REPORT 1046

A GENERAL INTEGRAL FORM OF THE BOUNDARY-LAYER EQUATION FOR INCOMPRESSIBLE FLOW WITH AN APPLICATION TO THE CALCULATION OF THE SEPARATION POINT OF TURBULENT BOUNDARY LAYERS [1]

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SUMMARY

A general integral form of the boundary-layer equation is derived from the Prandtl partial-differential boundary-layer equation. The general integral equation, valid for either laminar or turbulent incompressible boundary-layer flow, contains the Von Kármán momentum equation, the kinetic-energy equation, and the Loitsianskii equation as special cases.

In an attempt to obtain a practical method for the calculation of the development of the turbulent boundary layer, use is made of the experimental finding that all the velocity profiles of the turbulent boundary layer form essentially a single-parameter family. The general equation is thereby changed to a simpler one from which an equation for the space rate of change of the shape parameter of the turbulent boundary layer can be obtained.

The resulting equation for the space rate of change of the velocity-profile parameter is restricted by the assumption that the velocity profiles of the turbulent boundary layer can be approximated by power profiles. Two of the resulting equations are used to calculate the distribution of the profile shape parameter over an airfoil for one experimentally determined pressure distribution. Although different assumptions were tried for the shearing stress across the boundary layer, the calculated distribution of the profile shape parameter did not agree exactly with the experimental distribution.

An examination is made of the effect of using the experimentally determined single-parameter family of velocity profiles instead of the power profiles on certain functions that occur in the equation for the space rate of change of the velocity-profile parameter. One calculation of the distribution of the profile shape parameter over an airfoil is also made for the experimentally determined pressure distribution by using the single-parameter family of velocity profiles found from experiment. A comparison of the results with those of a calculation made with the same assumptions except for the use of power profiles shows some difference near the separation point. It is believed, however, that the apparent lack of reliability of the specific equations used to make the calculations is caused mainly by the lack of precise knowledge concerning the surface shear and the distribution of the shearing stress across the turbulent boundary layer. The present analysis emphasizes the need for information concerning the shearing stresses in turbulent boundary layers.

INTRODUCTION

An outstanding problem in aerodynamic theory is to calculate whether the flow will separate from the surface of a specific body and, if so, where the separation will occur. The concept of the boundary layer and the equations that describe the flow in it, introduced by Prandtl (reference 1) and first worked out in some detail by Blasius (reference 2), reduce the problem to solving the Prandtl boundary-layer equation when the flow is laminar. Because of the mathematical difficulty of solving the equation, approximate methods were developed for the calculation of the properties of the laminar boundary layer (reference 3). In some of these methods, for example, the Pohlhausen method (reference 3) and the Wieghardt method (reference 4), a functional form is chosen for the velocity distribution through the boundary layer and is combined with either the Von Kármán momentum equation alone (reference 5) or with both the Von Kármán momentum equation and the kinetic-energy equation (reference 4). The result is the replacement of the Prandtl partial-differential equation by one ordinary differential equation in the Pohlhausen method and by two ordinary differential equations in the Wieghardt method. A solution of the ordinary differential equation or equations provides the boundary-layer velocity profiles along the body. These and other approximate methods that use only the Von Kármán momentum equation, or the momentum and kinetic-energy equations together, do not satisfy exactly the Prandtl boundary-layer equation.

Because the flow in the boundary layer is more often turbulent than laminar in cases encountered in engineering, the problem of calculating the separation point is of even more importance for turbulent than for laminar boundary layers. In spite of the importance of the problem, however, less progress has been made in the development of methods for the calculation of the behavior of turbulent boundary layers than for laminar boundary layers. The lack of progress stems from the absence of an explicit independent equation for the shearing stress that is accurate enough to lead to a description of the flow when used with the Prandtl equation.

The main attempts to obtain methods for the calculation of the behavior of the incompressible turbulent boundary

layer in the presence of pressure gradients are those of references 6 to 12. The results of these attempts are unsatisfactory either because the assumptions upon which they rest are incorrect or because the equations used to make calculations were not derived from the boundary-layer equations.

The analysis of reference 6 is based on the assumption that the velocity profile is a single-valued function of the ratio of the pressure gradient to the skin friction, an assumption shown to be incorrect by later investigators (for example, see reference 12). In the analyses of references 7, 10, 11, and 12 the momentum equation is used, together with an auxiliary equation, to calculate the distribution of velocity profiles over a surface. In each of these four methods the auxiliary equation is not derived from the boundary-layer equations but is empirical.

In reference 8, the equation that gives the variation of the mixing length across a pipe (reference 3) was used to calculate the velocity profiles. The fact that the mixing-length distribution across the boundary layer is not the same as across pipes is shown in references 13 to 15.

Reference 9 does not provide a method for the calculation of the distribution of turbulent velocity profiles along a surface. It does, however, suggest that separation of the turbulent boundary layer always occurs when the numerical value of the nondimensional pressure gradient reaches an empirical constant.

The purpose of the present investigation is to begin with the boundary-layer equation for incompressible flow and to proceed as closely to a method for the calculation of the behavior of the turbulent boundary layer as the present knowledge of the turbulent boundary layer permits.

At first it might appear that the use of empirical auxiliary equations in methods for the calculation of the behavior of turbulent boundary layers can be avoided by developing a method similar to the Pohlhausen method which requires the solution of only the Von Kármán momentum equation. For turbulent flow, however, in contrast with laminar flow, the conditions on the behavior of the velocity profile at the surface that can be obtained from the boundary-layer equation seem to be of little or no use for the determination of the shape of the velocity profile across the boundary layer. This difference between laminar and turbulent flow makes inapplicable the Pohlhausen process in which a type of function is chosen to represent the velocity profiles, the function for the velocity profiles is combined with the Von Kármán momentum equation, and the resulting ordinary differential equation for the space rate of change of the profile shape parameter is solved.

An auxiliary equation for the calculation of the behavior of the turbulent boundary layer can, however, be obtained from the boundary-layer equation by making use of the

experimentally verifiable fact (references 7, 10, 11, 14, and 15) that all velocity profiles of the turbulent boundary layer form essentially a single-parameter family of curves. In the present analysis the Loitsianskii equation (reference 16) is generalized by multiplying the Prandtl boundary-layer equation not only by an arbitrary power of the velocity in the boundary layer but also by an arbitrary power of the distance from the surface. The resulting equation is then integrated across the boundary layer and provides a general integral form of the boundary-layer equation, valid for either laminar or turbulent flow. This general integral form of the boundary-layer equation reduces to the Loitsianskii equation when the distance from the surface is raised to the zeroeth power, to the Von Kármán momentum equation when both the distance from the surface and the velocity are raised to the zeroeth power, and to the kinetic-energy equation when the distance from the surface is raised to the zeroeth power and the velocity is raised to the first power.

When use is made of the assumption of a single-parameter family of velocity profiles, the general integral form of the boundary-layer equation becomes a general equation for the rate of change along the surface of the velocity-profile shape parameter. This equation for the rate of change of the velocity-profile shape parameter is the desired auxiliary equation.

The assumption of the single-parameter family of velocity profiles changes the problem from one of finding a solution of a partial-differential equation, the Prandtl boundary-layer equation, to one of finding a solution of two simultaneous ordinary differential equations, the equation for the rate of change of the shape parameter and the Von Kármán momentum equation. The differential equation for the rate of change of the shape parameter, however, cannot result in a solution of the problem in the present analysis because a knowledge of the shearing stress is lacking. In the present analysis various assumptions are made for the distribution of shearing stress through the boundary layer, and the distribution of the shape parameter over the surface of an airfoil is then calculated. Because of the arbitrary assumptions for the shear distribution and the use of a flat-plate skin-friction formula, precise agreement between the calculated and experimentally obtained distributions of the shape parameter is not obtained.

The problem of finding the shearing stress in the turbulent boundary layer remains. It is believed, however, that, if suitable approximations are found for the shear and surface friction, the equations presented herein should enable the development of the turbulent boundary layer to be calculated with an accuracy sufficient for engineering purposes.

The present work was begun while Dr. Lin was temporarily at the Langley Laboratory and was continued by correspondence.

SYMBOLS

A — arbitrary positive integer in shear polynomial

a, b, c — coefficients in polynomial for $\int_0^1 \zeta \frac{\partial g}{\partial \zeta} d\zeta$

B — exponent in expression for shear

c — reference chord

F_0, F_1 — functional notation

$f = \frac{u}{U}$

$g = \frac{\tau}{\tau_0}$

g_1 — derivative of shear polynomial for $\lambda = 0$

g_2 — coefficient of λ in expression for $\frac{\partial g}{\partial \zeta}$

$H = \frac{\delta^*}{\theta}$

H_0 — equilibrium value of H for $\omega = 0$

$I = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}(1-f^{m+1}) \left(\int_0^y \frac{\partial f}{\partial H} dy \right) dy$

$I_1 = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}(1-f^{m+1}) \left[\int_0^y \frac{\partial(1-f)}{\partial x} dy \right] dy$

$J = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}(1-f^{m+1}) \left[\int_0^y (1-f) dy \right] dy$

j, q, l — coefficients in polynomial for $\int_0^1 \zeta^{\frac{H-1}{2}} \frac{\partial g}{\partial \zeta} d\zeta$

K — ratio of kinetic-energy thickness to momentum thickness $\left(\frac{1}{\theta} \int_0^\delta (1-f^2)f\, dy \right)$

$K' = \frac{dK}{dH}$

k — function of H $\left(\frac{-(H-1)K}{K'} \right)$

$L = \frac{1}{\theta^{n+1}} \int_0^\delta f(1-f^{m-1})y^n dy$

$M = \frac{1}{\theta^{n+1}} \int_0^\delta y^n(1-f^{m+1}) dy$

m — exponent of u in derivation of general equation

$N = \frac{1}{\theta^{n+1}} \int_0^\delta y^n(1-f^{m+1})f\, dy$

$N' = \frac{dN}{dH}$

n — exponent of y in derivation of general equation

P — coefficient of ω in equation for $\theta \frac{dH}{dx}$

p — exponent in equation for power profiles $f = \zeta^p$

p_1 — static pressure

$Q = \frac{1}{\theta^n} \int_0^\delta y^{n-1}(1-f^{m+1}) dy$

$R_\theta = \frac{\theta U \rho}{\mu}$

r_0 — radius of body of revolution

S — coefficient of ϕ in equation for $\theta \frac{dH}{dx}$

U — velocity parallel to surface and at outer edge of boundary layer

u — velocity parallel to surface and inside boundary layer, positive in direction of positive x

v — velocity perpendicular to surface and inside boundary layer, positive in direction of positive y

v_0 — value of v at $y = 0$

x — coordinate parallel to surface, positive in direction from leading to trailing edge

y — coordinate perpendicular to surface, positive outward from surface

δ — smallest value of y for which the difference $U-u$ is negligible

δ^* — displacement thickness $\left(\int_0^\delta (1-f) dy \right)$

$\zeta = \frac{y}{\delta}$

$\eta = \frac{y}{\theta}$

θ — momentum thickness $\left(\int_0^\delta f(1-f) dy \right)$

$\lambda = \frac{\delta}{\tau_0} \frac{dp_1}{dx}$

μ — viscosity

$\xi = -\frac{\omega}{\phi}$

$\xi_0 = \frac{1}{H-1} \left(1 - \frac{2}{K} \int_0^1 f\, dg \right)$

ρ — density

τ — shearing stress

τ_0 — surface shearing stress

$\phi = \frac{\tau_0}{\rho U^2}$

$\psi = \frac{v_0}{U}$

$\omega = \frac{\theta}{U} \frac{dU}{dx}$

ANALYSIS

DERIVATION OF GENERAL EQUATION

The general equation is derived for the body of revolution because the equation for two-dimensional flow can be obtained from this equation by letting the radius of a transverse section of the body of revolution become infinite.

