TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 827

ANALYTICAL DETERMINATION OF CONTROL SYSTEM PULLEY-AXIS ANGLES

By I. H. Driggs
Bureau of Aeronautics, Navy Department

Washington
September 1941

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 827

ANALYTICAL DETERMINATION OF CONTROL SYSTEM PULLEY-AXIS ANGLES

By I. H. Driggs

The ideas developed in this paper are presented as a means of saving the designer's time and as a method of reduction of control-system friction by an accurate calculation of pulley-axis angles without the errors and difficult checking incident to any graphical method.

The growing realization of the necessity for greater care and refinement in the design of control systems is justification for the more accurate analytical methods given. Saving in layout and checking time is also of importance, although probably secondary to the greater refinement obtainable by the use of a calculating machine even in the hands of inexperienced personnel. It is suggested that the formulas given here are worthy of careful trial by any engineering department that is not already using an equivalent method of design.

Two mathematical laws are employed in the derivation of the formulas for pulley-axis angles to give correct alinement with control cables and therefore avoid binding, and to reduce friction. These laws are:

1. Any two intersecting straight lines determine a single plane.

2. The first partial differentials of the equation of a surface are proportional to the direction cosines of the normal to that surface at any point chosen for investigation.

Since any pulley, to operate properly, must have its center plane coincident with the plane containing the cables loading around it, the location and angles of the pulley axis, with reference to the coordinate planes, are the values required to properly construct and locate a pulley bracket. The equation of the plane of the two cables (and of the pulley) is first determined and then the direction cosines of the pulley axis found from the first partial derivatives of this plane. To express mathematically:

$$f(x,y,z) = Ax + By + Cz = 0$$

This is the form for the equation of a plane surface passing through three points, one of them the origin of coordinates.

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C$$

The values A, B, and C are proportional to the direction cosines; L, M, and N of the normal to the plane, f(x,y,z).

Then

$$L = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad M = \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad N = \frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

If the coordinates of each end of two intersecting lines (or cables) are: P_1 (x_1, y_1, z_1) and P_3 (o, o, o) and P_2 (x_2, y_2, z_2), respectively, then the equation containing these three points and the two straight lines or cables is (fig. 1):

$$x(y_1 z_2 - y_2 z_1) + y(z_1 x_2 - x_1 z_2) + z(x_1 y_2 - y_1 x_2) = o$$

Then

$$A = y_1 z_2 - y_2 z_1$$

$$B = z_1 x_2 - x_1 z_2$$

$$C = x_1 y_2 - y_1 x_2$$

It is to be noted that this equation is true only if one of the three points used to determine the plane is taken as the origin and the coordinates of the other two points measured therefrom. (See fig. 1.) Positive directions and angles are also defined in this sketch.

It should be noted that the points P_1, P_2, P_3 must lie on the cables, that is, in figure 1. P_3 is the intersection of the lines $P_1 - P_5$ and $P_2 - P_3$. This point is not the center of the pulley. The sketch (fig. 2) illustrates this point.

Some confusion regarding the signs of the angles θ, ψ, and ∅ may be avoided if the direction cosines L, M, and N are considered to be the coordinates x_n, y_n, and z_n of a point on the normal to the pulley plane unit distance from the origin. The proper angle of this normal may be laid out on the drawing board by choosing any convenient length - say, 10 inches - as the unit distance and laying out the coordinates of the point x_n, y_n, and z_n.

Since

$$x_n = 10L$$

$$y_n = 10M$$

$$z_n = 10N$$

If it is desired to compute the angles for purposes of greater accuracy, the above graphical method suggests the following:

$$\tan\theta = \frac{N}{L}$$

$$\tan\psi = \frac{L}{M}$$

$$\tan\phi = \frac{M}{N}$$

It may be desired to find the angle between the two cables in plane containing them for purposes of layout of cable guides, etc. If this angle is α, then

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{(\sqrt{x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}}$$

The coordinates of the center of the pulley may also be determined by similar analytic means but no simple expression can be readily found, since the values of x_a, y_a, and z_a (the coordinates of the pulley center) depend upon the solution of three simultaneous equations. It is believed that sufficient accuracy can be obtained if a combination graphical and analytical solution is employed.

A layout is made to scale similar to figure 3, in which the two cables intersecting at the center of coordinates are shown in all three views. Bisectors of the angles between the cables are then drawn in each view by the usual graphical construction. We know that these three bisectors must contain the center of the pulley axis, which must likewise be equidistant from the two cables. The above solution for the true included angle, α, between the cables allow a solution to be reached either graphically or analytically for the true distances from the axis of coordinates (intersection of cables).

