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GENERAL THEORY OF AERODYNAMIC INSTABILITY AND THE MECHANISM OF FLUTTER

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SUMMARY

The aerodynamic forces on an oscillating airfoil or airfoil-aileron combination of three independent degrees of freedom have been determined. The problem resolves itself into the solution of certain definite integrals, which have been identified as Bessel functions of the first and second kind and of zero and first order. The theory, being based on potential flow and the Kutta condition, is fundamentally equivalent to the conventional wing-section theory relating to the steady case.

The air forces being known, the mechanism of aerodynamic instability has been analysed in detail. An exact solution, involving potential flow and the adoption of the Kutta condition, has been arrived at. The solution is of a simple form and is expressed by means of an auxiliary parameter \( k \). The mathematical treatment also provides a convenient cyclic arrangement permitting a uniform treatment of all subcases of two degrees of freedom. The flutter velocity, defined as the air velocity at which flutter starts, and which is treated as the unknown quantity, is determined as a function of a certain ratio of the frequencies in the separate degrees of freedom for any magnitudes and combinations of the airfoil-aileron parameters.

For those interested solely or particularly in the numerical solutions Appendix I has been prepared. The routine procedure in solving numerical examples is put down detached from the theoretical background of the paper. It first is necessary to determine a certain number of constants pertaining to the case, then to perform a few routine calculations as indicated. The result is readily obtained in the form of a plot of flutter velocity against frequency for any values of the other parameters chosen. The numerical work of calculating the constants is simplified by referring to a number of tables, which are included in Appendix I. A number of illustrative examples and experimental results are given in Appendix II.

INTRODUCTION

It has been known that a wing or wing-aileron structurally restrained to a certain position of equilibrium becomes unstable under certain conditions. At least two degrees of freedom are required to create a condition of instability, as it can be shown that vibrations of a single degree of freedom would be damped out by the air forces. The air forces, defined as the forces due to the air pressure acting on the wing or wing-aileron in an arbitrary oscillatory motion of several degrees of freedom, are in this paper treated on the basis of the theory of nonstationary potential flow. A wing-section theory and, by analogy, a wing theory shall be thus developed that applies to the case of oscillatory motion, not only of the wing as a whole but also to that of an aileron. It is of considerable importance that large oscillations may be neglected; in fact, only infinitely small oscillations about the position of equilibrium need be considered. Large oscillations are of no interest since the sole attempt is to specify one or more conditions of instability. Indeed, no particular type or shape of airfoil shall be of concern, the treatment being restricted to primary effects. The differential equations for the several degrees of freedom will be put down. Each of these equations contains a statement regarding the equilibrium of a system of forces. The forces are of three kinds: (1) The inertia forces, (2) the restraining forces, and (3) the air forces.

There is presumably no necessity of solving a general case of damped or divergent motion, but only the borderline case of a pure sinusoidal motion, applying to the case of unstable equilibrium. This restriction is particularly important as the expressions for the air force developed for oscillatory motion can thus be employed. Imagine a case that is unstable to a very slight degree; the amplitudes will then increase very slowly and the expressions developed for the air forces will be applicable. It is of interest simply to know under what circumstances this condition may obtain and cases in which the amplitudes are decreasing or increasing at a finite rate need not be treated or specified. Although it is possible to treat the latter cases, they are of no concern in the present problem. Nor is the internal or solid friction of the structure of primary concern. The fortunate situation exists that the effect of the solid friction is favorable. Knowledge is desired concerning the condition as existing in the absence of the internal friction, as this case constitutes a sort of lower limit, which it is not always desirable to exceed.
Owing to the rather extensive field covered in the paper it has been considered necessary to omit many elementary proofs, it being left to the reader to verify certain specific statements. In the first part of the paper, the velocity potentials due to the flow around the airfoil-aileron are developed. These potentials are treated in two classes: The noncircular flow potentials, and those due to the surface of discontinuity behind the wing, referred to as "circular" potentials. The magnitude of the circulation for an oscillating wing-aileron is determined next. The forces and moments acting on the airfoil are then obtained by integration. In the latter part of the paper the differential equations of motion are put down and the particular and important case of unstable equilibrium is treated in detail. The solution of the problem of determining the flutter speed is finally given in an equation expressing a relationship between the various parameters. The three subcases of two degrees of freedom are treated in detail.

The paper proposes to disclose the basic nature of the mechanism of flutter, leaving modifications of the primary results by secondary effects for future investigations. Such secondary effects are: The effects of a finite span, of section shape, of deviations from potential flow, including also modifications of results to include twisting and bending of actual wing sections instead of pure torsion and deflection as considered in this paper.

The supplementary experimental work included in Appendix II similarly refers to well-defined elementary cases, the wing employed being of a large aspect ratio, nondeformable, and given definite degrees of freedom by a supporting mechanism, with external springs maintaining the equilibrium positions of wing or wing-aileron. The experimental work was carried on largely to verify the general shape of and the approximate magnitudes involved in the theoretically predicted response characteristics. As the present report is limited to the mathematical aspects of the flutter problem, specific recommendations in regard to practical applications are not given in this paper.

VELOCITY POTENTIALS, FORCES, AND MOMENTS OF THE NONCIRCULATORY FLOW

We shall proceed to calculate the various velocity potentials due to position and velocity of the individual parts in the whole of the wing-aileron system. Let us temporarily represent the wing by a circle (fig. 1). The potential of a source \( \varepsilon \) at the origin is given by

\[
\varphi = \frac{\varepsilon}{4\pi} \log (x^2 + y^2)
\]

For a source \( \varepsilon \) at \((x_1, y_1)\) on the circle

\[
\varphi = \frac{\varepsilon}{4\pi} \log \left\{ (x - x_1)^2 + (y - y_1)^2 \right\}
\]

Putting a double source \( 2\varepsilon \) at \((x_1, y_1)\) and a double negative source \(-2\varepsilon\) at \((x_1, -y_1)\) we obtain for the flow around the circle

\[
\varphi = \frac{\varepsilon}{2\pi} \log \left\{ (x - x_1)^2 + (y - y_1)^2 \right\}
\]

The function \( \varphi \) on the circle gives directly the surface potential of a straight line \( pq \), the projection of the circle on the horizontal diameter. (See fig. 1.) In this case \( y = \sqrt{1 - x^2} \) and \( \varphi \) is a function of \( x \) only.

We shall need the integrals:

\[
\int^1_0 \frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_0)^2 + (y - y_0)^2} \, dx_1 = 2(\varepsilon - c) \log N - 2\sqrt{1 - x^2} \cos^{-1}c
\]

and

\[
\int^1_0 \log \left\{ \frac{(x - x_1)^2 + (y - y_1)^2}{(x - x_1)^2 + (y - y_1)^2} \right\} (x_1 - c) \, dx_1 = -\sqrt{1 - x^2} \sqrt{1 - c^2} - \cos^{-1}(c - x) \sqrt{1 - x^2} + (x - c)^2 \log N
\]

where

\[
N = 1 - cx - \sqrt{1 - x^2} \sqrt{1 - c^2}
\]

The location of the center of gravity of the wing-aileron \( x_a \) is measured from \( a \), the coordinate of the axis of rotation (fig. 2); \( x_a \) is the location of the center of gravity of the aileron. The location of the hinge is measured from \( \ell \), the coordinate of the hinge; and \( r_a \) and \( r_\ell \) are the radii of gyration of the wing-aileron referred to \( a \), and of the aileron referred to the hinge. The quantities \( x_b \) and \( y_b \) are "reduced" values, as defined later in the paper. The quantities \( a, \alpha, c, \) and \( k \) are positive toward the rear (right), \( h \) is the vertical coordinate of the axis of rotation at \( a \) with respect to a fixed reference frame and is positive downward. The angles \( \alpha \) and \( \beta \) are positive clockwise (right-hand turn). The wind velocity \( v \) is to

\[
\text{Figure 1.—Conformal representation of the wing profile by a circle.}
\]

\[
\text{Figure 2.—Parameters of the airfoil-aileron combination.}
\]
the right and horizontal. The angle (of attack) \( \alpha \) refers to the direction of \( v \), the aileron angle \( \beta \) refers to the undeflected position and not to the wind direction. The quantities \( r_a \) and \( r_b \) always occur as squares. Observe that the leading edge is located at \(-1\), the trailing edge at \(+1\). The quantities \( a, c, x_a, x_b, r_a, r_b \), and \( r_b \), which are repeatedly used in the following treatment, are all dimensionless with the half chord \( b \) as reference unit.

The effect of a flap bent down at an angle \( \beta \) (see fig. 2) is seen to give rise to a function \( \varphi \) obtained by substituting \(-v\beta b\) for \( \epsilon \); hence

\[
\varphi = \varphi b \sqrt{1 - x^2} \cos \epsilon - (x - c) \log N
\]

To obtain the effect of the flap going down at an angular velocity \( \dot{\beta} \), we put \( \epsilon = -(x_1 - x)\dot{b} \) and get

\[
\varphi = \frac{\dot{b}}{2\pi} \left[ \sqrt{1 - c^2} \sqrt{1 - x^2} + \cos c(x - 2c) \right] \sqrt{1 - x^2}
\]

\[
- (x - c)^3 \log N
\]

To obtain the effect of an angle \( \alpha \) of the entire airfoil, we put \( \epsilon = -1 \) in the expression for \( \varphi_\beta \), hence

\[
\varphi = c(\alpha - b) \sqrt{1 - x^2}
\]

To depict the airfoil in downward motion with a velocity \( \dot{b} \) (+ down), we need only introduce \( \frac{\dot{b}}{b} \) instead of \( c \).

