REPORT 1119

RECIPROCITY RELATIONS IN AERODYNAMICS [1]

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SUMMARY

Reverse-flow theorems in aerodynamics are shown to be based on the same general concepts involved in many reciprocity theorems in the physical sciences. Reciprocal theorems for both steady and unsteady motion are found as a logical consequence of this approach. No restrictions on wing plan form or flight Mach number are made beyond those required in linearized compressible-flow analysis. A number of examples are listed, including general integral theorems for lifting, rolling, and pitching wings and for wings in nonuniform downwash fields. Correspondence is also established between the buildup of circulation with time of a wing starting impulsively from rest and the buildup of lift of the same wing moving in the reverse direction into a sharp-edged gust.

INTRODUCTION

Some of the most important results in the recent study of wing theory have been achieved through the development of reverse-flow relations. The theorems already obtained are of outstanding practical utility and it appears obvious that the fullest exploitation of the methods has yet to be accomplished, either from a purely theoretical standpoint or in the routine calculation of wing characteristics. Attention to such problems in aerodynamics was initiated by von Kármán (ref. 1) who first announced the invariance of drag with forward and reversed directions of flight for a nonlifting symmetrical wing at supersonic speed. Subsequently, advances in the theory were made by Munk, Hayes, Brown, Harmon, and Flax (refs. 2 through 7). Up to the present time, the most general results have been expressed by Ursell and Ward (ref. 8) and by R. T. Jones (ref. 9) in his attack on the study of wing shapes of minimum drag. In the forms given in the two latter papers, the derived equivalences could be termed reciprocal or reciprocity relations rather than reverse-flow relations; in fact, this change in terminology divorces attention, momentarily, from the purely aerodynamic aspects of the results and, in this way, suggests a reorientation in terms of the various similar relations appearing in other engineering fields. In the theory of elasticity, for example, a reciprocity theorem for small displacements of an elastic medium is so expressed as to appear in formal agreement with the statement of the result given by Ursell and Ward (see, e. g., ref. 10). This theorem is attributed to E. Betti and was published in 1872. A generalization was given by Lord Rayleigh in 1873, and in various sections of his two volumes on the theory of sound (ref. 11) discussions of reciprocal relations in an elastic medium and for acoustic sources are given. In 1886, von Helmholtz (ref. 12) obtained, by means of variational methods applied to Hamilton's characteristic function, a reciprocal theorem for small changes in the momenta and coordinates of a general dynamical system in forward and reversed motion. This result was commented on, in turn, by Lamb (ref. 13) and an independent proof based upon Lagrange's generalized equations was given. The paper by Lamb is of particular interest since it contains the essential idea underlying the development of reverse-flow theorems in wing theory. Thus, Lamb remarks, as had Lord Rayleigh previously, that reciprocity relations between sound sources do not apply directly in a moving atmosphere. He points out, however, that the reciprocity can be restored if the direction of the wind is also reversed. Further examples of reciprocal theorems appear in the theories of electricity and magnetism (in particular, reference should be made to Maxwell's discussion of the subject in ref. 14) and of optics.

The generality in the statement of reciprocity relations appears, almost universally, to have held back their application to problems for which they are obviously, in retrospect, particularly fitted. This generality is even more apparent in some of the conclusions of Lord Rayleigh and von Helmholtz which apply to nonconservative systems.

The purpose of the present paper is twofold. First, a close connection will be established between reverse-flow theorems in subsonic and supersonic, steady-state wing theory and known reciprocity relations between two solutions of the equation governing the flow field. In this way, machinery will be provided whereby extensions of existing results to the case of unsteady motion follow directly. Second, a number of particular problems in wing theory in steady and unsteady flow will be considered. It will be shown that, provided attention is limited to force and moment characteristics, the complexity of many solutions involving nonuniform flow fields, control-surface deflections, and unsteady motion can be reduced considerably. In some cases, previously obtained solutions will be calculated. Comparison with the original calculations will almost invariably highlight the economy of effort in obtaining the final result. The utility of reverse-flow theorems is based on the fact that they build from known solutions and thus avoid the necessity of starting each problem anew.

[1] Supersedes NACA TN 2700, "Reciprocity Relations in Aerodynamics" by Max. A. Heaslet and John R. Spreiter, 1952.

GENERAL ANALYSIS

RECIPROCITY RELATIONS FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, integral relations associated with linear partial differential equations will be reviewed from the standpoint of relating independent solutions. The subject matter is precisely Green's theorem and, in common with the usual expression of the theorem, it is preferable to treat the variables initially as abstract quantities. Consider, therefore, a class of linear partial differential equations of second order with independent variables $X_1, X_2, \ldots, X_m$ that may be thought of as rectangular coordinates in a space of m dimensions. Denote differentiation of the function $\psi(X_1, X_2, \ldots, X_m)$ with respect to the variables X_i and X_j by the subscript notation

$$(\psi)_{ij} \equiv \frac{\partial^2 \psi}{\partial X_i \partial X_j}$$

and consider the differential equation

$$L(\psi) \equiv \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}(\psi)_{ij} + B\psi = 0 \tag{1}$$

where, for the purposes at hand, $A_{ij}=A_{ji}$ are made independent of X_i and X_j, and B is a constant. Such equations fall within the class of self-adjoint equations.

By Green's theorem (see, e. g., ref. 15), it is possible to relate two arbitrary functions ψ and Ω by the integral expression

$$\iiint [\Omega L(\psi) - \psi L(\Omega)] dV = -\iint [\Omega D_n \psi - \psi D_n \Omega] dS \tag{2}$$

where the left member is a volume integral over a prescribed region in m-dimensional space and the right member is a surface integral over the hypersurface S enclosing the given region. Equation (2) certainly holds for any region in which ψ and Ω and their first and second derivatives are continuous. The directional derivaties D_n are defined in terms of the direction cosines $n_1, n_2, \ldots, n_m$ of the normal to the surface S with the stipulation that the normal is directed into the given region. Thus, setting

$$\sum_{j=1}^{m} n_j A_{ij} = N\nu_i$$

where $\nu_1, \nu_2, \ldots, \nu_m$ are the direction cosines of a line termed the "conormal," the directional derivative is defined by the expression

$$D_n \psi = N \sum_{i=1}^{m} \nu_i (\psi)_i = N(\psi)_\nu = N \frac{\partial \psi}{\partial \nu} \tag{3}$$

If, finally, ψ and Ω are assumed to satisfy equation (1), the left side of equation (2) vanishes and the resulting expression

$$\iint \Omega D_n \psi dS = \iint \psi D_n \Omega dS \tag{4}$$

is a general reciprocity relation expressing the functional dependence between two arbitrary solutions of equation (1).

An interesting interpretation of equation (4) has been given by several writers (see, e. g., ref. 16, p. 46) and applies to the particular case when ψ and Ω are identified with the perturbation velocity potential φ in the theory of incompressible-fluid flow. The governing equation is Laplace's equation in three dimensions

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{5}$$

and the reciprocity relation takes the form, for two possible solutions φ and φ',

$$\iint \varphi \frac{\partial \varphi'}{\partial n} dS = \iint \varphi' \frac{\partial \varphi}{\partial n} dS \tag{6}$$

where the directional derivatives are now along the true normals to the surface S enclosing a three-dimensional volume. It is known that any actual state of motion of a liquid for which a single-valued velocity potential exists can be produced instantaneously by the application of a properly chosen system of impulsive pressures. These impulsive pressures are directly proportional to the velocity potential plus an arbitrary constant which may, in the present case, be associated with the pressure of the uniform stream. Equation (6) is thus seen to represent summations of the cross products of impulse and normal velocity in two possible motions of a conservative system and is a special case of the dynamical theorem (ref. 11, p. 98)

$$\sum_{r=1}^{\infty} p_r \dot{q}_r' = \sum_{r=1}^{\infty} p_r' \dot{q}_r \tag{7}$$

where p_r, $\dot{q}_r$ and p_r', $\dot{q}_r'$ are generalized components of impulse and velocity in any two possible motions of a system which starts from rest.

The interpretation of equation (6) that leads to equation (7) provides an indication of the close connection between reciprocal theorems based upon the principles of least action and the symmetric character of Green's theorem for certain second-order differential equations. In the subsequent applications it will be convenient to proceed directly from equation (4) and seek to establish reciprocal relations between flow fields in wing theory. Such a process is well known when ψ or Ω is replaced by the elementary solution associated with a unit source and, in this case, establishes a general solution in terms of source and doublet distributions determined by arbitrary boundary conditions. The present objective is, however, different in that one wishes to get a symmetrical dependence between two general solutions. Moreover, the apparent symmetry of equation (4) must be consistent with physical considerations so that, for example, the velocity at point A induced by a single source at point B is equal to the velocity at point B induced by a source at point A. If, in the two systems, the effective free streams or flight directions are opposed, a fore and aft symmetry occurs and the possibility of maintaining symmetry in the reciprocal relations becomes feasible.

RECIPROCAL THEOREMS IN WING THEORY

REVERSE FLOW FOR SUBSONIC WINGS

Consider a thin wing at possibly a small angle of attack and situated in the immediate vicinity of a flat surface which is designated the xy plane. For sufficiently small thickness and angle of attack, the perturbation velocity potential or any of

the perturbation velocity components of the wing satisfy the linearized partial differential equation of compressible flow.

$$\beta^2\psi_{xx}+\psi_{yy}+\psi_{zz}=0 \tag{8}$$

where $\beta^2=1-M_0^2=1-(U_0/a_0)^2$, and U_0,a_0 are, respectively, the flow velocity and speed of sound in the free stream. Equation (8) applies in forward or reverse flow, provided the corresponding free-stream Mach numbers M_0 and M_0' are equal. In figure (1) a lifting wing with plan form P is indicated along with the vortex wakes as the configuration would appear if the two flow fields were superimposed. It will be assumed that the wing chord is finite and that the profiles are closed.

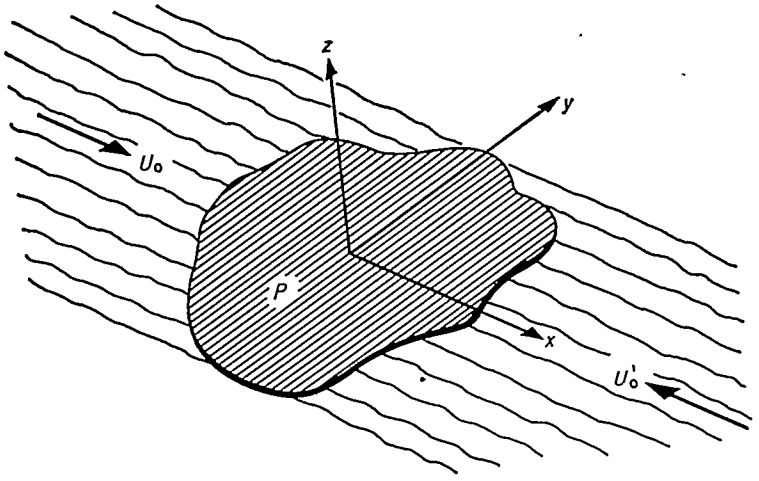

FIGURE. 1.—View of wing in combined flow field.