Analytically

$$d = \frac{R}{\sin \frac{\alpha}{2}} \qquad R = \text{radius of pulley to cable center line}$$

If the direction cosines of the bisector as laid out above are determined, then the coordinates x_a, y_a, and z_a can be calculated. The direction cosines of this bisector are found by choosing any point, such as P_b (fig. 3), on the line and projecting through the three views. If the coordinates of this point are x_b, y_b, and z_b, then

$$L_b = \frac{x_b}{\sqrt{(x_b^2 + y_b^2 + z_b^2)}}$$

$$M_b = \frac{y_b}{\sqrt{(x_b^2 + y_b^2 + z_b^2)}}$$

$$N_b = \frac{z_b}{\sqrt{(x_b^2 + y_b^2 + z_b^2)}}$$

Then

$$x_a = d\ L_b$$

$$y_a = d\ M_b$$

$$z_a = d\ N_b$$

Example

Figure 3 is a line sketch that might be made up giving the locations of three points through which cables must pass for a given airplane installation for use in finding the pulley-axis angles.

Choose the center point, P_3, as the origin. (Note: Any one of the three points might have been taken but it is somewhat clearer to place the origin at the point of intersection of the two cables where the pulley is to be located.)

$$x_1 = -40 \text{ in.} \qquad x_2 = 20 \text{ in.}$$

$$y_1 = 25 \text{ in.} \qquad y_2 = -8 \text{ in.}$$

$$z_1 = 10 \text{ in.} \qquad z_2 = 40 \text{ in.}$$

$$A = y_1\ z_2 - y_2\ z_1 = 25 \times 40 + 8 \times 10 = 1080$$

$$B = z_1\ x_2 - x_1\ z_2 = 10 \times 20 + 40 \times 40 = 1800$$

$$C = x_1\ y_2 - y_1\ x_2 = 40 \times 8 - 25 \times 20 = -180$$

Then

$$\sqrt{A^2 + B^2 + C^2} = 2104$$

The equation of the plane containing the two cables of figure 3 is then $1080x + 1800y - 180z = 0$ or $6x + 10y - z = 0$.

$$L = \frac{1080}{2104} = 0.5138 \qquad x_n = 5.138 \text{ in.}$$

$$M = \frac{1800}{2104} = .8560 \qquad y_n = 8.56 \text{ in.}$$

$$N = \frac{-180}{2104} = -.0856 \qquad z_n = -.856 \text{ in.}$$

$$\tan\theta = -0.1667, \quad \tan\psi = 0.600, \quad \tan\phi = -10.00$$

$$\theta = -9^\circ\ 30' \quad \psi = 31^\circ \quad \phi = 95^\circ\ 40'$$

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\left(\sqrt{x_1^2 + y_1^2 + z_1^2}\right)\left(\sqrt{x_2^2 + y_2^2 + z_2^2}\right)}$$

$$= \frac{-40 \times 20 - 25 \times 8 + 10 \times 40}{\left(\sqrt{(-40)^2 + (25)^2 + 10^2}\right)\left(\sqrt{20^2 + (-8)^2 + (40)^2}\right)}$$

$$= \frac{-600}{48.2 \times 45.5}$$

$$\cos\alpha = \frac{-600}{2190} = -0.274$$

$$\alpha = 105^\circ\ 50' +$$

Let

$R = 1.50$ in. - radius of pulley to cable center

Then

$$d = \frac{1.5}{\sin 52^\circ\ 40'} = \frac{1.5}{0.7951} = 1.886 \text{ in.}$$

The bisectors of the angles between the cables are drawn and the point, P_b, is chosen at random. The co-ordinates of this point are scaled as:

$$x_b = -11.6$$

$$y_b = 13.75$$

$$z_b = 25.00$$

$$L_b = \frac{-11.6}{30.95} = -0.375$$

$$M_b = \frac{13.75}{30.95} = 0.4245$$

$$N_b = \frac{25.0}{30.95} = 0.808$$

$$x_b = -1.886 \times 0.375 = -0.7075$$

$$y_b = 1.886 \times 0.4245 = 0.801$$

$$z_b = 1.886 \times 0.808 = 1.525$$

From the above example, the steps necessary for the solution of this problem may be listed as follows:

1. Obtain a line sketch, similar to figure 3, giving the locations of the intersection point of the two cables for which a pulley bracket is to be designed and one other point on each cable. This sketch should be fully dimensioned to approximate scale from any convenient planes, as shown in figure 3.

2. Choose the origin at the intersection of the two cables, P_3, in figure 3, and then determine and tabulate x_1, y_1, z_1 and x_2, y_2, z_2 with due regard to signs.