Thus

\[
\varphi = \dot{b} \sqrt{1 - x^2}
\]

Finally, to describe a rotation around point \( a \) at an angular velocity \( \alpha \), we notice that this motion may be taken to consist of a rotation around the leading edge \( c = -1 \) at an angular velocity \( \alpha \) plus a vertical motion with a velocity \(-a(1 + \alpha)\dot{b}\). Then

\[
\varphi = \frac{\dot{b}}{2\pi} \left[ \pi(x + 2) \sqrt{1 - x^2} - a(1 + \alpha) \right] \sqrt{1 - x^2}
\]

\[
\frac{\dot{b}}{2\pi} \left[ \frac{1}{2} x - a \right] \sqrt{1 - x^2}
\]

The following tables give in succession the velocity potentials and a set of integrals with associated constants, which we will need in the calculation of the air forces and moments.

**Velocity Potentials**

\[
\varphi = v \alpha b \sqrt{1 - x^2}
\]

\[
\varphi = \dot{b} \sqrt{1 - x^2}
\]

\[
\varphi = \dot{b} \left( \frac{1}{2} x - a \right) \sqrt{1 - x^2}
\]

\[
\varphi = - \frac{1}{\pi} \beta \sqrt{1 - x^2} \cos \epsilon - (x - c) \log N
\]

\[
\varphi = \frac{1}{2\pi} \beta \sqrt{1 - x^2} \cos \epsilon - (x - c) \log N
\]

where

\[
N = \frac{1 - 2c - \sqrt{1 - x^2} \sqrt{1 - c^2}}{x - c}
\]

**Integrals**

\[
\int_1^{+1} \varphi \, dx = \frac{b}{2} \alpha T_1
\]

\[
\int_1^{+1} \varphi \, dx = \frac{b}{2} \beta T_4
\]

\[
\int_1^{+1} \varphi \, dx = \frac{b}{2} \pi T_5
\]

\[
\int_1^{+1} \varphi \, dx = \frac{b}{2} \pi \alpha T_6
\]

\[
\int_1^{+1} \varphi \, dx = \frac{b}{2} \pi \beta T_7
\]

\[
\int_1^{+1} \varphi \, dx = -\frac{b}{2} \pi \alpha T_8
\]

\[
\int_1^{+1} \varphi \, dx = -\frac{b}{2} \pi \beta T_9
\]

**Constants**

\[
T_1 = -\frac{1}{3} \sqrt{1 - c^2}(2 + c^2) + c \cos \epsilon
\]

\[
T_2 = c(1 - c^2) - \frac{1}{3} c^2(1 + c^2) \cos \epsilon + c(\cos \epsilon)^2
\]

\[
T_3 = -\frac{1}{3} c^2(1 + c^2) - \frac{1}{2} c \sqrt{1 - c^2} \cos \epsilon + c(\cos \epsilon)^2
\]

\[
T_4 = \frac{b}{2} \pi \alpha T_1 - \frac{b}{2} \pi \alpha T_2
\]

\[
T_5 = \frac{b}{2} \pi \beta T_3 - \frac{b}{2} \pi \beta T_4
\]

\[
T_6 = \frac{b}{2} \pi \beta T_5 - \frac{b}{2} \pi \beta T_6
\]

\[
T_7 = \frac{b}{2} \pi \beta T_7 - \frac{b}{2} \pi \beta T_8
\]

**Forces and Moments**

The velocity potentials being known, we are able to calculate local pressures and by integration to obtain the forces and moments acting on the airfoil and aileron.

\[\text{(some of the more difficult integral evaluations are given in Appendix III.}\]
Employing the extended Bernoulli Theorem for unsteady flow, the local pressure is, except for a constant

\[ p = -\rho \left( \frac{\omega^2}{2} + \frac{\partial \varphi}{\partial t} \right) \]

where \( \omega \) is the local velocity and \( \varphi \) the velocity potential at the point. Substituting \( \omega = v + \frac{\partial \varphi}{\partial x} \), we obtain ultimately for the pressure difference between the upper and lower surface at \( x \)

\[ p = -2\rho \left( v \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} \right) \]

where \( v \) is the constant velocity of the fluid relative to the airfoil at infinity. Putting down the integrals for the force on the entire airfoil, the moment on the flap around the hinge, and the moment on the entire airfoil, we obtain by means of partial integrations

\[ P = -2\rho b \int_{-1}^{+1} \varphi dx \]

\[ M_\beta = -2\rho b \int_{-1}^{1} \frac{1}{2} \varphi(x-c) dx + 2\rho b \int_{c}^{1} \varphi dx \]

\[ M_a = -2\rho b \int_{-1}^{+1} \varphi(x-c) dx + 2\rho b \int_{1}^{+1} \varphi dx \]

Or, on introducing the individual velocity potentials from page 5,

\[ P = -\rho b^2 \left[ v \pi \alpha + \pi \beta \cdot b \pi \alpha \cdot v T \dot{\beta} - b T_0 \dot{\beta} \right] \] (I)

\[ M_\beta = -\rho b^2 \left[ -v T_0 \alpha - T_0 \dot{\beta} + 2 T_0 b \beta - \frac{1}{\pi} v T_0 \beta - \frac{1}{\pi} T_0 \dot{\beta} \right] + \rho b^2 \left[ -v T_0 \alpha - T_0 \dot{\beta} + 2 T_0 b \beta \frac{1}{\pi} v T_0 \beta - \frac{1}{\pi} T_0 \dot{\beta} \right] \]

\[ = -\rho b^2 \left[ T_0 v^2 \alpha - (2 T_0 + T_1) b \dot{\alpha} + 2 T_0 b \dot{\beta} + \frac{1}{\pi} T_0 v^2 \beta \right. \]

\[ + \left( \frac{1}{\pi} T_0 - \frac{1}{\pi} T_3 \right) b \dot{\beta} + \frac{1}{\pi} b T_0 b \dot{\beta} - T_0 \dot{\beta} \] (II)

\[ M_a = -\rho b^2 \left[ -v \pi \alpha + \pi \beta \cdot b \pi \alpha \cdot \dot{\beta} \cdot \left( T_1 - T_0 \right) \right. \]

\[ - (c-a) T_1 b \beta \alpha + ( - T_1 - (c-a) \cdot \dot{\beta} \cdot b \dot{\beta} \right] \]

\[ - b \pi \alpha - \pi \beta \dot{\beta} \] (III)

**VELOCITY POTENTIALS, FORCES, AND MOMENTS OF THE CIRCULATORY FLOW**

In the following we shall determine the velocity potentials and associated forces and moments due to a surface of discontinuity of strength \( U \) extending along the positive \( z \) axis from the wing to infinity. The velocity potential of the flow around the circle (fig. 3) resulting from the vortex element \(-\Delta \Gamma\) at \((X_0, 0)\) is

\[ \varphi = \frac{\Delta \Gamma}{2\pi} \left[ \tan^{-1} \frac{Y}{X-X_0} - \tan^{-1} \frac{Y}{X-\frac{1}{X_0}} \right] \]

\[ = \frac{\Delta \Gamma}{2\pi} \tan^{-1} \left( \frac{-\frac{1}{X_0} + X_0}{Y + \frac{1}{X_0} X + \frac{1}{X_0} Y} \right) \]

where \((X, Y)\) are the coordinates of the variable and \(X_0\) is the coordinate of \(-\Delta \Gamma\) on the \(x\) axis.

Introducing \(X_0 + \frac{1}{X_0} = 2x_0\)

or \(X_0 = x_0 - \sqrt{x_0^2 - 1}\) on the \(x\) axis

and \(X = x\) and \(Y = \sqrt{1-x^2}\) on the circle

the equation becomes

\[ \varphi_{zz} = \frac{\Delta \Gamma}{2\pi} \tan^{-1} \frac{\sqrt{1-x^2}}{x_0 - 1} \]

This expression gives the clockwise circulation around the airfoil due to the element \(-\Delta \Gamma\) at \(x_0\).

We have: \(P = -2\rho \left( \frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} \right)\)

But, since the element \(-\Delta \Gamma\) will now be regarded as moving to the right relative to the airfoil with a velocity \(v\)

\[ \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial x} \]

Hence, \(p = -2\rho \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x_0} \right)\)

Further

\[ \frac{2\pi}{\Delta \Gamma} \frac{\partial \varphi}{\partial x} = \frac{x}{\sqrt{x_0^2 - 1} (1 - x_0)} \]

\[ = \frac{1}{\sqrt{1 - x^2}} \left( \frac{1}{x_0 - 2} \right) \]

and

\[ \frac{2\pi}{\Delta \Gamma} \frac{\partial \varphi}{\partial x_0} = \frac{1}{\sqrt{x_0^2 - 1}} \left( \frac{1}{x_0 - 2} \right) \]

By addition:

\[ \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x_0} = \frac{\Delta \Gamma}{2\pi} \frac{x_0 + x}{\sqrt{1 - x^2} \sqrt{x_0^2 - 1}} \]
To obtain the force on the aileron, we need the integral
\[ \int_c \left( \frac{\partial \sigma}{\partial z} + \frac{\partial \sigma}{\partial x} \right) dx = \frac{\Delta \Gamma}{2\pi} \int_c \frac{x_0 + x}{\sqrt{x_0^2 - 1} \sqrt{1 - x^2}} dx \]
\[ = \frac{\Delta \Gamma}{2\pi} \left[ \frac{x_0}{\sqrt{x_0^2 - 1}} \cos^{-1}c + \frac{1}{\sqrt{x_0^2 - 1}} \right] \]
Thus, for the force on the aileron
\[ \Delta P_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \left( \frac{x_0}{\sqrt{x_0^2 - 1}} \cos^{-1}c + \frac{1}{\sqrt{x_0^2 - 1}} \sqrt{1 - c^2} \right) \] or
\[ \Delta P_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \left( \frac{x_0}{\sqrt{x_0^2 - 1}} \cos^{-1}c - \sqrt{1 - c^2} \right) + \frac{\sqrt{x_0^2 + 1}}{\sqrt{x_0 - 1}} \sqrt{1 - c^2} \]

Integrated, with \( \Delta \Gamma = Ud\alpha \)
\[ P_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \left( \cos^{-1}c - \sqrt{1 - c^2} \right) \int_1^\infty \frac{x_0}{\sqrt{x_0^2 - 1}} Ud\alpha \]
\[ + \sqrt{1 - c^2} \int_1^\infty \frac{x_0}{\sqrt{x_0 - 1}} Ud\alpha \]
for \( c = 1 \) we obtain the expression for \( P \), the force on the whole airfoil
\[ P = -\rho \nu \int_1^\infty \frac{x_0}{\sqrt{x_0^2 - 1}} Ud\alpha \quad (IV) \]

Since \( U \) is considered stationary with respect to the fluid elements
\[ U = f(\rho t - x_0) \]
where \( t \) is the time since the beginning of the motion. \( U \) is thus a function of the distance from the location of the first vortex element or, referred to a system moving with the fluid, \( U \) is stationary in value.

