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THEORY OF LIFTING SURFACES.

PART I.

by

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PART I.

R E S U M E

A. GENERAL BASES OF THE THEORY.

The problem of the flow of a fluid about a lifting surface of infinite span was solved some time ago for certain of these profiles by taking as a basis for the determination of lift the following formula of Kutta-Joukowski:

$$A = \rho V l \Gamma$$

where A is the lift, V the velocity of the fluid to infinitude, l the span, and Γ the CIRCULATION OF THE VELOCITY.

This way of examining the problem necessarily assumes the existence of a polytropic function of the speed potential. But if we wish to treat the problem of the permanent flow of a fluid about a surface of fixed span, we must admit the existence of vortexes in the current. As a matter of fact, according to the theory of Stokes, the circulation along a closed contour moving in a fluid remains constant. Now, if a closed contour surrounding a wing leaves it by slipping along the wing itself, the circulation along this contour, which had a certain value so long as the contour surrounded the wing, must be nullified; therefore this contour must cut VORTEX LINES.

The existence of vortexes in a NON-VISCOUS fluid seems to be in contradiction with the laws of Lagrange and Helmholtz. But, as the author has shown in a work on "Flow of Fluids in the Case of a very slight Friction," we must consider the problem of the flow of

NON-VISCOUS fluids along solid bodies as the limit of flow of a very slightly viscous fluid. In this case the fluid leaves the body along a line called the LINE OF SEPARATION, forming the generation of a SURFACE OF SCISSION such that the speeds are not the same on both sides of the surface, thus forming a vortical layer.

The classical hydrodynamics of perfect fluids must therefore be completed by the following axioms and corollaries:

AXIOM I. Vortical layers may take their rise at lines of separation.

AXIOM II. Along the sharp edges of a body, infinite speeds can only occur exceptionally.

COROLLARY I. In any body, the edges perpendicular to the direction of the current are always lines of separation and are therefore the causes of the formation of a vortical layer.

COROLLARY II. The intensity of the vortexes in the vortical layer is such that the velocity along the edge has a constant value, or such that the tendency of the velocities to assume infinite values in this place is limited as far as possible.

The determination of the velocities due to a certain system of vortexes may be simplified by assuming, in place of the body, "ATTACHED" VORTEXES, the action of which is such that the flow remains the same as when the body is present, and by then determining the form of the corresponding body.

The relations of Euler may then be maintained on condition of replacing the forces acting on the body by a system of forces acting in the fluid.

The condition of continuity can also be maintained.

The conditions at the limits necessitating the coincidence of the normal to the speed with the normal to the surface of the body allows of determining the form of the body.

B. GENERAL THEORY OF PERMANENT FLOW.

In an infinite, homogeneous, and incompressible fluid, flowing along an ensemble of wings and forming only one vortical layer behind each wing, we may write for each point of the fluid the equation of Bernouilli:

$$p + \frac{\rho v^2}{2} = p_o + \frac{\rho V^2}{2} = p_o + q \quad . \quad . \quad . \qquad (1)$$

where V and p_o are the velocity and pressure of the fluid at infinitude, and v and p the velocity and pressure of the fluid at the point considered.

If we introduce the vectorial notations of Gibbs, so that the vectors are designated by heavy characters, and in which a.b. indicates the product of scalar values, a x b the product of the vectors, Euler's equation becomes:

$$\rho v') . \nabla v + \nabla \text{[illegible]} \qquad (2)$$

where k is the force acting on the unit of volume of the fluid and is due to the system of wings considered. We know that

$$v \, . \, \nabla v = \text{grad} \, \frac{v^2}{2} + \text{rot } v \times v$$

and that, in consequence of equation (1)

$$\text{grad} \, \frac{\rho v^2}{2} + p = 0$$

Consequently

$$k = \rho \text{ rot } v \times v \qquad (3)$$

The speed of rotation v is due to the attached vortexes and to the free vortexes; therefore

$$\text{rot } v = \gamma + \varepsilon \tag{4}$$

The k forces depend only on the attached vortexes γ whilst the movement of the ε vortexes is not governed by external forces. The relation (3) is thus divided into the two following relations:

$$\rho\,(\gamma \times v) = k \tag{5}$$

$$\varepsilon \times v = 0 \tag{5a}$$

The first represents the law of Kutta-Joukowski for the unit of volume. The second shows that the lines of vortexes are identical with the fluid screw. The resultant of the forces exerted by the wings on the fluid in a lifting space R is:

$$K = \iiint^R k\, d\tau = \rho \iiint^R (\gamma \times v)\, d\tau \tag{8}$$

The speed v is composed of the speed of the fluid at infinitude V and the speed v^o due to the vortexes $\gamma + \varepsilon$ and which, by the formula of Biot-Savart is

$$v = V + v^o = V + \iiint \frac{(\gamma + \varepsilon) \times r}{4\pi r^3}\, d\tau \tag{9}$$