It is proposed to apply the general reciprocity relation (4) to these flow fields in a manner similar to the approach used in developing the basic solutions of the differential equation (see, e. g., refs. 17 and 18). Thus, for subsonic flow, hemispherical regions of large radius and lying first above and then below the plane of the wing will be chosen as the volumes of integration. The surface integrals will therefore extend over a hemisphere with center at the wing and a flat surface that lies immediately adjacent to the $z=0$ plane. This latter surface is subsequently to be brought into coincidence with the plane of the wing but must be considered first in its displaced position since only then can the flow be assumed free of possible singularities in perturbation velocities and their gradients. As in wing theory, in general, the attenuation of the perturbation potential and its gradients may be assumed of such a nature that the integrated contributions of the wing and its wake over the hemispherical surfaces vanish in the limit as the radius becomes infinitely large. It remains, therefore, to consider the integrals over the surfaces at $z=0+\epsilon$ and $z=0-\epsilon$. Denote these surfaces as σ_u and σ_l, respectively, where the subscripts u and l specify values above and below the $z=0$ plane. Equation (4) can then be written as

$$\iint_{\sigma_u}\psi\frac{\partial\psi'}{\partial n}\,dS=\iint_{\sigma_u}\psi'\frac{\partial\psi}{\partial n}\,dS; \qquad n \text{ directed upward} \tag{9a}$$

and

$$\iint_{\sigma_l}\psi\frac{\partial\psi'}{\partial n}\,dS=\iint_{\sigma_l}\psi'\frac{\partial\psi}{\partial n}\,dS; \qquad n \text{ directed downward} \tag{9b}$$

where the primes denote conditions for the reversed flow.

In equations (9), ψ is now replaced by the perturbation velocity potential $\varphi(x,y,z)$ and ψ' by the x or streamwise component of perturbation velocity $u(x,y,z)$. By virtue of the irrotationality of the flow, the gradients of u and w are related by the expression

$$\frac{\partial u}{\partial n}=\pm\frac{\partial w}{\partial x}$$

and equations (9) can both be written in the form

$$\iint\varphi\frac{\partial w'}{\partial x}\,dS=\iint u'w\,dS$$

Integration by parts over either of the surfaces σ_u or σ_l gives

$$\int_{-\infty}^{\infty}\varphi w'\Big|_{-\infty}^{\infty}dy-\iint uw'\,dS=\iint u'w\,dS \tag{10}$$

At $x=-\infty$ the potential φ for the forward flow vanishes and at $x=+\infty$ the upwash w' in the reversed flow vanishes so that the first term on the left is zero. The remaining double integrals have for their surfaces of integration the displaced planes σ_u and σ_l. In order to obtain a concise form of the reverse-flow theorem, it suffices to subtract the integrals extended over σ_l from the integrals over σ_u and let the planes approach coincidence with the plane of the wing. Since w,w' and u,u' are continuous everywhere except possibly in the immediate vicinity of the wing, the integration region can be restricted to planes slightly above and below the wing but extending beyond the wing edges. Provided the singularities at the edges can be disregarded, the analytical expression of the steady-state reverse-flow theorem of Ursell and Ward (ref. 8) becomes

$$-\iint_P(u_uw_u'-u_lw_l')\,dS=\iint_P(u_u'w_u-u_l'w_l)\,dS \tag{11}$$

for either lifting surfaces or symmetrical wings where P is the plan form of the wing in the $z=0$ plane.

It remains to discuss the effect of edge singularities. In the case of a lifting surface, square-root singularities in both u and w can occur at the leading and trailing edges just on and off the wing, respectively. In the combined flow fields, the limiting process would then yield residue terms analogous to the leading-edge thrust of a lifting plate. If, however, the Kutta condition is imposed at the trailing edge for both the flow in forward and reverse direction, a combination of singularities does not occur and equation (11) is valid. If the leading edge of a symmetrical wing is rounded so as to produce a square-root singularity in w on the wing, a square-root singularity in u occurs just off the wing and a term corresponding to the leading-edge drag (ref. 19) appears. If the geometry of the wing is fixed in the forward and reverse flow, however, the effect of these terms is canceled and equa-

tion (11) still applies. It is important to point out that if the Kutta condition is not satisfied, then the area of integration of equation (11) cannot be restricted to the plan form.

The two sides of equation (11) are expressed in terms of the same coordinate system but it is usually preferable to associate with each of the two streams an x axis extending in the stream direction. To this end introduce now the subscripts 1 and 2 to denote the forward and reverse flow and the two coordinate systems. Thus, in general,

$$x_1=-x_2+\xi, \qquad y_1=-y_2+\eta, \qquad z_1=z_2 \tag{12}$$

where ξ and η are arbitrary constants, and equation (11) then becomes

$$\iint_{P_1}[u_{u_1}(x_1,y_1)w_{u_2}(x_1,y_1)-u_{l_1}(x_1,y_1)w_{l_2}(x_1,y_1)]dx_1dy_1=$$

$$\iint_{P_2}[u_{u_2}(x_2,y_2)w_{u_1}(x_2,y_2)-u_{l_2}(x_2,y_2)w_{l_1}(x_2,y_2)]dx_2dy_2 \tag{13}$$

In the case of a symmetrical nonlifting wing, the relations

$$u_u=u_l, \qquad w_u=-w_l$$

must apply and, in the case of a lifting surface, linearized theory yields

$$u_u=-u_l, \qquad w_u=w_l$$

It follows that in either case, equation (13) reduces to the form

$$\iint_{P_1}u_1(x_1,y_1)w_2(x_1,y_1)dx_1dy_1=\iint_{P_2}u_2(x_2,y_2)w_1(x_2,y_2)dx_2dy_2 \tag{14}$$

where the velocity components can be evaluated on either the upper or lower surface of the plan form. If, furthermore, the linearized pressure relation

$$p-p_0=-\rho_0U_0u \tag{15}$$

is used where p is local static pressure, p_0 is static pressure in the free stream, and the wing profiles are assumed closed, equation (14) becomes

$$\iint_{P_1}p_1(x_1,y_1)w_2(x_1,y_1)dx_1dy_1=\iint_{P_2}p_2(x_2,y_2)w_1(x_2,y_2)dx_2dy_2 \tag{16}$$

If, instead of specifying boundary conditions in a single plane, it is necessary to treat boundary conditions for a system of planes, the expression of the reverse-flow theorem is of the same general form as equation (11). Provided the Kutta condition is imposed at the trailing edges of all lifting surfaces, the relation becomes

$$\iint_{\Sigma}(V_n)_1u_2dS=\iint_{\Sigma}(V_n)_2u_1dS \tag{17}$$

where the area of integration Σ extends over both sides of all the wing surfaces, V_n is the component of perturbation velocity normal to and directed away from each wing, and the subscripts 1 and 2 refer to forward and reverse flow in the two axial systems.

REVERSE FLOW FOR SUPERSONIC WINGS

The development of a reverse-flow theorem for supersonic wings parallels closely the analysis for the subsonic case. For either planar or multiplanar systems, the conormal in equation (4) is, in fact, the normal so long as the surface of integration is a plane parallel to the x axis. In the case of the single wing, for example, equations (9) apply where the surfaces σ are slightly removed from the plane of the wing and where ψ satisfies the differential equation

$$(M_0^2-1)\psi_{xx}-\psi_{yy}-\psi_{zz}=0$$

In the limit as σ approaches the $z=0$ plane, the reversibility theorem takes the form of equations (11) and (16), provided the integration extends beyond the edges of the wing. It is necessary to include these edges for wings with subsonic leading and trailing edges since singularities occur in the perturbation components and the solutions are not necessarily unique. For supersonic-type edges, the area of integration can be confined to the plan form of the wing and this is also true for subsonic edges, provided the Kutta condition holds for all subsonic trailing edges in both the forward and reverse flow. Equation (17) relates the two possible flows in the case of multiplanar systems.

REVERSE UNSTEADY MOTION

In the case of unsteady motion at either subsonic or supersonic flight speed, the basic equation may be taken in the form

$$\psi_{tt}-\psi_{xx}-\psi_{yy}-\psi_{zz}=0 \tag{18}$$

where $t=a_0t'$, a_0 is the speed of sound in the undisturbed region of the field, t' is time, and ψ is the perturbation velocity potential or any of the perturbation velocity components. Equation (18) is the acoustic equation for small disturbances in three space dimensions and holds for a system of Cartesian coordinates fixed relative to the undisturbed air. In applications to wing theory, therefore, the wings move relative to fixed axes.

In the derivation of a useful theorem it is convenient to treat thin wings at small angles of attack and to assume that the motion takes place in the xy plane. The visualization of the time and geometry relations is relatively easy for two-dimensional wings moving at a uniform speed, as indicated in figure 2. The airfoil starts at time $t'=0$ and moves to the left at a constant velocity U_0 so that the trace of the leading edge in the xt plane is $x=-U_0t'=-M_0t$, and the trailing-edge trace is $x=2a-M_0t$. The lines $x=\pm t$ and $x=2a\pm t$ are the traces of the extremeties of the regions affected by the acoustic waves set in motion at $t'=0$ by the leading and trailing edges. In figure 2, the wing has traveled a time $T'=T/a_0$ and the boundary condition determining the wing shape during the motion will be fixed by prescribing the value of vertical induced velocity w over the region "swept out" of the xt plane by the wing. In order to determine a reciprocal theorem, a second wing is assumed to start at the final position of the first wing and to move with negative velocity until it has reached the initial position of the first wing. With these

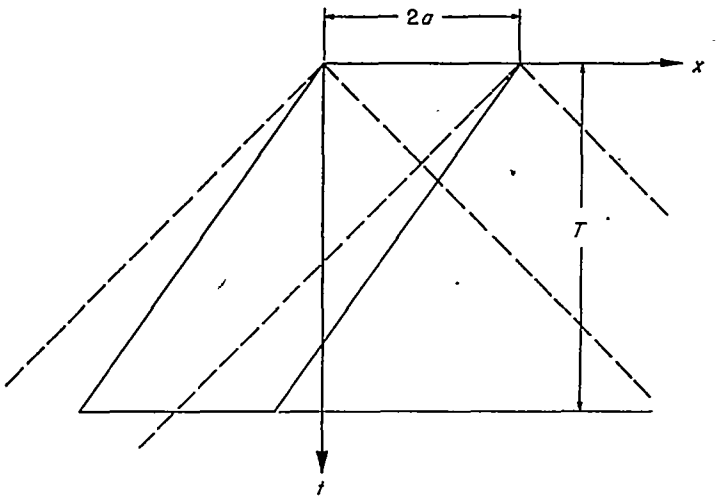

FIGURE 2.—Two-dimensional wing in xt plane.

concepts in mind, it follows from equation (4) that the relation

$$\iint \varphi \frac{\partial^2 \varphi'}{\partial z \partial t} dx dt = \int\int \frac{\partial \varphi'}{\partial t} w dx dt \tag{19}$$

holds where φ and φ' are, respectively, the perturbation potentials for the forward and reverse motions. The region of integration is determined by the area occupied in the xt plane by the wing, and the Kutta condition is assumed to apply to the trailing edge when in subsonic flight.

If the left side of equation (19) is integrated by parts, the general relation becomes

$$-\iint \frac{\partial \varphi}{\partial t} w' dx dt = \iint \frac{\partial \varphi'}{\partial t} w dx dt \tag{20}$$

If the motion of a three-dimensional wing is to be studied, equations (19) and (20) must be modified to include an integration with respect to y.

Two further changes in equation (20) serve to simplify applications. In the first place, asymmetry is restored to the expression if two distinct systems of axes are used as in equation (14); in the second place, the pressure relation

$$p - p_0 = -\rho_0 \frac{\partial \varphi}{\partial t'} = -\rho_0 a_0 \frac{\partial \varphi}{\partial t} \tag{21}$$

where p_0 denotes undisturbed pressure, permits the introduction of pressure p in the integrands. The final expression for the three-dimensional case is, therefore,

$$\iiint p_1(x_1, y_1, t_1) w_2(x_1, y_1, t_1) dx_1 dy_1 dt_1 = \iiint p_2(x_2, y_2, t_2) w_1(x_2, y_2, t_2) dx_2 dy_2 dt_2 \tag{22}$$

where the two motions now follow the same path in reverse directions but are referred to the two sets of oppositely oriented axes satisfying the relations

$$x_1 = -x_2 + \xi,\ y_1 = -y_2 + \eta,\ z_1 = z_2,\ t_1 = -t_2 + \tau \tag{23}$$

where ξ, η, and τ fix the relative positions of the two origins. Figure 3 indicates one possible orientation of the axes.