3. Calculate A, B, and C as:

$$A = y_1 z_2 - y_2 z_1$$

$$B = z_1 x_2 - x_1 z_2$$

$$C = x_1 y_2 - y_1 x_2$$

4. Find $\sqrt{A^2 + B^2 + C^2}$ and then,

L, M, N as:

$$L = \frac{A}{\sqrt{A^2 + B^2 + C^2}}$$

$$M = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

$$N = \frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

5. Determine x_n, y_n, and z_n from formulas

$$x_n = 10L$$

$$y_n = 10M$$

$$z_n = 10N$$

These dimensions are then laid out to determine axis angles in three views as the angles may be computed from:

$$\tan \theta = N/L$$

$$\tan \psi = L/M$$

$$\tan \phi = M/N$$

6. If the angle α between the two cables is desired, compute from

$$\cos \alpha = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{(\sqrt{x_1^2 + y_1^2 + z_1^2})(\sqrt{x_2^2 + y_2^2 + z_2^2})}$$

Find α from a table of trigometric functions.

7. Bisect the angles between cables in three views on scale layout, choosing any point, such as P_b (fig. 3) on this bisector.

8. Determine the coordinates of this point x_b, y_b, and z_b.

9. Find distance from cable intersection to pulley axis along this bisector in true view from formula:

$$d = \frac{R}{\sin \frac{\alpha}{2}}$$

10. Find direction cosines of bisector as

$$L_b = \frac{x_b}{\sqrt{x_b^2 + y_b^2 + z_b^2}}$$

$$M_b = \frac{y_b}{\sqrt{x_b^2 + y_b^2 + z_b^2}}$$

$$N_b = \frac{z_b}{\sqrt{x_b^2 + y_b^2 + z_b^2}}$$

11. Find coordinates of pulley center as:

$$x_a = d\ L_b$$

$$y_a = d\ M_b$$

$$z_a = d\ N_b$$

Bureau of Aeronautics, Navy Department,
Washington, D. C., September 17, 1941.

