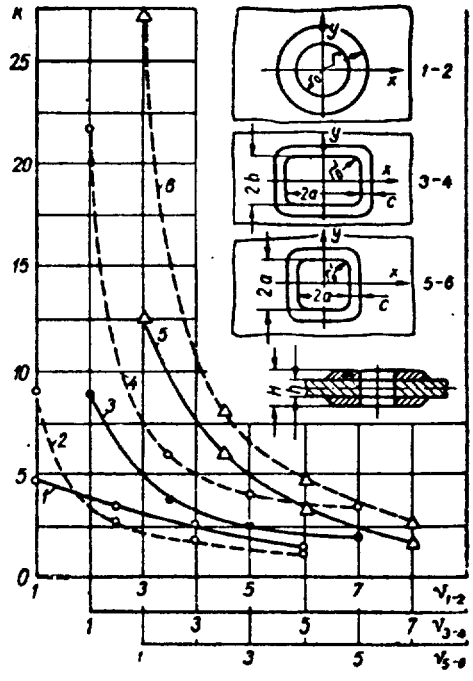


Figure A.44.

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Curves 1, 2 and 3 in Figure A.45 correspond to duraluminum shells with the parameters $\omega_1 = 0.142$, $\omega_2 = 2.0$ and $\omega_3 = 4.0$, and curves 4 and 5, to steel shells with the parameters $\omega_4 = 4.0$ and $\omega_5 = 10.0$.

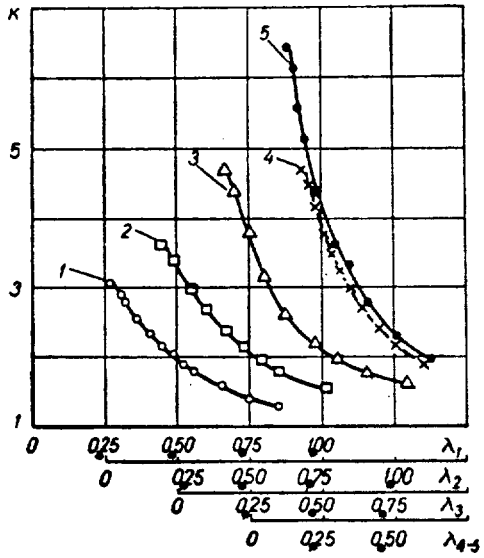


Figure A.45.

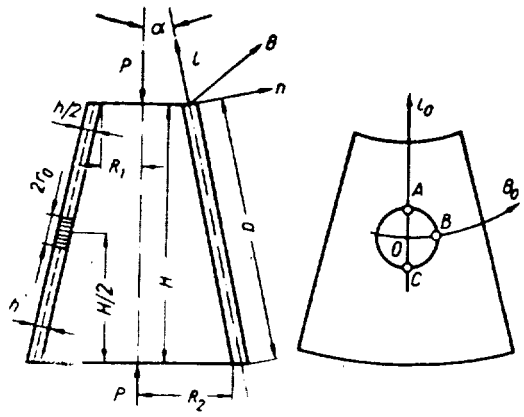


Figure A.46.

§9. Conical Shell Weakened by a Single Hole

Axial Compression of Shell with a Circular Hole. Two conical shells made of ED-6M material, which is optically active, were investigated by Yu. I. Vologzhaninov [1].

The meridional cross section passing through the center of the hole and element of the shell, containing a round hole of radius r_0, is shown in Figure A.46. The parameters (in mm) of the examined shells are presented in Table A.3, where the force P is given in Newtons and the angle α, in degrees.

TABLE A.3

Number of shells	r_0	R_1	R_2	H	h	$\alpha°$	P	ν
1	5,25	17	32	90	3	9,5	50,2	0,5
2	3,0	20	31	70	2	9,0	50,0	0,5

Tr. Note: Commas indicate decimal points.

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The stress state in the shell was recorded by "freezing." The determination of the stress state was accomplished by normal through illumination, which made it possible to find the forces T_s along the contour of the hole and the difference $T_\ell - T_\theta$ between the circumferential and meridional forces beyond the range of influence of the hole, i.e., in the zone of the basic stress state.