The boundary-layer equation of motion for the body of revolution, also valid for two-dimensional flow, from reference 3 is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_1}{dx}+\frac{1}{\rho}\frac{\partial \tau}{\partial y} \qquad (1)$$

After multiplying through by u^m, making use of the equation of continuity that is valid in the boundary layer of a body of revolution (reference 3)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{r_0}\frac{dr_0}{dx}=0$$

and noting that

$$\rho U\frac{dU}{dx}=-\frac{dp_1}{dx}$$

equation (1) becomes

$$\frac{u^{m+1}}{m+1}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{r_0}\frac{dr_0}{dx}\right)+\frac{u}{m+1}\frac{\partial u^{m+1}}{\partial x}+\frac{v}{m+1}\frac{\partial u^{m+1}}{\partial y}$$
$$=u^m\left(U\frac{dU}{dx}+\frac{1}{\rho}\frac{\partial \tau}{\partial y}\right) \qquad (2)$$

After equation (2) is written in a form in which each term vanishes at the outer edge of the boundary layer, each term of the equation is multiplied by y^n and integrated from $y=0$ to $y=\delta$. The result (see appendix A for detailed development) is

$$(n+1)N\frac{d\theta}{dx}+\theta\left(\frac{dN}{dx}-nI_1\right)+\frac{\theta}{U}\frac{dU}{dx}[N(m+2)-n(J-M)-$$
$$L(m+1)]+\frac{\theta}{r_0}\frac{dr_0}{dx}[N-n(J-M)]-nQ\frac{v_0}{U}$$
$$=-(m+1)\frac{\tau_0}{\rho U^2}\int_0^{\delta/\theta}f^m\eta^n\frac{\partial \frac{\tau}{\tau_0}}{\partial \eta}d\eta \qquad (3)$$

Equation (3) is the general integral form of the boundary-layer equation.

The Von Kármán momentum equation is obtained from equation (3) by letting $m=0$ and $n=0$, the equation for kinetic energy is obtained by letting $m=1$ and $n=0$, and the equation for moment of momentum is obtained by letting $m=0$ and $n=1$.

In the case $m=n=0$.

$$N\theta=\int_0^\delta(1-f)f\,dy=\theta$$

or

$$N=1$$

Also,

$$L\theta=\int_0^\delta(f-1)dy=-\delta^*$$

or

$$L=-\frac{\delta^*}{\theta}=-H$$

It can be easily verified that all the integrals, except Q, involved in equation (3) have finite integrands as n approaches 0. The limit nQ, however, approaches unity as

n approaches 0; thus

$$nQ\theta^n=n\int_0^\delta y^{n-1}(1-f^{m+1})dy$$
$$=\left[y^n(1-f^{m+1})\right]_0^\delta-\int_0^\delta y^n\frac{\partial}{\partial y}(1-f^{m+1})dy$$

The first term drops out if $n\neq 0$. Then, by taking the limit $n\to 0$, the result is

$$\lim_{n\to 0}nQ\theta^n=1$$

Hence, when $m=n=0$, equation (3) becomes

$$\frac{d\theta}{dx}+\frac{\theta}{U}\frac{dU}{dx}(H+2)+\frac{\theta}{r_0}\frac{dr_0}{dx}-\frac{v_0}{U}=\frac{\tau_0}{\rho U^2} \qquad (4)$$

Equation (4) is the momentum equation for flow over a body of revolution with flow through the surface. For two-dimensional flow, equation (4) becomes

$$\frac{d\theta}{dx}+\frac{\theta}{U}\frac{dU}{dx}(H+2)-\frac{v_0}{U}=\frac{\tau_0}{\rho U^2} \qquad (5)$$

when the value of $\frac{d\theta}{dx}$ from equation (4) is substituted into equation (3), the result is

$$\theta\left(\frac{dN}{dx}-nI_1\right)+\frac{\theta}{U}\frac{dU}{dx}\{m(N-L)-n[(J-M)+N(H+2)]-$$
$$L-NH\}-\frac{\theta}{r_0}\frac{dr_0}{dx}n[(J-M)+N]-\frac{v_0}{U}[nQ-N(n+1)]$$
$$=-\frac{\tau_0}{\rho U^2}\left[(m+1)\int_0^{\delta/\theta}f^m\eta^n\frac{\partial g}{\partial \eta}d\eta+(n+1)N\right] \qquad (6)$$

where

$$g=\frac{\tau}{\tau_0}$$

The assumptions contained in equation (6) are the usual boundary-layer assumptions. Equation (6) is valid for both laminar and turbulent flow.

FORM OF EQUATION (6) FOR SINGLE-PARAMETER FAMILY OF VELOCITY PROFILES

Equation (6) is now to be placed in a form valid when the velocity profiles form a single-parameter family of curves $(f=f(\eta,H))$. For this purpose the term I_1 of equation (6) is modified in the following manner:

By definition,

$$I_1\theta^{n+1}=\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y\frac{\partial(1-f)}{\partial x}dy\right]dy$$

Because f depends only on η and H,

$$\frac{\partial(1-f)}{\partial x}=\frac{\partial(1-f)}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial(1-f)}{\partial H}\frac{dH}{dx}$$

From the definition of η

$$\frac{\partial \eta}{\partial x}=-\frac{\eta}{\theta}\frac{d\theta}{dx}$$

then

$$\int_0^y \frac{\partial(1-f)}{\partial x}\,dy = -\frac{1}{\theta}\frac{d\theta}{dx}\int_0^y \eta\frac{\partial(1-f)}{\partial \eta}\,dy + \frac{dH}{dx}\int_0^y \frac{\partial(1-f)}{\partial H}\,dy$$

or, after an integration by parts of the first term on the right-hand side, the result is

$$\int_0^y \frac{\partial(1-f)}{\partial x}\,dy = -\frac{1}{\theta}\frac{d\theta}{dx}\left[y(1-f)-\int_0^y(1-f)\,dy\right]-\frac{dH}{dx}\int_0^y \frac{\partial f}{\partial H}\,dy$$

The expression for $I_1\theta^{n+1}$ then becomes

$$I_1\theta^{n+1} = -\frac{1}{\theta}\frac{d\theta}{dx}\int_0^\delta y^n(1-f^{m+1})\,dy + \frac{1}{\theta}\frac{d\theta}{dx}\int_0^\delta y^n f(1-f^{m+1})\,dy +$$

$$\frac{1}{\theta}\frac{d\theta}{dx}\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y(1-f)\,dy\right]dy -$$

$$\frac{dH}{dx}\int_0^\delta y^{n-1}(1-f^{m+1})\left(\int_0^y \frac{\partial f}{\partial H}\,dy\right)dy$$

But

$$\int_0^\delta y^n(1-f^{m+1})\,dy = M\theta^{n+1}$$

$$\int_0^\delta y^n(1-f^{m+1})f\,dy = N\theta^{n+1}$$

and

$$\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y(1-f)\,dy\right]dy = J\theta^{n+1}$$

Then with

$$\int_0^\delta y^{n-1}(1-f^{m+1})\left(\int_0^y \frac{\partial f}{\partial H}\,dy\right)dy = I\theta^{n+1}$$

the expression for I_1 can be written as

$$I_1 = \frac{1}{\theta}\frac{d\theta}{dx}(-M+N+J)-I\frac{dH}{dx} \tag{7}$$

When the expression for I_1 from equation (7) is substituted into equation (6) and equation (4) for $\frac{d\theta}{dx}$ is used, the following equation is obtained:

$$\theta\frac{dH}{dx}\left(\frac{dN}{dH}+nI\right) = \frac{\theta}{U}\frac{dU}{dx}[-n(H+1)(J-M)+L(m+1)+$$

$$N(H-m)]+\frac{\tau_0}{\rho U^2}\left[n(J-M)-N-\right.$$

$$\left.(m+1)\int_0^{\delta/\theta} f^m\eta^n\frac{\partial g}{\partial \eta}\,d\eta\right]+$$

$$\frac{v_0}{U}[n(J-M)-N+nQ] \tag{8}$$

where $\frac{dN}{dx}=\frac{dN}{dH}\frac{dH}{dx}$ has been used. Equation (8) for $\theta\frac{dH}{dx}$ is applicable both to two-dimensional flow and to flow over a body of revolution.

In equation (8) all the integrals, except the one involving the shear ratio g, are functions of H, m, and n only. For the present no restrictive assumptions regarding the shear are made. The form of the kinetic-energy equation for a single-parameter family of velocity profiles is obtained from equation (8) by placing $m=1$ and $n=0$ and dividing by $N'(H)=K'(H)$. Thus,

$$\theta\frac{dH}{dx}=\frac{K(H-1)}{K'}\frac{\theta}{U}\frac{dU}{dx}-\left(\frac{K+2\int_0^{\delta/\theta} f\frac{\partial g}{\partial \eta}\,d\eta}{K'}\right)\frac{\tau_0}{\rho U^2}-$$

$$\left(\frac{K-1}{K'}\right)\frac{v_0}{U} \tag{9}$$

The symbol K represents the ratio of the kinetic-energy thickness $\int_0^\delta (1-f^2)f\,dy$ to the momentum thickness. Note that in the derivation of equation (9) from equation (8), the assumption of a single-parameter family of curves is not restricted to the case $\frac{v_0}{U}=0$.

RESTRICTION OF GENERAL EQUATION TO POWER PROFILES

The data in figure 1 show that the power profiles defined by $f=\zeta^p$ are a good approximation to the "standard" profiles derived by fairing experimental data (reference 10). Equation (8) can be further developed by using the assumption that $f=\zeta^p$. After some fairly lengthy calculations (see appendix B), equation (8) becomes

$$\theta\frac{dH}{dx}=\frac{-4p(p+1)(2p+1)[p(m+2)+n+1]}{pm+n+1}\frac{\theta}{U}\frac{dU}{dx}+$$

$$\frac{2p[p(m+2)+n+1]}{pm+n}\left\{(2p+1)+\right.$$

$$\left.[p(m+2)+n+1]\int_0^1 \zeta^{pm+n}\frac{\partial g}{\partial \zeta}\,d\zeta\right\}\frac{\tau_0}{\rho U^2}+$$

$$\frac{2p(p+1)[p(m+2)+n+1]}{p(m+1)+n}\frac{v_0}{U} \tag{10}$$

As a first approximation the assumption has been made that $f=\zeta^p$ even when $\frac{v_0}{U}\neq 0$.

The occurrence of the arbitrary positive integers m and n in equations (8) and (10) requires an explanation. In order to determine why m and n appear, equation (8) is written in a different form. By making use of the definitions for N, I, J, M, L, and Q and integrating by parts where necessary in order to eliminate terms that contain η^{n-1}, the result is

$$\frac{dN}{dH}+nI=-(m+1)\int_0^{\delta/\theta} \eta^n f^m\left(f\frac{\partial f}{\partial H}-\frac{\partial f}{\partial \eta}\int_0^\eta \frac{\partial f}{\partial H}\,d\eta\right)d\eta$$

$$-n(H+1)(J-M)+L(m+1)+N(H-m)$$

$$=(m+1)\int_0^{\delta/\theta} \eta^n f^m\left[(H+1)\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta+f^2-1\right]d\eta$$

$$n(J-M)-N-(m+1)\int_0^{\delta/\theta} f^m\eta^n\frac{\partial g}{\partial \eta}\,d\eta$$

$$=-(m+1)\int_0^{\delta/\theta} \eta^n f^m\left(\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta+\frac{\partial g}{\partial \eta}\right)d\eta$$

and

$$n(J-M)-N+nQ=(m+1)\int_0^{\delta/\theta}\eta^n f^m \frac{\partial f}{\partial \eta}\left(1-\int_0^\eta f\,d\eta\right)d\eta$$

Equation (8) then becomes

$$\theta\frac{dH}{dx}=\frac{\int_0^{\delta/\theta}\eta^n f^m\left\{\frac{\tau_0}{\rho U^2}\left(\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta+\frac{\partial g}{\partial \eta}\right)+\frac{\theta}{U}\frac{dU}{dx}\left[1-f^2-(H+1)\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta\right]+\frac{v_0}{U}\left[\frac{\partial f}{\partial \eta}\left(\int_0^\eta f\,d\eta-1\right)\right]\right\}d\eta}{\int_0^{\delta/\theta}\eta^n f^m\left(f\frac{\partial f}{\partial H}-\frac{\partial f}{\partial \eta}\int_0^\eta\frac{\partial f}{\partial H}\,d\eta\right)d\eta}\tag{11}$$

By using the assumption of a single-parameter family of curves directly in the partial-differential equation (1), the following ordinary differential equation is obtained:

$$\theta\frac{dH}{dx}=\frac{\frac{\tau_0}{\rho U^2}\left(\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta+\frac{\partial g}{\partial \eta}\right)+\frac{\theta}{U}\frac{dU}{dx}\left[1-f^2-(H+1)\frac{\partial f}{\partial \eta}\int_0^\eta f\,d\eta\right]+\frac{v_0}{U}\left[\frac{\partial f}{\partial \eta}\left(\int_0^\eta f\,d\eta-1\right)\right]}{f\frac{\partial f}{\partial H}-\frac{\partial f}{\partial \eta}\int_0^\eta\frac{\partial f}{\partial H}\,d\eta}\tag{12}$$

The concept of a single-parameter family of velocity profiles is consistent with equation (1) and with particular functions for $\tau_0/\rho U^2$, g, and f when the right-hand side of equation (12) is independent of η. When the right-hand side of equation (12) is independent of η, the right-hand side of equation (11) is independent of m and n. Equations (11) and (12) are then identical.