Equation (22) reduces to a much simpler form, provided further restrictions are imposed on the upwash functions

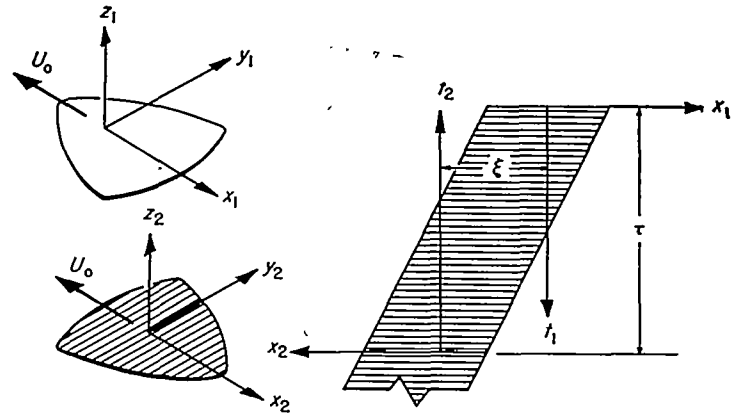

FIGURE 3.—Coordinate systems for three-dimensional wing in unsteady motion.

$w_2(x_1, y_1, t_1)$ and $w_1(x_2, y_2, t_2)$. In order to fix the idea, consider the case in which the wings have traveled a time T', $(T' = T/a_0)$ and a distance $U_0 T' = M_0 T$. Let the two systems of coordinate axes be placed such that $t_1 = 0$ sets the starting time of the forward motion and $t_2 = 0$ sets the starting time of the reverse motion; the two origins are furthermore oriented such that they are at opposite ends of the root chord of the common plan form. Equation (22) then becomes

$$\int_0^T dt_1 \int_{P_1(t_1)} \int p_1(x_1, y_1, t_1) w_2(x_1, y_1, t_1) dx_1 dy_1 = \int_0^T dt_2 \int_{P_2(t_2)} \int p_2(x_2, y_2, t_2) w_1(x_2, y_2, t_2) dx_2 dy_2$$

where the functions w_1 and w_2 have an implicit dependence upon T. If w_1 and w_2 remain constant for $x_1 + M_0 t_1 = \text{const.}$ or $x_2 + M_0 t_2 = \text{const.}$, the expression

$$\int_{P_1(T)} \int p_1(x_1, y_1, T) w_2(x_1, y_1, T) dx_1 dy_1 = \int_{P_2(T)} \int p_2(x_2, y_2, T) w_1(x_2, y_2, T) dx_2 dy_2 \tag{24}$$

follows after taking a derivative with respect to T of the original equality. Equations (24) and (14) are now equivalent in form, with T taking the role of an auxiliary parameter. In this way, certain classes of unsteady motions can be treated simultaneously with steady motions.

In the applications to follow it will be convenient to introduce into equation (24) upwash functions of the indicial type; that is, functions that are zero up to a fixed time and, after experiencing a finite discontinuity, remain constant for all subsequent values of time. Such indicial or step variations can be assumed, say, for angle of attack, rate of pitch, and rate of roll since they satisfy the requirements underlying the derivation of equation (24). This choice of functions will prove to be advantageous in that the integrals of the responsive pressures will yield results relating the wing characteristics. Theorems to be given later will speak specifically of steady and indicial motions. It is to be understood, however, that the indicial results can be further extended when the same wing is assumed to be executing

identical motions in forward and reverse flight. Thus, by means of Duhamel's integral (see, e. g., ref. 20), if $f(t)$ is the response in the wing characteristic to a step variation in w at time $t=0$, the response to an arbitrary variation with time of w can be written

$$F(t)=\frac{\partial}{\partial t}\int_0^t f(t-\tau)w(\tau)d\tau \tag{25}$$

If it is known, for example, that the lift per unit angle of attack is the same at corresponding values of time for a wing experiencing an indicial angle-of-attack change in forward and reverse flight, it follows that the build-up of lift is the same at corresponding values of time for all forward and reverse motions, provided the time histories of the motions are the same. The equivalence of lift would thus be established, for instance, for oscillatory variations in angle of attack.

An alternative study of reverse-flow theorems for oscillatory motions could be based upon the modified wave equation

$$\chi_{xx}+\chi_{yy}+\chi_{zz}+\omega^2\chi=0$$

which results from setting $\psi=e^{i\omega t}\chi$ in equation (18). Such a study would corroborate the conclusions drawn from equations (24) and (25).

APPLICATIONS

The results of the foregoing analysis may be employed to determine a number of special theorems that are particularly useful in the calculation of the aerodynamic characteristics of twisted wings and of wings in nonuniform downwash fields. The theorems apply equally to wings acting either alone or, in certain cases, in combination with other wings or with cylindrical bodies having their generators alined with the x axis. Moreover, they apply not only to wings in steady motion but also to wings performing unsteady motions of the indicial type, or unsteady motions derivable therefrom. For wings in more complex unsteady motions, however, it will be necessary to refer to the more general results of equation (22). Some problems of this nature will be described at the end of this section.

The applications to be included are exact within the framework of linear theory and involve no further rectrictions on the wing plan form or Mach number except in certain indicated cases where it will be convenient to use results based on slender-wing theory. The examples are intended to be representative in nature.

REVERSAL THEOREMS—STEADY AND INDICIAL MOTIONS

Reversal theorems are defined here as relations between the aerodynamic characteristics of identical wings executing the same type of motions in forward and reverse flight. The results presented in this section apply not only to single wings in steady motion but also to combinations of wings, as in cascades or multiplanes, performing either steady motion or motions of the indicial type.

DRAG OF SYMMETRICAL NONLIFTING WINGS

The drag of a symmetrical, sharp-edged wing in linear theory may be determined by integrating over the plan form the product of the pressure and the slope in the x direction of the wing surface; when the wing has blunt edges with slopes having square-root singularities, these singularities yield an added contribution (ref. 19). In general, therefore, the drag D of a symmetrical section is given by

$$D=D_e+2\iint_P p\left(\frac{dz}{dx}\right)_u dS \tag{26}$$

where D_e is the drag attributable to the edges.

If the subscripts 1 and 2 refer to the same wing in forward and reverse flow, respectively, and with the two systems of axes introduced in equation (23), local slopes are related as follows

$$\left.\frac{dz_1(x_1,y_1,T)}{dx_1}\right|_u=-\left.\frac{dz_2(x_2,y_2,T)}{dx_2}\right|_u \tag{27}$$

Equations (26) and (27), together with the reciprocal relation (24), yield

$$D_1-(D_e)_1=2\iint_{P_1} p_1\left(\frac{dz_1}{dx_1}\right)_u dS_1=-2\iint_{P_1} p_1\left(\frac{dz_2}{dx_2}\right)_u dS_1=$$

$$-2\iint_{P_2} p_2\left(\frac{dz_1}{dx_1}\right)_u dS_2=2\iint_{P_2} p_2\left(\frac{dz_2}{dx_2}\right)_u dS_2=D_2-(D_e)_2 \tag{28}$$

Since the geometry of the wing is fixed, the edge contributions are the same,

$$(D_e)_1=(D_e)_2 \tag{29}$$

and, consequently,

$$D_1=D_2 \tag{30}$$

which confirms the relation stated in reference 9.

THEOREM: The pressure drag in steady or indicial motion of symmetrical nonlifting wings is the same in forward and reverse flight.

LIFT ON FLAT-PLATE WINGS

The lift L of a wing may be determined by integrating the differential pressure $\Delta p=p_l-p_u$ over the wing plan form, thus

$$L=\iint_P \Delta p dS \tag{31}$$

For flat-plate wings, the local angle of attack of the wing surface is a constant

$$\alpha_1(x_1,y_1,T)=\alpha_1=\text{const.},\quad \alpha_2(x_2,y_2,T)=\alpha_2=\text{const.} \tag{32}$$

Application of equations (31), and (32), and (24) yields the following:

$$L_1\alpha_2=\iint_{P_1}\Delta p_1\alpha_2 dS_1=\iint_{P_2}\Delta p_2\alpha_1 dS_2=L_2\alpha_1$$

or

$$L_1/\alpha_1=L_2/\alpha_2 \tag{33}$$

THEOREM: The lift per unit angle of attack of flat-plate wings in steady or indicial motion is the same in forward and reverse flight.

This theorem generalizes the relation previously given by Brown (ref. 5) for steady motion.

DAMPING IN ROLL OF FLAT-PLATE WINGS

The rolling moment L' exerted on a wing, following the usual sign convention, is given by

$$L'=-\iint_P y\Delta p\,dS \tag{34}$$

The local angle of attack due to rotation about the x axis is

$$\alpha_1=\frac{p_1'y_1}{U_0},\qquad \alpha_2=\frac{p_2'y_2}{U_0}=-\frac{p_2'y_1}{U_0} \tag{35}$$

where p' is the angular velocity of roll, assumed constant. Application of equations (34), (35), and (24) yields the following:

$$\frac{p_2'L_1'}{U_0}=-\frac{p_2'}{U_0}\iint_{P_1} y_1\Delta p_1\,dS_1=\frac{p_2'}{U_0}\iint_{P_1} y_2\Delta p_1\,dS_1$$

$$=\frac{p_1'}{U_0}\iint_{P_2} y_1\Delta p_2\,dS_2=-\frac{p_1'}{U_0}\iint_{P_2} y_2\Delta p_2\,dS_2=\frac{p_1'L_2'}{U_0}$$

or

$$L_1'/p_1'=L_2'/p_2' \tag{36}$$

THEOREM: The rolling moment per unit angular rolling velocity of flat-plate wings in steady or indicial motion is the same in forward and reverse flight.

DAMPING IN PITCH OF FLAT-PLATE WINGS

Consider a wing, first, in forward flight and pitching with a uniform angular velocity q_1 about a lateral axis; second, in reverse flight and pitching with angular velocity q_2 about another lateral axis. Place each wing in a coordinate system such that the y axis coincides with the axis of rotation and designate the distances to the moment axes by x_0 with proper subscripts, as shown in figure 4. In such a coordinate system, the pitching moment M_{x_0} exerted on a wing, following the usual sign convention, is

$$M_{x_0}=-\iint_P (x-x_0)\Delta p\,dS \tag{37}$$

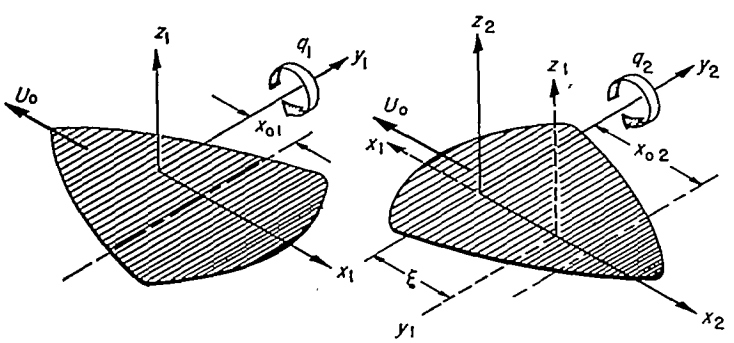

FIGURE 4.—Coordinate systems and symbols used in discussion of reverse-flow theorems for pitching wings.