The basic stress state in conical shell No. 1 (Table A.3), under the effect of compressive forces P (Figure A.46), found[1] with the aid of numerical integration on the BESM-2M computer, in the basic part of the shell, not in contact with the edges, i.e., when $0.2\ a < \ell < 0.7\ a$, will, for all practical purposes, be momentless, i.e., M_ℓ, M_θ and T_θ in this part of the shell will be absent.

Thus, the experimentally determined difference $T_\ell - T_\theta$ in the middle part of the shell, on the parallel passing through the center of the hole, but at a sufficient distance from it, will practically correspond to the force T_ℓ.

The distribution of forces T_s (n/cm) along the contours of the holes in shells No. 1 and 2 are presented in Figure A.47, a and b, respectively. The solid line in Figure A.47,a corresponds to the upper half of the contour of the hole ($t_0 \geqslant 0$), while the broken line corresponds to the lower half (Figure A.46) of the contour of the hole ($\ell_0 < 0$).

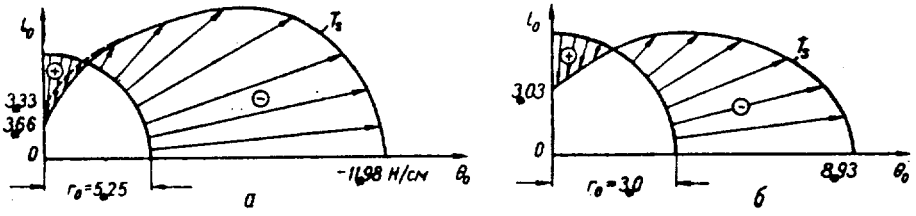


Figure A.47.

The values of forces T_s on the contours of the holes in the most characteristic points A, B and C (Figure A.46), as well as the values $T_\ell - T_\theta$ of the basic stress state on the parallel line passing through the center of the holes, and the concentration coefficients k^{exp} of forces, calculated on the basis of these data, are presented in Table A.4; here also are the values of k^{th}, calculated under the condition that a conical shell with a round hole is subjected to tension by the corresponding forces along the generatrices. /873

[1]See A. D. Kovalenko, Ya. M. Grigorenko, L. A. Il'in [1].

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TABLE A.4.

Number of shells	Point of contour	T_s	$T_l - T_\theta$	$K^{exp} = \frac{T_s}{T_l - T_\theta}$	K^{th}
1	A	3,66	−3,16	−1,16	−1,275
	B	−11,98		3,79	3,206
	C	3,33		−1,05	−1,137
2	A	3,03	−2,94	−1,03	−1,135
	B	−8,93		3,04	3,098
	C	3,03		−1,03	−1,061

Tr. Note: Commas indicate decimal points.

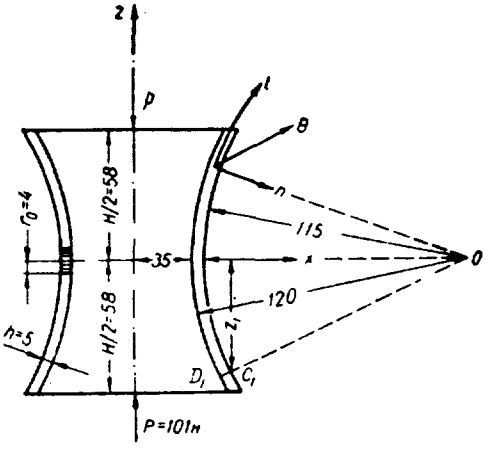


Figure A.48.

We see from the data presented in Table A.4 that k^{exp} and k^{th} agree satisfactorily for a small hole (shell No. 2). This can be attributed to the considerations given above, specifically the greater degree of accuracy of observation of the momentlessness of the basic stress state in the middle parts of the shell[1] during the experiment. The greater divergence of k^{exp} and k^{th} for a large hole (shell No. 1) can be explained by the fact that the perturbation zone around the hole in this case extends beyond the middle zone of the shell, in which the condition of momentlessness of the basic stress state was satisfied with a certain degree of accuracy.

§10. Shells of Negative Gauss Curvature

Circular Hole. Axial Compression of Shell. The stress concentration around "circular" hole[2] is examined in the works of Yu. I. Vologzhaninov, V. I. Savchenko, M. D. Fenchak [1] and Yu. I. Vologzhaninov [1] by the photoelasticity method with the aid of deformation "freezing."

A single-cavity hyperboloid, the outer surface of which is defined by the equation $x^2 + y^2 - z = a^2$, where a is the radius (outer) of the neck of the hyperboloid (z = 0) equal to 55 mm, is examined in the first of these works.