To obtain an equation for $\theta\frac{dH}{dx}$ that does not contain either m or n or both, the functions $\tau_0/\rho U^2$, g, and f must therefore be such that the right-hand side of equation (12) is independent of η; the solution of the equation for $\theta\frac{dH}{dx}$ then provides a solution of equation (1). Note that the problem is to find a solution not of equation (1) alone but of equation (1) and the independent relation for the shearing stress in turbulent boundary layers; this relation is at present unknown.

The nature of the approximation made in the present analysis, in order to obtain a specific equation for $\frac{dH}{dx}$, may be clarified by noting that a specific equation for $\frac{dH}{dx}$ is obtained from equation (8) by choosing the functions $\tau_0/\rho U^2$, g, and f and substituting an arbitrary positive integer for m and an arbitrary positive integer for n. The calculated distribution of H over a body for arbitrarily chosen functions for $\tau_0/\rho U^2$, g, and f is then consistent with the momentum equation and one of the integral equations for $\frac{dH}{dx}$. For example, if $m=1$ and $n=0$, both the momentum and the kinetic-energy equations are satisfied but no other ones. If $m=0$ and $n=1$, only the momentum and the moment of momentum equations are satisfied. In the present analysis only the momentum, the kinetic-energy, and the moment of momentum equations—equations which have familiar physical meaning—are used.

As noted previously, equation (11) is independent of m and n if the functions $\tau_0/\rho U^2$, g, and f are such that the right-hand side of equation (12) is independent of η. In this case a solution of equation (1) results and the functions $\tau_0/\rho U^2$, g, and f and the calculated distribution of H satisfy every particular equation obtainable from equation (11), (10), or (8) by assigning positive integers to m and n.

Note that m and n cannot both be made zero in equation (8) because $\frac{dN}{dH}+nI=0$ for $m=n=0$. If m and n are both

(a) $H=1.310$, 1.518, and 1.710.
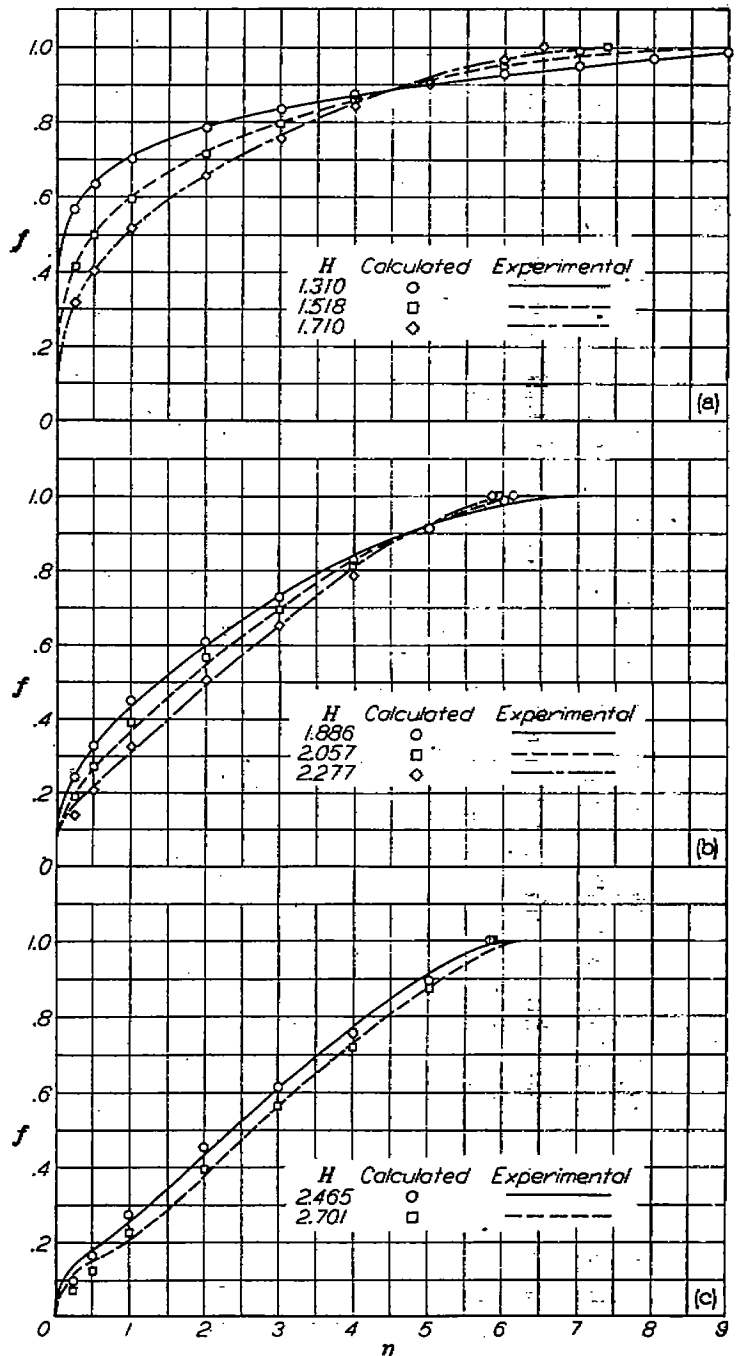(b) $H=1.886$, 2.057, and 2.277.
(c) $H=2.465$ and 2.701.

FIGURE 1.—Comparison of experimental and power velocity profiles.

zero, equation (8) becomes $0=0$. It is also noted that equations (8) and (10) are valid both for flow over a body of revolution and for two-dimensional flow.

For $m=1$ and $n=0$, equation (10) leads to the equation for kinetic energy

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1)\frac{\theta}{U}\frac{dU}{dx} +$$
$$(3H-1)\left(H+\frac{3H-1}{2}\int_0^1 \zeta^p \frac{\partial g}{\partial \zeta}\,d\zeta\right)\frac{\tau_0}{\rho U'^2} +$$
$$\frac{(H+1)(3H-1)}{4}\frac{v_0}{U} \tag{13}$$

where the relation for power profiles $2p+1=H$ has been introduced. This form of the energy equation can also be obtained from equation (9) by noting that, from the definition of K and the equation for power profiles,

$$K = \frac{2(2p+1)}{3p+1} = \frac{4H}{3H-1}$$

A comparison of the values of K obtained from this formula and obtained from the standard profiles is given in figure 2.

The equation of moment of momentum for power profiles is obtained from equation (10) by letting $m=0$ and $n=1$; it is

$$\theta \frac{dH}{dx} = -\frac{H(H+1)(H^2-1)}{2}\frac{\theta}{U}\frac{dU}{dx} +$$
$$(H^2-1)\left[H+(H+1)\int_0^1 \zeta \frac{\partial g}{\partial \zeta}d\zeta\right]\frac{\tau_0}{\rho U^2}+(H^2-1)\frac{v_0}{U} \tag{14}$$

In this equation the term involving the shear distribution may be rewritten as follows:

$$\int_0^1 \zeta \frac{\partial g}{\partial \zeta}\,d\zeta = -\int_0^1 g\,d\zeta$$

It then involves the mean shear inside the boundary layer.

ATTEMPTS TO DERIVE A RELATION GOVERNING THE CHANGE OF THE FORM PARAMETER

In most of the recent analyses of the development of a turbulent boundary layer, an empirical relation governing the change of the form parameter H is usually introduced.
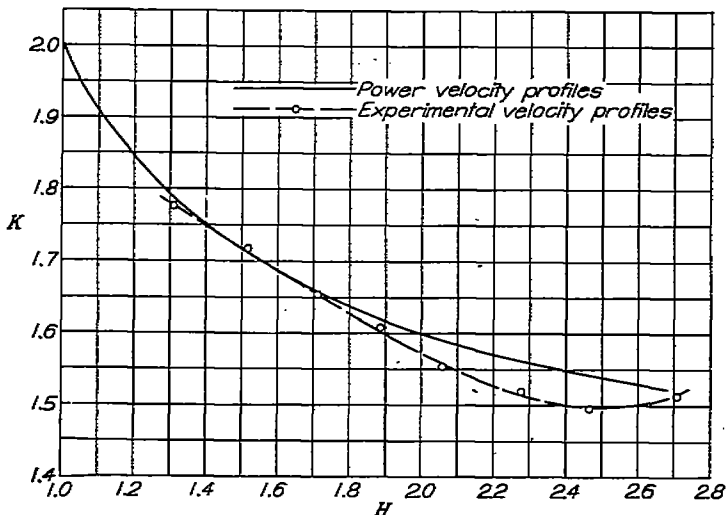

FIGURE 2.—Values of K for experimental and power velocity profiles.

It is clear that equation (10) automatically furnishes such relations if the shear distribution is known. In this section, three attempts are described to establish such a relation. These attempts are based on the following simple assumptions for the shear distribution:

(a) The shear distribution depends only on $\frac{\delta}{\tau_0}\frac{dp_1}{dx}$, which is equal to the Pohlhausen parameter multiplied by a factor (reference 3).

(b) The shear is constant across the boundary layer.

(c) The shear distribution depends only on the form parameter of the velocity distribution.

The first two assumptions are used either with the energy equation in the forms given by equations (9) and (13) or with equation (14) for the moment of momentum. The last assumption is used with equations (13) and (14) jointly.

(a) **Shear distribution depending only on the Pohlhausen parameter.**—The first assumption follows the original idea of the method of Von Kármán and Pohlhausen in using polynomial approximations together with the boundary conditions obtained by successive differentiation of the equations of motion (reference 3). Fediaevsky (reference 8) appears to have been the first to introduce it into the investigation of turbulent boundary layers. When the shear stress through the turbulent boundary layer is assumed to be a polynomial of fifth degree in $\zeta = \frac{y}{\delta}$ satisfying the following boundary conditions:

at $y=0$

$$\tau=\tau_0 \qquad \frac{\partial \tau}{\partial y}=\frac{dp_1}{dx} \qquad \frac{\partial^2 \tau}{\partial y^2}=0$$

at $y=\delta$

$$\tau=0 \qquad \frac{\partial \tau}{\partial y}=0 \qquad \frac{\partial^2 \tau}{\partial y^2}=0$$

the following expression is obtained:

$$g=(1-\zeta)^3[1+(3+\lambda)\zeta+3(2+\lambda)\zeta^2] \tag{15}$$

The shear distribution g is a function of ζ and λ, where

$$\lambda = \frac{\delta}{\tau_0}\frac{dp_1}{dx} = -\frac{\theta}{U}\frac{dU}{dx}\frac{\delta}{\theta}\frac{\rho U^2}{\tau_0}$$

The particular boundary conditions at $y=0$ restrict this development to the case $\frac{v_0}{U}=0$.

From the shear distribution (equation (15)) the calculation may be made of the coefficients P and S. The attempt to calculate P and S by using the standard profiles together with equations (9) and (15) was, however, unsuccessful for two reasons. First, the ratio δ/θ, which must be known, could not be accurately determined from the standard profiles. Second, for reasonable values of δ/θ, the calculated values of P were positive for values of H for which P should be negative.

