REPORT 1005

ANALYTICAL DETERMINATION OF COUPLED BENDING-TORSION VIBRATIONS OF CANTILEVER BEAMS BY MEANS OF STATION FUNCTIONS[1]

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SUMMARY

A method based on the concept of Station Functions is presented for calculating the modes and the frequencies of nonuniform cantilever beams vibrating in torsion, bending, and coupled bending-torsion motion. The method combines some of the advantages of the Rayleigh-Ritz and Stodola methods, in that a continuous loading function for the beam is used, with the advantages of the influence-coefficient method, in that the continuous loading function is obtained in terms of the displacements at a finite number of stations along the beam.

The Station Functions were derived for a number of stations ranging from one to eight. The deflections were obtained in terms of the physical properties of the beam and Station Numbers, which are general in nature and which have been tabulated for easy reference. Examples were worked out in detail; comparisons were made with exact theoretical results. For a uniform cantilever beam with n stations, the first n modes and frequencies were in good agreement with the theoretically exact values. The effect of coupling between bending and torsion was shown to reduce the first natural frequency to a value below that which it would have if there were no coupling.

INTRODUCTION

The failure of turbine and compressor blades due to vibrations has led to an increased interest in the study of the vibrations of these blades and in the determination of the natural modes and frequencies. In such theoretical studies, it is usually assumed that the compressor or turbine blade acts as a cantilever beam. The calculation of the uncoupled modes of arbitrarily shaped cantilever beams has been extensively investigated (references 1 to 4), but little work has as yet been done on calculating the coupled modes of such beams. If the geometry of the beam is such that coupling exists, the coupled modes are the actual vibrational modes that must be calculated.

Four general methods are currently in use for calculating uncoupled modes and frequencies of nonuniform beams. These methods are the Rayleigh-Ritz or energy method (reference 1), the Stodola method (references 5 and 6), the influence-coefficient method (references 4 and 7), and the integral-equation method (references 8 and 9). For each of these methods, computational work can usually be carried out in several ways. For example, by the use of influence coefficients the modes and frequencies can be determined by Mykelstad's iteration procedure (reference 7) or by matrix methods (reference 4).

Any one of these methods can be extended to the calculation of coupled bending-torsion modes. The Rayleigh-Ritz method usually requires that the uncoupled modes be determined before the coupled modes can be computed. In applying either the Rayleigh-Ritz or the Stodola method, great difficulty is encountered in accurately determining the higher modes, because the lower modes must first be "swept out" by the use of exact orthogonality conditions (reference 10); the process will otherwise always converge back to the lowest mode. The same difficulties are encountered in the integral-equation method.

The influence-coefficient method reduces the problem to one having a finite number of degrees of freedom. The beam is divided into n intervals and a concentrated loading is assumed at the center of gravity of each interval. The solution of the resultant determinantal equation gives the first n modes. The accuracy of the higher modes is, however, very poor; only the first third of the modes and the first half of the frequencies are obtained within the usual engineering accuracy. Carrying along so many useless modes greatly increases the labor involved.

A straightforward accurate method for determining the coupled bending-torsion modes and the frequencies of nonuniform cantilever beams, together with applications of this method, was developed at the NACA Lewis laboratory during 1949 and is presented herein. This method is based on the use of Station Functions as first discussed in reference 11. Incorporated in the method are the advantages of the continuous-function deflections of the Rayleigh-Ritz and Stodola methods together with the advantages of the finite number of degrees of freedom of the influence-coefficient method. When the method is applied to a uniform beam, the first n roots of the resultant determinantal equation are amply accurate for engineering purposes.

The final determinantal equation is solved herein by matrix-iteration methods (reference 4). Any other convenient method may, however, be used and no knowledge of matrix algebra is needed to carry out the calculations by the matrix method. The work can be done by an inexperienced computer, as the only operations necessary for determining each mode are cumulative multiplication and division. In addition, for the case in which the coupling coefficient remains constant along the beam, a simple quadratic

[1] Supersedes NACA TN 2185, "Analytical Determination of Coupled Bending-Torsion Vibrations of Cantilever Beams by Means of Station Functions" by Alexander Mendelson and Selwyn Gendler, 1950.

formula and a series of curves are presented for determining the first coupled mode in terms of the uncoupled modes. Examples are developed in detail and comparisons with exact theoretical results are included.

THEORY

In the usual influence-coefficient methods for solving dynamical problems, a continuous body having an infinite number of degrees of freedom is replaced by a body having a finite number of degrees of freedom. Two principal assumptions are then made that introduce inaccuracies into the solutions, particularly in the higher modes: (1) The resultant of the inertia loads of all the infinitesimal masses in a finite interval passes through the center of gravity of that interval; and (2) a concentrated load that is the resultant of a distributed load produces the same deflection as the distributed load. An attempt has been made to reduce the error due to the second of these assumptions by the use of weighting matrices (reference 12). Although the accuracy is thereby increased, the effect of the first assumption is still great enough to introduce serious errors (reference 11).

In order to eliminate these assumptions, Rauscher (reference 11) introduced the concept of Station Functions. Instead of assuming the inertia loads to be concentrated at the centers of gravity of the intervals, the inertia loads and, consequently, the deflections are assumed to be continuous functions along the beam. The values of these continuous deflection functions at the reference stations must equal the deflections of the reference stations. The loading on the beam is therefore a continuous function of the deflections of the reference stations. Inasmuch as the deflections of the reference stations can be computed from the loading on the beam, which in turn is available from the deflections, the deflections are therefore obtained as functions of themselves. This procedure gives n homogeneous equations in the n deflections of the reference stations. The resultant determinantal equation has n roots for the frequency; it will be shown that for a uniform beam all these roots are sufficiently accurate for engineering purposes if the deflection functions are properly chosen. (For coupled bending-torsion vibrations, $2n$ homogeneous equations and $2n$ roots are obtained for n stations.)

The deflection functions used must satisfy the boundary conditions of the problem and also the condition that, at any reference station, the value of the function must equal the deflection of the reference station. Although it is always possible to find directly a single function that will satisfy these conditions, it is more convenient to obtain different component functions at each station and to add all these component functions together to give the complete deflection function. Rauscher (reference 11) calls these component deflection functions Station Functions. For example, the complete torsional deflection function for the beam will have the following form:

$$\theta(z)=\sum_{j=1}^{n} f_j(z)\theta_j$$

where

z dimensionless distance along beam

$\theta(z)$ torsional deflection at distance z from root

θ_j torsional deflection at j^{th} station

$f_j(z)$ Station Function in torsion associated with j^{th} station

(All symbols are defined in appendix A.)

Each Station Function must satisfy the boundary conditions of the problem and the following additional conditions: (1) At the reference station with which it is associated, the Station Function equals the deflection of that reference station; and (2) at all other reference stations, the Station Function equals zero. The sum of all these Station Functions will then give the complete deflection function for the beam. The Station Functions and corresponding loading functions are derived in appendix B for torsional vibrations,

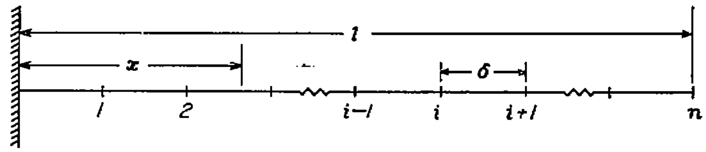


FIGURE 1.—Cantilever beam with n stations.

bending vibrations, and coupled bending-torsion vibrations of an arbitrary cantilever beam.