In the second work is examined a shell whose outer surface is formed by the rotation of part of an arc of a circle of radius 115 mm around the z axis (Figure A.48). The outer radius of the neck of the hyperboloid is a = 40 mm.

[1]We will recall that in the theoretical solution of this problem it was assumed that the shell was subjected to tension by forces ph along the generatrices.

[2]Such a hole is found by intersecting each of the surfaces given below with a circular cylinder, the axis of which is directed along the normal to the surface (Figure A.48).

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The thickness of the shell in both cases was constant, h = 5 mm, and the height of these shells was also identical, H = 116 mm. The radii (outer) of the cylinders were r_0 = 4 mm, and at the intersection of these cylinders with the above-mentioned surfaces, "circular" holes were formed on the neck. This makes it possible to compare the results of experimental studies for two shells.

In the investigation of the meridionals ζ_n and sections θ_n of the "frozen" basic stress state orthogonal to them, it was shown (in the second work) that the distribution of stresses σ_ζ, σ_θ along the lines normal to the median surface (lines C_iD_i in Figure A.48), i.e., through the thickness of the shell, has a linear nature. The analogous conclusion was derived through investigation of a single-cavity hyperboloid in the first of the above-mentioned works.

The values of stresses σ_ζ^{Di}, σ_θ^{Di}, σ_θ^{Ci}, C_ζ^{Ci} are given in Table A.5 for the interior and outer surfaces, and for shell No. 2 for various z_i (Figure A.48). The forces (in n/cm) in the direction of generatrix T_ζ and in the circumferential direction T_θ were calculated on the basis of these stresses (n/cm^2).

TABLE A.5

z^i, мм	σ_ζ^{Di}	σ_ζ^{Ci}	σ_θ^{Di}	σ_θ^{Ci}	T_ζ	T_θ
0	—7,85	—8,14	—5,2	0	—4,0	—1,27
±10	—7,55	—8,44	—5,1	—0,10	—4,0	—1,27
±20	—6,28	—9,70	—4,02	—1,18	—4,0	—1,27
±30	—2,35	—13,1	—1,86	—2,45	—3,92	—1,08
±40	+0,98	—16,5	+1,96	—2,75	—3,92	—0,20

Tr. Note: Commas indicate decimal points.

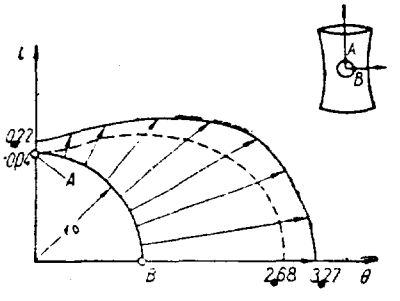


Figure A.49.

Analysis of the data presented in Table A.5 shows that the deflecting moments M_ζ, determined by stresses σ_ζ, are insignificant when z = 0, but increase as z increases. The moments M_θ, on the other hand, are greatest when z = 0, and then decrease as z increases. Forces T_ζ and T_θ are constant when $-20 \leqslant z \leqslant 20$. As the distance to the edges of the shell decreases, the forces T_ζ continue to be about constant, while forces T_θ change signs. This indicates that part of the shell, that which lies near the supporting edges, is subjected to tension in the circumferential direction during compression (Figure A.48). /874

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Contour forces T_s were determined by normal (to the median surface of the shell) through illumination in a row of points of the contour of the hole. The solid line in Figure A.49 corresponds to concentration coefficients $k = T_s/T_l$ of shell No. 2, where T_s are forces along the contour of the hole, and T_l is meridional force of the basic stress state on the parallel line passing through the center of the hole ($z = 0$). The broken lines indicate the concentration coefficients k^{pl} for a plate with the same circular hole, subjected to load "at infinity" by mutually perpendicular forces, corresponding to forces T_l and T_θ of the basic stress state of a shell when $z = 0$ (see Table A.5).

For the shell examined in the first work, the ratio of forces T_l and T_θ for $z = 0$ was 3.15, while for the shell examined in the second work, this ratio was only 1.15. In spite of such a great difference, the character of distribution of the concentration coefficients on the contour of the hole in both cases, for shells and for plates, coincided. However, the numerical value of the concentration coefficients k for the shells was somewhat higher.