The local angle-of-attack distributions due to rotations about the y axis are

$$\alpha_1=\frac{q_1}{U_0}x_1,\qquad \alpha_2=\frac{q_2}{U_0}x_2=\frac{q_2}{U_0}(\xi-x_1) \tag{38}$$

Application of equations (37), (38), and (24) yields the following:

$$\frac{q_2}{U_0}[(M_1)_{x_{01}}+(\xi-x_{01})L_1]=\frac{q_2}{U_0}\iint_{P_1}[(x_{01}-x_1)+(\xi-x_{01})]\Delta p_1\,dS_1=$$

$$\frac{q_2}{U_0}\iint_{P_1}(\xi-x_1)\Delta p_1\,dS_1=\frac{q_2}{U_0}\iint_{P_1}x_2\Delta p_1\,dS_1=$$

$$\frac{q_1}{U_0}\iint_{P_2}x_1\Delta p_2\,dS_2=\frac{q_1}{U_0}[(M_2)_{x_{02}}+(\xi-x_{02})L_2]$$

or

$$\frac{(M_1)_{x_{01}}+(\xi-x_{01})L_1}{q_1}=\frac{(M_2)_{x_{02}}+(\xi-x_{02})L_2}{q_2} \tag{39}$$

This equation indicates that the pitching moment due to pitching velocity is, in general, not the same for wings in forward and reverse flight. However, if $x_{01}=x_{02}=\xi$, the pitching moment per unit angular pitching velocity of flat-plate wings in steady or indicial motion is invariant.

SPECIAL RECIPROCAL THEOREMS AND APPLICATIONS

In the following section, several special reciprocal theorems will be derived and applications will be illustrated. Reciprocal theorems, in contrast to reversal theorems treated in the preceding section, are defined here as relations between the aerodynamic properties of wings in forward and reverse flight that have dissimilar camber, twist, and thickness distributions but have the same plan forms. The motions may or may not be similar, although it is assumed in this section that both wings are in either steady motion or unsteady motion of the indicial type. As noted in the preceding section, the results apply equally to wings acting alone or in combination.

SYMMETRIC NONLIFTING WINGS—STEADY MOTION

The problems of paramount interest in the application of the general relations are found from considerations of pressure integrals over lifting surface; such problems will be given detailed treatment later. In the present section, a brief indication is given of the manner in which useful results can be derived for symmetric wings. The discussion will be limited to steady-state, two-dimensional, subsonic pressure fields although fairly obvious extensions can be carried out.

If the geometry of a real symmetric airfoil is prescribed the theoretical pressure distribution exists and is unique. If, however, the pressure distribution is prescribed, a real airfoil does not necessarily exist, but by means of reciprocal relations it is possible to derive certain conditions of compatibility that need to be imposed. Consider, therefore, the two subsonic solutions

$$u_1(x_1)=0,\qquad w_1(x_1)=U_0/\sqrt{a^2-x_1^2};\qquad -a<x_1<a$$

and

$$u_2(x_2),\qquad w_2(x_2)$$

The first solution has square-root singularities in w at each end of the airfoil and, correspondingly, singularities in u occur just ahead of the point $x=-a$ and just behind $x=a$. On the other hand, equation (14) certainly applies if w_2 is zero at $x=\pm a$. If the origins of the two systems of axes are at the same position, it follows from equation (14) that $u_2(x)$ must satisfy the relation

$$0=\int_{-a}^{a}\frac{u_2(x_2)dx_2}{\sqrt{a^2-x_2^2}} \tag{40}$$

This result is useful in the calculation of airfoil shapes involving a change in pressure distribution from that of a known reference profile. The restriction on w_2 at the nose and tail implies that the derived and reference profiles have the same slope and radius of curvature at those points. The restriction on u_2, as given in equation (40), can be interpreted as a condition that must exist by virtue of the fact that the drag of an airfoil in two-dimensional potential flow in zero.

As a second example, consider the solutions

$$u_1(x_1)=\frac{tU_0}{2a},\ w_1(x_1)=\frac{-tx_1U_0}{2a\sqrt{a^2-x_1^2}};\ -a<x_1<a$$

that represent velocity and slope of a thin ellipse of thickness t and chord $2a$. If w_2 is chosen as above, such that it vanishes at the nose and tail of the airfoil, if u_2 is the corresponding velocity distribution, and if the two sets of axes are as before, equation (14) yields

$$\int_{-a}^{a}\frac{dz_2}{dx_1}dx_1=-\int_{-a}^{a}\frac{u_2(x_2)x_2dx_2}{\sqrt{a^2-x_2^2}}$$

From this result, together with the general closure condition,

$$\int_{-a}^{a}\frac{dz}{dx}dx=\frac{1}{U_0}\int_{-a}^{a}w(x)dx=0 \tag{41}$$

a necessary condition for the closure of the second airfoil is

$$\int_{-a}^{a}\frac{u_2(x_2)x_2dx_2}{\sqrt{a^2-x_2^2}}=0 \tag{42}$$

As a final example, consider the solutions for $-a<x_1<a$

$$u_1(x_1)=\frac{2tU_0}{3\sqrt{3}a^2}(a-2x_1),$$

$$w_1(x_1)=\frac{2tU_0}{3\sqrt{3}a^2}\frac{2x_1^2-ax_1-a^2}{\sqrt{a^2-x_1^2}},$$

representing velocity and slope of a thin Joukowsky type airfoil. In this case, w_1 vanishes at the tail and the downwash distribution w_2 for the reverse wing may have a square-root singularity at the nose. The nose of the first wing is, however, blunt and for equation (14) to apply the second wing must have a cusped tail. Under these conditions, equation (14) yields

$$\frac{2tU_0^2}{3\sqrt{3}a^2}\int_{-a}^{a}(a-2x_1)\frac{dz_2}{dx_2}dx_1 = \frac{2tU_0}{3\sqrt{3}a^2}\int_{-a}^{a}\frac{2x_1^2-ax_1-a^2}{\sqrt{a^2-x_1^2}}u_2(x_2)dx_2$$

Making the substitution $x_1=-x_2$ in this equation and integrating the left side by parts, one has

$$U_0\left\{\left[(2x_1-a)z_2\right]_{-a}^{a}-2\int_{-a}^{a}z_2(x_1)dx_1\right\} = \int_{-a}^{a}\frac{2x_2^2+ax_2-a^2}{\sqrt{a^2-x_2^2}}u_2(x_2)dx_2$$

For all real airfoils with cusped trailing edges, therefore, the area A_2 can be expressed as

$$A_2=-\int_{-a}^{a}\frac{2x_2^2+ax_2-a^2}{\sqrt{a^2-x_2^2}}\frac{u_2(x_2)}{U_0}dx_2 \tag{43}$$

LIFT—STEADY AND INDICIAL MOTION

The reciprocal theorems offer considerable advantage in the calculation of the lift of wings having a nonuniform angle-of-attack distribution or of wings in a stream having nonuniform flow directions. For these applications, it is convenient to consider a special form of the reciprocal theorem which relates the lift on a wing having arbitrary distribution of local angle of attack to that of the flat-plate wing of identical plan form in flight in the reverse direction. Since the solution of this latter problem is often known or can be found relatively easily, the solution of the original problem is facilitated in many instances.

Lift of arbitrarily cambered wings.—Consider two wings of identical plan form in flight in opposite directions, as shown in figure 5. Wing 1 is arbitrarily cambered and twisted and wing 2 is flat.

$$\alpha_1=\alpha_1(x_1,y_1,T),\ \alpha_2=\text{const.} \tag{44}$$

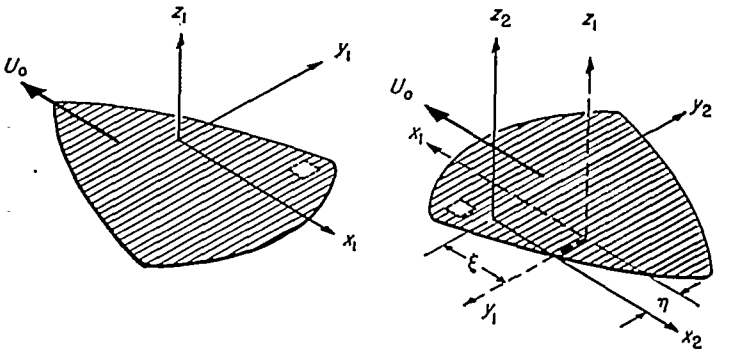

FIGURE 5.—Coordinate systems and symbols used in discussion of relation between lift of arbitrarily cambered wings and loading on flat-plate wings.

Application of equations (44) and (24) yields the following:

$$\alpha_2L_1=\iint_{P_1}\alpha_2\Delta p_1dS_1=\iint_{P_2}\alpha_1\Delta p_2dS_2$$

or

$$L_1=\iint_{P_2}\alpha_1\left(\frac{\Delta p_2}{\alpha_2}\right)dS_2 \tag{45}$$

THEOREM: The lift in steady or indicial motion of a wing having arbitrary twist and camber is equal to the integral over the plan form of the product of the local angle of attack and the loading per unit angle of attack at the corresponding point of a flat-plate wing of identical plan form in flight in the reverse direction.

Equation (45) may be used to derive Munk's integral formula for the lift of an arbitrarily curved airfoil in subsonic flow. Consider airfoils 1 and 2 placed in their respective coordinate systems, as indicated in figure 6. The angle-of-attack distributions on the two airfoils are given by

$$\alpha_1=-\frac{dz_1}{dx_1}, \qquad \alpha_2=\text{const.} \tag{46}$$

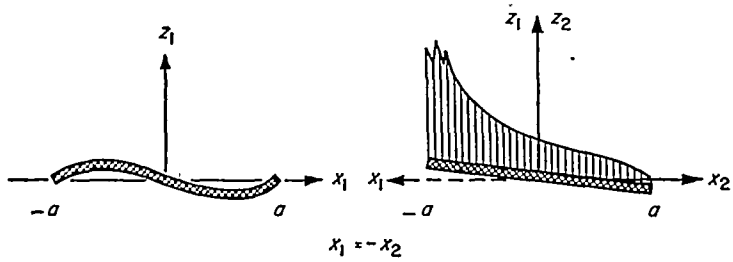

FIGURE 6.—Sketch of arbitrarily cambered airfoil illustrating symbols used in equations (46) through (50).