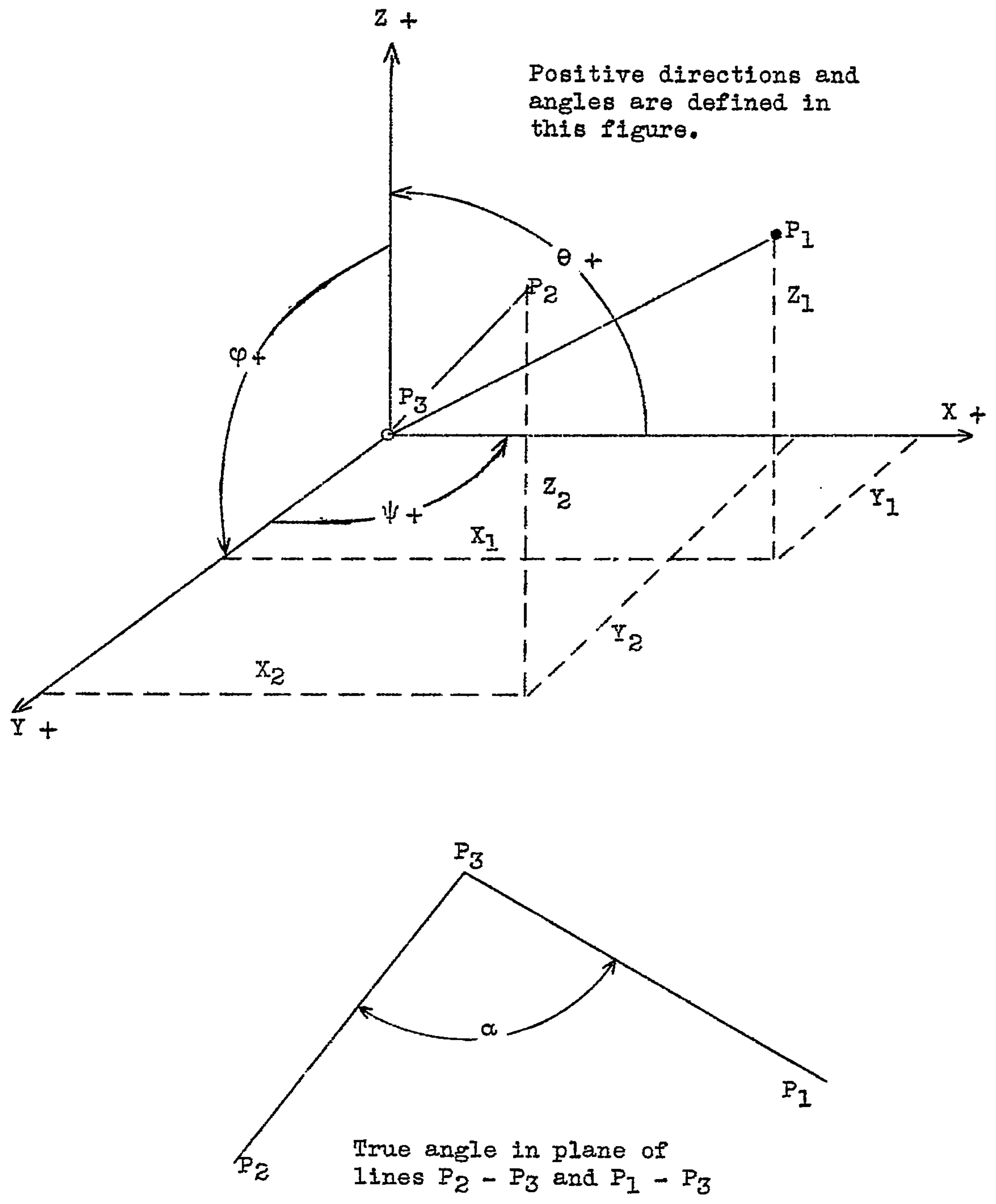

Z +
Positive directions and angles are defined in this figure.
P1
θ +
P2
Z1
φ +
P3
X +
Z2
ψ +
Y1
X1
Y2
X2
Y +
P3
α
P1
P2
True angle in plane of lines P2 - P3 and P1 - P3