The calculation of the part of P independent of the shear profile was then made both for the standard profiles and the power profiles by making use of the kinetic-energy equations (equations (9) and (13), respectively); the comparison is shown in figure 3. The closeness of the results suggests

that it is permissible to use power profiles as an approximation for calculating P and S. From equations (13) and (15),

$$P=-H(3H-1)\left[H-1-\frac{96(3H-1)}{(H+5)(H+7)(H+9)}\right] \quad (16)$$

and

$$S=(3H-1)\left[H-\frac{240(3H-1)}{(H+5)(H+7)(H+9)}\right] \quad (17)$$

The functions P and S, given by equations (16) and (17), respectively, are shown in figure 4.

The fact that the equation

$$\theta\frac{dH}{dx}=P\omega+S\phi$$

where P and S are obtained from equations (16) and (17), respectively, does not predict the behavior of the turbulent boundary layer is shown as follows: Let $\omega=0$; then, for H greater than approximately 1.5, $\frac{dH}{dx}$ should be negative. Because S from equation (17) is positive for $H>1.2$, it follows that $\frac{dH}{dx}$ is positive. This conclusion is incorrect;
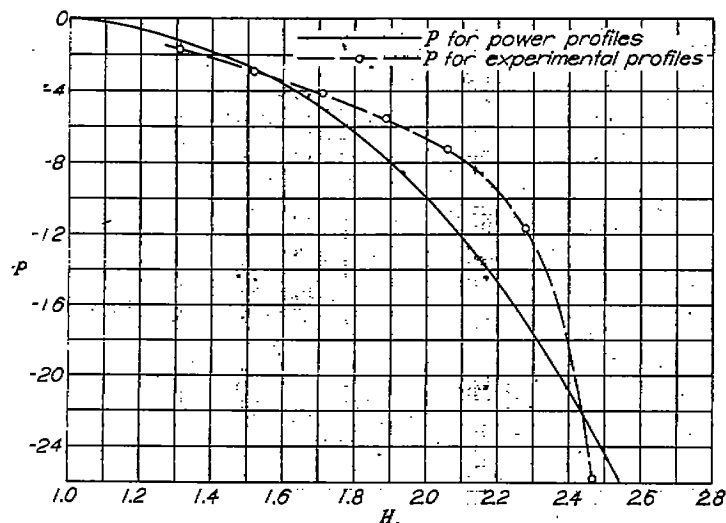
FIGURE 3.—Values of P from energy equation. For power profiles, $P=-H(H-1)(3H-1)$; for experimental profiles, $P=\frac{K(H-1)}{K'}$.
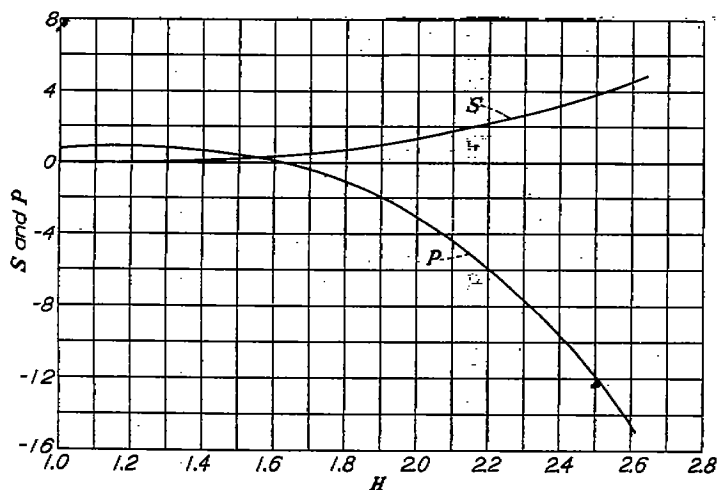

FIGURE 4.—Values of P from equation (16) and values of S from equation (17).

therefore, the function for S (equation (17)) is inconsistent with the known behavior of turbulent boundary layers.

To show that the function for P (equation (16)) is inconsistent with the known behavior of turbulent boundary layers, let $H\approx1.4$. By making $\frac{dU}{dx}$ positive and large, $\frac{dH}{dx}$ becomes positive and large because P given by equation (16) is positive. For positive values of $\frac{dU}{dx}$, however, it is known that $\frac{dH}{dx}$ should be negative. The function for P (equation (16)) is therefore inconsistent with the known behavior of turbulent boundary layers.

In order to determine whether functions for P and S that do not result in obviously incorrect conclusions can be obtained by making the shear polynomial satisfy a greater number of boundary conditions at the outer edge of the boundary layer, the shear polynomial is generalized by writing

$$g=(1-\zeta)^A\left[1+A\zeta+\frac{A(A+1)}{2}\zeta^2\right]+\lambda\zeta(1-\zeta)^A(1+A\zeta) \quad (18)$$

The boundary conditions at the surface that are satisfied by equation (18) are

$$g=1 \qquad \text{or} \qquad \tau=\tau_0$$

$$\frac{\partial g}{\partial\zeta}=\lambda \qquad \text{or} \qquad \frac{\partial\tau}{\partial y}=\frac{dp_1}{dx}$$

$$\frac{\partial^2 g}{\partial\zeta^2}=0 \qquad \text{or} \qquad \frac{\partial^2\tau}{\partial y^2}=0$$

At $y=\delta$, the conditions that are satisfied are

$$g=0$$

$$\frac{\partial g}{\partial\zeta}=0$$

$$\frac{\partial^2 g}{\partial\zeta^2}=0$$

$$\cdots\cdots$$

$$\frac{\partial^{A-1}g}{\partial\zeta^{A-1}}=0$$

In order to evaluate the integral $\int_0^1\zeta^p\frac{\partial g}{\partial\zeta}d\zeta$ in equation (13) the term $\frac{\partial g}{\partial\zeta}$ is written as

$$\frac{\partial g}{\partial\zeta}=g_1+\lambda g_2$$

where

$$g_1=(1-\zeta)^A[A+A(A+1)\zeta]-A(1-\zeta)^{A-1}\left[1+A\zeta+\frac{A(A+1)}{2}\zeta^2\right]$$

and, for $A\geqq1$,

$$g_2=(1-\zeta)^A(1+2A\zeta)-\zeta A(1+A\zeta)(1-\zeta)^{A-1}$$

By using the expression for g_1, the equation obtained for S is

$$S=2(3p+1)^2\left\{\frac{2p+1}{3p+1}-\frac{A!p}{(A+p)(A+p-1)(A+p-2)\ldots(p+1)}\left[\frac{1}{p}+\frac{A}{A+1+p}+\frac{A(A+1)(p+1)}{2(A+2+p)(A+1+p)}\right]\right\} \quad (19)$$

By using the expression for g_2 the equation for P is found to be

$$P=-2(2p+1)(3p+1)^2\left\{\frac{2p}{3p+1}-\frac{A!(p+1)}{(A+p)(A+p-1)(A+p-2)\ldots(p+1)}\left[\frac{1}{A+1+p}+\frac{A(p+1)}{(A+2+p)(A+1+p)}\right]\right\} \quad (20)$$

To avoid positive values for S obtained from equation (19) for $H<3$, it is found that A must be 1 in the expression for g_1. It is also found that, to avoid positive values for P in equation (20) for $H>0$, A must be ∞ in the expression for g_2. The values for S and P then become

$$S=\frac{(3H-1)(H-1)(H-3)}{H+5} \quad (21)$$

$$P=-H(H-1)(3H-1) \quad (22)$$

The expression for P (equation (22)) is the same as the coefficient of $\frac{\theta}{U}\frac{dU}{dx}$ in equation (13); letting $A\to\infty$ makes the coefficient of λ in equation (18) become zero. The shear profile then contributes nothing to the coefficient of $\frac{\theta}{U}\frac{dU}{dx}$ in equation (13).

Equations (21) and (22) for S and P, respectively, were tested by making a computation of H and θ for the pressure distribution given in table I of reference 10. The computation began at $\frac{x}{c}=0.075$ with the values given in table I of reference 10. The equations used are

$$\theta\frac{dH}{dx}=-H(H-1)(3H-1)\omega+\frac{(3H-1)(H-1)(H-3)}{H+5}\phi$$

and

$$\frac{d\theta}{dx}=-(H+2)\omega+\phi$$

The equation for $\frac{d\theta}{dx}$ is the Von Kármán momentum equation. The equation for ϕ was obtained from reference 17 and is

$$\phi=\frac{0.006535}{R_\theta^{1/6}}$$

The calculated distribution of H along x was far from the experimental curve.

In an attempt to reduce the sensitivity of the equation for $\theta\frac{dH}{dx}$ to the shear distribution, the moment of momentum equation (equation (14)), in which the shear appears in the coefficient of ϕ only as a mean value, is used. When the generalized expression (equation (18)) is used for the shear distribution g, the result obtained is

$$\theta\frac{dH}{dx}=\left[-\frac{H(H-1)(H+1)^2}{2}+\frac{3H(H+1)^3}{(A+2)(A+3)}\right]\omega+$$

$$(H^2-1)\left[H-\frac{3(H+1)}{A+3}\right]\phi \quad (23)$$

where $\psi=0$ as required by equation (18). To keep the coefficient of ω negative for all positive values of H, A must equal ∞ in the coefficient of ω. The shear distribution is then independent of the pressure gradient. To make the coefficient of ϕ negative for values of H near 3, A must have the smallest value that it can take; therefore, let $A=1$ in the coefficient of ϕ. Equation (23) then becomes

$$\theta\frac{dH}{dx}=\frac{-H(H^2-1)(H+1)}{2}\omega+\frac{(H^2-1)(H-3)}{4}\phi \quad (24)$$

A calculation for the example in table I of reference 10 with equation (24) resulted in a computed curve for H that was far from the experimental curve.

(b) Assumption of constant shear across the boundary layer.—All the computations of H have led to values of H much larger than the experimental values. Therefore, in order to reduce the calculated values of H it is necessary to increase S. In order to increase S, the assumption of constant shear across the boundary layer is made. For constant shear it can be shown that

$$\int_0^1 \zeta\frac{\partial g}{\partial \zeta}d\zeta=-1$$

by letting $g=(1-\zeta)^B$ and taking the limit of the integral as $B\to0$. Equation (14), after the assumption of constant shear is introduced, becomes

$$\theta\frac{dH}{dx}=\frac{-H(H+1)(H^2-1)}{2}\omega-(H^2-1)\phi+(H^2-1)\psi$$

In order to make $\frac{dH}{dx}=0$ at $H=1.286$ for $\frac{dU}{dx}=0$, the coefficient of ϕ was arbitrarily changed to $H^2-1.286^2$. The equation then becomes

$$\theta\frac{dH}{dx}=\frac{-H(H+1)(H^2-1)}{2}\omega-(H^2-1.653)\phi+(H^2-1)\psi \quad (25)$$

This equation was used for the computation of H with $\psi=0$, and the results for the example given in table I of reference 10 are shown in figure 5.

The assumption of constant shear across the boundary layer was also combined with the kinetic-energy equation. When the power profiles and the assumption of constant shear are used in equation (13), the kinetic-energy equation becomes

$$\theta\frac{dH}{dx}=-H(H-1)(3H-1)\omega-\frac{(H-1)(3H-1)}{2}\phi+$$
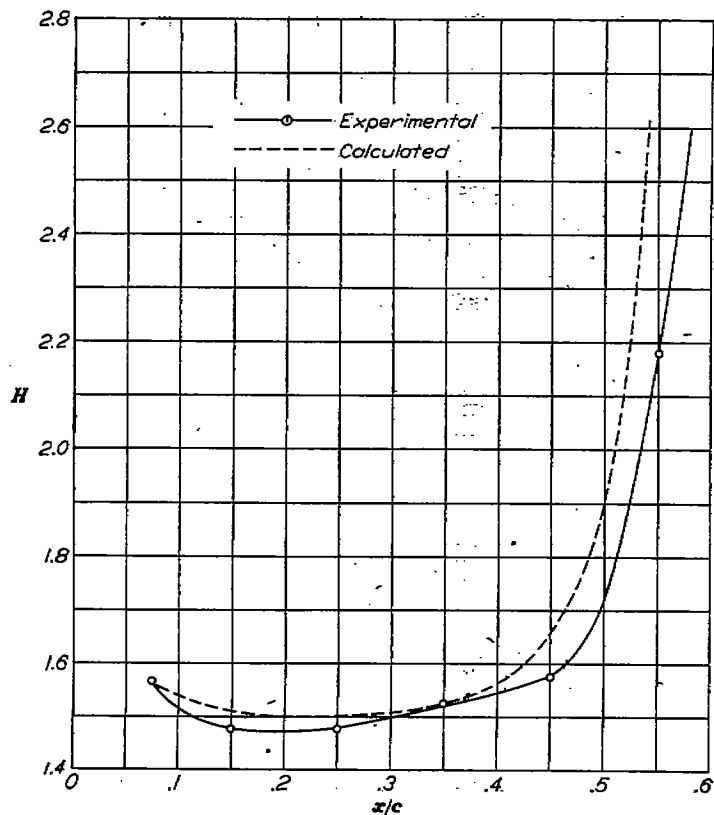
$$\frac{(H+1)(3H-1)}{4}\psi$$

FIGURE 5.—Comparison of experimental values of H and values calculated by equation (25). $P=-\dfrac{H(H+1)(H^2-1)}{2}$; $S=H^2-1.653$; $\psi=0$.