The loading per unit angle of attack on airfoil 2 is

$$\frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{\beta}\sqrt{\frac{a-x_2}{a+x_2}}=\frac{4q_0}{\beta}\sqrt{\frac{a+x_1}{a-x_1}} \tag{47}$$

where q_0 is free-stream dynamic pressure ($\frac{1}{2}\rho_0 U_0^2$). Substitution into equation (47) yields the lift formula

$$L_1=-\frac{4q_0}{\beta}\int_{-a}^{a}\frac{dz_1}{dx_1}\sqrt{\frac{a+x_1}{a-x_1}}\,dx_1 \tag{48}$$

The corresponding formula for the lift of a tier of curved airfoils may also be derived similarly from the expression for the loading on an equivalent tier of flat airfoils. Consider, for example, an unstaggered lattice of flat-plate airfoils arranged vertically. If the gap distance between the plates is h, the loading per unit angle of attack is

$$\frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{\beta}\operatorname{sech}\left(\frac{\pi a}{\beta h}\right)\sqrt{\frac{\sinh\left[\pi(a-x_2)/\beta h\right]}{\sinh\left[\pi(a+x_2)/\beta h\right]}}$$

The formula for the lift on one of a lattice of identically cambered airfoils [2] is therefore

$$L_1=-\frac{4q_0}{\beta}\operatorname{sech}\left(\frac{\pi a}{\beta h}\right)\int_{-a}^{a}\frac{dz_1}{dx_1}\sqrt{\frac{\sinh\left[\pi(a+x_1)/\beta h\right]}{\sinh\left[\pi(a-x_1)/\beta h\right]}}$$

The load distribution per unit angle of attack for a two-dimensional supersonic wing is

$$\frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{\beta} \tag{49}$$

and, from equation (47), the lift is

$$L_1=-\frac{4q_0}{\beta}\int_{-a}^{a}\frac{dz_1}{dx_1}\,dx_1 \tag{50}$$

The extension of this result to include supersonic-edged wings straight trailing edges leads to a result given originally by Lagerstrom and Van Dyke (ref. 21). If, as in figure 7, the sweep angle of the straightedge is Λ, the load distribution per unit angle of attack of the reversed wing is

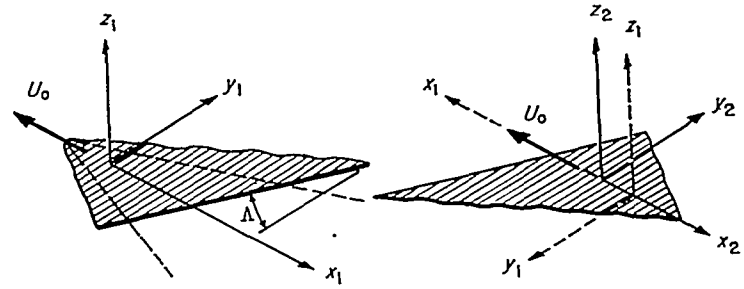

FIGURE 7.—Sketch of supersonic-edged wing illustrating symbols used in equations (51) and (52).

$$\frac{\Delta p_2}{\alpha_2}=\frac{4q_0\cos\Lambda}{\sqrt{1-M_0^2\cos^2\Lambda}} \tag{51}$$

and the lift is

$$L_1=\frac{-4q_0}{\sqrt{\sec^2\Lambda-M_0^2}}\iint_{P_1}\frac{dz_1}{dx_1}\,dx_1dy_1 \tag{52}$$

A less obvious application yields the build-up of lift with time of an arbitrarily cambered two-dimensional supersonic airfoil starting impulsively from rest at a constant speed. Rewriting equation (45), lift is

$$L_1=-\int_{-M_0T}^{2a-M_0T}\frac{dz_1(x_2,T)}{dx_1}\,\frac{\Delta p_2(x_2,T)}{\alpha_2}\,dx_2 \tag{53}$$

Figure 8 presents sketches showing the final positions of the airfoils relative to each other for various values of T. In reference 22 the expressions for the loading in equation (53) are given. Over the intervals denoted by ①, ②, and ③ in the sketch, these expressions are

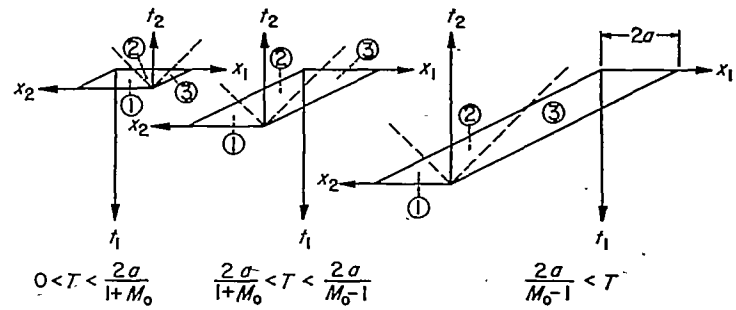

FIGURE 8.—Regions in which different forms of equation (54) apply.

$$\left.\begin{aligned}
&\text{Region ①}\quad \frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{M_0}\\
&\text{Region ②}\quad \frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{\pi\sqrt{M_0^2-1}}\left[\arccos\frac{M_0x+T}{x+M_0T}+\frac{\sqrt{M_0^2-1}}{M_0}\left(\frac{\pi}{2}+\arcsin\frac{x}{T}\right)\right]\\
&\text{Region ③}\quad \frac{\Delta p_2}{\alpha_2}=\frac{4q_0}{\sqrt{M_0^2-1}}
\end{aligned}\right\} \tag{54}$$

Lift on a wing in a nonuniform downwash field.—The reciprocal theorem of equation (45) can also provide a particularly good method of determining the lift on a wing in

[2] This result, as well as the detailed pressure distribution, has been derived by Mr. Paul F. Byrd of the Ames Aeronautical Laboratory by means of a direct inversion of the singular integral equation relating the aerodynamic pressure and geometry of the airfoil. His work has also been used in deriving equations (80) and (81).

certain nonuniform downwash fields of known structure. Such problems arise whenever a wing acts in the presence of other wings, bodies, or propellers but is always of prime concern in the determination of the lift on a tail acting in the downwash field of a wing. In most problems, the downwash velocities at the position of the tail may be considered to be constant in the longitudinal direction and to vary in the spanwise direction, thus

$$\alpha_1=\alpha_1(y_1,T), \quad \alpha_2=\text{const.} \tag{55}$$

and

$$L_1=\iint_{P_2}\alpha_1\left(\frac{\Delta p_2}{\alpha_2}\right)dS_2=\int_{-s}^{s}\alpha_1\left(\frac{l_2}{\alpha_2}\right)dy_2 \tag{56}$$

where l_2 is the span load distribution associated with the load distribution Δp_2. Summarizing, the lift in steady or indicial motion of a wing in a downwash field which varies across the span is equal to the integral over the span of the product of the local angle of attack and the span loading per unit angle of attack at the corresponding spanwise station of a flat-plate wing of identical plan form in flight in the reverse direction. This statement generalizes the result given recently by Alden and Schindel (ref. 23) for steady flow about wings having supersonic leading and trailing edges and streamwise side edges.

As for example, consider the problem of determining the lift on a wing at a geometrical angle of attack of zero resulting from the presence of an infinite line vortex of strength Γ extending in the flight direction. The wing will be considered to have such a plan form that its span loading when in flight in the reverse direction is elliptic. The notation is as shown in figure 9. For this problem, therefore, the span

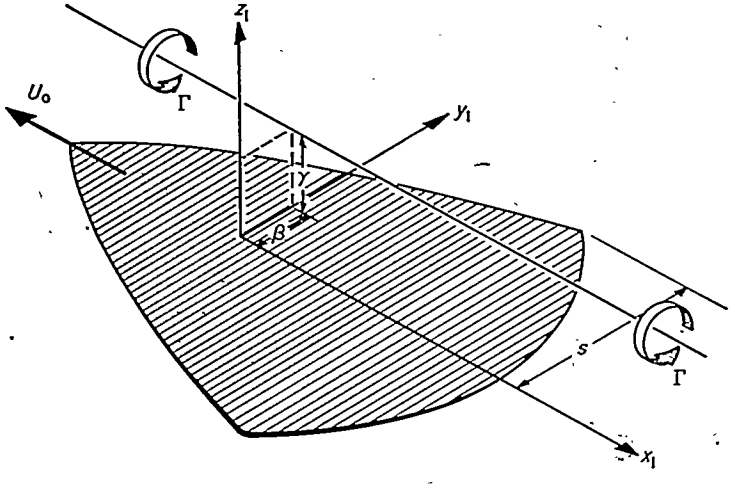

FIGURE 9.—View of wing and neighboring vortex.

loading of the wing in reverse flight is given by

$$\frac{l_2}{\alpha_2}=\frac{2L_2}{\pi s\alpha_2}\sqrt{1-\frac{y_2^2}{s^2}} \tag{57}$$

The local angle of attack of the original wing due to the presence of the vortex is given by

$$\alpha_1=\frac{\Gamma}{2\pi U_0}\frac{y_1-\beta}{(y_1-\beta)^2+\gamma^2} \tag{58}$$

Substitution of equations (57) and (58) into equation (56) yields the following formula for the lift:

$$L_1=\frac{\Gamma}{\pi U_0 s}\frac{L_2}{\alpha_2}\left\{\frac{-\beta}{s}+\sqrt{\frac{1}{2}\left[\sqrt{\left(1+\frac{\gamma^2}{s^2}-\frac{\beta^2}{s^2}\right)^2+\frac{4\beta^2\gamma^2}{s^4}}-\left(1+\frac{\gamma^2}{s^2}-\frac{\beta^2}{s^2}\right)\right]}\right\} \tag{59}$$

The lift on a wing in the vicinity of a number of such vortices may be found by superposition.

Lift due to deflection of a portion of the wing surface.—Let a portion P' of the surface of wing 1 be deflected a constant angle δ and the remainder of the wing be a flat plate alined with the free-stream direction. Let wing 2 be a flat-plate wing inclined at an angle of attack α, thus

$$\alpha_1=\begin{cases}\delta \text{ on } P'\\ 0 \text{ elsewhere}\end{cases} \qquad \alpha_2=\text{const.} \tag{60}$$

Substitution of equation (60) into (45) yields the following result:

$$\frac{L_1}{\delta}=\iint_{P_2'}\left(\frac{\Delta p_2}{\alpha_2}\right)dS_2 \tag{61}$$

The lift in steady or indicial motion per unit angular deflection of a portion of the wing surface is thus equal to the lift per unit angle of attack on the corresponding portion of a flat-plate wing in flight in the reverse direction. This generalizes a result given previously by Morikawa and Puckett (ref. 24) for steady flow about low-aspect-ratio wings.

This rule is very useful in the determination of the lift resulting from the deflection of a flap or control surface. This is particularly true for supersonic speeds since the loading on the related flat wing is often a constant over a large portion of the area of integration.

As a further example, consider the case of a low-aspect-ratio wing having a straight trailing edge and mounted on an infinite cylindrical body of revolution. The entire wing-body combination is at zero angle of attack except for the flaps on the rear of the wing that are deflected an angle δ. The problem is to determine the lift on the entire wing-body combination due to the deflection of the flaps. Slender-wing-theory results of reference 25 are to be used. The notation is indicated in figure 10.

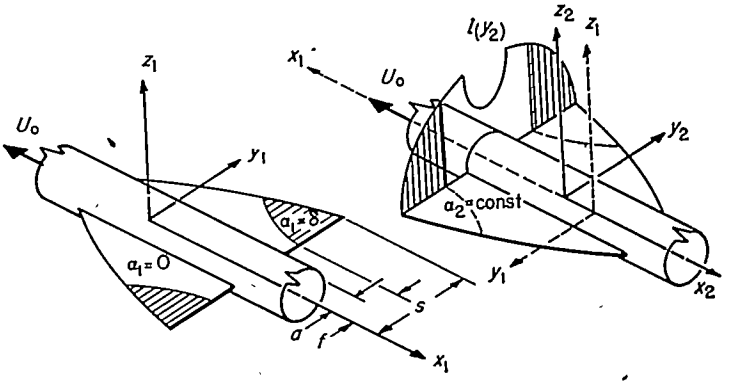

FIGURE 10.—View of lifting slender wing-body combinations.