Torsional vibrations.—It is shown in appendix B that the torsional deflections of the reference stations for a beam divided into n intervals of length δ, as shown in figure 1, are given by the following system of equations:

$$\theta_i=\omega^2\delta^2\frac{I_0}{C_0}\sum_{j=1}^{n}\alpha_{ij}\theta_j \tag{1}$$

where

$$\alpha_{ij}\equiv\sum_{k=1}^{i}\frac{1}{C_k}\left[I_kN_{jk}-(k-1)I_kM_{jk}+\sum_{r=k+1}^{n}I_rM_{jr}\right] \tag{2}$$
$$i \text{ and } j=1,2,\ldots n$$

ω frequency of vibration

δ length of interval

I_0 mass moment of inertia per unit length about elastic axis at root section

I_k ratio of average mass moment of inertia per unit length of k^{th} interval to mass moment of inertia per unit length at root section

C_0 torsional stiffness of root section

C_k ratio of average torsional stiffness of k^{th} interval to torsional stiffness at root section

The Station Numbers N_{jk} and M_{jk} are functions only of the integers k, j, and n and are defined as

$$\left.\begin{aligned}N_{jk}&\equiv\int_{k-1}^{k}zf_j(z)\,dz\\M_{jk}&\equiv\int_{k-1}^{k}f_j(z)\,dz\end{aligned}\right\} \tag{3}$$

where $f_j(z)$ represents the Station Functions derived in appendix B and is given by

$$f_j(z)=a_{1j}z+a_{2j}z^2+\ldots+a_{(n+1)j}z^{(n+1)} \tag{4}$$

The coefficients a_{ij} are determined in appendix B by satisfying the conditions on the Station Functions. The integrals in equations (3) are thus seen to be integrals of simple polynomials and the limits of integration are integers. The Station Numbers N_{jk} and M_{jk} are therefore rational numbers, functions only of the integers n, k, and j. These numbers have been evaluated and are listed in tables I to VIII.

If the physical properties of the beam under consideration are known for each of the n intervals, C_k and I_k will be known. The Station Numbers N_{jk} and M_{jk} can be obtained from tables I to VIII. From equation (2), α_{ij} can then be easily calculated.

Equation (1) actually represents n homogeneous equations in the n unknown deflections θ_i. With $\frac{1}{\omega^2}\frac{C_0}{I_0\delta^2}\equiv\lambda$, these equations can be written as follows:

$$\left.\begin{array}{l}(\alpha_{11}-\lambda)\theta_1+\alpha_{12}\theta_2+\alpha_{13}\theta_3+\ldots+\alpha_{1n}\theta_n=0\\ \alpha_{21}\theta_1+(\alpha_{22}-\lambda)\theta_2+\alpha_{23}\theta_3+\ldots+\alpha_{2n}\theta_n=0\\ \alpha_{31}\theta_1+\alpha_{32}\theta_2+(\alpha_{33}-\lambda)\theta_3+\ldots+\alpha_{3n}\theta_n=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_{n1}\theta_1+\alpha_{n2}\theta_2+\alpha_{n3}\theta_3+\ldots+(\alpha_{nn}-\lambda)\theta_n=0\end{array}\right\} \tag{5}$$

For a nontrivial solution, the determinant of the coefficients must vanish and the characteristic equation becomes

$$\begin{vmatrix}\alpha_{11}-\lambda & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22}-\lambda & \alpha_{23} & \ldots & \alpha_{2n}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}-\lambda & \ldots & \alpha_{3n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \ldots & \alpha_{nn}-\lambda\end{vmatrix}=0 \tag{6}$$

or

$$|\lambda I-[\alpha_{ij}]|=0 \tag{6a}$$

where I is the identity matrix, and $[\alpha_{ij}]$ is the dynamical matrix.

Equation (6) can be solved for the n values of λ by any method available. The method used herein was to obtain the values of λ as the latent roots of the matrix $[\alpha_{ij}]$, which is actually the dynamical matrix for the problem. The mode shapes are obtained at the same time.

Bending vibrations.—The bending deflections for the beam shown in figure 1 are given by the following system of equations (appendix B):

$$y_i=\omega^2\delta^4\frac{m_0}{B_0}\sum_{j=1}^{n}\beta_{ij}y_j \tag{7}$$

where

$$\beta_{ij}\equiv\sum_{k=1}^{i}\frac{1}{B_k}\left(m_k(iP'_{jk}-Q'_{jk})+\sum_{r=k+1}^{n}m_r\left\{\left(i-k+\frac{1}{2}\right)N'_{jr}+\left[\frac{k^3-(k-1)^3}{3}-\frac{(2k-1)i}{2}\right]M'_{jr}\right\}\right) \tag{8}$$

$$i \text{ and } j=1,2,\ldots n$$

m_0 mass per unit length of beam at root section

m_k ratio of average mass per unit length of k^{th} interval to mass per unit length at root section

B_0 bending stiffness at root section

B_k ratio of average bending stiffness of k^{th} interval to bending stiffness at root section

The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are functions only of the integers k, j, and n and are defined by

$$\left.\begin{array}{l}P'_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^2}{2}-(k-1)z+\frac{1}{2}(k-1)^2\right]g_j(z)dz\\ Q'_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^3}{6}-\frac{1}{2}(k-1)^2z+\frac{1}{3}(k-1)^3\right]g_j(z)dz\\ M'_{jk}\equiv\int_{k-1}^{k}g_j(z)dz\\ N'_{jk}\equiv\int_{k-1}^{k}z\,g_j(z)dz\end{array}\right\} \tag{9}$$

The Station Functions $g_j(z)$ are derived in appendix B and are given by

$$g_j(z)=b_{2j}z^2+b_{3j}z^3+b_{4j}z^4+\ldots+b_{(n+3)j}z^{(n+3)} \tag{10}$$

The integrals in equations (9) are thus seen to be integrals of simple polynomials. The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are rational numbers, functions only of the integers j, k, and n. These numbers have been evaluated and are listed in tables I to VIII.

If the physical properties of the beam are known for each of the n intervals, m_k and B_k will be known. The Station Numbers M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} are obtained from tables I to VIII; β_{ij} can then easily be calculated by equation (8).

The determinantal equation is:

$$\begin{vmatrix}\beta_{11}-\lambda & \beta_{12} & \beta_{13} & \ldots & \beta_{1n}\\ \beta_{21} & \beta_{22}-\lambda & \beta_{23} & \ldots & \beta_{2n}\\ \beta_{31} & \beta_{32} & \beta_{33}-\lambda & \ldots & \beta_{3n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \ldots & \beta_{nn}-\lambda\end{vmatrix}=0 \tag{11}$$

or

$$|\lambda I-[\beta_{ij}]|=0 \tag{11a}$$

where

$$\lambda\equiv\frac{B_0}{m_0\delta^4}\frac{1}{\omega^2}$$

The dynamical matrix is $[\beta_{ij}]$.

Coupled bending-torsion vibration.—The torsional and bending deflections due to coupled bending-torsion vibrations of a cantilever beam are given by (appendix B):

$$\left.\begin{array}{l}\theta_i=\omega^2\frac{m_0}{B_0}\delta^4\sum_{j=1}^{n}\left(\Gamma\alpha_{ij}\theta_j+\epsilon\Gamma\gamma_{ij}\frac{y_j}{r_0}\right)\\ \frac{y_i}{r_0}=\omega^2\frac{m_0}{B_0}\delta^4\sum_{j=1}^{n}\left(\delta_{ij}\theta_j+\beta_{ij}\frac{y_j}{r_0}\right)\end{array}\right\} \tag{12}$$

where

$$\epsilon\equiv\frac{r_0^2}{r_{g0}^2}$$

$$\Gamma\equiv\frac{1}{\delta^2}\frac{I_0}{C_0}\frac{B_0}{m_0}$$

r_0 absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction for root section

r_{g0} radius of gyration about elastic axis at root section

The quantities α_{ij} and β_{ij} are defined by equations (2) and (8). The quantities γ_{ij} and δ_{ij} are given by

$$\left.\begin{aligned}\gamma_{ij}&\equiv\sum_{k=1}^{i}\frac{1}{C_k}\left[S_kN'_{jk}-(k-1)S_kM'_{jk}+\sum_{r=k+1}^{n}S_rM'_{jr}\right]\\ \delta_{ij}&\equiv\sum_{k=1}^{i}\frac{1}{B_k}\left(S_k(iP_{jk}-Q_{jk})+\sum_{r=k+1}^{n}S_r\left\{\left(i-k+\frac{1}{2}\right)N_{jr}+\right.\right.\\ &\qquad\left.\left.\left[\frac{k^3-(k-1)^3}{3}-\frac{(2k-1)i}{2}\right]M_{jr}\right\}\right)\end{aligned}\right\}\tag{13}$$

where

$$P_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^2}{2}-(k-1)z+\frac{1}{2}(k-1)^2\right]f_j(z)dz$$

$$Q_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^3}{6}-\frac{1}{2}(k-1)^2z+\frac{1}{3}(k-1)^3\right]f_j(z)dz$$

and S_k is the ratio of the average static mass unbalance of the k^{th} interval to the static mass unbalance at the root section.