Similarly we obtain for the moment on the aileron
\[ \int_c \left( \frac{\partial \sigma}{\partial z} + \frac{\partial \sigma}{\partial x} \right) dx \int_1^\infty \frac{x_0 + x}{\sqrt{x_0^2 - 1} \sqrt{1 - x^2}} dx \]
\[ = \frac{\Delta \Gamma}{2\pi} \left[ \frac{x_0}{\sqrt{x_0^2 - 1}} \frac{\sqrt{1 - x^2}}{2} - c \sqrt{1 - x^2} + \frac{1}{2} \right] \cos^{-1}c \]
\[ = \frac{\Delta \Gamma}{2\pi} \left[ \frac{x_0}{\sqrt{x_0^2 - 1}} \left( \sqrt{1 - c^2} - c \cos^{-1}c \right) \right] \]
\[ + \frac{1}{2\pi} \left( \sqrt{1 - c^2} - c \cos^{-1}c \right) \]

Finally
\[ \Delta M_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \left( \sqrt{\frac{z_0}{x_0^2 - 1}} \right) \frac{\sqrt{1 - c^2}}{1 + \frac{c}{2}} \]
\[ - \cos^{-1}c \left( 1 + \frac{1}{2} \right) \frac{\sqrt{\frac{z_0 + 1}{x_0^2 - 1}} \left( \cos^{-1}c - c \sqrt{1 - c^2} \right) }{\sqrt{\frac{z_0 + 1}{x_0^2 - 1}} \left( \cos^{-1}c - c \sqrt{1 - c^2} \right) } \]

Putting \( \Delta \Gamma = Ud\alpha \) and integrating
\[ M_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \left[ \sqrt{\frac{1 - c^2}{1 + \frac{c}{2}}} \right] \]
\[ - \cos^{-1}c \left( 1 + \frac{1}{2} \right) \int_1^\infty \frac{z_0}{\sqrt{x_0^2 - 1}} Ud\alpha \]
\[ + \left( \cos^{-1}c - c \sqrt{1 - c^2} \right) \frac{1}{2} \int_1^\infty \frac{z_0 + 1}{\sqrt{x_0^2 - 1}} Ud\alpha \]

Further, for the moment on the entire airfoil around \( \alpha \)
\[ \int_1^\infty \left( \frac{\partial \sigma}{\partial z} + \frac{\partial \sigma}{\partial x} \right) (x-a) dx = \frac{\Delta \Gamma}{2\pi} \frac{1}{\sqrt{x_0^2 - 1}} \left[ \left( (z_0 + \frac{z}{2} - \alpha) \sqrt{1 - z^2} \right) \right] \]
\[ + \left( \frac{1}{2} - z_0 \right) \cos^{-1}z \left[ \frac{1}{2} - z_0 \alpha \right] = \frac{\Delta \Gamma}{2\pi} \frac{1}{\sqrt{x_0^2 - 1}} \left( \frac{1}{2} - z_0 \alpha \right) \]

and
\[ \Delta M_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \frac{1}{\sqrt{x_0^2 - 1}} \]

Integrated, this becomes
\[ M_a = -\rho \nu \frac{\Delta \Gamma}{\pi} \int_1^\infty \frac{1}{\sqrt{x_0^2 - 1}} Ud\alpha \]
\[ = -\rho \nu \frac{\Delta \Gamma}{\pi} \int_1^\infty \left( \frac{1}{2} + \frac{1}{2} \right) z_0 \left( \frac{a + 1}{2} \right) Ud\alpha \]
\[ = -\rho \nu \frac{\Delta \Gamma}{\pi} \int_1^\infty \left( \frac{1}{2} \right) \frac{z_0 + 1}{\sqrt{x_0^2 - 1}} \left( \frac{a + 1}{2} \right) Ud\alpha \]

THE MAGNITUDE OF THE CIRCULATION

The magnitude of the circulation is determined by the Kutta condition, which requires that no infinite velocities exist at the trailing edge, or, at \( z = 1 \)
\[ \frac{\partial}{\partial x} (\varphi_n + \varphi_s + \varphi_i + \varphi_a + \varphi_p + \varphi_l) = \text{finite} \]

Introducing the values of \( \varphi_n \), etc. from page 5 and \( \varphi_\alpha \) from \( \frac{\partial \varphi}{\partial x} \) page 6 gives the important relation:
\[ \frac{1}{2\pi} \int_1^\infty \sqrt{\frac{z_0 + 1}{z_0 - 1}} Ud\alpha = \nu + k + b \left( \frac{1}{2} - \alpha \right) \]
\[ + \frac{T_{\text{le}}}{\pi} b + b \frac{T_{\text{le}}}{2\pi} \beta \quad (VII) \]

This relation must be satisfied to comply with the Kutta condition, which states that the flow shall leave the airfoil at the trailing edge.

It is observed that the relation reduces to that of the Kutta condition for stationary flow on putting \( x_0 = \infty \),
and in subsequence omitting the variable parameters \( \alpha, \beta, \) and \( h. \)

Let us write

\[
\frac{1}{2\pi} \int_{1}^{\infty} \sqrt{\frac{2z+1}{2z-1}} Ud\Sigma = u + \frac{1}{2} \left( 1 - \frac{1}{2} \right) x
\]

\[
+ \frac{T_{10}}{\pi} y + \frac{T_{11}}{2\pi} \beta = Q
\]

Introduced in (IV)

\[
P = -2\pi \rho b^2 \left( \frac{1}{2} \right) z \frac{1}{\sqrt{z^2 - 1}} \frac{Ud\Sigma}{\int_{1}^{\infty} \sqrt{\frac{2z+1}{2z-1}} Ud\Sigma}
\]

from (V)

\[
M_{\beta} = -2\pi \rho b^2 \left[ \left( \sqrt{1 - c^2} \left( 1 + \frac{c}{2} \right) - \cos^{-1} c \left( c + \frac{1}{2} \right) \right) \int_{1}^{\infty} \frac{z_0}{\sqrt{z_0^2 - 1}} \frac{Ud\Sigma}{\int_{1}^{\infty} \sqrt{\frac{2z_0+1}{2z_0-1}} Ud\Sigma} \right]
\]

\[
\frac{1}{2} \left( \cos^{-1} c - c \sqrt{1 - c^2} \right) Q
\]

\[
= \frac{1}{2} \left( \cos^{-1} c - c \sqrt{1 - c^2} \right) Q
\]

\[
= Q
\]

\[
= Q
\]

\[
= Q
\]

where \( Q \) is given above and \( C = C(k) \) will be treated in the following section.

**VALUE OF THE FUNCTION \( C(u) \)**

Put \( U = U_{0} e^{i \left( -k \left( x - x_0 \right) + \nu \right)} \)

where \( s = \nu t \left( s \to \infty \right) \), the distance from the first vortex element to the airfoil, and \( k \) a positive constant determining the wave length, then

\[
C(k) = \int_{1}^{\infty} \frac{z_0}{\sqrt{z_0^2 - 1}} e^{-iks_0 \beta} \frac{Ud\Sigma}{\int_{1}^{\infty} \sqrt{\frac{2z_0+1}{2z_0-1}} Ud\Sigma}
\]

These integrals are known, see next part, formulas (XIV)–(XVII) and we obtain

\[
C(k) = \frac{-\pi J_{1} + i\pi J_{1}}{2 J_{2} - \pi J_{2} + i\pi J_{2} - \pi J_{1}} \frac{1}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{(-J_{1} + iJ_{1})[(-J_{1} + J_{2}) - i(J_{1} - J_{2})] \left( (J_{1} + J_{2})^2 + (J_{1} - J_{2})^2 \right)}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{J_{1}(J_{1} + J_{2}) + J_{2}(J_{1} - J_{2})}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{1}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{1}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{1}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

\[
= \frac{1}{(J_{1} + J_{2})^2 + (J_{1} - J_{2})^2}
\]

These functions, which are of fundamental importance in the theory of the oscillating airfoil are given graphically against the argument \( \frac{1}{k} \) in figure 4.

**SOLUTION OF THE DEFINITE INTEGRALS IN \( C \) BY MEANS OF BESSEL FUNCTIONS**

We have

\[
K_{\alpha}(z) = \int_{0}^{z} e^{-z \cosh t} \cosh nt \ dt
\]

(Formula (34), p. 51—Gray, Mathews & MacRobert: Treatise on Bessel Functions. London, 1922)

where

\[
K_{\alpha}(t) = e^{t \alpha} G_{\alpha}(it)
\]

(Eq. (28), sec. 3, p. 23, same reference)

and

\[
G_{\alpha}(z) = -\frac{1}{2} Y_{\alpha}(z) + \left( \log 2 - \gamma + i \frac{\pi}{2} \right) J_{\alpha}(z)
\]

but

\[
Y_{\alpha}(z) = \frac{1}{2} Y_{\alpha}(z) + (\log 2 - \gamma) J_{\alpha}(z)
\]

(where \( Y_{\alpha}(z) \) is from N. Nielsen: Handbuch der Theorie der Cylinderfunktionen. Leipzig, 1904).