The solution of this problem is particularly facilitated by the fact that slender-wing theory indicates that the loading on wing 2 is concentrated on the leading edge, as shown in the figure. Therefore, the lift of wing 1 is found by integrating the span loading curve of wing 2 over the portion of the span between $y=f$ and $y=s$. Thus

$$\frac{L_1}{\delta}=\int\int_{flap}\left(\frac{\Delta p_2}{\alpha_2}\right)dS_2=2\int_f^s\left[\frac{l_2(y_2)}{\alpha_2}\right]dy_2 \tag{62}$$

The span loading on wing 2 is given by

$$l_2(y_2)=4q_0\alpha_1 a\sqrt{\left(1-\frac{y^2}{s^2}\right)\left(\frac{s^2}{a^2}-\frac{a^2}{y^2}\right)} \tag{63}$$

The lift due to the deflection of the flap is therefore

$$L_{flap}=4q_0\delta s^2\left\{\left(1-\frac{a^2}{s^2}\right)^2\frac{\pi}{4}-\sqrt{\left(1-\frac{f^2}{s^2}\right)\left(\frac{f^2}{s^2}-\frac{a^4}{s^4}\right)}+\right.$$
$$\frac{1}{2}\left(1+\frac{a^4}{s^4}\right)\arcsin\frac{1-2(f/s)^2+(a/s)^4}{1-(a/s)^4}+$$
$$\left.\frac{a^2}{s^2}\arcsin\frac{[1+(a/s)^4](f/s)^2-2(a/s)^4}{(f/s)^2[1-(a/s)^4]}\right\} \tag{64}$$

A plot of the results is shown in figure 11. The lift per unit angle of flap deflection $(L/\delta)_{flap}$ has been nondimensionalized by dividing by the lift per unit angle of attack $(L/\alpha)_{W-B}$ of a slender wing-body combination of identical plan form. From reference 25, $(L/\alpha)_{W-B}$ is given by

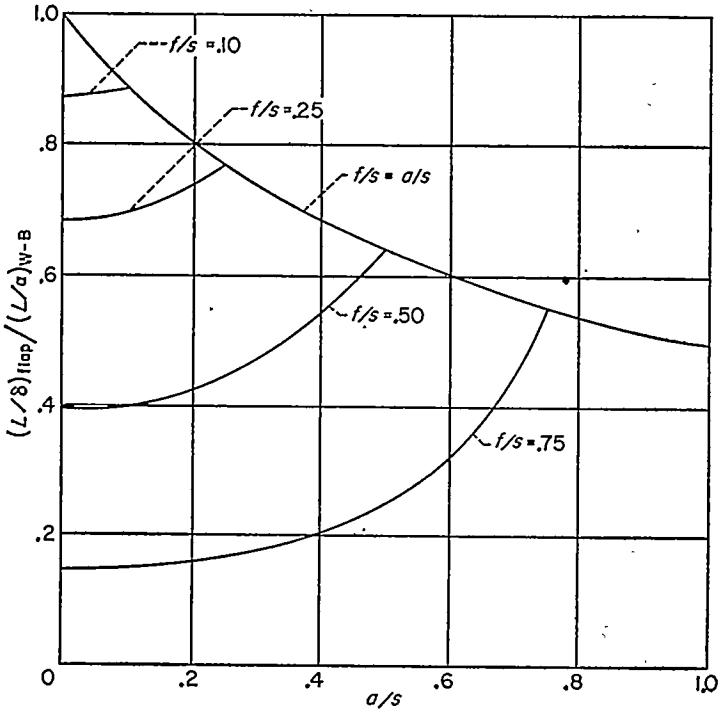

FIGURE 11.—Lift of a slender wing-body combination resulting from flap deflection.

$$(L/\alpha)_{W-B}=2\pi q_0 s^2\left(1-\frac{a^2}{s^2}\right)^2 \tag{65}$$

ROLLING MOMENT—STEADY AND INDICIAL MOTION

The calculation of the rolling-moment characteristics of wings having a nonuniform angle-of-attack distribution or of wings in a stream with nonuniform-flow directions can be performed in many cases through use of the reciprocal theorem in a manner analogous to that described for the lift characteristics in the preceding section. In every case, the rolling moment of the given wing will be related to the lift on a rolling flat-plate wing. For the sake of simplicity, all the present examples will be confined to the case where the rolling moments are evaluated about the x axis, considered to lie in the plane of symmetry.

Rolling moment of arbitrarily cambered wings.—Consider two wings of identical plan form in flight in opposite directions, as shown in figure 12. If the local angle-of-attack

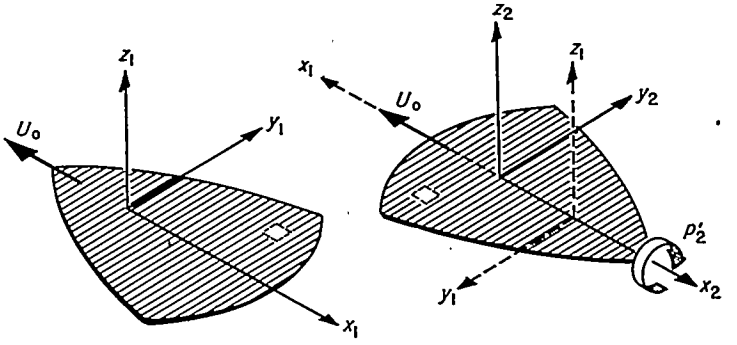

FIGURE 12.—Coordinate systems and symbols used in discussion of relation between rolling moment of arbitrarily cambered wings and the loading on rolling flat-plate wings.

distribution of wing 1 is arbitrary and that of wing 2 varies linearly with y (which might be likened to either a wing with linear twist distribution or to a flat-plate wing rolling about the x axis with constant angular velocity p_2'),

$$\alpha_1=\alpha_1(x_1,y_1,T), \qquad \alpha_2=\frac{p_2'y_2}{U_0} \tag{66}$$

and the following relations can be written:

$$L_1'=-\iint_{P_1}y_1\Delta p_1 dS_1=\iint_{P_1}y_2\Delta p_1 dS_1=\iint_{P_1}\frac{U_0}{p_2'}\alpha_2\Delta p_1 dS_1=$$
$$\iint_{P_2}\frac{U_0}{p_2'}\alpha_1\Delta p_2 dS_2=\iint_{P_2}\alpha_1\left(\frac{\Delta p_2}{p_2'/U_0}\right)dS_2 \tag{67}$$

THEOREM: The rolling moment in steady or indicial motion of a wing having arbitrary twist and camber is equal to the integral over the plan form of the product of the local angle of attack and the loading per unit (p_2'/U_0) at the corresponding point of a rolling flat-plate wing of identical plan form in flight in the reverse direction.

Applications of this theorem follow in a manner very similar to that described previously for the corresponding theorem regarding lift.

Rolling moment on a wing in a nonuniform downwash field.—Consider a wing placed in a flow field in which the downwash velocities at the position of the wing are constant in the longitudinal direction and vary in the spanwise direction. The related wing is again a flat-plate wing rolling with an angular velocity p_2' as described in the preceding section, thus

$$\alpha_1=\alpha_1(y_1,T), \qquad \alpha_2=\frac{p_2y_2}{V_0} \tag{68}$$

The rolling moment of the first wing is then given by

$$L_1'=\iint_{P_2}\alpha_2\left(\frac{\Delta p_2}{p_2'/U_0}\right)dS_2=\int_{-s}^{s}\alpha_1\left(\frac{l_2}{p_2'/U_0}\right)dy_2 \qquad (69)$$

or, in words, the rolling moment in steady or indicial motion of a wing in a downwash field which varies across the span is equal to the integral over the span of the product of the local angle of attack and the span loading per unit (p_2'/U_0) at the corresponding spanwise station of a rolling flat-plate wing of identical plan form in flight in the reverse direction.

Rolling moment due to deflection of a portion of the wing surface.—Let a portion P' of the surface of a wing be deflected a constant angle δ and the remainder of the wing be a flat-plate alined with the free-stream direction. The related wing is a flat-plate wing rolling with angular velocity p_2'

$$\alpha_1=\begin{cases}\delta \text{ on } P'\\ 0 \text{ elsewhere}\end{cases} \qquad \alpha_2=\frac{p_2'y_2}{U_0} \qquad (70)$$

Substitution from equation (70) into (67) yields the following result:

$$\frac{L_1'}{\delta}=\iint_{P_2'}\left(\frac{\Delta p_2}{p_2'/U_0}\right)dS_2 \qquad (71)$$

Thus, the rolling moment in steady or indicial motion due to a given angular deflection of a portion of the wing surface is equal to the lift per unit (p_2'/U_0) on the corresponding portion of a rolling flat-plate wing of identical plan form in flight in the reverse direction.

As an example, consider a wing-body combination consisting of a low-aspect-ratio wing having a straight trailing edge mounted on an infinite circular cylinder, as shown in figure 13. The body is at zero angle of attack, the right wing

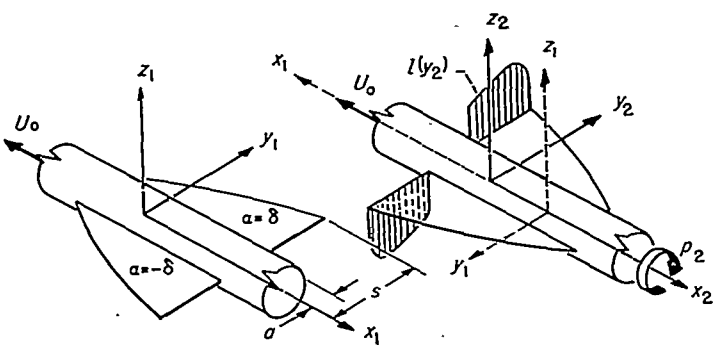

FIGURE 13.—Sketch illustrating symbols used in discussion of rolling moment resulting from differential deflection of wings of slender wing-body combination.

P' is deflected an angle δ, and the left wing P'' is deflected $-\delta$. The problem is to determine by means of slender-wing theory the rolling moment exerted on the entire wing-body combination. The notation is indicated in figure 13.

$$\alpha_1=\begin{cases}-\delta \text{ on left wing, } P''\\ 0 \text{ on body}\\ +\delta \text{ on right wing, } P'\end{cases} \qquad \alpha_2=\frac{p_2'y_2}{U_0} \qquad (72)$$

Since slender-wing theory indicates that the loading on wing 2 is concentrated on the leading edge, the rolling moment of wing 1 can be found by integrating the span loading on wing 2.

$$L_1'=\iint_{P_2'}\alpha_1\left(\frac{\Delta p_2}{p_2'/U_0}\right)dS_2=-\delta\int_{-s}^{-a}\left(\frac{l_2}{p_2'/U_0}\right)dy_2+\delta\int_{a}^{s}\frac{l_2}{p_2'/U_0}\,dy_2=2\delta\int_{a}^{s}\left(\frac{l_2}{p_2'/U_0}\right)dy_2$$

where, from reference 26,

$$\frac{l_2}{p_2'/U_0}=q_0s^2\left[\left(1+\frac{2}{\pi}\arccos\frac{2as}{s^2+a^2}\right)\left(y+\frac{a^2}{y}\right)\sqrt{s^2+\frac{a^4}{s^2}-y^2-\frac{a^4}{y^2}}+\frac{2}{\pi}\left(y-\frac{a^2}{y}\right)^2\operatorname{arc\,cosh}\frac{(y^2+a^2)(s^2-a^2)}{(y^2-a^2)(s^2+a^2)}\right] \qquad (73)$$

The resulting expression for the rolling moment is

$$L_1'=4q_0\delta s^2\left\{F(\varphi,k)\left[\frac{R^2}{2}\left(1-\frac{2R^2}{3}+R^4\right)\left(1+\frac{2}{\pi}\arccos\frac{2R}{1+R^2}\right)-\frac{2R^3}{3\pi}(1-R^2)\right]+E(\varphi,k)\left[\frac{1}{2}\left(\frac{1}{3}-2R^2+\frac{R^4}{3}\right)\left(1+\frac{2}{\pi}\arccos\frac{2R}{1+R^2}\right)+\frac{2}{3\pi}R(1-R^2)\right]+\frac{8R^3}{3\pi}\ln\frac{2R}{1+R^2}+\frac{R^2}{3}(1-R^2)\left(1+\frac{2}{\pi}\arccos\frac{2R}{1+R}\right)\right\} \qquad (74)$$

where

$$R=a/s$$

$$\varphi=\arcsin\frac{1}{\sqrt{1+R^2}}$$

$$k=\sqrt{1-R^4}$$

A plot of the results is shown in figure 14. The rolling moment has been nondimensionalized by dividing by the value corresponding to that of the wing alone $(R=0)$.