The Station Numbers P_{jk} and Q_{jk} are listed in tables I to VIII with the other Station Numbers. The determinantal equation becomes

$$\begin{vmatrix}\Gamma\alpha_{11}-\lambda & \Gamma\alpha_{12} & \ldots & \Gamma\alpha_{1n} & \epsilon\Gamma\gamma_{11} & \epsilon\Gamma\gamma_{12} & \ldots & \epsilon\Gamma\gamma_{1n}\\ \Gamma\alpha_{21} & \Gamma\alpha_{22}-\lambda & \ldots & \Gamma\alpha_{2n} & \epsilon\Gamma\gamma_{21} & \epsilon\Gamma\gamma_{22} & \ldots & \epsilon\Gamma\gamma_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \Gamma\alpha_{n1} & \Gamma\alpha_{n2} & \ldots & \Gamma\alpha_{nn}-\lambda & \epsilon\Gamma\gamma_{n1} & \epsilon\Gamma\gamma_{n2} & \ldots & \epsilon\Gamma\gamma_{nn}\\ \delta_{11} & \delta_{12} & \ldots & \delta_{1n} & \beta_{11}-\lambda & \beta_{12} & \ldots & \beta_{1n}\\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} & \beta_{21} & \beta_{22}-\lambda & \ldots & \beta_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \delta_{n1} & \delta_{n2} & \ldots & \delta_{nn} & \beta_{n1} & \beta_{n2} & \ldots & \beta_{nn}-\lambda\end{vmatrix}=0\tag{14}$$

or

$$|\lambda I-[\eta_{ij}]|=0\tag{14a}$$

where $[\eta_{ij}]$ is the dynamical matrix and I is the identity matrix.

The first n roots of equation (14) will give the first n coupled frequencies.

APPLICATIONS AND RESULTS

In applying the previously discussed method, it is necessary to determine for a given beam the elements α_{ij}, β_{ij}, γ_{ij}, and δ_{ij} of the dynamical matrices. These quantities will depend on the physical properties of the beam and on the number of stations chosen. If the physical properties of the beam are known, the quantities α_{ij}, β_{ij}, γ_{ij}, and δ_{ij} can be directly calculated from equations (2), (8), and (13). The numbers M_{jk}, N_{jk}, P_{jk}, Q_{jk}, M'_{jk}, N'_{jk}, P'_{jk}, and Q'_{jk} appearing in these equations depend on the number of stations n that are used and can be read directly from tables I to VIII for any given number of stations up to eight. Once these quantities have been calculated, equations (6), (11), or (14) can be solved for the frequencies by any method desired. The matrix-iteration method used herein is simple and rapid and requires no particular computing skill. As will be indicated, however, the accuracy of equations (6), (11), and (14) is such that relatively few stations need be used, in which case it may be convenient to expand the determinants and to solve the resultant low-order algebraic equation.

In order to illustrate the accuracy, this method was applied to torsional vibrations, bending vibrations, and coupled vibrations of a uniform cantilever beam. The exact theoretical values for torsional vibrations and bending vibrations of uniform cantilevers are well known. The exact theoretical values for the coupled bending-torsion vibration of a uniform beam were calculated (appendix D). A comparison was then made between the values obtained by the method presented and the exact theoretical values. The number of stations used was 1, 2, and 3 ($n=1$, $n=2$, and $n=3$). The comparisons are summarized in table IX.

Torsional vibration.—For the case of a uniform beam, $C_k=I_k=1$ and equation (2) becomes

$$\alpha_{ij}=\sum_{k=1}^{i}\left[N_{jk}-(k-1)M_{jk}+\sum_{r=k+1}^{n}M_{jr}\right]\tag{15}$$

The values of N_{jk} and M_{jk} are given in tables I to VIII. The table to be used depends on the choice of the number of stations.

Let $n=1$;

$$\therefore\alpha_{11}=N_{11}$$

From table I, $N_{11}=5/12$,

$$\therefore\alpha_{11}=5/12$$

and

$$\theta_1=\frac{5}{12}l^2\frac{I_0}{C_0}\omega^2\theta_1$$

or

$$\omega^2=\frac{12}{5}\frac{C_0}{I_0l^2}=2.400\frac{C_0}{I_0l^2}$$

$$\omega=1.549\sqrt{\frac{C_0}{I_0l^2}}$$

The exact theoretical value for the first torsional frequency is

$$\omega=1.571\sqrt{\frac{C_0}{I_0l^2}}$$

The percentage error is −1.4 when only one station is used.

The mode shape obtained by the method of Station Functions agrees well with the theoretical mode shape, as is shown in figure 2 (a).

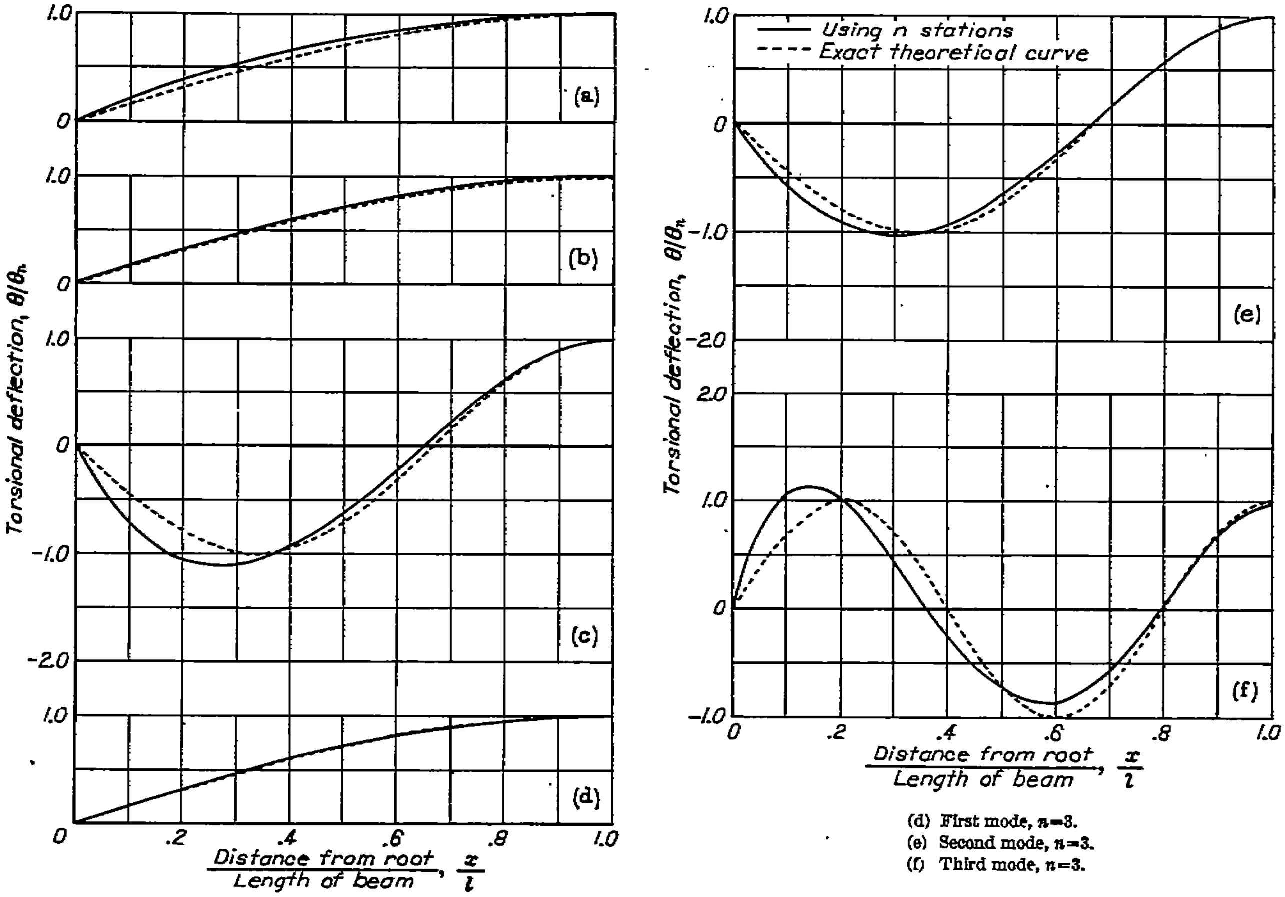
Let $n=2$; then by equation (15) and table II,

$$\alpha_{11}=N_{11}+M_{12}=\frac{8}{15}+\frac{5}{12}=\frac{57}{60}$$

$$\alpha_{12}=N_{21}+M_{22}=-\frac{31}{240}+\frac{29}{48}=\frac{57}{120}$$

$$\alpha_{21}=N_{11}+N_{12}=\frac{8}{15}+\frac{8}{15}=\frac{16}{15}$$

$$\alpha_{22}=N_{21}+N_{22}=-\frac{31}{240}+\frac{239}{240}=\frac{13}{15}$$



(a) First mode, $n=1$.
(b) First mode, $n=2$.
(c) Second mode, $n=2$.