---

1 This may also be expressed in Hankel functions, \( C = \frac{H_{\alpha}(\xi)}{\xi} \).
Thus,

\[ G_m(x) = -\frac{\pi}{2} \left[ Y_m(x) - iJ_m(x) \right] \]

We have

\[ K_0(-ik) = \int_0^\infty e^{ikt} \cos t \, dt = \int_0^\infty \frac{e^{ikt}}{\sqrt{x^2 - 1}} \, dx \]

or

\[ -\frac{\pi}{2} Y_0(k) + i \frac{\pi}{2} J_0(k) = \int_0^\infty \cos kx \, dx \sqrt{\frac{x^2}{x^2 - 1}} + i \int_0^\infty \sin kx \, dx \sqrt{\frac{x^2}{x^2 - 1}} \]

Thus,

\[ \int_0^\infty \cos kx \, dx \sqrt{\frac{x^2}{x^2 - 1}} = -\frac{\pi}{2} Y_0(k) \] (XIV)

\[ \int_0^\infty \sin kx \, dx \sqrt{\frac{x^2}{x^2 - 1}} = \frac{\pi}{2} J_0(k) \] (XV)

Further,

\[ K_1(-ik) = \int_0^\infty e^{ikt} \cosh t \, dt = \int_0^\infty \frac{e^{ikt}}{\sqrt{x^2 - 1}} \, dx \]

\[ i \Omega_1(k) = -i \frac{\pi}{2} Y_1(k) - \frac{\pi}{2} J_1(k) \]

\[ = \int_0^\infty \frac{x \cos kx + i \sin kx}{\sqrt{x^2 - 1}} \, dx \]

Thus,

\[ \int_0^\infty \frac{x \cos kx}{\sqrt{x^2 - 1}} = -\frac{\pi}{2} J_1(k) \] (XVI)

\[ \int_0^\infty \frac{x \sin kx}{\sqrt{x^2 - 1}} = \frac{\pi}{2} Y_1(k) \] (XVII)

**TOTAL AERODYNAMIC FORCES AND MOMENTS**

**TOTAL FORCE**

From equations (I) and (VIII) we obtain

\[ P = -\rho b^2 \left( v \rho \alpha + \rho \beta - \rho b \alpha - \rho T \beta - \rho T b \beta \right) \]

\[ -2\pi \rho b \rho C \left( v \alpha + \rho \beta + \rho \left( \frac{1}{2} - a \right) \alpha + \frac{1}{2} \rho T \beta \right) \]

\[ + b \frac{1}{2} \rho T \beta \] (XVIII)

**TOTAL MOMENTS**

From equations (II) and (IX) we obtain similarly

\[ M_a = -\rho b^2 \left( -2T_a - T_1 + T_2 \left( a - \frac{1}{2} \right) \right) \rho b \alpha + 2T_1 \rho b \alpha a \]

\[ + \frac{1}{2} \rho T_1 \beta \right) \]

\[ - T_1 \beta \left( \rho b \alpha + \rho \beta + \rho \left( \frac{1}{2} - a \right) \alpha \right) \]

\[ + b \frac{1}{2} \rho T_1 \beta \] (XIX)

**DEFINITION OF PARAMETERS USED IN EQUATIONS**

\[ k = \frac{\pi \rho b^3}{M} \]

the ratio of the mass of a cylinder of air of a diameter equal to the chord of the wing to the mass of the wing, both taken for equal length along span.
\[ r_a = \sqrt{\frac{I_a}{M_b}} \quad \text{the radius of gyration divided by } b. \]
\[ x_a = \frac{S_a}{M_b} \quad \text{the center of gravity distance of the wing from } a, \text{ divided by } b. \]
\[ \omega_a = \sqrt{\frac{C_a}{I_a}} \quad \text{the frequency of torsional vibration around } a. \]
\[ r_b = \sqrt{\frac{I_b}{M_b^2}} \quad \text{reduced radius of gyration of aileron divided by } b, \text{ that is, the radius at which the entire mass of the airfoil would have to be concentrated to give the moment of inertia of the aileron } I_b. \]
\[ x_b = \frac{S_b}{M_b} \quad \text{reduced center of gravity distance from } c. \]
\[ \omega_b = \sqrt{\frac{C_b}{I_b}} \quad \text{frequency of torsional vibration of aileron around } c. \]
\[ \omega_s = \sqrt{\frac{C_s}{M}} \quad \text{frequency of wing in deflection.} \]

**Final Equations in Nondimensional Form**

On introducing the quantities \( M_a, M_b, \) and \( P, \) replacing \( T_9 \) and \( T_{10} \) from page 5, and reducing to nondimensional form, we obtain the following system of equations:

\[
\begin{align*}
\alpha \left[ r_a^3 + k \left( \frac{1}{a} + a^2 \right) \right] + \alpha \beta \kappa \left( \frac{1}{a} - a \right) + \alpha \frac{C_a}{M_b} \beta \left[ r_a^3 + (c-a) x_a - \frac{T_1}{\pi} \kappa - (e-a) \frac{T_2}{\pi} \kappa \right] + \frac{1}{\pi} \beta \kappa \frac{v}{b} \left[ \frac{T_9}{\pi} + \frac{1}{\beta} \beta \right] &= 0 \\
\beta \left[ \frac{C_a}{M_b} + \frac{1}{\beta} \kappa \right] + \beta \left[ \frac{C_a}{M_b} \frac{1}{\beta} \kappa \left( x_a - \frac{T_1}{\pi} \right) \right] + \frac{1}{\beta} \beta \left( \frac{1}{\pi} T_9 \right) \frac{v}{b} \left[ \frac{T_9}{\pi} + \frac{1}{\beta} \beta \right] &= 0 \\
\alpha \left( x_a - \alpha \right) + \alpha \beta \kappa + \beta \left( x_a - \frac{1}{\pi} T_1 \right) - \frac{1}{\beta} \beta \kappa \frac{v}{b} \left[ \frac{T_9}{\pi} + \frac{1}{\beta} \beta \right] &= 0
\end{align*}
\]

**Solution of Equations**

As mentioned in the introduction, we shall only have to specify the conditions under which an unstable equilibrium may exist, no general solution being needed. We shall therefore introduce the variables \( \alpha, \) \( \beta, \) \( \kappa, \) etc., as functions of the distance \( s \) or, in complex form with \( \frac{1}{\kappa} \) as an auxiliary parameter, giving the ratio of the wave length to \( 2\pi \) times the half chord \( b: \)

\[
\alpha = \alpha_0 e^{i \frac{k s}{b}}
\]

\[
\beta = \beta_0 e^{i \left( \frac{k s}{b} + \nu_1 \right)}
\]

and

\[
\kappa = \kappa_0 e^{i \left( \frac{1}{2} \frac{k s}{b} + \nu_2 \right)}
\]

where \( s \) is the distance from the airfoil to the first vortex element, \( \frac{d s}{d t} = v, \) and \( \nu_1 \) and \( \nu_2 \) are phase angles of \( \beta \) and \( \kappa \) with respect to \( \alpha. \)

Having introduced these quantities in our system of equations, we shall divide through by \( \left( \frac{v}{b} \right)^2. \)

We observe that the velocity \( v \) is then contained in only one term of each equation. We shall consider this term containing \( v \) as the unknown parameter \( \Omega X. \) To distinguish terms containing \( X \) we shall employ a bar; terms without bars do not contain \( X. \)

We shall resort to the following notation, taking care to retain a perfectly cyclic arrangement. Let the letter \( A \) refer to the coefficients in the first equation not containing \( C(k) \) or \( X, B \) to similar coefficients of the second equation, and \( C \) to those in the third equation. Let the first subscript \( a \) refer to the first variable \( \alpha, \) the subscript \( b \) to the second, and \( h \) to the third. Let the second subscripts 1, 2, 3 refer to the second derivative, the first derivative, and the argument of each variable, respectively. \( A_{ab} \) thus refers to the coefficient in the first equation associated with the second derivative of \( \alpha \) and not containing \( C(k) \) or.
The solution of the instability problem as contained in the system of three equations $A$, $B$, and $C$ is given by the vanishing of a third-order determinant of complex numbers representing the coefficients. The solution of particular subcases of two degrees of freedom is given by the minors involving the particular coefficients. We shall denote the case torsion-aileron $(\alpha, \beta)$ as case $3$, aileron-deflection $(\beta, h)$ as case $2$, and deflection-torsion $(h, \alpha)$ as case $1$. The determinant form of the solution is given in the major case and in the three possible subcases, respectively, by:

$$
D = \begin{vmatrix}
R_{\alpha\alpha} + iI_{\alpha\alpha}, R_{\beta\alpha} + iI_{\beta\alpha}, R_{\alpha h} + iI_{\alpha h} \\
R_{\beta\alpha} + iI_{\beta\alpha}, R_{\alpha h} + iI_{\alpha h}, R_{\beta h} + iI_{\beta h} \\
R_{\alpha h} + iI_{\alpha h}, R_{\beta h} + iI_{\beta h}, R_{h h} + iI_{h h}
\end{vmatrix} = 0
$$

and

$$
M_{\alpha h} = \begin{vmatrix}
R_{\alpha h} + iI_{\alpha h}, R_{\beta h} + iI_{\beta h} \\
R_{\alpha h} + iI_{\alpha h}, R_{\beta h} + iI_{\beta h}, R_{h h} + iI_{h h}
\end{vmatrix} = 0 \text{ Case 3}
$$

$$
M_{\alpha h} = \begin{vmatrix}
R_{\beta h} + iI_{\beta h}, R_{h h} + iI_{h h} \\
R_{\beta h} + iI_{\beta h}, R_{h h} + iI_{h h}, R_{h h} + iI_{h h}
\end{vmatrix} = 0 \text{ Case 2}
$$

$$
M_{\alpha h} = \begin{vmatrix}
R_{h h} + iI_{h h}, R_{h h} + iI_{h h} \\
R_{h h} + iI_{h h}, R_{h h} + iI_{h h}, R_{h h} + iI_{h h}
\end{vmatrix} = 0 \text{ Case 1}
$$

**REAL EQUATIONS**

$$
\begin{align*}
\bar{R}_{\alpha\alpha} - I_{\alpha\alpha} & = 0 \\
\bar{R}_{\beta\alpha} - I_{\beta\alpha} & = 0 \\
\bar{R}_{\alpha h} - I_{\alpha h} & = 0
\end{align*}
$$