On the basis of the above experimental studies of shells with holes of negative gauss curvature, the following conclusions can be made:

1) the perturbation zone around a hole is of local (encompassing the hole) character and is propagated from the edge of the hole to a distance not exceeding 3 radii of the hole;

2) the maximum concentration coefficient of forces T_s in a shell with a hole will be (with a free hole) on the contour of the hole, and will not be greater than 1.3 times the concentration coefficient for a plate with the same hole, but loaded "at infinity" by forces that correspond to the basic stress state in a shell at the point $z = 0$ (see Figure A.48).

§11. Effect of Creep of Material on Stress Concentration Around a Circular Hole

The stress concentration with consideration of the creep of a material at points of a circular hole (Figure A.50), located within a plate, which plate is subjected to forces of tension $\sigma_x^{(\infty)} = p$, $\sigma_y^{(\infty)} = q$ ($p \geqslant q$), was analyzed by the optic method by I. I. Bugakov [1].

The change in stresses σ_θ at the point A (Figure A.50) in time was /875
analyzed by the photocreep method on models made of two different "age" lots of technically transparent celluloid. The first model was made of celluloid with an "age" of 1-1/2 years, and the second with an age of half a year.

It was established experimentally that the dependence

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$$\varepsilon_i = s_i \varphi(t) e^{bs_i} \quad \left(s_i = \frac{3}{2}\sqrt{s_{\alpha\beta}s_{\alpha\beta}}\right), \tag{A.3}$$

where t is time; b is a constant of the material; s_i is the intensity of tangential stresses; $s_{\alpha\beta}$ are stress deviator components; ε_i is the intensity of displacement deformations, is valid for these celluloids in the case of the biaxial stress state under the conditions of steady, or more accurately, quasi-steady creep.

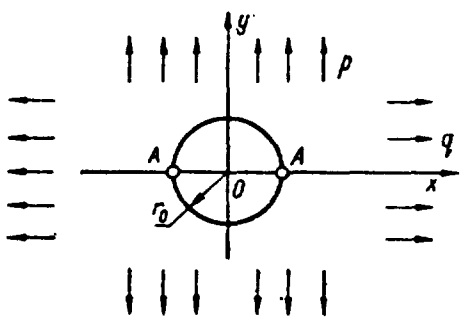


Figure A.50.

For small ranges of change of the stresses it is more convenient to use a simpler function:

$$\varepsilon_i = Bs_i^m, \tag{A.4}$$

where B is the time function t.

Samples of these materials were tested under uniaxial tension at a temperature of 20°C (the basic tests were also conducted at this temperature). The tests showed that for the material of the first model the constant b = 0.011, where s_i changes in the range 85-210 kg/cm^2, and m = 2.2. For the material of the second model, however, the constant b = 0.016, where s_i changes in the range 100-180 kg/cm^2 and m = 3.0.

Both models were taken in the form of a cross of thickness 4 mm. In the central part of the crosses, which measured 60 × 60 mm, a circular hole of diameter 7 mm was made. The values used for the parameters p, q and α in the tests are presented in Table A.6.

TABLE A.6

	100	114	114	100
q	0	38	76	100
$\alpha = q/p$	0	1/3	2/3	1

TABLE A.7

α	0	1/3	2/3	1	Model
k	2,00	1,82	1,67	1,50	First
k	1,72	1,57	1,43	1,30	Second

Tr. Note: Commas indicate decimal points.

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The intensity of the tangential stresses at "infinity" $s_i^{(\infty)} = \sqrt{p^2 - pq + q^2}$ was 100 kg/cm^2 in all tests.

The stress concentration coefficient k at point A (Figure A.50) was determined by the formula $k = \sigma/p$, where σ are stresses at point A, found by the optical method.

The studies revealed that the redistribution of stresses around holes was practically concluded for the first model after about 25 hr, and after only about 5 hr for the second model, after their loading. After the passing of the above-mentioned time, the models were subjected to the conditions of steady creep, for which the values found for k are presented in Table A.7.

Simple (approximate) formulas were found by I. I. Bugakov [1] for the concentration coefficient k at point A (Figure A.50):

$$k = 1 + \frac{2 - \alpha}{m}, \quad k = 1 + \frac{2 - \alpha}{1 + s_0},$$

the first of which is suggested for use with dependence (A.3) and the second, /876
for use with dependence (A.4).