(d) First mode, $n=3$.
(e) Second mode, $n=3$.
(f) Third mode, $n=3$.

FIGURE 2.—Comparison of theoretical mode shapes with mode shapes obtained by taking n stations along the beam for torsional vibrations.

The determinantal equation then becomes

$$\begin{vmatrix} \frac{57}{60}-\lambda & \frac{57}{120} \\ \frac{16}{15} & \frac{13}{15}-\lambda \end{vmatrix}=0$$

which gives

$$\lambda_1=1.6214$$

$$\lambda_2=0.1953$$

Therefore

$$\omega_1=1.571\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_2=4.526\sqrt{\frac{C_0}{I_0l^2}}$$

The exact theoretical values are

$$\omega_1=1.571\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_2=4.712\sqrt{\frac{C_0}{I_0l^2}}$$

The precentage errors of the first two modes, for only two stations, are found to be 0 and −4.

The mode shapes are shown in figures 2 (b) and 2 (c). Agreement of the first mode with the exact theoretical shape is excellent; the second mode agrees fairly well.

Let $n=3$; then by equation (15) and table III,

$$\alpha_{11}=N_{11}+M_{12}+M_{13}=0.945833$$
$$\alpha_{12}=N_{21}+M_{22}+M_{23}=0.958333$$
$$\alpha_{13}=N_{31}+M_{32}+M_{33}=0.520834$$
$$\alpha_{21}=N_{11}+N_{12}+2M_{13}=1.033333$$
$$\alpha_{22}=N_{21}+N_{22}+2M_{23}=1.883333$$
$$\alpha_{23}=N_{31}+N_{32}+2M_{33}=1.011113$$
$$\alpha_{31}=N_{11}+N_{12}+N_{13}=1.012500$$
$$\alpha_{32}=N_{21}+N_{22}+N_{23}=2.025000$$
$$\alpha_{33}=N_{31}+N_{32}+N_{33}=1.387501$$

The determinantal equation is

$$\begin{vmatrix} 0.945833-\lambda & 0.958333 & 0.520834 \\ 1.033333 & 1.883333-\lambda & 1.011113 \\ 1.012500 & 2.025000 & 1.387501-\lambda \end{vmatrix}=0$$

The solutions are

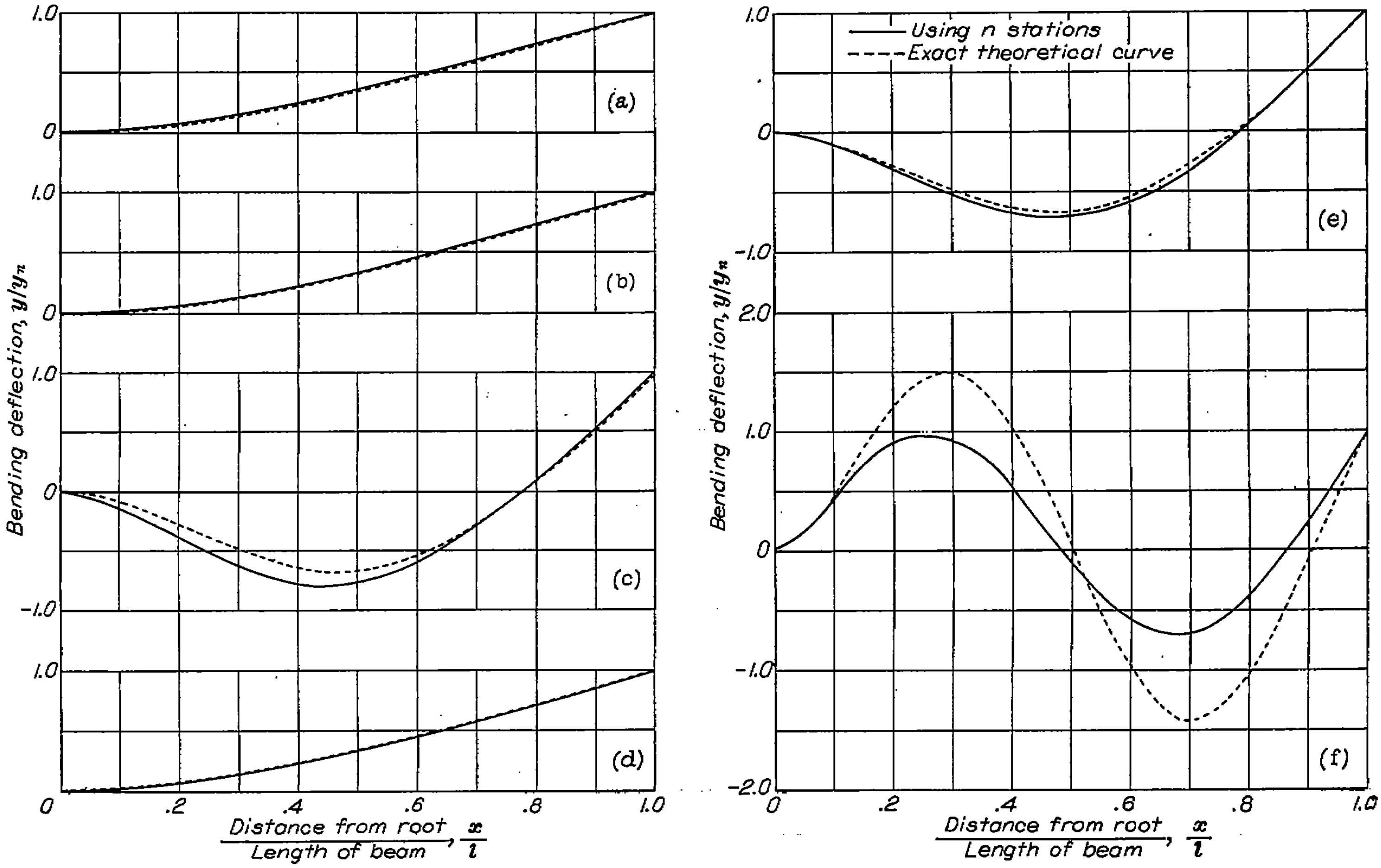
$$\lambda_1=3.6474$$
$$\lambda_2=0.4093$$
$$\lambda_3=0.1599$$

Therefore

$$\omega_1=1.571\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_2=4.689\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_3=7.502\sqrt{\frac{C_0}{I_0l^2}}$$



(a) First mode, $n=1$.
(b) First mode, $n=2$.
(c) Second mode, $n=2$.
(d) First mode, $n=3$.
(e) Second mode, $n=3$.
(f) Third mode, $n=3$.

FIGURE 3.—Comparison of theoretical mode shapes with mode shapes obtained by taking n stations along the beam for bending vibrations.

The exact theoretical values are

$$\omega_1=1.571\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_2=4.712\sqrt{\frac{C_0}{I_0l^2}}$$

$$\omega_3=7.854\sqrt{\frac{C_0}{I_0l^2}}$$

The percentage errors of the first three modes, calculated by use of three stations, are found to be 0, −0.5, and −4.5, respectively.

The mode shapes are shown in figures 2 (d) to 2 (f). The first two modes agree very well with the theoretical shapes; agreement of the third mode is fair.

This procedure can be carried out as shown for any number of stations desired.