**IMAGINARY EQUATIONS**

$$
\begin{align*}
I_{\alpha\alpha} - I_{\beta\alpha} + I_{\alpha h} & = 0 \\
I_{\beta\alpha} - I_{\alpha h} + I_{\beta h} & = 0 \\
I_{\alpha h} - I_{\beta h} + I_{h h} & = 0
\end{align*}
$$

**NOTES.**—Terms with bars contain $X$; terms without bars do not contain $X$.

The $9$ quantities $R_{\alpha\alpha}, R_{\beta\alpha},$ etc., refer to the real parts and the $9$ quantities $I_{\alpha\alpha}, I_{\beta\alpha},$ etc., to the imaginary parts of the coefficients of the $3$ variables $\alpha, \beta,$ and $h$ in the $3$ equations $A$, $B$, $C$ on page 10. Denoting the coefficients of $\bar{a}$, $\bar{a}$, and $a$ in the first equation by $p$, $q$, and $r$,

$$
R_{\alpha\alpha} + iI_{\alpha\alpha} = \frac{1}{K} \left[ -p + iq \frac{b}{K} + r \left( \frac{b}{K} \right)^2 \right]
$$

which, separated in real and imaginary parts, gives the quantities $R_{\alpha\alpha}$ and $I_{\alpha\alpha}$. Similarly, the remaining quantities $R$ and $I$ are obtained. They are all functions of $k$ or $C(k)$. The terms with bars $\bar{R}_{\alpha\alpha}, \bar{R}_{\beta\alpha},$ and $\bar{R}_{\alpha h}$ are seen to be the only ones containing the unknown $X$. The quantities $\Omega$ and $X$ will be defined shortly. The quantities $R$ and $I$ are given in the following list:

---

*The factor $\frac{1}{K}$ or $\frac{1}{K}$ is not included in these constants. See the expressions for the $R$'s and $I$'s on next page.*
Complex cubic equation in $X$:

$$\Omega_0 \Omega_0 X^3 + (\Omega_0 \Omega_0 A_{sh} + \Omega_0 A_{se} A_{ae} + \Omega_0 A_{se} A_{sb}) X^2 + (\Omega_0 M_{sa} + \Omega_0 M_{sa} + \Omega_0 M_{ca}) X + D = 0 \quad (XXI)$$

Case 3, $(\alpha, \beta)$:

$$\Omega_0 \Omega_0 X^3 + (\alpha \alpha A_{sh} + \alpha \beta A_{se} A_{ae}) X + M_{sh} = 0 \quad (XXII)$$

Case 2, $(\beta, \gamma)$:

$$\Omega_0 \Omega_0 X^3 + (\beta \gamma A_{sh} + \alpha \gamma A_{se} A_{ae}) X + M_{se} = 0 \quad (XXIII)$$

Case 1, $(\gamma, \alpha)$:

$$\Omega_0 \Omega_0 X^3 + (\gamma \alpha A_{sh} + \gamma \alpha A_{se} A_{ae}) X + M_{sa} = 0 \quad (XXIV)$$

$$\Omega_0 X = \frac{C_0}{k^2 \eta_0^2 k^2 \mu_0^2 k} \left( \frac{\omega_0 x_{-\alpha}}{\omega_0 x_{-\beta}} \right)^2 \left( \frac{\omega_0 x_{-\gamma}}{\omega_0 x_{-\delta}} \right)^2$$

and finally

$$X = \frac{1}{k} \left( \frac{b r_{\gamma} r_{\delta}}{v k} \right)^2$$

We are at liberty to introduce the reference parameters $\omega_r$ and $r_r$, and the convention adopted is: $\omega_r$ is the last $\omega$ in cyclic order in each of the subcases 3, 2, and 1.

Then $\Omega_r = \left( \frac{\omega_r x_{-\gamma}}{\omega_{r+1} x_{-\beta+1}} \right)^2$ and $\Omega_{r+1} = 1$, thus for

Case 3, $\Omega_r = \left( \frac{\omega_r x_{-\gamma}}{\omega_{r+1} x_{-\beta+1}} \right)^2$ and $\Omega_{r+1} = 1$

Case 2, $\Omega_r = \left( \frac{\omega_r x_{-\gamma}}{\omega_{r+1} x_{-\beta+1}} \right)^2$ and $\Omega_{r+1} = 1$

Case 1, $\Omega_r = \left( \frac{\omega_r x_{-\gamma}}{\omega_{r+1} x_{-\beta+1}} \right)^2$ and $\Omega_{r+1} = 1$

To treat the general case of three degrees of freedom (equation (XXI)), it is observed that the real part of the equation is of third degree while the imaginary part furnishes an equation of second degree. The problem is to find values of $X$ satisfying both equations. We shall adopt the following procedure: Plot graphically $X$ against $k$ for both equations. The points of intersection are the solutions. We are only concerned with positive values of $\frac{1}{k}$ and positive values of $X$. Observe that we do not have to solve for $k$, but may reverse the process by choosing a number of values of $k$ and solve for $X$. The plotting of $X$ against $k$ for the second-degree equation is simple enough, whereas the task of course is somewhat more laborious for the third-degree equation. However, the general case is of less practical importance than the three subcases. The equation simplifies considerably, becoming of second degree in $X$. 

The solution as given by the three-row determinant shall be written explicitly in $X$. We are immediately able to put down for the general case a cubic equation in $X$ with complex coefficients and can easily segregate the three subcases. The quantity $D$ is as before the value of the determinant, but with the term containing $X$ missing. The quantities $M_{sa}$, $M_{sb}$, and $M_{sh}$ are the minors of the elements in the diagonal squares $a_\alpha$, $b_\beta$, and $c_\delta$, respectively. They are expressed explicitly in terms of $R$ and $I$ under the subcases treated in the following paragraphs.

$$D = \begin{vmatrix} A_{sa} + \Omega_0 X & A_{sd} & A_{sh} \\ A_{sa} & A_{sb} + \Omega_0 X & A_{sb} \\ A_{sa} & A_{sd} & A_{sh} + \Omega_0 X \end{vmatrix} = 0$$

where $A_{sa} = R_{sa} + iL_{sa}$ etc.
We shall now proceed to consider these three subcases. By virtue of the cyclic arrangement, we need only consider the first case \((\alpha, \beta)\). The complex quadratic equations (XXII)–(XXIV) all resolve themselves into two independent statements, which we shall for convenience denote “Imaginary equation” and “Real equation,” the former being of first and the latter of second degree in \(X\). All constants are to be resolved into their real and imaginary parts, denoted by an upper index \(R\) or \(I\), respectively.

Let \(M_{\alpha}=M_{\alpha R}+iM_{\alpha I}\) and let similar expressions denote \(M_{\beta}\) and \(M_{\beta I}\).

Case 3, \((\alpha, \beta)\). Separating equation (XXII) we obtain:

(1) Imaginary equation:

\[
(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})X+M_{\beta I}ch=0
\]

\[
X=-\frac{M_{\beta I}ch}{\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I}}
\]

(2) Real equation:

\[
\Omega_{\alpha}\omega_{\beta}X^{2}+(\omega_{\alpha}R_{\alpha R}+\omega_{\beta}R_{\alpha I})X+M_{\beta R}ch=0
\]

Eliminating \(X\) we get:

\[
\Omega_{\alpha}\omega_{\beta}(M_{\beta I}ch)^{2}-(\omega_{\alpha}R_{\alpha R}+\omega_{\beta}R_{\alpha I})(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})M_{\beta I}ch
\]

\[
+M_{\alpha R}(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})^{2}=0
\]

By the convention adopted we have in this case:

\[
\omega_{\alpha}^{\ast}=\omega_{\beta}, \quad \omega_{\alpha}=\left(\frac{\omega_{\alpha}}{\omega_{\beta}}\right)^{2}, \quad \text{and} \quad \Omega_{\alpha}=1
\]

Arranging the equation in powers of \(\omega_{\alpha}\) we have:

\[
\Omega_{\alpha}^{2}[-M_{\alpha R}(R_{\alpha R}I_{\alpha I})+M_{\alpha I}I_{\alpha R})+(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})M_{\beta I}ch
\]

\[
+M_{\alpha R}(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})^{2}=0
\]

But we have:

\[
(M_{\beta I}ch)^{2}=-M_{\alpha R}(R_{\alpha I}I_{\alpha R}+I_{\alpha R}R_{\alpha I})
\]

\[
=M_{\alpha R}[R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}-R_{\alpha R}I_{\alpha R}-R_{\alpha I}I_{\alpha I}]
\]

\[
=\pm M_{\alpha R}(R_{\alpha I}I_{\alpha R}+I_{\alpha R}R_{\alpha I})
\]

Finally, the equation for Case 3 \((\alpha, \beta)\) becomes:

\[
\Omega_{\alpha}^{2}(M_{\beta R}ch)^{2}-(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})M_{\alpha R}ch
\]

\[
+2M_{\alpha R}I_{\alpha R}I_{\alpha I}+M_{\alpha I}ch(I_{\alpha I}R_{\alpha R}+I_{\alpha R}R_{\alpha I})=0 \quad (XXV)
\]

where:

\[
M_{\alpha R}=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}-I_{\alpha I}R_{\alpha R}-I_{\alpha R}R_{\alpha I}
\]

\[
M_{\alpha I}ch=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}+I_{\alpha R}R_{\alpha I}-I_{\alpha I}R_{\alpha R}
\]

The remaining cases may be obtained by cyclic rearrangement:

Case 2, \((\beta, \alpha)\):

\[
\omega_{\beta}=\omega_{\alpha}, \quad \Omega_{\beta}=\left(\frac{\omega_{\beta}}{\omega_{\alpha}}\right)^{2}, \quad \Omega_{\alpha}=1
\]

\[
\Omega_{\beta}(M_{\alpha R}I_{\alpha I}+M_{\alpha I}I_{\alpha R})+\omega_{\alpha}[M_{\alpha R}R_{\alpha I}I_{\alpha R}+I_{\alpha R}R_{\alpha I}]
\]

\[
+2M_{\alpha R}I_{\alpha R}I_{\alpha I}+M_{\alpha I}ch(I_{\alpha I}R_{\alpha R}+I_{\alpha R}R_{\alpha I})=0 \quad (XXVI)
\]

where:

\[
M_{\alpha R}=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}-I_{\alpha I}R_{\alpha R}-I_{\alpha R}R_{\alpha I}
\]

\[
M_{\alpha I}ch=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}+I_{\alpha R}R_{\alpha I}-I_{\alpha I}R_{\alpha R}
\]

Case 1, \((\theta, \alpha)\):

\[
\Omega_{\theta}=\left(\frac{\omega_{\alpha}}{\omega_{\beta}}\right)^{3} \quad \Omega_{\alpha}=1
\]

\[
\Omega_{\theta}(M_{\beta R}ch)^{2}+(\omega_{\alpha}R_{\alpha R}+\omega_{\beta}R_{\alpha I})(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})M_{\beta R}ch
\]

\[
+M_{\alpha R}(\omega_{\alpha}I_{\alpha R}+\omega_{\beta}I_{\alpha I})^{2}=0 \quad (XXVII)
\]

where:

\[
M_{\beta R}=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}-I_{\alpha I}R_{\alpha R}-I_{\alpha R}R_{\alpha I}
\]

\[
M_{\beta I}ch=R_{\alpha I}I_{\alpha R}+R_{\alpha R}I_{\alpha I}+I_{\alpha R}R_{\alpha I}-I_{\alpha I}R_{\alpha R}
\]

Equations (XXV), (XXVI), and (XXVII) thus give the solutions of the cases: torsion-aileron, aileron-deflection, and deflection-torsion, respectively. The quantity \(\Omega\) may immediately be plotted against \(1/k\) for any value of the independent parameters.

The coefficients in the equations are all given in terms of \(R\) and \(I\), which quantities have been defined above. Routine calculations and graphs giving \(\Omega\) against \(1/k\) are contained in Appendix I and Appendix II.

Knowing related values of \(\Omega\) and \(1/k\), \(X\) is immediately expressed as a function of \(\Omega\) by means of the first-degree equation. The definition of \(X\) and \(\Omega\) for each subcase is given above. The cyclic arrangement of all quantities is very convenient as it permits identical treatment of the three subcases.

It shall finally be repeated that the above solutions represent the border case of unstable equilibrium. The plot of \(X\) against \(\Omega\) gives a boundary curve between the stable and the unstable regions in the \(X\Omega\) plane.

It is preferable, however, to plot the quantity \(1/k\cdot X\) instead of \(X\), since this quantity is proportional to the square of the flutter speed. The stable area can easily be identified by inspection as it will contain the axis \(1/k\cdot X=0\), if the combination is stable for zero velocity.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., May 3, 1934.
APPENDIX I

PROCEDURE IN SOLVING NUMERICAL EXAMPLES

(1) Determine the $R$'s and $I$'s, nine of each for a major case of three degrees of freedom, or those pertaining to a particular subcase, 4 $R$'s and 4 $I$'s. Refer to the following for the $R$'s and $I$'s involved in each case:

The numerals 1 to 9 and 11 to 19 are used for convenience.

(Major case) Three degrees of freedom

1 $R_{ca}$ $I_{ca}$ 11
2 $R_{db}$ $I_{db}$ 12
3 $R_{cb}$ $I_{cb}$ 13
4 $R_{ba}$ $I_{ba}$ 14
5 $R_{bb}$ $I_{bb}$ 15
6 $R_{sb}$ $I_{sb}$ 16
7 $R_{ab}$ $I_{ab}$ 17
8 $R_{cb}$ $I_{cb}$ 18
9 $R_{ch}$ $I_{ch}$ 19

(Case 3) Torsional-aileron (α, β)

1 $R_{ba}$ $I_{ba}$ 11
2 $R_{bb}$ $I_{bb}$ 12
4 $R_{ba}$ $I_{ba}$ 14
5 $R_{bb}$ $I_{bb}$ 15

(Case 2) Aileron-deflection (β, h)

5 $R_{bb}$ $I_{bb}$ 15
6 $R_{bh}$ $I_{bh}$ 10
8 $R_{dh}$ $I_{dh}$ 18
9 $R_{ch}$ $I_{ch}$ 19

(Case 1) Deflection-torsion (h, α)

7 $R_{ca}$ $I_{ca}$ 17
9 $R_{ch}$ $I_{ch}$ 19
1 $R_{ca}$ $I_{ca}$ 11
3 $R_{ch}$ $I_{ch}$ 13

It has been found convenient to split the $R$'s in two parts $R = R' + R''$, the former being independent of the argument $\frac{1}{K}$. The quantities $I$ and $R''$ are func-

The formulas are given in the following list.

$$R''_{ca} = \frac{1}{K} 2 \left( a + \frac{1}{2} \right) \left( \frac{1}{2} - a \right) G - \frac{F}{K}$$ (1)

$$R''_{ab} = \frac{1}{K} \pi (T_{a} + T_{b}) \left( \frac{1}{2} - a \right) G - \frac{F}{K}$$ (2)

$$R''_{cb} = \frac{1}{K} 2 \left( a + \frac{1}{2} \right) G$$ (3)

$$R''_{ba} = -\frac{1}{K} \pi \left( \frac{1}{2} - a \right) G - \frac{F}{K}$$ (4)

$$R''_{bb} = -\frac{1}{K} \pi \frac{T_{b}}{2} \left( T_{11} G - \frac{2}{K} T_{b} F \right) - \frac{1}{K} (T_{11} - T_{11} T_{11})$$ (5)

$$R''_{bh} = -\frac{1}{K} \pi G$$ (6)

$$R''_{ca} = -\frac{1}{K} 2 \left( a + \frac{1}{2} \right) G - \frac{F}{K}$$ (7)

$$R''_{cb} = -\frac{1}{K} 2 \left( T_{11} G - 2 T_{b} F \right)$$ (8)

$$R''_{cb} = -\frac{1}{K} 2 G$$ (9)

$$I_{aa} = -2 \left( a + \frac{1}{2} \right) \left[ \left( \frac{1}{2} - a \right) F + \frac{1}{K} G \right] + \frac{1}{2} - a$$ (11)

$$I_{ab} = -\frac{1}{\pi} \left[ \left( a + \frac{1}{2} \right) \left( T_{11} F + \frac{2}{K} T_{b} G \right) + \frac{1}{2} - \frac{1}{2} T_{b} \right]$$ (12)

$$I_{ab} = -2 \left( a + \frac{1}{2} \right) F$$ (13)

$$I_{ba} = \frac{T_{b}}{\pi} \left( \frac{1}{2} - a \right) F + \frac{1}{K} G \right] + \frac{1}{\pi} \left[ \left( p - T_{11} \right) \right.$$ (14)

Where $p = -\frac{1}{3} (1 - e)^{5/2}$

$$I_{ba} = \frac{1}{2} \pi \frac{T_{b}}{\pi} \left[ T_{11} \left( T_{11} F + \frac{2}{K} T_{b} G \right) - T_{11} \right]$$ (15)

$$I_{bb} = \frac{T_{b}}{\pi} F$$ (16)

$$I_{ca} = 2 \left( \left( \frac{1}{2} - a \right) F + \frac{1}{K} G \right] + 1$$ (17)

$$I_{cb} = \frac{1}{\pi} \left( T_{11} F + \frac{2}{K} T_{b} G \right) - T_{b}$$ (18)

$$I_{cb} = 2 F$$ (19)

The quantities $I$ given in the appendix and used in the following calculations are seen to differ from the $I$'s given in the body of the paper by the factor $\frac{1}{K}$. It may be noticed that this factor drops out in the first-degree equations.
Choosing certain values of \( a \) and \( c \) and employing
the values of the \( T \)'s given by the formulas of the report
(p. 5) or in table I and also using the values of \( F \) and \( G \)
(formulas XII and (XIII)) or table II, we evaluate
the quantities \( I \) and \( R'' \) for a certain number of \( \frac{1}{k} \)
values. The results of this evaluation are given in
tables III and IV, which have been worked out for
\( a = 0, -0.2, \) and \(-0.4, \) and for \( c = 0.5 \) and \( c = 0. \) The
range of \( \frac{1}{k} \) is from 0 to 40. These tables save the work
calculation the \( I \)'s and \( R'' \)'s for almost all cases of
practical importance. Interpolation may be used for
intermediate values. This leaves the quantities \( R' \)
to be determined. These, being independent of \( \frac{1}{k} \), are as
a result easy to obtain. Their values, using the same
system of numbers for identification, and referring to
the definition of the original independent variables on
pages 9 and 10, are given as follows:

\[
R'_{ab} = -\frac{r_a^2}{\kappa} - \left( \frac{1}{S} + a^2 \right)
\]

(1)

\[
R'_{ab} = -\frac{r_a^2}{\kappa} \left( c-a \right) \frac{2a}{\kappa} + \frac{T_1}{\pi} + \left( c-a \right) \frac{T_1}{\pi}
\]

(2)

\[
R'_{bb} = \frac{2a}{\kappa} + a
\]

(3)

\[
R'_{sa} = \text{same as } R'_{ab}
\]

(4)

\[
R'_{sb} = \frac{r_a^2}{\kappa} + \frac{T_1}{\pi} + T_1
\]

(5)

\[
R'_{aa} = \frac{2a}{\kappa} + \frac{T_1}{\pi}
\]

(6)

\[
R'_{ab} = \text{same as } R'_{ab}
\]

(7)

\[
R'_{bb} = \text{same as } R'_{bb}
\]

(8)

\[
R'_{aa} = \frac{1}{\kappa} - 1
\]

(9)

Because of the symmetrical arrangement in the
determinant, the 9 quantities are seen to reduce to
6 quantities to be calculated. It is very fortunate,
indeed, that all the remaining variables segregate them-
seventy in the 6 values of \( R' \) which are independent of \( \frac{1}{k} \),
while the more complicated \( I \) and \( R'' \) are functions
solely of \( c \) and \( a \). In order to solve any problem it is
therefore only necessary to refer to tables III and IV
and then to calculate the 6 values of \( R' \).