The greatest stresses during the biaxial stretching of a plate with a round hole, illustrated in Figure A.50 ($p \geqslant q$), will not always be at point A of the contour of the hole, i.e., not for all values s_0 and m. Studies show that starting with certain values s_0 and m, the greatest stressed points are displaced from the contour of the hole to within the region in the direction of the Ox axis (Figure A.50). Apparently, these maximum tension stresses will differ only slightly from stresses σ_θ at point A. Thus, in the axisymmetric case ($p = q$), the greatest stress points near the hole for the "limit state of creep" will located[1] on a circle of radius $r = 2.07\ r_0$ (where r_0 is the radius of the hole) and the stress acting within them (the greatest, principal) will be 15% greater than stress σ_θ at point A. In the experimental determination of the values of k, presented in Table A.7, the greatest tension stresses were found at point A (Figure A.50).

§12. Glass Plastic Plate Weakened by a Circular Hole[2]

The geometric dimensions of the tested samples and the orientation of the principal stresses of anisotropy in relation to the direction of tension are presented in Figure A.51. The arrows with the numbers 1 and 2 indicate the principal stresses of anisotropy, where, for samples of type A, the moduli of elasticity along both principal directions of anisotropy are nearly equal

[1]See L. M. Kachanov [1], p. 227.
[2]The data are taken from the works of T. Hayashi [1, 2].

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(Table A.8), while for samples of type B, the ratio of the moduli is 3.5 (Table A.9). The elastic constants of these samples, given in Table A.10, were found by the tensometry method on samples subjected to tension. /877

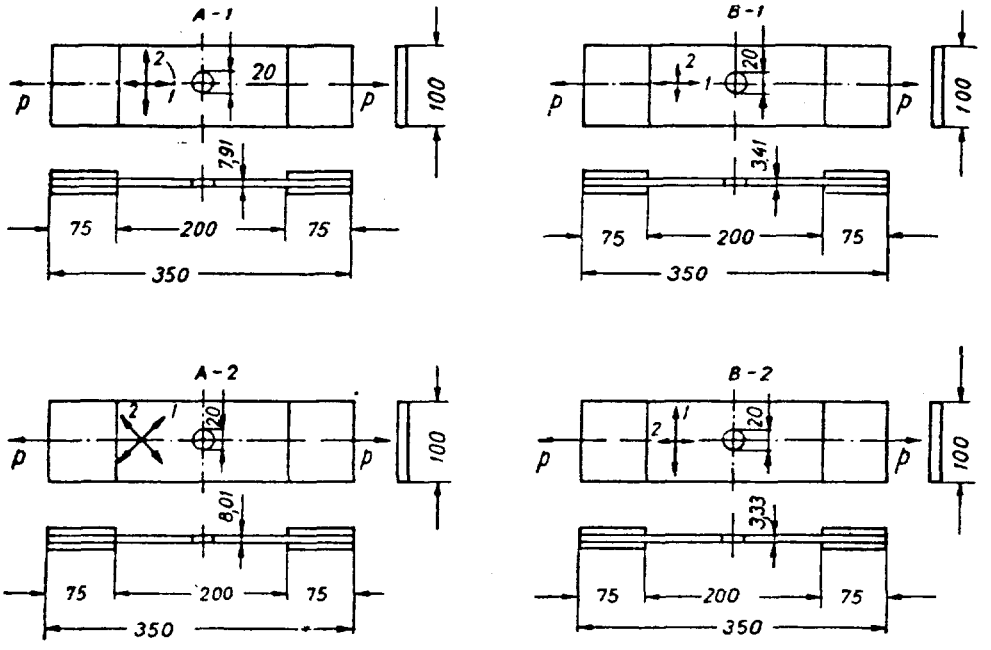


Figure A.51.

TABLE A.8

Action of force	Young's modulus kg/mm²	Poisson's ratio
In direction 1 (0°)	E_1=971	ν_1=0,188
In direction 2 (90°)	E_2=957	ν_2=0,171
At angle 45° . .	E_{45}=519	ν_{45}=0,547

TABLE A.9

Action of force	Young's modulus kg/mm²	Poisson's ratio
In direction 1 (0°)	E_1=2110	ν_1=0,260
In direction 2 (90°)	E_2=604	ν_2=0,104
At angle 45° . .	E_{45}=602	ν_{45}=0,421

Tr. Note: Commas indicate decimal points.