PITCHING MOMENT—STEADY AND INDICIAL MOTION

A number of useful relations regarding the pitching-moment characteristics of wings may be found through application of the reciprocal theorem. Since the general procedure is closely analogous to that of the preceding sections, the following discussion will be brief.

Pitching moment of arbitrarily cambered wing.—Consider the problem of determining the pitching moment M_1 about the origin of wing 1 possessing an arbitrary distribution of camber. The related wing in flight in the reverse direction, wing 2, is a flat-plate wing of identical plan form pitching about the moment axis of wing 1, as indicated in figure 15, thus

$$\alpha_1=\alpha_1(x_1,y_1,T), \qquad \alpha_2=-\frac{q_2x_1}{U_0}=\frac{q_2(x_2-\xi)}{U_0} \qquad (75)$$

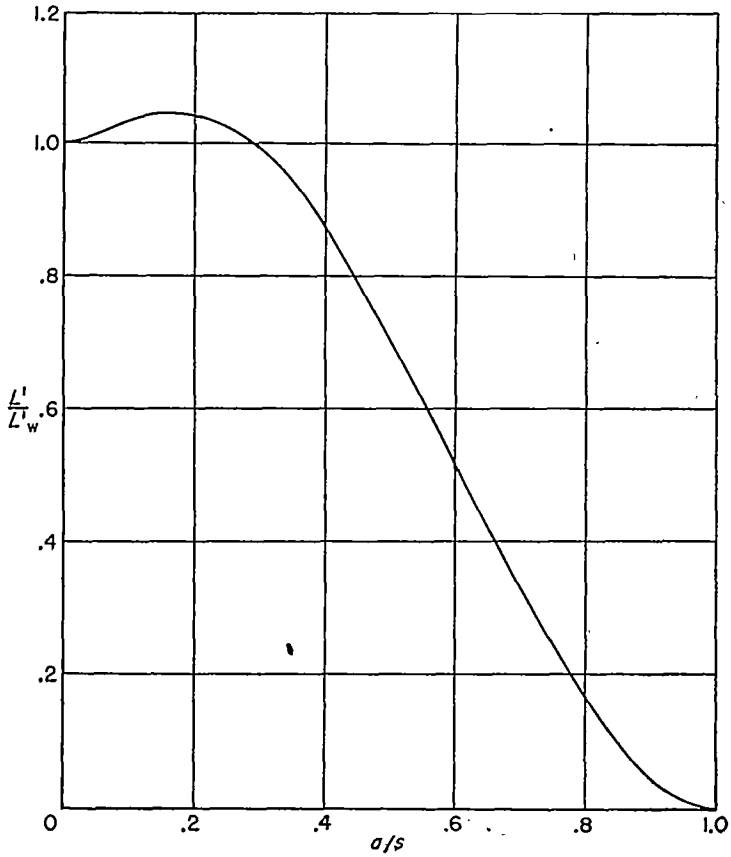

FIGURE 14.—Rolling moment of slender wing-body combination resulting from differential deflection of wings.

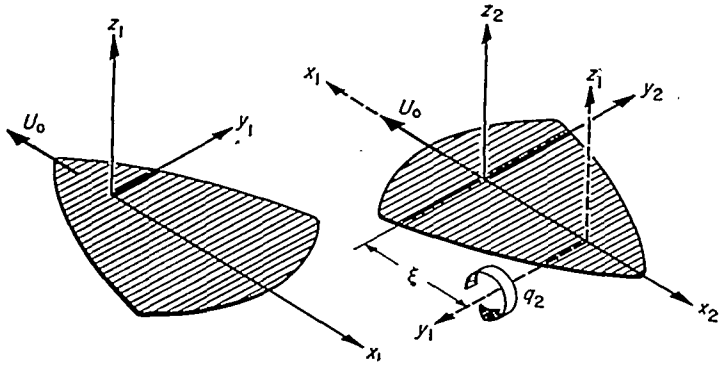

FIGURE 15.—Coordinate systems and symbols used in discussion of relation between pitching moment on arbitrarily cambered wings and the loading on pitching flat-plate wings.

The pitching moment of wing 1 is given by

$$M_1=-\iint_{P_1} x_1\Delta p_1 dS_1=\iint_{P_1}\alpha_2\left(\frac{\Delta p_1}{q_2/U_0}\right)dS_1$$

$$=\iint_{P_2}\alpha_1\left(\frac{\Delta p_2}{q_2/U_0}\right)dS_2 \quad (76)$$

THEOREM: The pitching moment in steady or indicial motion of a wing having arbitrary twist and camber is equal to the integral over the plan form of the product of the local angle of attack and the loading per unit (q_2/U_0) at the corresponding point of a flat-plate wing of identical plan form in flight in the reverse direction and pitching about the moment axis of the first wing.

The necessity for pitching wing 2 about the moment axis of wing 1 may be removed by considering wing 2 to be re-expressed in terms of two component wings having angle-of-attack distributions given by

$$\alpha_{2'}=\frac{q_{2'}(x_{2'}-x_{02'})}{U_0}, \qquad \alpha_{2''}=\frac{q_{2'}(x_{02'}-\xi)}{U_0}=\text{const.} \quad (77)$$

Wing 2′ is thus pitching with angular velocity $q_{2'}$ about an axis at $x_{2'}=x_{02'}$ and wing 2″ is a flat-plate wing at a constant angle of attack. The pitching moment on wing 1 is then given by

$$M_1=\iint_{P_{2'}}\alpha_1\left(\frac{\Delta p_{2'}}{q_{2'}/U_0}\right)dS_{2'}+\iint_{P_{2''}}\alpha_1\left(\frac{\Delta p_{2''}}{q_{2'}/U_0}\right)dS_{2''}$$

$$=\iint_{P_{2'}}\alpha_1\left(\frac{\Delta p_{2'}}{q_{2'}/U_0}\right)dS_{2'}+(x_{02'}-\xi)\iint_{P_{2''}}\alpha_1\left(\frac{\Delta p_{2''}}{\alpha_{2''}}\right)dS_{2''} \quad (78)$$

Applications of pitching-moment theorem.—The application of equation (76) or (78) to problems analogous to those discussed in the preceding section can be carried out in a straightforward manner. Consider, first, unstaggered lattices on airfoils such that the airfoils in lattice 1 have arbitrary camber distributions and those in lattice 2 are flat plates pitching about their midchord positions. The angles of attack in the two lattice systems are

$$\alpha_1(x_1,z_1)=-\frac{dz_1}{dx_1}, \qquad \alpha_2(x_2,z_2)=\frac{q_2x_2}{U_0} \quad (79)$$

and the load distribution on each airfoil in lattice 2 is, in subsonic steady flow,

$$\Delta p_2(x_2)=q_0\left(\frac{-4q_2h}{\pi U_0}\right)\arccos\left[\operatorname{sech}\left(\frac{\pi a}{\beta h}\right)\cosh\left(\frac{\pi x_2}{\beta h}\right)\right] \quad (80)$$

where $2a$ is chord length. Equation (76) yields, for pitching moment of the first airfoil about its midchord point, the result

$$M_1=-\frac{4q_0h}{\pi}\int_{-a}^{a}\frac{dz_1}{dx_1}\arccos\left[\operatorname{sech}\left(\frac{\pi a}{\beta h}\right)\cosh\left(\frac{\pi x_1}{\beta h}\right)\right]dx_1 \quad (81)$$

A second example, illustrating unsteady effects, is the following: Let wing 1 be a flat-plate wing, then α_1 is constant, and equation (78) simplifies to

$$\frac{M_1}{\alpha_1}=\frac{L_{2'}}{q_{2'}/U_0}+(x_{02'}-\xi)\frac{L_{2''}}{\alpha_{2''}} \quad (82)$$

where $L_{2'}$ is the lift on wing 2′ pitching about $x_{2'}=x_{02'}$, and $L_{2''}$ is the lift on an inclined flat-plate wing. Equation (82) may be expressed in terms of conventional stability derivatives as follows:

$$(C_{m_\alpha})_1=(C_{L_q})_{2'}+\left(\frac{x_{02'}-\xi}{c_0}\right)(C_{L_\alpha})_{2''} \quad (83)$$

An application of this result to unsteady-flow problems is indicated in figure 16 obtained from indicial-lift and pitching-moment results of reference 27. This figure shows the growth of lift and pitching moment on triangular wings with

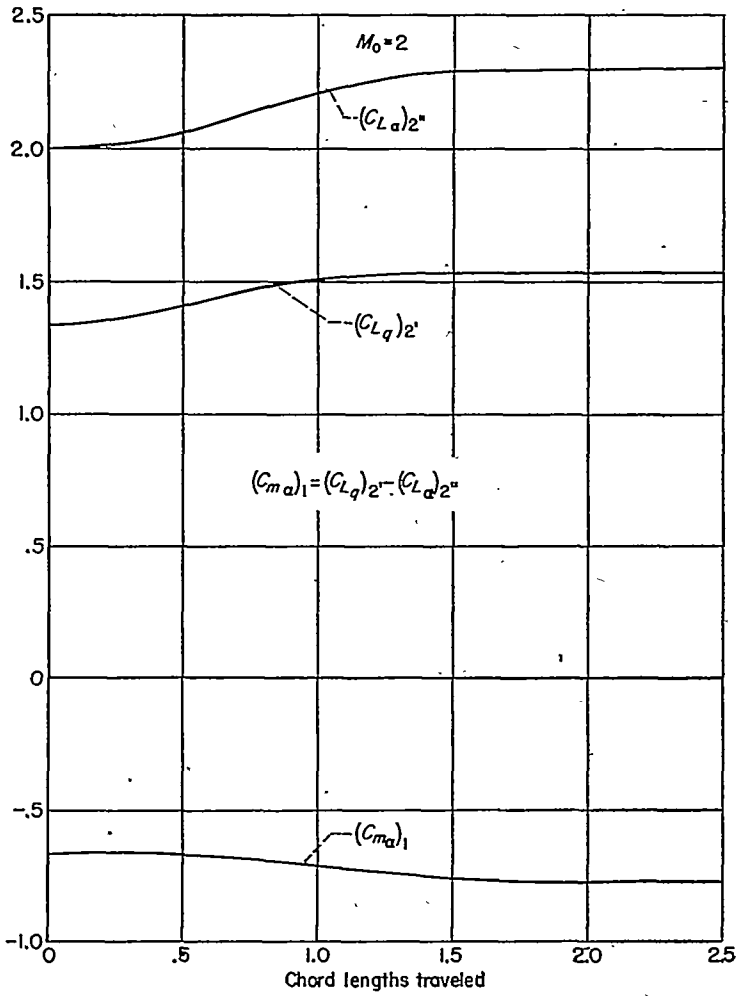

FIGURE 16.—Indicial lift and pitching moment on triangular wings with supersonic edges.

supersonic edges at a Mach number of 2 following indicial angle-of-attack and pitching-velocity changes. In these results, the rotation and moment axes are always at the leading edge or apex, therefore, $x_{02'}=0$ and $\xi=c_0$. It may be seen that the three curves are related in the simple linear manner indicated by equation (83).