FIGURE 6.—Values of S from kinetic-energy equation with the assumption of constant shear. For power profiles, $S=-\dfrac{(H-1)(3H-1)}{2}$; for experimental profiles, $S=-\dfrac{K-2}{K'}$.
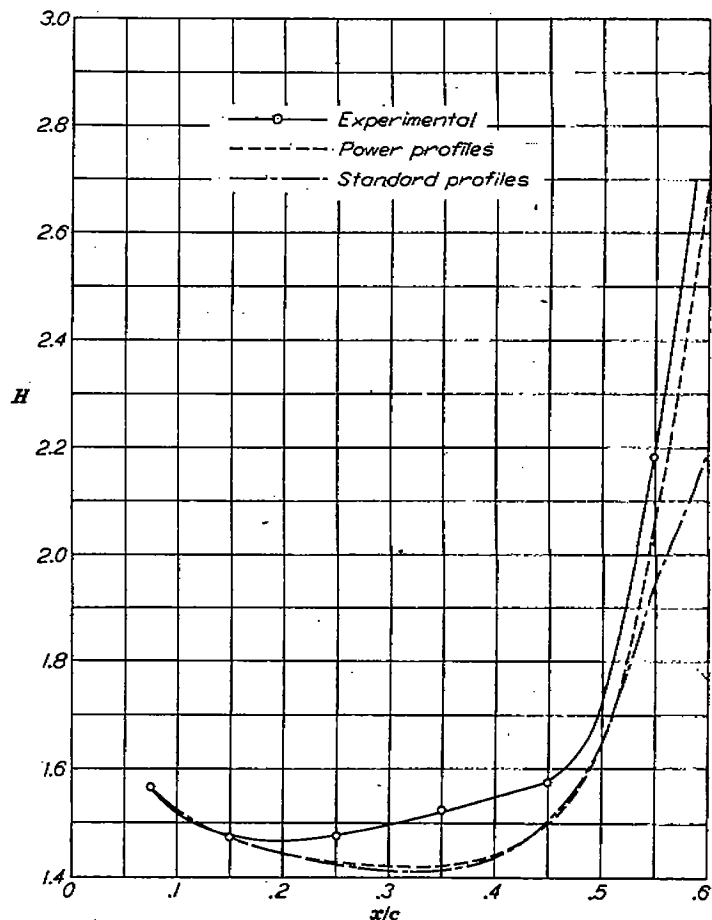
FIGURE 7.—Comparison of experimental values of H and values of H calculated by kinetic energy equation with constant shear across boundary layer.

The function $-H(H-1)(3H-1)$ is shown in figure 3 and the function $\dfrac{-(H-1)(3H-1)}{2}$, in figure 6. When the standard profiles are substituted for the power profiles and the assumption of constant shear is made, the kinetic-energy equation (equation (9)) becomes

$$\theta\frac{dH}{dx}=\frac{K(H-1)}{K'}\omega-\frac{K-2}{K'}\phi-\frac{K-1}{K'}\psi$$

where the function $\dfrac{K(H-1)}{K'}$ is shown in figure 3 and the function $-\dfrac{K-2}{K'}$ is shown in figure 6. The results of these calculations of H (with $\psi=0$) are shown in figure 7. In this case, the use of power profiles makes the result somewhat different from that obtained by using the standard profiles.

(c) **Determination of S by the simultaneous use of the energy and moment of momentum equations.**—It seems obvious that, if equations (13) and (14) were exact, the coefficients of ω, ϕ, and ψ in equation (13) would be equal to the coefficients of ω, ϕ, and ψ in equation (14). The ratio of the coefficients of ω is

$$\frac{-\dfrac{H(H+1)(H^2-1)}{2}}{-H(H-1)(3H-1)}=\frac{(H+1)^2}{2(3H-1)}$$

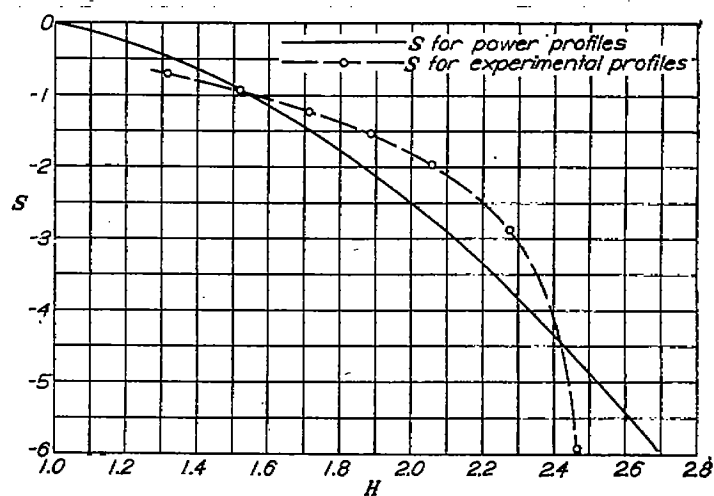The curve of $\dfrac{(H+1)^2}{2(3H-1)}$ is given in figure 8 and is seen to be close to unity.

The ratio of the coefficients of ψ is

$$\frac{4(H^2-1)}{(H+1)(3H-1)}=\frac{4(H-1)}{3H-1}$$

The curve of $\dfrac{4(H-1)}{3H-1}$ is also given in figure 8. The values are far from unity for small values of H but become equal to unity for $H=3$.
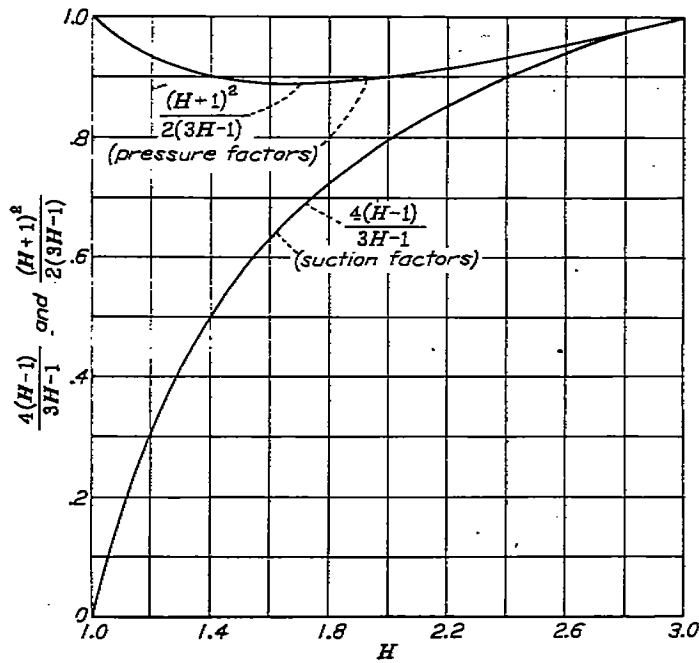
FIGURE 8.—Ratios of pressure and suction factors in energy and moment of momentum equations.

Equating the coefficients of ϕ results in

$$(H^2-1)\left[H+(H+1)\int_0^1 \zeta \frac{\partial g}{\partial \zeta} d\zeta\right]$$

$$=(3H-1)\left[H+\frac{3H-1}{2}\int_0^1 \zeta^{\frac{H-1}{2}} \frac{\partial g}{\partial \zeta} d\zeta\right]$$

or

$$(H^2-1)(H+1)\int_0^1 \zeta \frac{\partial g}{\partial \zeta} d\zeta-\frac{(3H-1)^2}{2}\int_0^1 \zeta^{\frac{H-1}{2}} \frac{\partial g}{\partial \zeta} d\zeta$$

$$=H^2(3-H) \qquad (26)$$

Now let

$$\int_0^1 \zeta \frac{\partial g}{\partial \zeta} d\zeta=a+bH+cH^2$$

and

$$\int_0^1 \zeta^{\frac{H-1}{2}} \frac{\partial g}{\partial \zeta} d\zeta=j+qH+lH^2$$

When the shear distribution is assumed to depend only on H, the integrals in equation (26) are functions of H alone. Because equation (26) is then an identity, the coefficients of the various powers of H can be equated to zero. The resulting equations are:

For H^0

$$-a-\frac{j}{2}=0$$

for H^1

$$-a-b+3j-\frac{q}{2}=0$$

for H^2

$$a-b-c-\frac{9}{2}j+3q-\frac{l}{2}=3$$

for H^3

$$a+b-c-\frac{9}{2}q+3l=-1$$

for H^4

$$b+c-\frac{9}{2}l=0$$

for H^5

$$c=0$$

The results obtained are:

$$a=\frac{28}{128}$$

$$b=-\frac{180}{128}$$

$$c=0$$

$$j=-\frac{56}{128}$$

$$q=-\frac{32}{128}$$

$$l=-\frac{40}{128}$$

Therefore,

$$\int_0^1 \zeta \frac{\partial g}{\partial \zeta} d\zeta=\frac{7-45H}{32}$$

and

$$\int_0^1 \zeta^{\frac{H-1}{2}} \frac{\partial g}{\partial \zeta} d\zeta=-\frac{7+4H+5H^2}{16}$$

Equation (14), the moment of momentum equation, becomes

$$\theta\frac{dH}{dx}=-\frac{H(H^2-1)(H+1)}{2}\omega-$$

$$\frac{(H-1)(3H-1)(7+22H+15H^2)}{32}\phi+(H^2-1)\psi \quad (27)$$

and equation (13), the energy equation, becomes

$$\theta\frac{dH}{dx}=-H(H-1)(3H-1)\omega-$$

$$\frac{(H-1)(3H-1)(7+22H+15H^2)}{32}\phi+\frac{(H+1)(3H-1)}{4}\psi$$

$$(28)$$

The variation of H with x for the initial values and the pressure distribution given in table I of reference 10 was computed by using a modified form of equation (28). In order that $\frac{dH}{dx}=0$ at values of H in agreement with experiment when $\omega=0$, the coefficient of ϕ in equation (28) was replaced by

$$\frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}$$

where

$$H_0=H_0(R_\theta)$$

The variation of H_0 with R_θ was calculated from the equation

$$\log_{10} H_0=0.5990-0.1980 \log_{10} R_\theta-0.0189 (\log_{10} R_\theta)^2$$

which was derived to represent a faired curve through the

experimental data (see fig. 9); the data were obtained from reference 13 and from British results that are not generally available. The result of a computation of H for $\psi=0$ and with equation (28) modified as follows

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1)\,\omega -$$

$$\frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}\phi + \frac{(H+1)(3H-1)}{4}\psi$$

$$(29)$$

is given in figure 10.