Figure 1

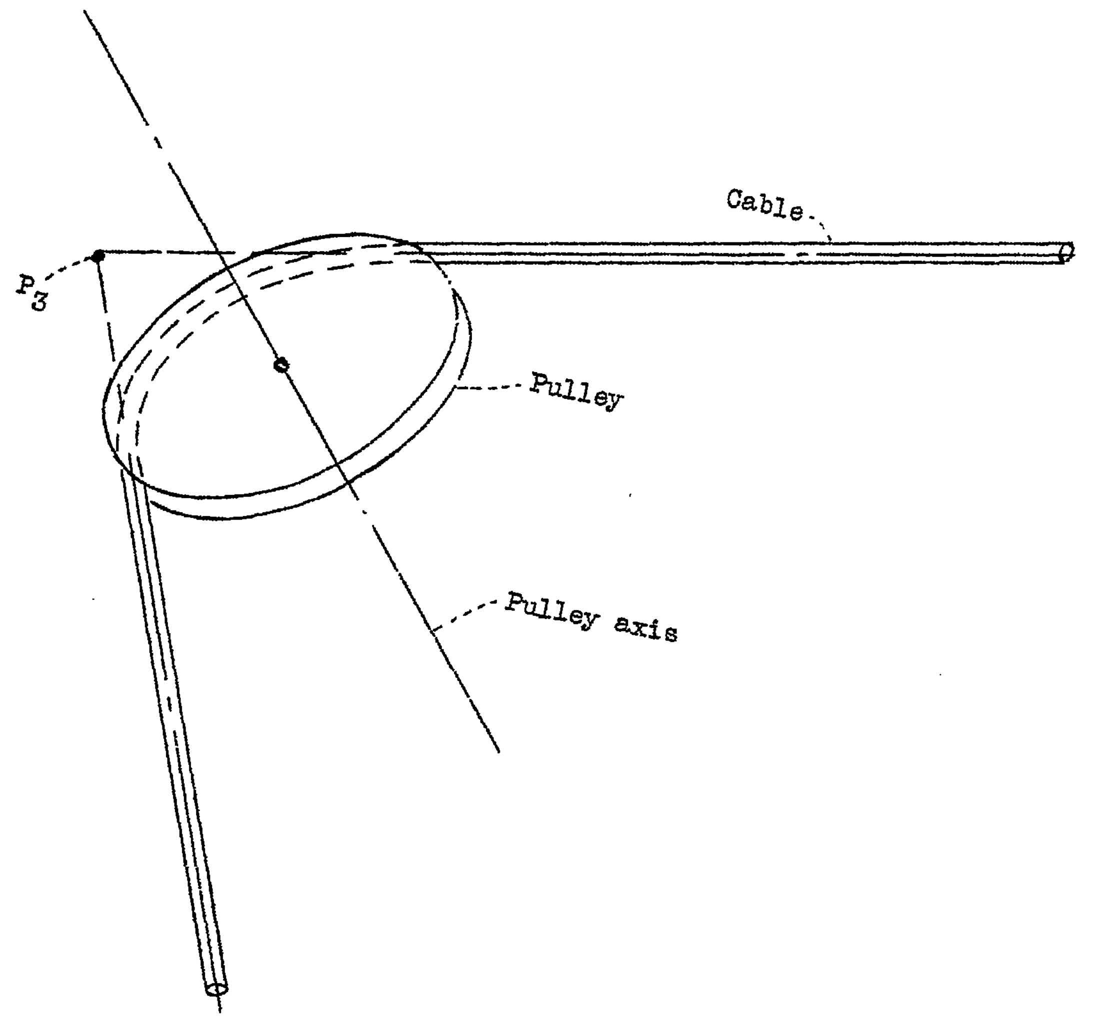
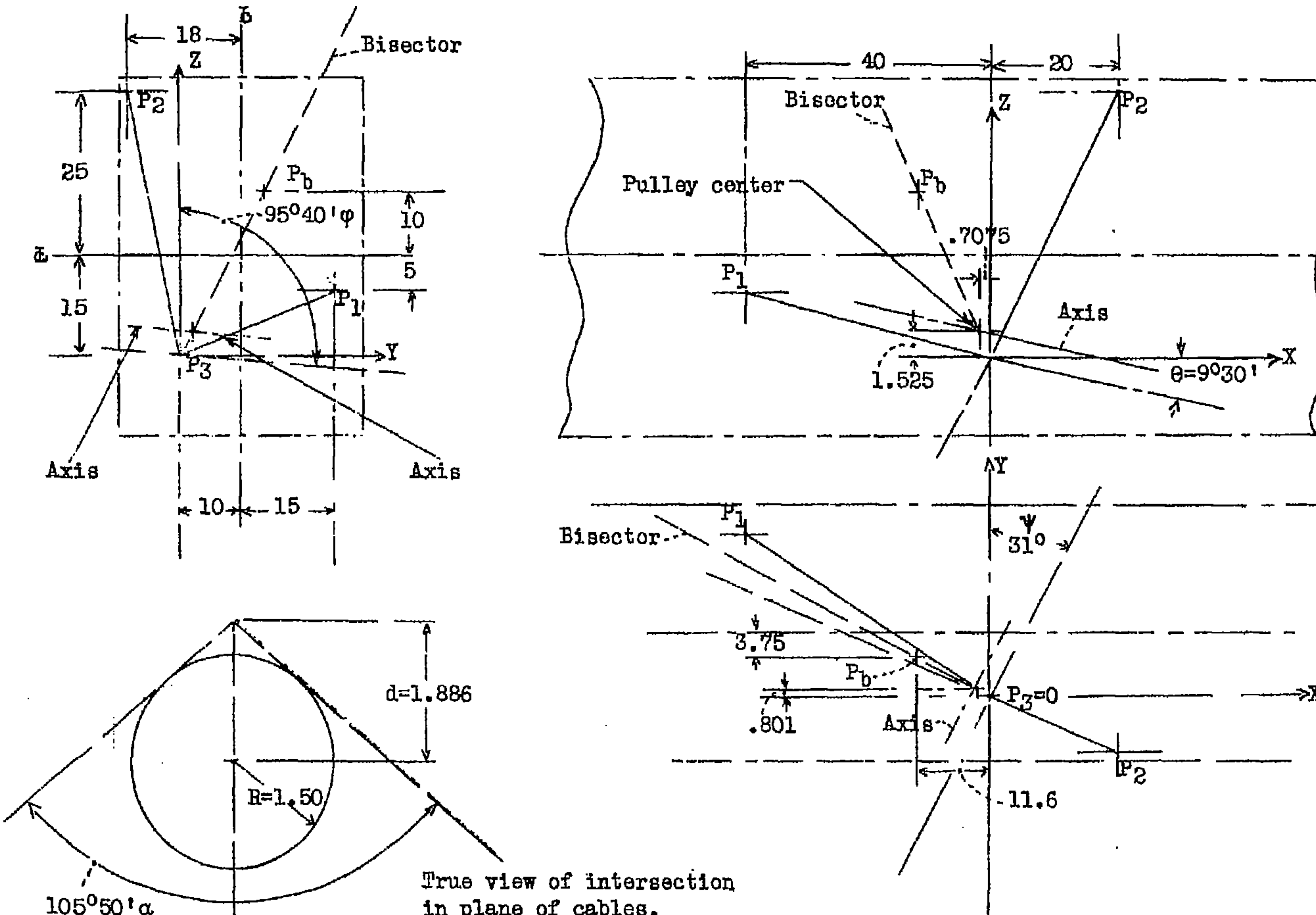


Figure 2



True view of intersection in plane of cables.

Figure 3