Bending vibrations.—For a uniform beam, $B_k=m_k=1$ and equation (8) becomes

$$\beta_{ij}=\sum_{k=1}\left\{iP'_{jk}-Q'_{jk}+\sum_{r=k+1}^{n}\left[\left(i-k+\frac{1}{2}\right)N'_{jr}+\left(\frac{k^3-(k-1)^3}{3}-\frac{(2k-1)}{2}\,i\right)M'_{jr}\right]\right\} \tag{16}$$

Let $n=1$;

$$\therefore \beta_{11}=P'_{11}-Q'_{11}$$

and from table I

$$\beta_{11}=\frac{71}{630}-\frac{31}{1008}=\frac{59}{720}$$

Therefore, from equation (7),

$$\omega=3.493\sqrt{\frac{B_0}{m_0l^4}}$$

The exact theoretical value is

$$\omega=3.516\sqrt{\frac{B_0}{m_0l^4}}$$

The precentage error for just one station is found to be −0.65.

The mode shape is shown in figure 3 (a) and is seen to agree very well with the theoretically exact shape.

Let $n=2$; then by equation (16) and table II,

$$\beta_{11}=P'_{11}-Q'_{11}+\frac{1}{2}N'_{12}-\frac{1}{6}M'_{12}=0.422745$$

$$\beta_{12}=P'_{21}-Q'_{21}+\frac{1}{2}N'_{22}-\frac{1}{6}M'_{22}=0.295925$$

$$\beta_{21}=2P'_{11}+2P'_{12}-Q'_{11}-Q'_{12}+\frac{3}{2}N'_{12}-\frac{2}{3}M'_{12}=1.145167$$

$$\beta_{22}=2P'_{21}+2P'_{22}-Q'_{21}-Q'_{22}+\frac{3}{2}N'_{22}-\frac{2}{3}M'_{22}=0.905530$$

The characteristic equation is

$$\begin{vmatrix} 0.422745-\lambda & 0.295925 \\ 1.145167 & 0.905530-\lambda \end{vmatrix}=0$$

The roots are

$$\lambda_1=1.2943$$

$$\lambda_2=0.0339$$

$$\therefore \omega_1=3.516\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=21.71\sqrt{\frac{B_0}{m_0l^4}}$$

The exact theoretical values are

$$\omega_1=3.516\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=22.04\sqrt{\frac{B_0}{m_0l^4}}$$

The percentage errors for two stations are therefore found to be 0 for the first mode and −1.5 for the second mode. The mode shapes are plotted in figures 3 (b) and 3 (c). The first mode agrees excellently with the theoretically exact shape; the second mode agrees fairly well.

Let $n=3$; then by equation (16) and table III,

$$\beta_{11}=P'_{11}-Q'_{11}+\frac{1}{2}N'_{12}+\frac{1}{2}N'_{13}-\frac{1}{6}M'_{12}-\frac{1}{6}M'_{13}=0.270604$$

$$\beta_{12}=P'_{21}-Q'_{21}+\frac{1}{2}N'_{22}+\frac{1}{2}N'_{23}-\frac{1}{6}M'_{22}-\frac{1}{6}M'_{23}=1.009943$$

$$\beta_{13}=P'_{31}-Q'_{31}+\frac{1}{2}N'_{32}+\frac{1}{2}N'_{33}-\frac{1}{6}M'_{32}-\frac{1}{6}M'_{33}=0.487441$$

$$\beta_{21}=2P'_{11}+2P'_{12}-Q'_{11}-Q'_{12}+\frac{3}{2}N'_{12}+2N'_{13}-\frac{2}{3}M'_{12}-\frac{4}{3}M'_{13}=0.648170$$

$$\beta_{22}=2P'_{21}+2P'_{22}-Q'_{21}-Q'_{22}+\frac{3}{2}N'_{22}+2N'_{23}-\frac{2}{3}M'_{22}-\frac{4}{3}M'_{23}=3.266250$$

$$\beta_{23}=2P'_{31}+2P'_{32}-Q'_{31}-Q'_{32}+\frac{3}{2}N'_{32}+2N'_{33}-\frac{2}{3}M'_{32}-\frac{4}{3}M'_{33}=1.689891$$

$$\beta_{31}=3P'_{11}+3P'_{12}+3P'_{13}-Q'_{11}-Q'_{12}-Q'_{13}+\frac{5}{2}N'_{12}+4N'_{13}-\frac{7}{6}M'_{12}-\frac{10}{3}M'_{13}=0.985135$$

$$\beta_{32}=3P'_{21}+3P'_{22}+3P'_{23}-Q'_{21}-Q'_{22}-Q'_{23}+\frac{5}{2}N'_{22}+4N'_{23}-\frac{7}{6}M'_{22}-\frac{10}{3}M'_{23}=5.822852$$

$$\beta_{33}=3P'_{31}+3P'_{32}+3P'_{33}-Q'_{31}-Q'_{32}-Q'_{33}+\frac{5}{2}N'_{32}+4N'_{33}-\frac{7}{6}M'_{32}-\frac{10}{3}M'_{33}=3.204301$$

The characteristic equation is

$$\begin{vmatrix} 0.270604-\lambda & 1.009943 & 0.487441 \\ 0.648170 & 3.266250-\lambda & 1.689891 \\ 0.985135 & 5.822852 & 3.204301-\lambda \end{vmatrix}=0$$

The roots are

$$\lambda_1=6.5521$$

$$\lambda_2=0.1667$$

$$\lambda_3=0.0223$$

Therefore

$$\omega_1=3.516\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=22.04\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_3=60.20\sqrt{\frac{B_0}{m_0l^4}}$$

The exact values are

$$\omega_1=3.516\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=22.04\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_3=61.70\sqrt{\frac{B_0}{m_0l^4}}$$

The percentage errors for three stations are found to be 0, 0, and −2.4, respectively. The modes are plotted in figures 3 (d) to 3 (f). The first two modes are seen to agree very well with the theoretical mode shape; agreement of the third mode is fair.

Coupled bending-torsion vibrations.—A uniform beam with the following constants was chosen:

$$\gamma=\frac{\omega_t^2}{\omega_b^2}=38.56$$

$$\epsilon=0.8$$

$$\Gamma=\frac{n^2}{193.2}$$

$$\epsilon\Gamma=\frac{n^2}{241.5}$$

The values of α_{ij} and β_{ij} are obtained as previously and are the same as given before for $n=1$, $n=2$, and $n=3$. Also, because $S_k=B_k=C_k=m_k=I_k=1$, equations (13) become

$$\gamma_{ij}=\sum_{k=1}^{i}\left[N'_{jk}-(k-1)M'_{jk}+\sum_{r=k+1}^{n}M'_{jr}\right]$$

$$\delta_{ij}=\sum_{k=1}^{i}\left\{iP_{jk}-Q_{jk}+\sum_{r=k+1}^{n}\left[\left(i-k+\frac{1}{2}\right)N_{jr}+\left(\frac{k^3-(k-1)^3}{3}-\frac{2k-1}{2}\,i\right)M_{jr}\right]\right\}$$

Let $n=1$; then the determinant is

$$\begin{vmatrix} \Gamma\alpha_{11}-\lambda & \epsilon\Gamma\gamma_{11} \\ \delta_{11} & \beta_{11}-\lambda \end{vmatrix} = \begin{vmatrix} 0.002156-\lambda & 0.001196 \\ 0.111111 & 0.081944-\lambda \end{vmatrix} = 0$$

The roots are

$$\lambda_1=0.0837$$

$$\lambda_2=0.0005$$

$$\omega_1=3.46\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=44.7\sqrt{\frac{B_0}{m_0l^4}}$$

The procedure for calculating the exact theoretical values is derived in appendix D. The exact values are

$$\omega_1=3.49\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=20.6\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_3=49.1\sqrt{\frac{B_0}{m_0l^4}}$$

The percentage error for the first mode, calculated by use of one station, is -0.9.

Let $n=2$; then the determinant is

$$\begin{vmatrix} \Gamma\alpha_{11}-\lambda & \Gamma\alpha_{12} & \epsilon\Gamma\gamma_{11} & \epsilon\Gamma\gamma_{12} \\ \Gamma\alpha_{21} & \Gamma\alpha_{22}-\lambda & \epsilon\Gamma\gamma_{21} & \epsilon\Gamma\gamma_{22} \\ \delta_{11} & \delta_{12} & \beta_{11}-\lambda & \beta_{12} \\ \delta_{21} & \delta_{22} & \beta_{21} & \beta_{22}-\lambda \end{vmatrix}=0$$

Substituting the known values and solving for λ give for the first two roots

$$\lambda_1=1.3197$$

$$\lambda_2=0.0412$$

and the frequencies become

$$\omega_1=3.48\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=19.7\sqrt{\frac{B_0}{m_0l^4}}$$

The percentage errors for two stations are -0.3 for the first mode and -4.4 for the second mode.