Assumptions (b) and (c) lead to somewhat better results than assumption (a) although they are still not as satisfactory as those obtained from the purely empirical relations introduced in references 10 and 12. It is clear that this difference is caused partly by the inaccuracy of the simple assumptions about the shear distribution and can be improved by using better descriptions. However, in view of the limited present knowledge of the shear distribution, it does not seem worth while to make more complicated assumptions.
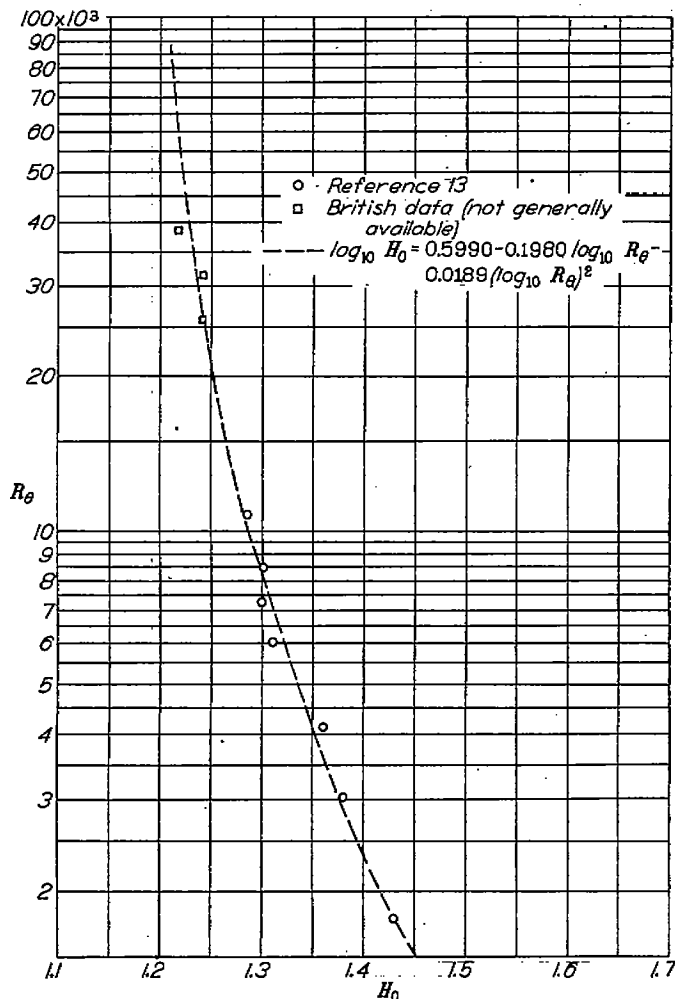
It may be noted that the final equations obtained for the change of the form parameter by the three assumptions are all of the form

$$\theta \frac{dH}{dx} = P(H)\,\omega + S(H)\,\phi$$

where $\omega = \frac{\theta}{U}\frac{dU}{dx}$ and $\phi = \frac{\tau_0}{\rho U^2}$. This form is used in reference 12, but a different form is used in reference 10.

INVESTIGATION OF ENERGY EQUATION

Since none of the three assumptions for the shear distribution results in a dependable equation for $\frac{dH}{dx}$, an investigation is made to determine whether a result common to the three assumptions—namely that the coefficient of $\frac{\tau_0}{\rho U^2}$ in the equations for $\frac{dH}{dx}$ is a function of H alone—is very far from true by using experimental data and the kinetic-energy equation without any assumption for the shear.

If no assumptions other than the boundary-layer assumptions are made and if in equation (6) $n=0$ and $m=1$, the result is

$$\theta \frac{dK}{dx} = -\phi\left(K+2\int_0^{\delta/\theta} f\frac{\partial g}{\partial \eta}d\eta\right) + \omega(H-1)K + \psi(1-K) \quad (30)$$

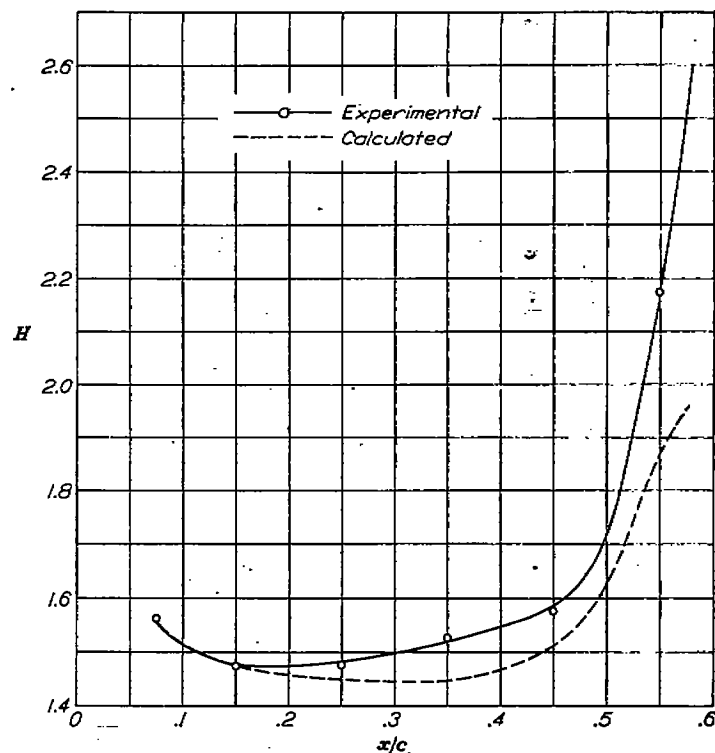If the assumption of a single-parameter family of curves is

FIGURE 9.—Variation of H_0 with R_θ.

FIGURE 10.—Comparison of experimental values of H with values calculated by equation (29). $P=-H(H-1)(3H-1)$; $S=-\frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}$; $\psi=0$.

made $(f=f(\eta,H))$, then $K=K(H)$, and equation (30) becomes

$$\theta\frac{dH}{dx}=-\phi\frac{K+2\int_0^{8/\theta}f\frac{\partial g}{\partial\eta}d\eta}{K'}+\omega\frac{H-1}{K'}K+\psi\frac{1-K}{K'}$$

or, for $\psi=0$,

$$\theta\frac{dH}{dx}=\phi k(H)(\xi+\xi_0) \qquad (31)$$

where

$$k(H)=\frac{-(H-1)K}{K'}$$

$$\xi=-\frac{\omega}{\phi}$$

$$\xi_0=\frac{1}{H-1}\left(1-\frac{2}{K}\int_0^1 f\,dg\right)$$

If the assumption is made that $g=g(\eta,H)$, then $f=f(g,H)$ and $\int_0^1 f\,dg$ is a function of H only. Therefore, $\xi_0=\xi_0(H)$. Equation (31) then becomes

$$\theta\frac{dH}{dx}=\phi k(H)[\xi+\xi_0(H)] \qquad (32)$$

In order to obtain an estimate of the quantity $1-\frac{2}{K}\int_0^1 f\,dg$ under the assumption that $f=f(g,\,H)$, reference 12 is used. Equation (7) of reference 12 may be written as

$$\theta\frac{dH}{dx}=\phi k_1(H)[\xi-2.065(H-1.4)] \qquad (33)$$

where

$$k_1(H)=e^{5(H-1.4)}$$

Note that Garner's equation (equation (33)) has the form the kinetic-energy equation takes when the assumptions that $f=f(\eta,H)$ and that $g=g(\eta,\,H)$ are used in the kinetic-energy equation. The kinetic-energy equation (equation (31)) can also be placed in the form of equation (32) when the more general assumption that $g=\frac{\omega}{\phi}\,F_0(\eta,H)+F_1(\eta,H)$ is made for the shear distribution. For the purpose of obtaining an estimate of the value of $1-\frac{2}{K}\int_0^1 f\,dg$, the quantity $\xi+\xi_0(H)$ in equation (32) is assumed to be identical with the quantity $\xi-2.065(H-1.4)$ in equation (33). Then

$$\xi_0(H)=-2.065(H-1.4)$$

and for $H=1.5$, for example,

$$\xi_0(H)=-0.2065$$

therefore,

$$\frac{1}{0.5}\left(1-\frac{2}{K}\int_0^1 f\,dg\right)=-0.2065$$

or

$$1-\frac{2}{K}\int_0^1 f\,dg=-0.1032$$

Therefore $1-\frac{2}{K}\int_0^1 f\,dg$ is the difference between two quantities, each of which is much larger than their difference. It follows that, in order to determine $\xi_0(H)$ for values of H not close to separation with any accuracy, f and g must be known with relatively good accuracy.

It may be noted that the moment of momentum equation is also sensitive to g. This sensitivity can be seen by writing the coefficient of ϕ in equation (14) as

$$(H^2-1)\left[(H+1)\left(1-\int_0^1 g\,d\zeta\right)-1\right]$$

When it is noted that the integral $\int_0^1 g\,d\zeta$ is of the order of unity and that H lies between 1.2 and 2.6, the sensitivity of the coefficient of ϕ to g becomes clear.

In an attempt to determine whether ξ_0 is determined mainly by H, all the data that were used in reference 10 were used to compute ξ_0 by making use of equation (32) in the form

$$\xi_0(H)=\frac{\theta\frac{dH}{dx}}{\phi k(H)}-\xi$$

The surface-friction coefficient ϕ was calculated by the formula (from reference 17)

$$\phi=\frac{0.006535}{R_\theta^{1/6}}$$

and $k(H)$ was calculated by the expression obtained from the moment of momentum equation

$$k(H)=\frac{H(H^2-1)(H+1)}{2}$$

The values of ξ_0 plotted against H are given in figure 11. The effort to determine whether ξ_0 is a function mainly of H is inconclusive. At least part of the scatter occurs because $\frac{dH}{dx}$ and $\frac{dU}{dx}$ were obtained from curves faired through experimental points. In addition, the calculation of ξ_0 requires the subtraction of ξ from $\frac{\theta\frac{dH}{dx}}{\phi k(H)}$, an operation which further decreases the accuracy of the calculated values of ξ_0.

DISCUSSION

Although equation (6) is valid whenever the boundary-layer assumptions are valid, the equations for $\frac{dH}{dx}$ that result after additional assumptions are made do not lead to
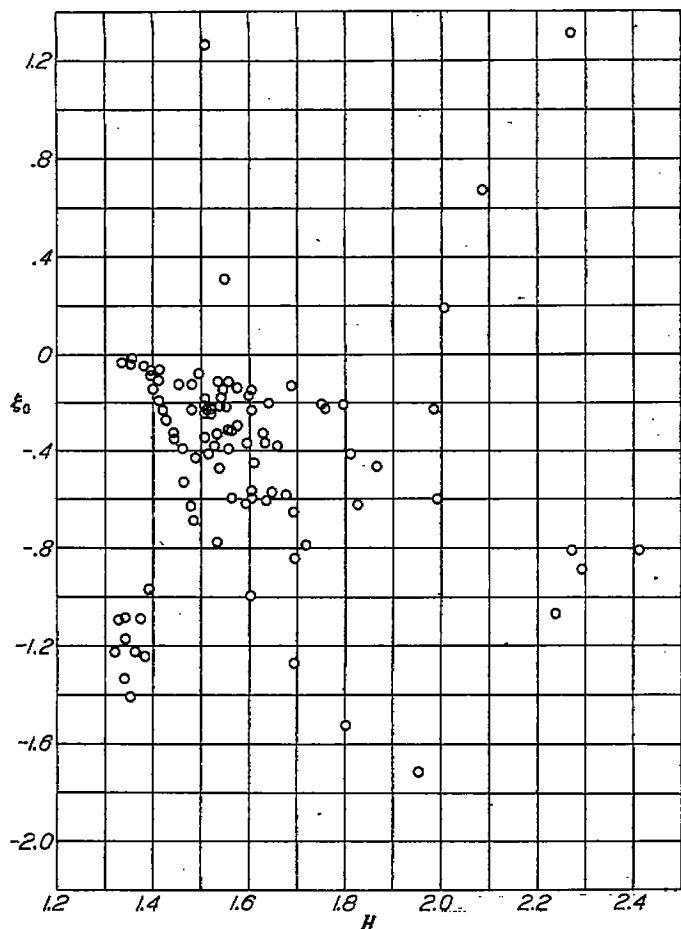
FIGURE 11.—Values of ξ_0 plotted against H.

good agreement with experiment. The first of the additional assumptions made is that all velocity profiles of the turbulent boundary layer belong to a single-parameter family of curves. The experimental data of references 7, 10, 11, 14, and 15 substantiate this assumption.

The second assumption is that the single-parameter family of curves can be approximated by power profiles. The data in figure 1, in which velocity profiles are compared, and also the data in figures 2 and 3, in which K and P are compared, show this assumption to be good, at least for $H < 1.8$.