The samples were cut from a material made of 10 parallel layered sheets of glass fabric, saturated with polystyrene resin, where, for samples of type A, oriented glass plastic, bearing the Japanese trademark BHVT EC-181 was used, and for samples of type B, nonoriented glass plastic with the trademark BHAAX ECF-18. In both cases, polystyrene resin of the Rigolac 1553z type was used as the bonding agent. The similar coefficients of refraction of the glass fabric and the bonding agent resulted in good transparency for the investigation /878

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in which the glass fabric, which causes orthotropy in properties, had no effect on the optical activity of the bonding agent.

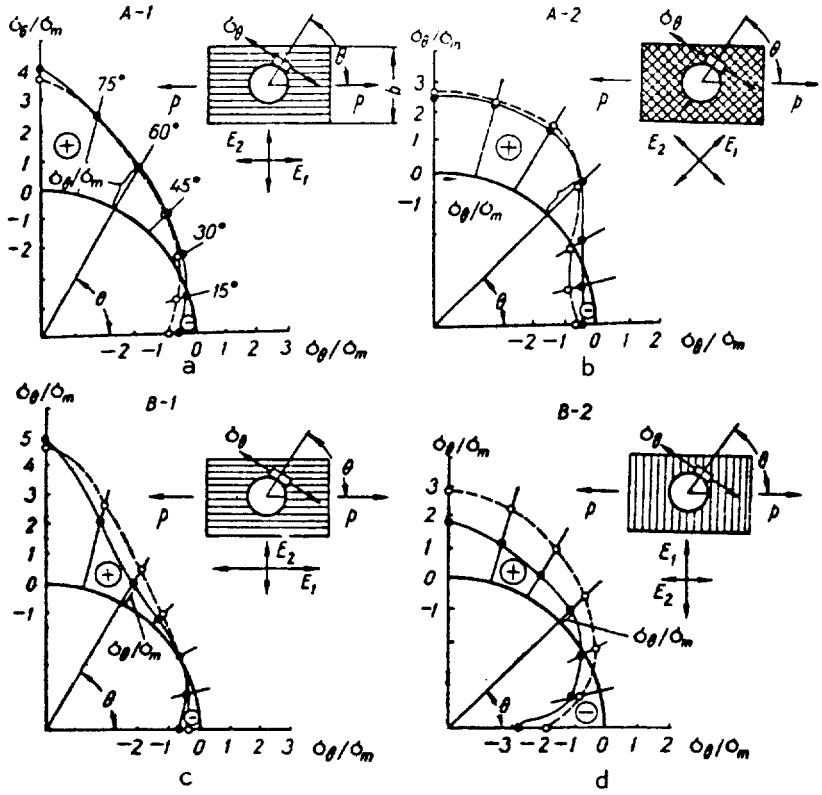


Figure A.52.

TABLE A.10

Elastic Constants kg/mm^2	for sample A	for sample B
E_1	964	2120
E_2	964	604
G	166	210
ν_1	0,188	0,260
ν_2	0,188	0,074

Tr. Note: Commas indicate decimal points.

Figure A.52 shows the diagrams of distribution of σ_θ/σ_m on the contour[1] of a circular hole for four types of analyzed samples. The solid line with the shaded circles represents experimental data and the broken line with the unshaded circles represents theoretical values of σ_θ/σ_m, calculated by T. Hayashi [1] by a formula[2] taken from the work of A. E. Green, G. J. Taylor [1]. This work is noteworthy in that due to

[1]We will note that $\sigma_m = P/bh$, where P is the force that stretches the plate along the axis; b is width, h is the thickness of the plate.
[2]This formula is the partial case of more general formula (III.25).

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the good agreement between the theoretical and experimental data, it becomes possible to analyze more complex problems of stress concentration around holes in glass plastics, for which the theoretical solutions are not yet known, and to use the photoelasticity method.

And so, from the data presented in Figure A.52, it is concluded that in determining the mean components of stresses in glass plastics, reinforced by glass fiber or glass fabrics, it possible[1] to replace them with a continuous elastic anisotropic body and to apply to them the corresponding formulas of elasticity theory of an anisotropic medium, presented in Chapter III.

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[1]It is reasoned that both the filler and the bonding agent should be sufficiently transparent, and the numerical values of their refraction coefficients should be about the same.

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