Consequently:

$$K = \rho \left(\iiint^R \gamma\, d\tau \right) \times V$$

$$+ \rho \iiint^R \iiint \frac{\gamma \times ((\gamma' + \varepsilon') \times r)}{4\pi r^3}\, d\tau\, d\tau' \tag{10}$$

where the values accompanied by the mark ' refer to the integration of the expression (9) in space filled with vortexes. In the case where all the attached vortexes are parallel,

$$\iiint \iiint \frac{\gamma \times (\gamma' \times r)}{4\pi r^3}\, d\tau\, d\tau' = 0 \tag{11}$$

In order to simplify, it is useful to consider a "lifting screw" having a section F one of which coincides with the vortex lines of the γ field.

We have then:

$$d\tau = dF \cdot ds \quad \text{and} \quad ds \parallel \gamma$$

and we may write, replacing v by its mean value $\bar{v}$ in section F

$$k = \rho \int^{s} ds \times \left(\bar{v} \cdot \iint^{F} dF \cdot \gamma \right) = \rho \int ds \times \bar{v} \cdot \Gamma \qquad (12)$$

which constitutes a generalization of the law of Kutta-Joukowski for a lifting screw.

The formation of a system of vortexes behind the lifting surfaces has the effect of leaving in the fluid a certain quantity of kinetic energy, corresponding for a unit of volume to the stress of the force K equal to

$$k \cdot V = -k \cdot v^{o} = \rho \gamma \times v \cdot v^{o} = \rho \cdot \gamma \times V \cdot v^{o}$$

The total power expended in consequence of the total resistance W, is

$$K \cdot V = W \cdot V = \iiint \gamma \times V \cdot v^{o} \, d\tau =$$

$$= \iiint^{R} \iiint \gamma \times V \cdot \frac{(\gamma' + \varepsilon') \times r}{4 \pi r^{3}} \, d\tau \, d\tau' \qquad (13)$$

When the lifting vortexes are parallel we can neglect the influence of ε on v. we then have:

$$W \cdot V = \iiint^{R} \iiint \gamma \times V \cdot \frac{\varepsilon' \times r}{4 \pi r^{3}} \, d\tau \, d\tau' \qquad (13a)$$

C. SIMPLIFICATIONS REQUIRED.

Since in the actual state of mathematics it is impossible to find a rigorously correct solution of the above relations, certain simplifications are necessary. These consist in assuming that the force exerted by the air is very slight, and in retaining only the lowest terms in the relations.

Thus, in all expressions containing the sum of $V + v^o$, the term v^o will not be considered. We shall also suppress the second term of the resultant K in (10) but, on the contrary, will keep this term in the expression of the resistance W, where it preponderates.

Another simplification consists in assuming for each wing one single vortex-screw passing through the centers of gravity of the different sections, the intensity of which in each section is equal to the circulation in the section.

Thus, if we admit a single lifting screw perpendicular to the speed V (non-staggered monoplane), we shall obtain for this case the following relations:

$$w(x) = -\frac{1}{4\pi} \int_{x'=a}^{x'=b} \int_{y=0}^{y=\infty} \frac{d\Gamma}{dx'} \frac{(x - x')\, dx' dy}{((x - x')^2 + y^2)^{3/2}} =$$

$$= \frac{1}{4\pi} \int_{a}^{b} \frac{d\Gamma}{dx'} \cdot \frac{dx'}{x - x'} \qquad (14)$$

$$A = \rho\, V \int_a^b \Gamma\, dx \qquad (15)$$

$$W = \rho \int_a^b \Gamma\, w\, dx = \frac{\rho}{4\pi} \int_a^b \int_a^b \Gamma(x) \frac{d\Gamma}{dx'} \cdot \frac{dx dx'}{x - x'} \qquad (16)$$

where w is the component of the speed v^o following the normal to the speed V and the span of the wing is comprised between x = a and x = b.

These formulas only hold good when w is small in relation to V; now this involves the annulling of Γ at the extremities of the lifting screw, for otherwise the speed w would be in inverse proportion to the distance of the point considered at the tip of the wing, and this is not so in fact.

If we further assume that the lift is constant along the greater part of the wing and diminishes rapidly to zero point at the tips, the vortexes at the tip of each wing may be replaced by a single vortex-screw.