This procedure can be carried out for any number of stations desired. For three stations, the frequencies obtained are

$$\omega_1=3.48\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_2=20.6\sqrt{\frac{B_0}{m_0l^4}}$$

$$\omega_3=48.2\sqrt{\frac{B_0}{m_0l^4}}$$

The precentage errors are -0.3 for the first mode, 0 for the second mode, and -1.8 for the third mode.

The results obtained by the method presented are seen to agree very well with the exact theoretical values.

These results are summarized in table IX, where a comparison is made with the results obtained for uncoupled bending and torsional vibrations by use of influence coefficients with weighted matrices (reference 12). The values using weighted matrices were taken from table I of reference 12. It can be seen that for a given number of stations, the results obtained by the method presented herein are considerably better than those obtained by using influence coefficients with weighted matrices. In general, it is indicated that for a uniform cantilever beam using n stations along the beam, the first $n-1$ frequencies and modes are in excellent agreement with exact theoretical values and even the n^{th} mode is given within the accuracy with which the physical properties of the material are known. For a tapered beam, more stations may be required, depending on the amount of taper. The number of stations required to give satisfactory accuracy is listed in table X. A comparison is made by using weighted influence coefficients; the values are taken from table II of reference 12.

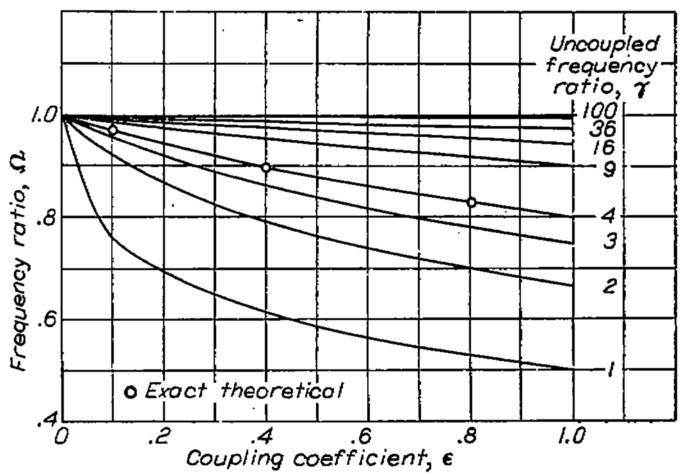


FIGURE 4.—Variation of frequency ratio Ω with coupling coefficient ϵ for several values of uncoupled frequency ratio γ.

The first vibrational frequency is given approximately by equation (C2) (appendix C) when coupling exists between bending and torsion; it is plotted in figure 4. In order to check these curves, the exact solution was obtained (appendix D) for the ratio $(\omega_t/\omega_b)^2$ equal to 4 and was plotted on the same figure. The values given by equation (C2) are seen to be in excellent agreement with the theoretically exact values.

The effect of the coupling between bending and torsion is to reduce the first natural frequency below that which would exist if there were no coupling. This effect is shown in figure 4, wherein the value of Ω is always less than 1. This decrease in the first natural frequency due to coupling is, however, relatively unimportant in the practical range of $(\omega_t/\omega_b)^2>4$ and $\epsilon<0.75$.

SUMMARY OF RESULTS

A method based on the use of Station Functions is presented for calculating uncoupled and coupled bending-torsion modes and frequencies of arbitrary continuous cantilever beams. The results of calculations made by this method indicated that by the use of Station Functions derived herein, n modes and frequencies can be obtained with sufficient accuracy by using just n stations along the beam if the beam is uniform. For a tapered beam, more stations may be required, depending on the amount of taper. The amount of computational labor is markedly less than for other methods. The use of Station Numbers tabulated herein further reduces the amount of calculation necessary. The effect of coupling between bending and torsion is shown to reduce the first natural frequency to a value below that which it would have if there were no coupling.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, *October 18, 1949.*

APPENDIX A

SYMBOLS

The following symbols are used in this report:

a_{ij}	coefficient in equation for Station Function in torsion
B	bending stiffness of beam, function of z
B_0	bending stiffness at root section of beam
B_k	ratio of average bending stiffness of k^{th} interval to bending stiffness of root section
b_{ij}	coefficient in equation for Station Function in bending
C	torsional stiffness of beam, function of z
C_0	torsional stiffness of root section of beam
C_k	ratio of average torsional stiffness of k^{th} interval to torsional stiffness at root section
c_1, c_2, c_3	constants defined in appendix B
$f_j(z)$	Station Function in torsion for j^{th} station (defined in text)
$g_j(z)$	Station Function in bending for j^{th} station (defined in text)
I	mass moment of inertia per unit length of beam about elastic axis, function of z, except where otherwise defined
I_0	mass moment of inertia per unit length of beam about elastic axis at root section
I_k	ratio of average mass moment of inertia per unit length of k^{th} interval to mass moment of inertia per unit length at root section
i,j,k,n	station indices
j,k,r	summation indices
l	length of beam
$M_{jk}, N_{jk}, P_{jk}, Q_{jk}, M'_{jk}, N'_{jk}, P'_{jk}, Q'_{jk}$	Station Numbers (defined in text); function of indices j, k, and n
m	mass per unit length of beam, function of z
m_0	mass per unit length of beam at root section
m_k	ratio of average mass per unit length of k^{th} interval to mass per unit length at root section
n	number of stations along beam
$q_b(z)$	bending loading function on beam
$q_t(z)$	torsional loading function on beam
r	absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction
r_{g0}	radius of gyration about elastic axis at root section
r_0	absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction for root section
S	static mass unbalance, function of z, mr
S_0	static mass unbalance at root section, m_0r_0
S_k	ratio of average of static mass unbalance at k^{th} section to static mass unbalance at root section
x	distance from root of beam, except where otherwise defined
y	bending deflection, function of z
y_i	bending deflection at i^{th} station
z	dimensionless distance along beam, x/δ
$\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \eta_{ij}$	elements of dynamical matrix defined in text
Γ	$\frac{1}{\delta^2}\frac{I_0}{C_0}\frac{B_0}{m_0}$
γ	uncoupled frequency ratio, $(\omega_t/\omega_b)^2$
δ	length of interval along beam between two stations
ϵ	coupling coefficient, $(r_0/r_{g0})^2$
θ	torsional deflection, function of z
θ_i	torsional deflection at i^{th} station
λ	root of frequency equation or characteristic root of dynamical matrix
Ω	frequency ratio, $(\omega/\omega_b)^2$
ω	frequency of vibration
ω_b	frequency of uncoupled fundamental bending mode
ω_t	frequency of uncoupled fundamental torsional mode
$\ddot{}$	second derivative of deflection with respect to time

APPENDIX B

STATION FUNCTIONS AND DETERMINANTAL EQUATIONS

TORSIONAL VIBRATIONS

A schematic diagram of a cantilever beam divided into n intervals of length δ is shown in figure 1. The Station Functions for the torsional vibrations of such a beam must satisfy the following conditions:

At

$$z=0 \quad f_i(0)=0 \tag{B1}$$
$$z=n \quad f'_i(n)=0 \tag{B2}$$
$$z=i \quad f_i(i)=1 \tag{B3}$$
$$z=j \quad f_i(j)=0 \quad j\neq i \tag{B4}$$

where $f'(z)$ denotes the derivative with respect to z.