The quantities (1) to (9) and (11) to (19) thus
having been determined, the plot of \( \Omega \) against \( \frac{1}{k} \) which
constitutes our method of solution, is obtained by
solving the equation \( a \Omega + b \Omega + c = 0 \). The constants
\( a, b, \) and \( c \) are obtained automatically by computation
according to the following scheme:

<table>
<thead>
<tr>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = -M^R_{ab} (15.14 + 12.4) + M^R_{ab} (11.15) )</td>
</tr>
<tr>
<td>( c = M^S_{ab} (15.14 - M^R_{ab} (11.15) )</td>
</tr>
<tr>
<td>Solution: ( \frac{1}{X} = \frac{\Omega_m (15) + 11}{M^R_{ab}} )</td>
</tr>
</tbody>
</table>

Similarly

<table>
<thead>
<tr>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = -M^R_{ab} (15.19 + 15.0) - M^R_{ab} (9.19) )</td>
</tr>
<tr>
<td>( c = M^S_{ab} (15.19 - M^R_{ab} (9.19) )</td>
</tr>
<tr>
<td>Solution: ( \frac{1}{X} = \frac{\Omega_m (19) + 15}{M^S_{ab}} )</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>Case 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = -M^R_{ab} (11.15 + 11.3) + M^R_{ab} (9.19) )</td>
</tr>
<tr>
<td>( c = M^S_{ab} (11.15 - M^R_{ab} (9.19) )</td>
</tr>
<tr>
<td>Solution: ( \frac{1}{X} = \frac{\Omega_m (11) + 19}{M^S_{ab}} )</td>
</tr>
</tbody>
</table>

\( \Omega \) is defined as \( \left( \frac{a \omega_{aR}}{a \omega_{bR}} \right)^2 \) for case 3;

\( \Omega \) is defined as \( \left( \frac{b \omega_{aR}}{b \omega_{bR}} \right)^2 \) for case 2; and

\( \omega_b \) is defined as \( \left( \frac{b \omega_{aR}}{b \omega_{bR}} \right)^2 \) for case 1.

The quantity \( \frac{1}{X} \) is \( \left( \frac{\Omega}{\Omega_m \omega_{aR}} \right)^2 \) by definition.

Since both \( \Omega \) and \( \frac{1}{X} \) are calculated for each value of
\( \frac{1}{k} \) we may plot \( \frac{1}{k} \) directly as a function of \( \Omega \). This
quantity, which is proportional to the square of the
flutter speed, represents the solution.

We shall sometimes use the square root of the above
quantity, viz, \( \frac{1}{k} \sqrt{\frac{1}{X}} = \frac{\sqrt{\kappa \eta}}{b \omega_{aR}} \) and will denote this
quantity by $F$, which we shall term the “flutter factor.” The flutter velocity is consequently obtained as

$$ v = F \frac{b \omega_r \alpha}{\kappa} $$

Since $F$ is nondimensional, the quantity $\frac{b \omega_r \alpha}{\kappa}$ must obviously be a velocity. It is useful to establish the significance of this velocity, with reference to which the flutter speed, so to speak, is measured. Observing that $\kappa = \frac{\pi \rho b^2}{M}$ and that the stiffness in case 1 is given by

$$ \omega_a = \sqrt{\frac{C_a}{M b^2 \alpha^2}} $$

this reference velocity may be written:

$$ v_R = \frac{b \omega_r \alpha}{\sqrt{\frac{C_a}{\pi \rho b^2}}} = \frac{1}{b} \sqrt{\frac{C_a}{\pi \rho}} \text{ or } \pi \rho v_b^2 = C_a $$

The velocity $v_R$ is thus the velocity at which the total force on the airfoil $\pi \rho v_b^2 b$ attacking with an arm $\frac{b}{2}$ equals the torsional stiffness $C_a$ of the wing. This statement means, in case 1, that the reference velocity used is equal to the “divergence” velocity obtained with the torsional axis in the middle of the chord. This velocity is considerably smaller than the usual divergence velocity, which may be expressed as

$$ v_D = v_R \frac{1}{1 + a} $$

where $a$ ranges from 0 to $\frac{1}{2}$. We may thus express the flutter velocity as

$$ v_F = v_R F $$

In case 3 the reference velocity has a similar significance, that is, it is the velocity at which the entire lift of the airfoil attacking with a leverage $\frac{1}{2} b$ equals numerically the torsional stiffness $C_a$ of the aileron or movable tail surface.

In case 2, no suitable or useful significance of the reference velocity is available.

### Table I.—Values of $T$

<table>
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<tr>
<th>$c = 1$</th>
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<td>3.1410</td>
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</table>

### Table II.—Table of the Bessel Functions $J_0, J_1, Y_0, Y_1$, and the Functions $F$ and $G$

$$ F(x) = \frac{J_0(Y_0 + Y_1)(Y_1 - Y_0)}{(Y_2 - Y_0)(Y_2 - Y_1)} $$

$$ G(x) = \frac{J_1(Y_0 + Y_1)(Y_1 - Y_0)}{(Y_2 - Y_0)(Y_2 - Y_1)} $$

<table>
<thead>
<tr>
<th>$k$</th>
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<th>$J_0$</th>
<th>$J_1$</th>
<th>$Y_0$</th>
<th>$Y_1$</th>
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### Table II.—Table of the Bessel Functions $J_0, J_1, Y_0, Y_1$, and the Functions $F$ and $G$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{1}{k}$</th>
<th>$J_0$</th>
<th>$J_1$</th>
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### Table II.—Table of the Bessel Functions $J_0, J_1, Y_0, Y_1$, and the Functions $F$ and $G$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{1}{k}$</th>
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### TABLE III—VALUES OF R

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### TABLE IV—VALUES OF I

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1 Independent of c.  
2 Independent of $\alpha$.  

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*Note: The table data is presented in tabular format with columns and rows, each representing different values, but the specific values are not transcribed due to the limitations of this medium.*

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
APPENDIX II

NUMERICAL CALCULATIONS

A number of routine examples have been worked out to illustrate typical results. A "standard" case has been chosen, represented by the following constants:

\( \kappa = 0.1, \, c = 0.5, \, a = -0.4, \, x_n = 0.2 \),

\( r_a = 0.25, \, \tan \gamma = \frac{1}{80}, \, \rho_r = \frac{1}{160} \)

\( \omega_o, \omega_p, \omega_n \) variable.

We will show the results of a numerical computation of the three possible subcases in succession.

Case 3, Torsion-aileron \((\alpha, \beta)\): Figure 5 shows the \( \Omega_n \) against \( \frac{1}{k} \) relation and figure 6 the final curve

\[ F = \kappa \left( \frac{v}{\omega_n \beta} \right)^2 \text{ against } \Omega_n = \left( \frac{\omega_n \beta}{\omega_p \beta} \right)^2 = 40 \left( \frac{\omega_n}{\omega_p} \right)^2 \]

Case 2, Aileron-flutter \((\beta, \gamma)\): Figure 7 shows the \( \Omega_n \) against \( \frac{1}{k} \) relation and figure 8 the final curve

\[ \kappa \left( \frac{v}{\omega_n \beta} \right)^2 \text{ against } \Omega_n = \left( \frac{\omega_n}{\omega_n \beta} \right)^2 = \frac{1}{160} \left( \frac{\omega_n}{\omega_p} \right)^2 \]

It is realized that considerable care must be exercised to get these curves reasonably accurate.

The heavy line shows the standard case, while the remaining curves show the effect of a change in the value of \( x_n \) to \( \frac{1}{40} \) and \( \frac{1}{160} \).

Case 1, Flexure-torsion \((h, \alpha)\): Figure 9 shows again the \( \Omega_n \) against \( \frac{1}{k} \) relation and figure 10 the final result

\[ \kappa \left( \frac{v}{\omega_n \alpha} \right)^2 \text{ against } \Omega_n = \left( \frac{\omega_n}{\omega_n \alpha} \right)^2 = 4 \left( \frac{\omega_n}{\omega_n} \right)^2 \]

Case 1, which is of importance in the propeller theory, has been treated in more detail. The quantity \( F \) shown in the figures is \( \sqrt{k} \cdot \frac{v}{\omega_n \alpha} \).

Figure 11 shows the dependency on \( \frac{\omega_n}{\omega_n} \); figure 12 shows the dependency on the location of the axis \( \alpha \); figure 13 shows the dependency on the radius of gyration \( r_\alpha = r_\alpha \); and figure 14 shows the dependency on the location of the center of gravity \( z \), for three different combinations of constants.

EXPERIMENTAL RESULTS

Detailed discussion of the experimental work will not be given in this paper, but shall be reserved for a later report. The experiments given in the following are
restricted to wings of a large aspect ratio, arranged with two or three degrees of freedom in accordance with the theoretical cases. The wing is free to move parallel to itself in a vertical direction \((h)\); is equipped with an axis in roller bearings at \((a)\) (fig. 2) for torsion, and with an aileron hinged at \((c)\). Variable or exchangeable springs restrain the wing to its equilibrium position.

![Graph](image1)

**Figure 9.**—Case 1. Flexure-torsion \((h, a)\): Standard case. Showing \(\Omega\) against \(1/k\).

We shall present results obtained on two wings, both of symmetrical cross section 12 percent thick, and with chord \(2b = 12.7\) cm, tested at \(0^\circ\).

![Graph](image2)

**Figure 10.**—Case 1. Flexure-torsion \((h, a)\): Standard case. Showing flutter factor \(\Omega\) against \(\Omega_a\).