If α_1 is independent of x and varies only in the spanwise direction, that is, if $\alpha_1=\alpha_1(y_1)$, the pitching moment on wing 1 is given by the following equation, analogous to equation (56) for lift:

$$M_1=\int_{-s}^{s}\alpha_1\left(\frac{l_{2'}}{q_{2'}/U_0}\right)dy_{2'}+(x_{02'}-\xi)\int_{-s}^{s}\alpha_1\left(\frac{l_{2''}}{\alpha_{2''}}\right)dy_{2''} \quad (84)$$

If a portion P' of the surface of wing 1 is deflected a constant angle δ and the remainder of the wing is a flat plate alined with the free-stream direction, the following relations hold:

$$\alpha_1=\begin{cases}\delta \text{ on } P'\\ 0 \text{ elsewhere}\end{cases}$$

and

$$\frac{M_1}{\delta}=\iint_{P'_{2'}}\left(\frac{\Delta p_{2'}}{q_{2'}/U_0}\right)dS_{2'}+(x_{02'}-\xi)\iint_{P'_{2''}}\left(\frac{\Delta p_{2''}}{\alpha_{2''}}\right)dS_{2''} \quad (85)$$

RECIPROCAL RELATIONS INVOLVING MOTION INTO A GUST

All previous applications that have been considered were derived from equation (24). In the present section, the more general equation (22) will be used to develop two theorems which relate the build-up of lift on a wing entering a gust and the build-up of circulation on the same wing moving indicially but in the opposite direction. The relations to be obtained hold for the Mach number range for which the wave equation applies. Under the special assumptions of incompressible flow, the results in two dimensions establish a direct connection between the circulation function calculated by Wagner (ref. 28) and the gust lift curve calculated by Küssner (ref. 29) and von Kármán and Sears (ref. 30). A proof of the connection between these functions for two-dimensional incompressible flow has been given by Sears (ref. 31).

TWO-DIMENSIONAL FLOW

A flat plate is assumed to be moving in two modes of motion: In the motion associated with the x_2,z_2,t_2, axes, the wing starts at time zero ($t_2=0$) and moves at a constant velocity U_0 and at a constant angle of attack; the motion associated with the x_1,z_1,t_1, axes starts at time zero ($t_1=0$) with the wing moving in the opposite direction at a velocity U_0 and entering a sharp-edged gust. The gust exists for all x_1 less than zero and has a vertical velocity $w_g=-\alpha_g U_0$. The two wings, therefore, have angles of attack as follows:

$$\alpha_2(x_2,t_2)=\alpha_2=\text{const.} \quad \text{for} -M_0t_2<x_2<2a-M_0t_2, \qquad t_2>0$$

$$\alpha_1(x_1,t_1)=\alpha_g=\text{const.} \quad \text{for}\begin{cases}-M_0t_1<x_1<0 & 0\leqq t_1\leqq 2a/M_0\\ -M_0t_1<x_1<2a-M_0t_1 & 2a/M_0\leqq t_1\end{cases}$$

The two-dimensional form of equation (22) yields

$$\alpha_2\int_0^T dt_1\int_{2a-M_0t_1}^{-M_0t_1}\Delta p_g(x_1,t_1)dx_1=\rho_0a_0\alpha_g\iint_A\Delta\frac{\partial\varphi_2(x_2,t_2)}{\partial t_2}\,dS \quad (86)$$

where the region A is bounded by the lines $x_1=0$, $x_1=2a-M_0t_1$, $t_1=T$, and $x_1=-M_0t_1$. The integral on the right can be rewritten as a line integral by means of the identity

$$-\int P\cos(t,n)ds=\iint\frac{\partial P}{\partial t}\,dS$$

and equation (86) becomes

$$\alpha_2\int_0^T L_g(t_1)dt_1=\rho_0a_0\alpha_g\int_C\Delta\varphi_2\cos(t_2,n)ds$$

where the line integral extends around the boundary of the region A. Since $\Delta\varphi_2$ vanishes on the lines $x_2=-M_0t_2$ and $t_2=0$, the equation becomes

$$\alpha_2\int_0^T L_g(t_1)dt_1=\rho_0a_0\alpha_g\int_{2a-M_0T}^{2a}\Delta\varphi_2\left(x_2,\frac{2a-x_2}{M_0}\right)dx_2$$

Differentiation with respect to T yields

$$\alpha_2L_g(T)=\rho_0U_0\alpha_g\Delta\varphi_2(2a-M_0T,T)$$

The discontinuity in φ_2 is evaluated at the trailing edge at time T and is therefore equal to the circulation Γ_2 of the airfoil at time T. The equality thus becomes

$$\frac{L_g(T)}{\alpha_g}=\rho_0 U_0 \frac{\Gamma_2(T)}{\alpha_2} \tag{87}$$

THEOREM: The circulation per unit angle of attack of a flat plate moving indicially with a velocity U_0 is proportional to the lift per unit α_g of the plate entering a sharp-edged gust having a uniform vertical velocity equal to $w_g=-\alpha_g U_0$.

In figure 17, the time variation of these variables, as well as the lift of the indicial wing, is indicated for low speed and for flight Mach numbers equal to 0.8 and 1.46 as determined from references 22 and 27.

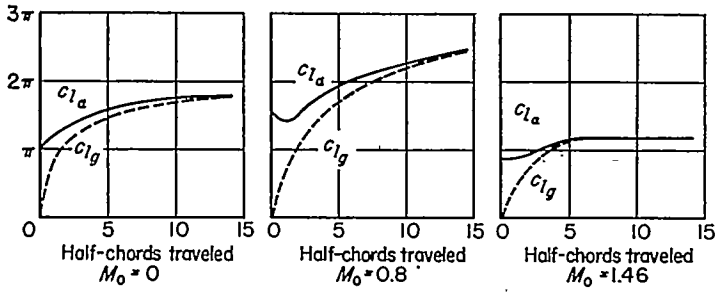

FIGURE 17.—Growth of c_{l_α} and c_{l_g} with chord lengths traveled.

THREE-DIMENSIONAL FLOW

The extension of the above results to three dimensions follows directly. The origin of the x_1, y_1, z_1, t_1 axes is assumed to be initially at the foremost point of the wing in reverse motion. The two wings have, respectively, angles of attack α_2=const. over the reverse moving plan form for all values of time and α_g=const. over the region occupied simultaneously by the forward moving wing and the gust. Equation (22) gives

$$\alpha_2\int_0^T dt_1 \iint_{P(t_1)} \Delta p_g(x_1,y_1,t_1)dx_1dy_1=$$

$$\rho_0 a_0 \alpha_g \iiint \Delta \frac{\partial \varphi_2(x_2,y_2,t_2)}{\partial t_2}\,dx_2dy_2dt_2 \tag{88}$$

The integral on the right can be rewritten as a two-dimensional surface integral by means of the identity

$$-\iint P\cos(t,n)dS=\iiint\frac{\partial P}{\partial t}\,dxdydt$$

and equation (88) becomes

$$\alpha_2\int_0^T L_g(t_1)dt_1=\rho_0 a_0\alpha_g\iint \Delta\varphi_2\cos(t_2,n)dS_2$$

where the integral on the right extends over the boundary of the volume in x_2, y_2, t_2 space occupied by the wing and the gust. The value of $\Delta\varphi_2$ must, of course, vanish on the leading edge of the wing and at $t_2=0$. In order to fix the limits of integration, suppose the wing is symmetrical about its longitudinal axis and let the leading edge of the forward wing be given by the equation

$$x_1=f(\pm y_1)-M_0t_1 \text{ or } y_1=\pm s(x_1+M_0t_1)$$

For the reverse wing and its coordinate system, this edge, which is now the trailing edge, is

$$x_2=c_r-M_0t_2-f(\pm y_2) \text{ or } y_2=\pm s(c_r-x_2-M_0t_2)$$

where c_r is the root chord. The reverse-flow integrals of equation (88) then become

$$\alpha_2\int_0^T L_g(t_1)dt_1=\rho_0a_0\alpha_g\int_{c_r-M_0T}^{c_r}dx_2\int_{-s(c_r-x_2)}^{s(c_r-x_2)}$$

$$\Delta\varphi_2\left[x_2,y_2,\frac{c_r-x_2-f(\pm y_2)}{M_0}\right]dy_2$$

where $s(x_2)$ is the local half-span of the wing. Differentiation with respect to T yields

$$\alpha_2L_g(T)=\rho_0U_0\alpha_g\int_{-s(M_0T)}^{s(M_0T)}\Delta\varphi_2\left[c_r-M_0T,y_2,T-\frac{f(\pm y_2)}{M_0}\right]dy_2 \tag{89}$$

The discontinuity in φ is thus to be integrated spanwise at the rearmost point of the indicial wing; this follows from the relation

$$\Delta\varphi_2\left[x,y,T-\frac{f(\pm y_2)}{M_0}\right]=\Delta\varphi_2(x,y,t);\qquad T-\frac{f(\pm y_2)}{M_0}<t<T$$

which fixes the vorticity in the wake of the wing once it is shed from the trailing edge.

It remains to mention the nature of the limits $\pm s(M_0T)$. As shown in figure 18, the span width of the vortex wake at the trailing edge is, during the early stages of the motion, dependent on the local span width of the wing. The width $2s(M_0T)$ of wake is, in fact, equal to the maximum width of the portion of the first wing that lies within the gust. After

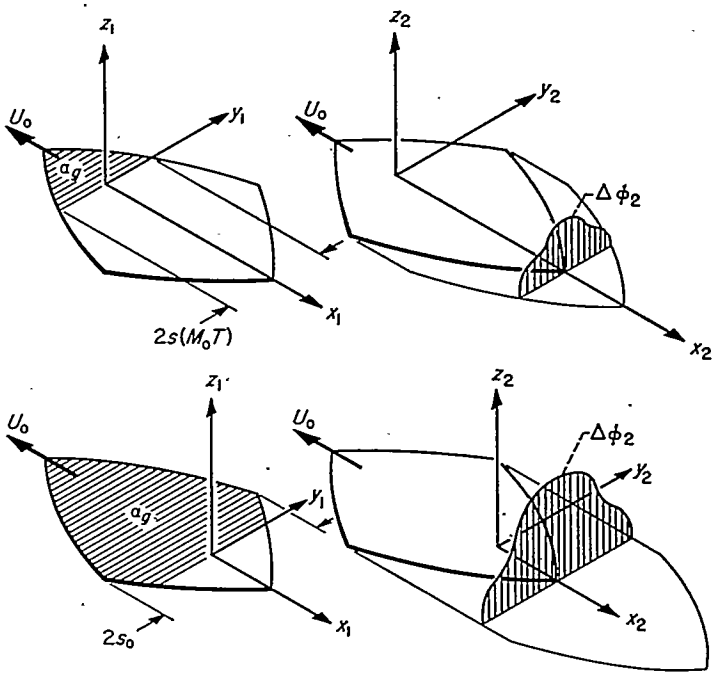

FIGURE 18.—Sketch illustrating nature of integration limits in equation (89).

sufficient time has passed for the vortex wake of the indicial wing to develop its full span width at the trailing edge, $s(M_0T)$ becomes s_0 or semispan of the wing. From equation (89) one may conclude the following:

> THEOREM: The lift per unit α_z, of a flat-plate wing entering a sharp-edged gust having a uniform vertical velocity equal to $w_z=-\alpha_z U_0$ is proportional, at each instant of time, to the spanwise integral at the trailing edge of the vorticity shed by the same wing moving indicially in the reverse direction with a velocity U_0.

As a direct example, this theorem has been used to confirm, from a knowledge of the indicial solution, the sharp-edged-gust lift of the rectangular-plan-form supersonic wing given by Miles in reference 32.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., *Feb. 19, 1952.*

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