Equations (B1) and (B2) represent the boundary conditions that must be satisfied by a cantilever beam vibrating in torsion; equations (B3) and (B4) represent the further conditions imposed upon the Station Functions. These conditions will be satisfied by a function of the type

$$f_i(z)=a_{1i}z+a_{2i}z^2+\ .\ .\ .\ +a_{(n+1)i}z^{(n+1)} \tag{B5}$$

where the coefficients a_{ij} must satisfy the following simultaneous equations obtained from conditions (B2), (B3), and (B4):

$$0=a_{1i}+2na_{2i}+3n^2a_{3i}+\ .\ .\ .\ +(n+1)n^na_{(n+1)i} \tag{B2a}$$

$$1=ia_{1i}+i^2a_{2i}+i^3a_{3i}+\ .\ .\ .\ +i^{(n+1)}a_{(n+1)i} \tag{B3a}$$

$$0=ja_{1i}+j^2a_{2i}+j^3a_{3i}+\ .\ .\ .\ +j^{(n+1)}a_{(n+1)i}\quad j\neq i \tag{B4a}$$

The coefficients a_{ij} can be obtained by solving equations (B2a) to (B4a) and the functions $f_i(z)$ determined for each station. Equation (B5), however, can also be written in the following form:

$$f_i(z)=\frac{\prod_{j\neq i}(z-j)z(z-c_1)}{\prod_{j\neq i}(i-j)i(i-c_1)} \tag{B5a}$$

where $\prod_{j\neq i}$ represents the product for all values of j except $j=i$. The function in equation (B5a) obviously satisfies conditions (B1), (B3), and (B4) because it has zeros at all points specified by equation (B4), it equals 1 at the point specified by equation (B3), and it equals zero at the point specified by equation (B1). In order to satisfy condition (B2), the constant c_1 is determined by substitution of equation (B5a) into equation (B2).

$$c_1=n \text{ for } i\neq n$$

$$c_1=n\left(1+\frac{1}{1+\sum_{j\neq n}\frac{n}{n-j}}\right) \text{ for } i=n$$

Equation (B5) can be obtained from equation (B5a) by carrying out the indicated multiplications. The complete deflection function is then given by

$$\theta(z)=f_1(z)\theta_1+f_2(z)\theta_2+\ .\ .\ .\ +f_n(z)\theta_n$$
$$=\sum_{j=1}^{n}f_j(z)\theta_j \tag{B6}$$

The continuous loading function $q_t(z)$ can now be written as

$$q_t(x)=I\omega^2\theta(z)=I\omega^2\sum_{j=1}^{n}f_j(z)\theta_j \tag{B7}$$

A continuous loading function, which is a function of the deflections at the reference stations, has thus been obtained.

BENDING VIBRATIONS

The Station Functions for the bending vibrations of the beam shown in figure 1 must satisfy the following conditions: at

$$z=0 \quad g_i(0)=0 \tag{B8}$$
$$z=0 \quad g'_i(0)=0 \tag{B9}$$
$$z=n \quad g''_i(n)=0 \tag{B10}$$
$$z=n \quad g'''_i(n)=0 \tag{B11}$$
$$z=i \quad g_i(i)=1 \tag{B12}$$
$$z=j \quad g_i(j)=0 \quad j\neq i \tag{B13}$$

where $g'(z)$, $g''(z)$, and $g'''(z)$ denote the first, second, and third derivatives, respectively, of $g(z)$ with respect to z.

Equations (B8) to (B11) represent the boundary conditions that must be satisfied by a cantilever beam vibrating in bending and equations (B12) and (B13) represent the additional conditions imposed upon the Station Functions. These conditions will be satisfied by functions of the type

$$g_i(z)=b_{2i}z^2+b_{3i}z^3+\ldots+b_{(n+3)i}z^{(n+3)} \tag{B14}$$

where the coefficients b_{ij} must satisfy the following equations obtained from conditions (B10) to (B13):

$$0=2b_{2i}+6n\,b_{3i}+\ldots+(n+3)(n+2)n^{(n+1)}b_{(n+3)i} \tag{B10a}$$

$$0=6b_{3i}+24n\,b_{4i}+\ldots+(n+3)(n+2)(n+1)n^nb_{(n+3)i} \tag{B11a}$$

$$1=i^2b_{2i}+i^3b_{3i}+\ldots+i^{(n+3)}b_{(n+3)i} \tag{B12a}$$

$$0=j^2b_{2i}+j^3b_{3i}+\ldots+j^{(n+3)}b_{(n+3)i}\quad j\neq i \tag{B13a}$$

The coefficients can therefore be obtained from equations (B10a) to (B13a) and the functions $g_i(z)$ determined for each station i. Equation (B14) can, however, be written in the following form:

$$g_i(z)=\frac{\prod\limits_{j\neq i}(z-j)z^2(z^2+c_2z+c_3)}{\prod\limits_{j\neq i}(i-j)i^2(i^2+c_2i+c_3)} \tag{B14a}$$

where $\prod\limits_{j\neq i}$ represents the product for all values of j except $j=i$. The function in equation (B14a) obviously satisfies conditions (B8), (B9), (B12), and (B13), because it has zeros at all points specified by conditions (B8), (B9), and (B13) and equals 1 at the point specified by equation (B12). In order to satisfy conditions (B10) and (B11), the constants c_2 and c_3 are determined by substitution of equation (B14a) into equations (B10) and (B11). The general forms for c_2 and c_3 are, however, complicated and it is easier to obtain the numerical values of these constants for each specific case. Equation (B14) can then be obtained from equation (B14a) by carrying out the indicated multiplications. The complete deflection function is then given by

$$y(z)=\sum_{j=1}^{n}g_j(z)y_j \tag{B15}$$

The continuous bending loading function $q_b(z)$ can now be written as

$$q_b(z)=m\omega^2y(z)=m\omega^2\sum_{j\neq i}^{n}g_j(z)y_j \tag{B16}$$

COUPLED BENDING-TORSION VIBRATIONS

The Station Functions for the coupled bending-torsion vibrations are the same as previously given for the bending vibrations and the torsion vibrations. The loading functions, however, are given as follows (reference 7):

$$\begin{aligned}q_t(z)&=I\omega^2\theta(z)+S\omega^2y(z)\\&=\omega^2\sum_{j=1}^{n}[If_j(z)\theta_j+Sg_j(z)y_j]\end{aligned} \tag{B17}$$

and

$$\begin{aligned}q_b(z)&=S\omega^2\theta(z)+m\omega^2y(z)\\&=\omega^2\sum_{j=1}^{n}[Sf_j(z)\theta_j+mg_j(z)y_j]\end{aligned} \tag{B18}$$

DETERMINANTAL EQUATIONS AND DYNAMICAL MATRICES

Once the Station Functions and the corresponding loading functions have been determined, the deflections at the reference stations can be obtained in terms of the loading function. A homogeneous equation in the reference-station deflections for each station is thereby obtained. The determinant of the coefficients of the resultant set of homogeneous equations can be set equal to zero; the determinantal frequency equation is thus derived. The deflections at the reference stations are obtained by the well-known equations for obtaining influence coefficients.

Torsion.—The deflection at the station i due to the continuous loading $q_t(z)$ on the beam is given by

$$\theta_i=\delta^2\int_0^i q_t(z)\int_0^z\frac{dz_1}{C}\,dz+\delta^2\int_i^n q_t(z)\int_0^i\frac{dz_1}{C}\,dz \tag{B19}$$

If C is assumed to have a constant value for each interval, these integrals may be written as the sum of integrals over each section. Equation (B19) then becomes

$$\theta_i=\frac{\delta^2}{C_0}\sum_{k=1}^{i}\frac{1}{C_k}\left[\int_{k-1}^{k}zq_t(z)\,dz+\int_{k-1}^{k}(1-k)q_t(z)\,dz+\int_k^n q_t(z)\,dz\right] \tag{B20}$$

By substituting the relation

$$q_t(z)=\omega^2I\sum_{j=1}^{n}f_j(z)\theta_j$$

and by assuming a constant value for I for each interval and changing the summation order,

$$\theta_i=\omega^2\delta^2\frac{I_0}{C_0}\sum_{j=1}^{n}\left\{\sum_{k=1}^{i}\frac{1}{C_k}\left[I_k\int_{k-1}^{k}zf_j(z)\,dz-(k-1)I_k\int_{k-1}^{k}f_j(z)dz+\sum_{r=k+1}^{n}I_r\int_{r-1}^{r}f_j(z)\,dz\right]\right\}\theta_j \tag{B21}$$

Let

$$\left.\begin{aligned}\int_{k-1}^{k}zf_j(z)\,dz&\equiv N_{jk}\\\int_{k-1}^{k}f_j(z)\,dz&\equiv M_{jk}\end{aligned}\right\} \tag{B22}$$

Then

$$\theta_i=\omega^2\frac{I_0}{C_0}\delta^2\sum_{j=1}^{n}\alpha_{ij}\theta_j \tag{B23}$$

where

$$\alpha_{ij}\equiv\sum_{k=1}^{i}\frac{1}{C_k}\left[I_kN_{jk}-(k-1)I_kM_{jk}+\sum_{r=k+1}^{n}I_rM_{jr}\right] \tag{B24}$$

If $C_k=I_k=1$ (constant cross section), then

$$\alpha_{ij}=\sum_{k=1}^{i}\left[N_{jk}-(k-1)M_{jk}+\sum_{r=k+1}^{n}M_{jr}\right] \tag{B25}$$

Let

$$\lambda\equiv\frac{C_0}{I_0\omega^2\delta^2} \tag{B26}$$

Then

$$\lambda\theta_i=\sum_{j=1}^{n}\alpha_{ij}\theta_j \tag{B23a}$$

and the characteristic equation is

$$|[\alpha_{ij}]-\lambda I|=0 \tag{B27}$$

where I is the identity matrix.