Wing \(A\), aluminum, with the following constants:

\[
\begin{align*}
\kappa &= \frac{1}{410}, \quad a = -0.4, \quad z_a = 0.31, 0.173, \text{ and } 0.038, \\
\Omega &= 0.33 \quad \text{and} \quad \omega = 7 \times 2\pi
\end{align*}
\]

![Graph](image3)

**Figure 11.**—Case 1. Flexure-torsion \((h, a)\): Showing dependency of \(\Omega\) on \(\Omega_a\). The upper curve is experimental. Airfoil with \(r = \frac{1}{2}\), \(a = -0.4; \quad z = 0.2; \quad z_a = 0.31; \quad \Omega = \text{variable.}

Wing \(A\), aluminum, with the following constants:
Wing B, wood, with flap, and the constants:

$$\kappa = \frac{1}{100}, \quad c = 0.5, \quad a = -0.4, \quad x_a = 0.192, \quad r_a = 0.178,$$
$$x_2 = 0.019, \quad r_2 = 0.0079, \quad \text{and} \quad \omega_a \text{ kept constant} = 17.6 \times 2\pi.$$

The results for wing A, case 1, are given in figure 15; and those for wing B, cases 2 and 3, are given in figures 16 and 17, respectively. The abscissae are the frequency ratios and the ordinates are the velocities in cm/sec. Compared with the theoretical results calculated for the three test cases, there is an almost perfect agreement in case 1 (fig. 15). Not only is the minimum velocity found near the same frequency ratio, but the experimental and theoretical values are, furthermore, very nearly alike. Very important is also the fact that the peculiar shape of the response curve in case 2, predicted by the theory, repeats itself experimentally. The theory predicts a range of instabilities extending from a small value of the velocity to a definite upper limit. It was very gratifying to observe that the upper branch of the curve not only existed but that it was remarkably definite. A small increase in speed near this upper limit would suffice to change the condition from violent flutter to complete rest, no range of transition being observed. The experimental cases 2 and 3 are compared with theoretical results given by the dotted lines in both figures (figs. 16 and 17).

The conclusion from the experiments is briefly that the general shapes of the predicted response curves repeat themselves satisfactorily. Next, that the influence of the internal friction\(^1\) obviously is quite appreci-

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\(^1\)This matter is the subject of a paper now in preparation.
able in case 3. This could have been expected since the predicted velocities and thus also the air forces on the aileron are very low, and no steps were taken to eliminate the friction in the hinge. The outline of the stable region is rather vague, and the wing is subject
to temporary vibrations at much lower speeds than that at which the violent flutter starts. The above experiments are seen to refer to cases of exaggerated unbalance, and therefore of low flutter speeds. It is evident that the internal friction is less important at larger velocities. The friction does in all cases increase the speed at which flutter starts.

**Figure 15.**—Case 1. Wing A. Theoretical and experimental curves giving flutter velocity \( v \) against frequency ratio \( \omega / \omega_k \). Deflection-torsion.

**Figure 16.**—Case 2. Wing B. Theoretical and experimental curves giving flutter velocity \( v \) against frequency ratio \( \omega / \omega_k \), Aileron-deflection (\( \delta, \beta \)).

**Figure 17.**—Case 3. Theoretical curve giving flutter velocity against the frequency ratio \( \omega / \omega_k \). The experimental unstable area is indefinite due to the importance of internal friction at very small velocities. Torsion-aileron (\( \alpha, \beta \)).
APPENDIX III

EVALUATION OF $\varphi_\beta$

$$\int^1 \log \left( \frac{(x-z_1)^2+(y-y_1)^2}{(x-z_1)^2+(y+y_1)^2} \right) dx_1$$
$$= -\left[ \frac{z_1 \log \left( \frac{x-z_1}{x+z_1} \right) + (y-y_1)^2}{y_1} \right]_1^1 + 2y \int_1^1 \frac{z_1 dx_1}{y_1 (x-z_1)}$$
$$= -2c \log \frac{1-z-x \sqrt{1-c^2}}{c-z} - 2y \int_1^1 \frac{z_1 dx_1}{\sqrt{1-z_1^2} (x-z_1)}$$
$$+ \int_1^1 \frac{x_1 dx_1}{\sqrt{1-x_1^2} (x-z_1)} + \int_1^1 \frac{x_1 dx_1}{\sqrt{1-x_1^2} (x-z_1)\sqrt{1-c^2}}$$
$$+ x \int_1^1 \frac{dx_1}{(x-x_1) \sqrt{1-x_1^2}} \text{[Putting } x_1 = \cos \theta \text{]}$$

$$= -\frac{z}{\sqrt{1-z^2}} \log \frac{1-x \cos \theta + \sqrt{1-x^2} \sin \theta}{x \cos \theta - 1} \left| \begin{array}{c} \cos \theta = 1 \\
\cos \theta = c \end{array} \right.$$
\begin{align*}
&= -\cos^{-1}c \left[ \frac{\cos^3 \theta \sin \theta}{4} + \frac{3}{4} \left( \frac{\theta}{2} + \sin \theta \cos \theta \right) \right] \\
&\quad + \frac{1}{3} \left( 3c \cos^{-1}c - \frac{3}{4} \sqrt{1-c^2} \right) \sin \theta (\cos^3 \theta + 2) \\
&\quad + \left( \cos^{-1}c - \frac{2 \cos^{-1}c c^2 \sqrt{1-c^2}}{4} \right) \left( \frac{\theta}{2} + \sin \theta \cos \theta \right) \\
&\quad + \left( -3c \cos^{-1}c + \sqrt{1-c^2} \cos^{-1}c + \frac{3c^2 \sqrt{1-c^2}}{4} \right) \cos \theta \\
&= \cos^{-1}c \left( \frac{3}{8} \pi + \frac{\pi}{2} - 3\pi \right) = -\frac{9}{4} \pi \cos^{-1}c \\
&\frac{2\pi}{c} \int_c^1 q_c(x-c) \, dx \\
&= \cos^{-1}c \left[ \frac{c^3 \sqrt{1-c^2}}{4} + \frac{3 \cos^{-1}c}{8} + \frac{3c^2 \sqrt{1-c^2}}{8} \right] \\
&\quad - \left[ c \cos^{-1}c - \frac{\sqrt{1-c^2}}{4} \right] \left( c^2 \sqrt{1-c^2} + 2 \sqrt{1-c^2} \right) \\
&\quad - \left( \cos^{-1}c - \frac{2 \cos^{-1}c c^2 \sqrt{1-c^2}}{4} \right) \left( \cos^{-1}c + c \sqrt{1-c^2} \right) \\
&\quad - \left( -3c \cos^{-1}c + \sqrt{1-c^2} + \frac{3c^2 \sqrt{1-c^2}}{4} \right) \sqrt{1-c^2} \\
&\quad - \left( 2c^3 \cos^{-1}c - \frac{c^2 \sqrt{1-c^2} \cos^{-1}c}{4} \right) \cos^{-1}c \\
&= -\frac{9}{8} \pi \cos^{-1}c \\
&\quad + \sqrt{1-c^2} \cos^{-1}c \left[ \frac{c^3}{4} + \frac{3c}{8} \sqrt{1-c^2} - 2c^2 - \frac{c}{2} + c^3 + 3c + c \right. \\
&\quad + \frac{c^3}{8} \right] + \frac{c^2 \sqrt{1-c^2}}{4} \left[ (1-c^2) - \frac{c^2 (1-c^2)}{8} \right] \\
&\quad - (1-c^2) - \frac{3c^2 (1-c^2)}{4} = -\frac{1}{8} \left( 8 + c^2 \right) \cos^{-1}c \\
&\quad + \frac{c \sqrt{1-c^2} \cos^{-1}c}{4} \left( 7 + 2c^2 \right) - \frac{(1-c^2)}{8} \left( 5c^3 + 4 \right) (= T) \\
&\int_c^1 \frac{2(x-c) \log \frac{1-cx-\sqrt{1-x^2} \sqrt{1-c^2}}{x-c}}{x-c} \, dx = T_2 = -2 \int \left( x-c \right) \log \left( x-c \right) \, dx \\
&\quad + 2 \int \left( x-c \right) \log \left( 1-cx-\sqrt{1-x^2} \sqrt{1-c^2} \right) \, dx \\
&\quad - 2 \cos^{-1}c \int \sqrt{1-x^2} \, dx = \frac{2(\pi-c^3)}{2} \log (x-c) \\
&\quad + \int \left( x-c \right) \, dx + 2 \cos^{-1}c \int \sin 2\theta \, d\theta \\
&\quad + (x-c) \log \left( 1-cx-\sqrt{1-x^2} \sqrt{1-c^2} \right) \\
&\quad - \cos^{-1}c \left( \frac{3}{8} \pi + \frac{\pi}{2} - 3\pi \right) = -\frac{9}{4} \pi \cos^{-1}c \\\n&\quad - \frac{c + x \sqrt{1-c^2}}{1-cx-\sqrt{1-x^2} \sqrt{1-c^2}} \, dx \\
&\quad = \int \left( 1 \right) \, dx + \sqrt{1-c^2} \int \frac{(x-c)}{\sqrt{1-x^2}} \, dx \\
&\quad + \int \left( x-c \right) \, dx + \sqrt{1-c^2} \int \frac{(x-c)}{\sqrt{1-x^2}} \, dx \\
&\quad = \pi \cos^{-1}c \int \frac{(x-c)}{\sqrt{1-x^2}} \, dx \\
&T_3 = -(x-c) \log (x-c) + 2 \cos^{-1}c \int \sin 2\theta \, d\theta \\
&\quad + (x-c) \log \left( 1-cx-\sqrt{1-x^2} \sqrt{1-c^2} \right) \\
&\quad + \sqrt{1-c^2} \int (\cos \theta - c) \, d\theta \\
&\quad = 2 \cos^{-1}c \frac{\theta - \sin \theta \cos \theta}{\cos \theta - \cos \theta} + \sqrt{1-c^2} \sin \theta \\
&\quad - \frac{c \sqrt{1-c^2} \theta}{\cos \theta} \left. \right|_{\cos \theta = 1} \\
&\quad - (1-c^2) - (\cos^{-1}c)^2 + 2c \sqrt{1-c^2} \cos^{-1}c \\
\end{align*}