All the first works of the author were based on this consideration of three vortex-screws. This notion may be successfully employed in examining the interaction of a system of wings.

D. APPLICATIONS TO A SINGLE WING.

Three problems may be solved by means of the considerations exposed above:

1st. Given the distribution of lift along the span, as well as ρ and V, it is required to determine the configuration and resistance of the wing.

2nd. To determine for a given wing the resistance and the distribution of lift.

3rd. Given the total lift, the span, ρ and V, it is required to determine the distribution of lift affording the least resistance.

As regards the first two cases, if we know the relation existing between the lift and the angle of attack of a wing, we can at present only determine the angle of attack required for a given chord, and inversely, the chord required for a given angle of attack.

As a matter of fact we may assume that, for the lift of an element of a wing of finite span to be the same as that of an element of a wing of infinite span attacked at an angle α', the former element must have an angle of attack

$$\alpha = \alpha' + \operatorname{arctg} w/V \qquad (17)$$

The author gives an example of the solution of the first two problems for a wing of infinite span where the lift is a recurring function of the section. A Fourier serial development gives terms in the form $\Gamma = \overline{\Gamma} \cos \mu x$. We then have (expression 14)

$$w = \frac{\mu \Gamma}{4}$$

and if we assume $\Gamma = Vtc_1 \alpha'$ where t is the chord of the wing, c_1 a numerical coefficient which, according to Mises is equal to $\overline{II}$, we obtain

$$\alpha = \alpha + \frac{w}{V} = \alpha' \left(1 + \frac{c_1}{4} \mu \Gamma \right)$$

and the resistance W for a given span l

$$W = \frac{\rho l \overline{\Gamma}^2 \mu}{8} \qquad (20)$$

The inverse problem will be solved by giving a Fourier serial development to the given function: $\alpha = f(x)$ and by determining for each term of this series the corresponding values of α' and of Γ, and then taking their sum in order to obtain the values of the total lift and resistance.

A more important example refers to a single wing of finite span for which

$$\Gamma = \sqrt{1 - \xi^2} \left(\Gamma_0 + \Gamma_2 \xi^2 + \Gamma_4 \xi^4 + \ldots \right)$$

where $\xi = x : b/2$

The application of formulas 14, 15, and 16 gives:

$$A = \frac{\pi}{4} \rho bV \left(\Gamma_0 + \frac{\Gamma_2}{4} + \frac{\Gamma_4}{8} + \ldots \right) \qquad (21a)$$

$$w = \frac{1}{2b}\left\{\Gamma_0 + \Gamma_2 (3\xi^2 - \frac{1}{2}) + \Gamma_4 (5\xi^4 - \frac{3}{2}\xi^2 - \frac{1}{16}) + \ldots\right\} \quad (22a)$$

$$W = \frac{\pi\rho}{4}\left\{\frac{1}{2}\Gamma_0^2 + \frac{1}{4}\Gamma_0\Gamma_2 + \frac{1}{8}\Gamma_0\Gamma_4 + \ldots + \frac{1}{8}\Gamma_2^2 + \frac{11}{64}\Gamma_2\Gamma_4 + \ldots + \frac{1}{256}\Gamma_4^2 + \ldots\right\} \quad (23a)$$

These formulas enable us to deal with the problem of minimum resistance for a given lift; we arrive at the conclusion that the distribution of lift along the span must, in this case, take an elliptical form and that the speed w is then constant: We have

$$w = \frac{2A}{\pi\rho\, Vb^2} \quad (24)$$

$$W = \frac{w}{V} \cdot A = \frac{2A^2}{\pi\rho\, V^2b^2} \quad (25)$$

The plane form of the wing should consist of two demi-ellipses joined on the line of their main axis equal to the span.

For such wings we can thus bring out the influence of the Aspect Ratio; in fact, we have

$$\alpha = \alpha' + \frac{c_a F}{\pi b^2} \quad (26)$$

where $c_a = A/qF$, $q = \frac{\rho V^2}{2}$, F is the surface of the wing.

The actual resistance W of a wing is equal to the sum of the INDUCED RESISTANCE W_1 and the PROFILE RESISTANCE W_0.

The coefficients of resistance are respectively:

$$c_w = c_{wi} + c_{wo}$$

where $c_{wi} = \dfrac{c_a^2 \cdot F}{\pi \; b^2}$

We can thus pass from a wing for which we know the aerodynamical characteristics for a given aspect ratio, to the same wing having any aspect ratio whatever.

MM. Betz and Munk, the author's collaborators, have shown that the above formulas determining the influence of the aspect ratio, may be applied to all wings, whatever be their plane form.

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