From the data in figures 1 to 3, it is inferred that power profiles can be substituted for the standard velocity profiles without greatly affecting the calculated distribution of H against x for $H < 1.8$. To test this inference, the kinetic-energy equation was used with the assumption of constant shear across the boundary layer; the result is shown in figure 7. As expected from the data of figures 1 to 3, the effect of the substitution of power profiles for the standard profiles is noticeable only for $H > 1.8$. It thus appears that the inaccuracy of the equations for $\dfrac{dH}{dx}$ that were tested is caused mainly by the surface-friction law that was used and by the assumed shear distributions rather than by the use of the power profiles.

The data of references 12 and 15 show skin frictions that increase strongly in the region upstream of the separation point before dropping to zero at the separation point. On the other hand, the skin-friction data presented in reference 14 indicate that the skin friction falls monotonically to zero as the separation point is reached. In the present analysis a skin-friction law obtained from experiments on flat plates is used. It is therefore probable that part of the inaccuracy in the equations used to calculate H is caused by the use of a relation for the skin friction that does not give correct values when there are pressure gradients along the surface.

The assumptions for the shear distribution that were made to obtain a specific equation for $\dfrac{dH}{dx}$ were

(a) The shear distribution depends only on the ratio of the pressure gradient to the skin friction $\dfrac{\delta}{\tau_0}\dfrac{dp_1}{dx}$ or $-\dfrac{\delta}{\theta}\dfrac{\omega}{\phi}$

(b) The shear is constant across the boundary layer

(c) The shear distribution depends only on the form parameter of the velocity distribution

Because none of these simple assumptions is derived from a knowledge of the details of the turbulent flow, it is not likely that any of them are valid. When it is recalled that the coefficient of ϕ in both the kinetic-energy and the moment of momentum equations is sensitive to the shear distribution, it is not surprising that a reliable equation for $\dfrac{dH}{dx}$ was not found.

In order to obtain a reliable equation for $\dfrac{dH}{dx}$ from equation (8) it thus seems necessary to calculate the surface shear and the shear distribution across the boundary layer more accurately than in the present analysis. Efforts should therefore be made to understand the mechanics of turbulent shear flow sufficiently well to provide an independent relation for the shearing stress that will predict the behavior of turbulent boundary layers when used with the Prandtl boundary-layer equation (equation (1)).

CONCLUDING REMARKS

A general integral form of the boundary-layer equation is derived from the Prandtl partial-differential boundary-layer equation. The general integral equation, valid for either laminar or turbulent incompressible boundary-layer flow, contains the Von Kármán momentum equation, the kinetic-energy equation, and the Loitsianskii equation as special cases.

In an attempt to obtain a practical method for the calculation of the development of the turbulent boundary layer, use is made of the experimental finding that all the velocity profiles of the turbulent boundary layer form essentially a single-parameter family. The general equation is thereby changed to a simpler one from which an equation for the space rate of change of the shape parameter of the turbulent boundary layer can be obtained.

The resulting equation for the space rate of change of the velocity-profile parameter is restricted by the assumption that the velocity profiles of the turbulent boundary layer can be approximated by power profiles. Two of the resulting equations are used to calculate the distribution of the profile shape parameter over an airfoil for one experimentally determined pressure distribution. Although different assumptions were tried for the shearing stress across the boundary layer, the calculated distribution of the profile shape parameter did not agree exactly with the experimental distribution.

An examination is made of the effect of using the experimentally determined single-parameter family of velocity profiles instead of the power profiles on certain functions that occur in the equation for the space rate of change of the velocity-profile parameter. One calculation of the distribution of the profile shape parameter over an airfoil is also made for the experimentally determined pressure distribution by using the single-parameter family of velocity profiles found from experiment. A comparison of the results with those of a calculation made with the same assumptions except for the use of power profiles shows some difference near the separation point. It is believed, however, that the apparent lack of reliability of the specific equations used to make the calculations is caused mainly by the lack of precise knowledge concerning the surface shear and the distribution of the shearing stress across the turbulent boundary layer. The present analysis emphasizes the need for information concerning the shearing stresses in turbulent boundary layers.

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LANGLEY FIELD, VA., *May 22, 1950.*

APPENDIX A

DETAILED DEVELOPMENT OF EQUATION (3)

Equation (2) can be written so that terms of the form $u^{m+1}-U^{m+1}$ appear explicitly; therefore, each term will vanish at the outer edge of the boundary layer. The resulting equation is

$$\frac{1}{m+1}\left[\frac{\partial}{\partial x}(u^{m+1}-U^{m+1})u+\frac{\partial}{\partial y}(u^{m+1}-U^{m+1})v\right]+$$
$$\frac{1}{m+1}\frac{\partial u U^{m+1}}{\partial x}+\frac{1}{m+1}\frac{\partial v U^{m+1}}{\partial y}+$$
$$\frac{1}{m+1}\frac{(u^{m+1}-U^{m+1})u}{r_0}\frac{dr_0}{dx}+\frac{1}{m+1}\frac{uU^{m+1}}{r_0}\frac{dr_0}{dx}$$
$$=u^m U\frac{dU}{dx}+\frac{u^m}{\rho}\frac{\partial\tau}{\partial y} \tag{A1}$$

or, after simplification,

$$-\frac{1}{m+1}\left[\frac{\partial}{\partial x}(U^{m+1}-u^{m+1})u+\frac{\partial}{\partial y}(U^{m+1}-u^{m+1})v\right]-$$
$$\frac{1}{m+1}\frac{(U^{m+1}-u^{m+1})u}{r_0}\frac{dr_0}{dx}-(u^m U-uU^m)\frac{dU}{dx}=\frac{u^m}{\rho}\frac{\partial\tau}{\partial y} \tag{A2}$$

Equation (A2) is now multiplied through by y^n and integrated with respect to y from $y=0$ to $y=\delta$. The resulting equation is

$$-\frac{1}{m+1}\int_0^\delta y^n\frac{\partial}{\partial x}(U^{m+1}-u^{m+1})u\,dy-$$
$$\frac{1}{m+1}\int_0^\delta y^n\frac{\partial}{\partial y}(U^{m+1}-u^{m+1})v\,dy-$$
$$\frac{1}{m+1}\frac{1}{r_0}\frac{dr_0}{dx}\int_0^\delta y^n(U^{m+1}-u^{m+1})u\,dy-$$
$$\frac{dU}{dx}\int_0^\delta y^n(u^m U-uU^m)\,dy=\frac{1}{\rho}\int_0^\delta u^m y^n\frac{\partial\tau}{\partial y}\,dy$$

or, after simplification and substitution of the formula for the differentiation of a definite integral

$$\int_0^\delta y^n\frac{\partial}{\partial x}(U^{m+1}-u^{m+1})u\,dy=\frac{d}{dx}\int_0^\delta y^n(U^{m+1}-u^{m+1})u\,dy$$

the following equation results:

$$-\frac{1}{m+1}\frac{d}{dx}U^{m+2}\int_0^\delta y^n\left[1-\left(\frac{u}{U}\right)^{m+1}\right]\frac{u}{U}\,dy-$$
$$\frac{1}{m+1}\int_0^\delta y^n\frac{\partial}{\partial y}(U^{m+1}-u^{m+1})v\,dy-$$
$$\frac{1}{m+1}\frac{U^{m+2}}{r_0}\frac{dr_0}{dx}\int_0^\delta y^n\left[1-\left(\frac{u}{U}\right)^{m+1}\right]\frac{u}{U}\,dy-$$
$$U^{m+1}\frac{dU}{dx}\int_0^\delta\left[\left(\frac{u}{U}\right)^m-\frac{u}{U}\right]y^n\,dy=\frac{1}{\rho}\int_0^\delta u^m y^n\frac{\partial\tau}{\partial y}\,dy \tag{A3}$$

By integration by parts,

$$\int_0^\delta y^n\frac{\partial}{\partial y}(U^{m+1}-u^{m+1})v\,dy=-n\int_0^\delta(U^{m+1}-u^{m+1})vy^{n-1}\,dy$$

and equation (A3) becomes

$$-\frac{1}{m+1}\frac{d}{dx}U^{m+2}\int_0^\delta y^n\left[1-\left(\frac{u}{U}\right)^{m+1}\right]\frac{u}{U}\,dy+$$
$$\frac{n}{m+1}\int_0^\delta(U^{m+1}-u^{m+1})vy^{n-1}\,dy-$$
$$\frac{1}{m+1}\frac{U^{m+2}}{r_0}\frac{dr_0}{dx}\int_0^\delta y^n\left[1-\left(\frac{u}{U}\right)^{m+1}\right]\frac{u}{U}\,dy-$$
$$U^{m+1}\frac{dU}{dx}\int_0^\delta\left[\left(\frac{u}{U}\right)^m-\frac{u}{U}\right]y^n\,dy=\frac{1}{\rho}\int_0^\delta u^m y^n\frac{\partial\tau}{\partial y}\,dy \tag{A4}$$

The velocity v can be eliminated from the term

$$\int_0^\delta (U^{m+1}-u^{m+1})vy^{n-1}\,dy$$

by the following development:

The velocity v may be written as

$$v=\int_0^y \frac{\partial v}{\partial y}\,dy+v_0$$

or, by use of the equation of continuity,

$$v=-\int_0^y \frac{\partial u}{\partial x}\,dy-\frac{1}{r_0}\frac{dr_0}{dx}\int_0^y u\,dy+v_0$$

$$=\int_0^y \frac{\partial(U-u)}{\partial x}\,dy-y\frac{dU}{dx}+\frac{1}{r_0}\frac{dr_0}{dx}\int_0^y (U-u)\,dy-\frac{U}{r_0}\frac{dr_0}{dx}y+v_0$$

$$=\int_0^y \left[U\frac{\partial\left(1-\frac{u}{U}\right)}{\partial x}+\frac{dU}{dx}\left(1-\frac{u}{U}\right)\right]dy-y\frac{dU}{dx}+$$
$$\frac{U}{r_0}\frac{dr_0}{dx}\int_0^y \left(1-\frac{u}{U}\right)dy-\frac{U}{r_0}\frac{dr_0}{dx}y+v_0$$

$$v=U\int_0^y \frac{\partial\left(1-\frac{u}{U}\right)}{\partial x}\,dy+\frac{dU}{dx}\int_0^y\left(1-\frac{u}{U}\right)dy-y\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)+$$
$$\frac{U}{r_0}\frac{dr_0}{dx}\int_0^y \left(1-\frac{u}{U}\right)dy+v_0$$

or, after terms are collected and f is substituted for u/U, the result is

$$v=U\int_0^y \frac{\partial(1-f)}{\partial x}\,dy+\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)\left[\int_0^y (1-f)\,dy-y\right]+v_0$$

The term $\int_0^\delta (U^{m+1}-u^{m+1})\,vy^{n-1}\,dy$ can now be written as

$$\int_0^\delta (U^{m+1}-u^{m+1})vy^{n-1}\,dy=\int_0^\delta (U^{m+1}-u^{m+1})y^{n-1}$$
$$\left\{U\int_0^y \frac{\partial(1-f)}{\partial x}\,dy+\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)\left[\int_0^y (1-f)\,dy-y\right]+v_0\right\}dy$$

or

$$\int_0^\delta (U^{m+1}-u^{m+1})vy^{n-1}\,dy$$
$$=U^{m+2}\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y \frac{\partial(1-f)}{\partial x}\,dy\right]dy+\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)U^{m+1}\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y (1-f)\,dy-y\right]dy+$$
$$v_0U^{m+1}\int_0^\delta y^{n-1}(1-f^{m+1})\,dy$$

Now let

$$\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y \frac{\partial(1-f)}{\partial x}\,dy\right]dy=I_1\theta^{n+1}$$

$$\int_0^\delta y^{n-1}(1-f^{m+1})\left[\int_0^y (1-f)\,dy\right]dy=J\theta^{n+1}$$

$$\int_0^\delta y^n(1-f^{m+1})\,dy=M\theta^{n+1}$$

and

$$\int_0^\delta y^{n-1}(1-f^{m+1})\,dy=Q\theta^n$$

The term $\int_0^\delta (U^{m+1}-u^{m+1})vy^{n-1}\,dy$ now becomes

$$\int_0^\delta (U^{m+1}-u^{m+1})vy^{n-1}\,dy=U^{m+2}I_1\theta^{n+1}+$$
$$\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)U^{m+1}(J-M)\theta^{n+1}+v_0U^{m+1}Q\theta^n$$

Now let

$$\int_0^\delta y^n(1-f^{m+1})f\,dy=N\theta^{n+1}$$

and

$$\int_0^\delta (f-f^m)y^n\,dy=L\theta^{n+1}$$

Equation (A4) can then be written as

$$-\frac{1}{m+1}\frac{d}{dx}(U^{m+2}N\theta^{n+1})+\frac{n}{m+1}U^{m+2}I_1\theta^{n+1}+$$
$$\frac{n}{m+1}\left(\frac{dU}{dx}+\frac{U}{r_0}\frac{dr_0}{dx}\right)U^{m+1}(J-M)\theta^{n+1}+\frac{n}{m+1}v_0U^{m+1}Q\theta^n-$$
$$\frac{1}{m+1}\frac{U^{m+2}}{r_0}\frac{dr_0}{dx}N\theta^{n+1}+U^{m+1}\frac{dU}{dx}L\theta^{n+1}=\frac{U^m}{\rho}\int_0^\delta f^m y^n \frac{\partial\tau}{\partial y}\,dy$$
$$\text{(A5)}$$

After $\frac{d}{dx}U^{m+2}N\theta^{n+1}$ is expanded and terms are collected, equation (A5) becomes

$$(n+1)N\frac{d\theta}{dx}+\theta\left(\frac{dN}{dx}-nI_1\right)+$$
$$\frac{\theta}{U}\frac{dU}{dx}[N(m+2)-n(J-M)-L(m+1)]+$$
$$\frac{\theta}{r_0}\frac{dr_0}{dx}[N-n(J-M)]-nQ\frac{v_0}{U}=-(m+1)\frac{\tau_0}{\rho U^2}\int_0^{\delta/\theta} f^m\eta^n\frac{\partial\frac{\tau}{\tau_0}}{\partial\eta}\,d\eta$$

where

$$\eta=\frac{y}{\theta}$$

APPENDIX B

SIMPLIFICATION OF TERMS IN EQUATION (8) FOR POWER PROFILES

CALCULATION OF $\frac{dN}{dH}+nI$

The definition of $N\theta^{n+1}$ is

$$N\theta^{n+1}=\int_0^\delta (1-f^{m+1})fy^n\,dy$$

therefore,

$$N=\int_0^{\delta/\theta}(1-f^{m+1})f\eta^n\,d\eta$$

and

$$\frac{dN}{dH}=\int_0^{\delta/\theta}\left[(1-f^{m+1})\frac{\partial f}{\partial H}-f(m+1)f^m\frac{\partial f}{\partial H}\right]\eta^n\,d\eta$$

or

$$\frac{dN}{dH}=\int_0^{\delta/\theta}\frac{\partial f}{\partial H}[1-(m+2)f^{m+1}]\eta^n\,d\eta$$

The definition of $I\theta^{n+1}$ is

$$I\theta^{n+1}=\int_0^\delta (1-f^{m+1})\left(\int_0^y\frac{\partial f}{\partial H}\,dy\right)y^{n-1}\,dy$$

therefore,

$$I=\int_0^{\delta/\theta}(1-f^{m+1})\left(\int_0^\eta\frac{\partial f}{\partial H}\,d\eta\right)\eta^{n-1}\,d\eta$$

Then

$$\frac{dN}{dH}+nI=\int_0^{\delta/\theta}\frac{\partial f}{\partial H}[1-(m+2)f^{m+1}]\eta^n\,d\eta+$$
$$n\int_0^{\delta/\theta}(1-f^{m+1})\left(\int_0^\eta\frac{\partial f}{\partial H}\,d\eta\right)\eta^{n-1}\,d\eta \qquad (B1)$$

By integration by parts,

$$n\int_0^{\delta/\theta}(1-f^{m+1})\left(\int_0^\eta\frac{\partial f}{\partial H}\,d\eta\right)\eta^{n-1}\,d\eta=$$
$$-\int_0^{\delta/\theta}\left[(1-f^{m+1})\frac{\partial f}{\partial H}-(m+1)f^m\frac{\partial f}{\partial \eta}\int_0^\eta\frac{\partial f}{\partial H}\,d\eta\right]\eta^n\,d\eta \qquad (B2)$$

When equation (B2) is substituted into equation (B1), the following equation is obtained:

$$\frac{dN}{dH}+nI=(m+1)\int_0^{\delta/\theta}\eta^n f^m\left(\frac{\partial f}{\partial \eta}\int_0^\eta\frac{\partial f}{\partial H}\,d\eta-f\frac{\partial f}{\partial H}\right)d\eta$$

Use is now made of the power-law assumption

$$f=\zeta^p=\eta^p\left(\frac{\theta}{\delta}\right)^p$$

Then

$$\frac{\partial f}{\partial H}=\frac{1-2p^2}{2(p+1)(2p+1)}\zeta^p+\frac{\zeta^p\log\zeta}{2}$$

and

$$\frac{\partial f}{\partial \eta}\int_0^\eta\frac{\partial f}{\partial H}\,d\eta-f\frac{\partial f}{\partial H}$$
$$=p\left\{\frac{1-2p^2}{2(p+1)^2(2p+1)}\zeta^{2p}+\frac{1}{2}\left[\frac{\zeta^{2p}\log\zeta}{p+1}-\frac{\zeta^{2p}}{(p+1)^2}\right]\right\}-$$
$$\left[\frac{1-2p^2}{2(p+1)(2p+1)}\zeta^{2p}+\frac{\zeta^{2p}\log\zeta}{2}\right]$$

After a lengthy manipulation, $\frac{dN}{dH}+nI$ is found to be

$$\frac{dN}{dH}+nI=-\frac{m+1}{2}\frac{(p+1)^n(2p+1)^n(pm+n)}{p^{n+1}[p(m+2)+n+1]^2}$$

where use has been made of the following equation:

$$\frac{\delta}{\theta}=\frac{(p+1)(2p+1)}{p}$$

CALCULATION OF N

The definition of $N\theta^{n+1}$ is

$$N\theta^{n+1}=\int_0^\delta (1-f^{m+1})fy^n\,dy$$

When $f=\zeta^p$ is used, the equation for N is

$$N=\left(\frac{\delta}{\theta}\right)^{n+1}\frac{p(m+1)}{[p(m+2)+n+1](p+n+1)}$$

or

$$N=\frac{(m+1)(p+1)^{n+1}(2p+1)^{n+1}}{p^n[p(m+2)+n+1](p+n+1)}$$

CALCULATION OF J

From the definition of $J\theta^{n+1}$,

$$J\theta^{n+1}=\int_0^\delta (1-f^{m+1})\left[\int_0^y(1-f)\,dy\right]y^{n-1}\,dy$$

When $f=\zeta^p$, the equation for J is

$$J=\left(\frac{\delta}{\theta}\right)^{n+1}\int_0^1[1-\zeta^{p(m+1)}]\left(\zeta-\frac{\zeta^{p+1}}{p+1}\right)\zeta^{n-1}\,d\zeta$$

or, after a lengthy manipulation,

$$J=\frac{(p+1)^n(2p+1)^{n+1}}{p^{n-1}}(m+1)$$
$$\left\{\frac{(p+1+n)[pm+n+2(p+1)]+p(m+1)+1+n}{(n+1)(p+1+n)[p(m+1)+n+1][p(m+1)+p+1+n]}\right\}$$

CALCULATION OF M

From the definition of $M\theta^{n+1}$

$$M\theta^{n+1} = \int_0^\delta (1-f^{m+1})y^n\,dy$$

When $f=\zeta^p$, the equation for M is

$$M = (m+1)\frac{(p+1)^{n+1}(2p+1)^{n+1}}{p^n(n+1)[p(m+1)+n+1]}$$

CALCULATION OF L

From the definition of $L\theta^{n+1}$

$$L\theta^{n+1} = \int_0^\delta (1-f^{m-1})fy^n\,dy$$

When $f=\zeta^p$, the equation for L is

$$L = (m-1)\frac{(p+1)^{n+1}(2p+1)^{n+1}}{p^n(p+n+1)(pm+n+1)}$$

CALCULATION OF $\dfrac{N(H-m)-n(J-M)(H+1)+(m+1)L}{\frac{dN}{dH}+nI}$

From the expressions for J and M, the expression for $J-M$ is

$$J-M = \frac{(p+1)^n(2p+1)^{n+1}}{p^{n-1}}(m+1)$$

$$\frac{(p+1+n)[pm+n+2(p+1)]+[p(m+1)+1+n]}{(n+1)(p+1+n)[p(m+1)+n+1][p(m+1)+p+1+n]} -$$

$$(m+1)\frac{(p+1)^{n+1}(2p+1)^{n+1}}{p^n(n+1)[p(m+1)+n+1]}$$

After a lengthy simplification, the result is

$$J-M = \frac{-(m+1)(p+1)^n(2p+1)^{n+1}}{p^n(p+1+n)[p(m+1)+p+1+n]}$$

or

$$-n(J-M)(H+1) = \frac{2n(m+1)(p+1)^{n+1}(2p+1)^{n+1}}{p^n(p+1+n)[p(m+1)+p+1+n]}$$

where $H=2p+1$ was used. The expression obtained for $N(H-m)+(m+1)L$ is

$$N(H-m)+(m+1)L$$
$$= \frac{(m+1)(p+1)^{n+1}(2p+1)^{n+1}2p(pm+m+n)}{p^n(p+n+1)[p(m+2)+n+1](pm+n+1)}$$

and the expression obtained for

$$N(H-m)-n(J-M)(H+1)+(m+1)L$$

is

$$N(H-m)-n(J-M)(H+1)+(m+1)L$$
$$= \frac{2(m+1)(p+1)^{n+1}(2p+1)^{n+1}[n(pm+n+1)+p(pm+m+n)]}{p^n(p+1+n)[p(m+2)+n+1](pm+n+1)}$$

By substitution and simplification

$$\frac{N(H-m)-n(J-M)(H+1)+(m+1)L}{\frac{dN}{dH}+nI}$$

$$= \frac{-4p(p+1)(2p+1)[p(m+2)+n+1]}{pm+n+1}$$

It can also be shown that

$$\frac{-N+n(J-M)-(m+1)\int_0^{\delta/\theta}\eta^n f^m\dfrac{\partial g}{\partial\eta}\,d\eta}{\frac{dN}{dH}+nI}$$

$$= \frac{2p}{pm+n}[p(m+2)+n+1]\left\{2p+1+[p(m+2)+n+1]\right.$$

$$\left.\int_0^1 \zeta^{pm+n}\frac{\partial g}{\partial\zeta}\,d\zeta\right\}$$

EVALUATION OF $-N+n(J-M)+nQ$

From the results for N and $J-M$

$$-N+n(J-M) = \frac{-(m+1)(p+1)^n(2p+1)^{n+1}}{p^n[p(m+2)+n+1]}$$

For Q, the development is

$$Q\theta^n = \int_0^\delta (1-f^{m+1})y^{n-1}\,dy$$

and with $f=\zeta^p$, the following expression is obtained for $n\neq 0$:

$$Q = \frac{m+1}{n}\frac{(p+1)^n(2p+1)^n}{p^{n-1}[p(m+1)+n]}$$

Then, by substitution and simplification, for $n\neq 0$,

$$-N+n(J-M)+nQ = \frac{-(m+1)(p+1)^{n+1}(2p+1)^n(pm+n)}{p^n[p(m+2)+n+1][p(m+1)+n]} \tag{B3}$$

If use is made of the previously derived result that $nQ=1$ for $n=0$, the following equation is obtained for $n=0$:

$$-N+n(J-M)+nQ = \frac{-m(p+1)}{p(m+2)+1} \tag{B4}$$

If n is placed equal to zero in equation (B3), equation (B4) results; therefore, equation (B3) is valid for $n=0$ as well as $n\neq 0$.

Then, for all values of n,

$$\frac{-N+n(J-M)+nQ}{\frac{dN}{dH}+nI} = \frac{2(p+1)p[p(m+2)+n+1]}{p(m+1)+n}$$

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