Bending.—The deflection at the station i due to the continuous loading $q_b(z)$ on the beam will be given by

$$y_i=\delta^4\int_0^i q_b(z)\int_0^z\frac{(z-z_1)(i-z_1)}{B}\,dz_1\,dz+\delta^4\int_i^n q_b(z)\int_0^i\frac{(z-z_1)(i-z_1)}{B}\,dz_1\,dz \tag{B28}$$

If B is assumed to have a constant value for each interval, these integrals may be written as the sum of integrals over each interval. Equation (B28) then becomes

$$y_i=\frac{\delta^4}{B_0}\sum_{k=1}^{i}\frac{1}{B_k}\Big\{i\int_{k-1}^{k}\left[\frac{z^2}{2}-(k-1)z+\frac{1}{2}(k-1)^2\right]q_b(z)dz-\int_{k-1}^{k}\left[\frac{z^3}{6}-\frac{1}{2}(k-1)^2z+\frac{1}{3}(k-1)^3\right]q_b(z)dz+i\int_{k}^{n}\left[z-\frac{1}{2}(2k-1)\right]q_b(z)dz+\int_{k}^{n}\left[\frac{1}{2}(2k-1)z-\frac{k^3-(k-1)^3}{3}\right]q_b(z)dz\Big\} \tag{B29}$$

By substituting the relation

$$q_b(z)=\omega^2 m\sum_{j=1}^{n}g_j(z)y_j \tag{B30}$$

and by assuming a constant average value for m in each interval and changing the summation order,

$$y_i=\frac{\omega^2 m_0\delta^4}{B_0}\sum_{j=1}^{n}\beta_{ij}y_j \tag{B31}$$

where

$$\beta_{ij}\equiv\sum_{k=1}^{i}\frac{1}{B_k}\Big\{m_k(iP'_{jk}-Q'_{jk})+\sum_{r=k+1}^{n}m_r\Big[\Big(i-k+\frac{1}{2}\Big)N'_{jr}+\Big(\frac{k^3-(k-1)^3}{3}-\frac{(2k-1)}{2}i\Big)M'_{jr}\Big]\Big\} \tag{B32}$$

$$\left.\begin{aligned}
P'_{jk}&\equiv\int_{k-1}^{k}\left[\frac{z^2}{2}-(k-1)z+\frac{1}{2}(k-1)^2\right]g_j(z)\,dz\\
Q'_{jk}&\equiv\int_{k-1}^{k}\left[\frac{z^3}{6}-\frac{1}{2}(k-1)^2z+\frac{1}{3}(k-1)^3\right]g_j(z)\,dz\\
N'_{jk}&\equiv\int_{k-1}^{k}z\,g_j(z)\,dz\\
M'_{jk}&\equiv\int_{k-1}^{k}g_j(z)\,dz
\end{aligned}\right\} \tag{B33}$$

For a uniform beam, $m_k=B_k=1$ and equation (B32) becomes

$$\beta_{ij}=\sum_{k=1}^{i}\Big(iP'_{jk}-Q'_{jk}+\sum_{r=k+1}^{n}\Big\{\Big(i-k+\frac{1}{2}\Big)N'_{jr}+\Big[\frac{k^3-(k-1)^3}{3}-\frac{(2k-1)}{2}i\Big]M'_{jr}\Big\}\Big) \tag{B32a}$$

Let

$$\lambda\equiv\frac{B_0}{\omega^2\delta^4 m_0} \tag{B34}$$

then the characteristic equation becomes

$$|[\beta_{ij}]-\lambda I|=0 \tag{B35}$$

where I is the identity matrix and β_{ij} is the dynamical matrix. In expanded form, equation (B35) becomes

$$\begin{vmatrix}\beta_{11}-\lambda & \beta_{12} & \dots & \beta_{1n}\\ \beta_{21} & \beta_{22}-\lambda & \dots & \beta_{2n}\\ \dots & \dots & \dots & \dots\\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn}-\lambda\end{vmatrix}=0 \tag{B35a}$$

where λ is a latent root of the matrix $[\beta_{ij}]$.

Coupled bending-torsion vibrations.—The deflections at station i are given as before by equations (B19) and (B28). The loading functions q_t and q_b are changed as follows:

$$\left.\begin{aligned}q_t(z)&=\omega^2[I\,\theta(z)+S\,y(z)]\\ q_b(z)&=\omega^2[S\,\theta(z)+m\,y(z)]\end{aligned}\right\} \tag{B36}$$

If these two equations are substituted into equations (B19) and (B28) and the integrations are performed as previously, the following relation is obtained:

$$\left.\begin{aligned}\theta_i&=\frac{\omega^2 m_0\delta^4}{B_0}\sum_{j=1}^{n}\Big(\Gamma\alpha_{ij}\theta_j+\epsilon\Gamma\gamma_{ij}\frac{y_j}{r_0}\Big)\\ \frac{y_i}{r_0}&=\frac{\omega^2 m_0\delta^4}{B_0}\sum_{j=1}^{n}\Big(\delta_{ij}\theta_j+\beta_{ij}\frac{y_j}{r_0}\Big)\end{aligned}\right\} \tag{B37}$$

where α_{ij} and β_{ij} are given in equations (B24) and (B32) and

$$\epsilon\equiv\frac{r_0^2}{r_{g0}^2}$$

$$\Gamma\equiv\frac{1}{\delta^2}\frac{I_0B_0}{C_0m_0}$$

$$\gamma_{ij}\equiv\sum_{k=1}^{i}\frac{1}{C_k}\Big[S_kN'_{jk}-(k-1)S_kM'_{jk}+\sum_{k=k+1}^{n}S_rM'_{jr}\Big]$$

$$\delta_{ij}\equiv\sum_{k=1}^{i}\frac{1}{B_k}\Big\{S_k[i\,P_{jk}-Q_{jk}]+\sum_{r=k+1}^{n}S_r\Big[\Big(i-k+\frac{1}{2}\Big)N_{jr}+\Big(\frac{k^3-(k-1)^3}{3}-\frac{2k-1}{2}i\Big)M_{jr}\Big]\Big\} \tag{B38}$$

where

$$P_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^2}{2}-(k-1)z+\frac{1}{2}(k-1)^2\right]f_j(z)\,dz$$

$$Q_{jk}\equiv\int_{k-1}^{k}\left[\frac{z^3}{6}-\frac{1}{2}(k-1)^2z+\frac{1}{3}(k-1)^3\right]f_j(z)\,dz$$

the determinantal equation therefore is

$$|\lambda I-[\eta_{ij}]|=0$$

where $[\eta_{ij}]$ is the dynamical matrix, the elements of which are as indicated in equation (B37). The matrix $[\eta_{ij}]$ is seen to be a $2n\times 2n$ matrix.

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APPENDIX C

QUADRATIC FORMULA FOR FIRST COUPLED MODE

If only the first vibrational mode is desired, it is possible to obtain this mode approximately by coupling together the fundamental uncoupled bending mode with the fundamental uncoupled torsional mode to obtain a simple quadratic equation for the first coupled frequency. This equation is valid when the coupling coefficient ϵ is constant along the beam. The differential equations obtained by coupling the fundamental uncoupled torsional mode with the fundamental uncoupled bending mode are:

$$\left.\begin{aligned} m\ddot{y}+S\ddot{\theta}+m\omega_b^2 y=0 \\ S\ddot{y}+I\ddot{\theta}+I\omega_t^2\theta=0 \end{aligned}\right\} \qquad \text{(C1)}$$

where

m mass per unit length of beam, function of z
S static mass unbalance, function of z
I mass moment of inertia about elastic axis, function of z
ω_b frequency of uncoupled fundamental bending mode
ω_t frequency of uncoupled fundamental torsional mode
.. denotes differentiation twice with respect to time

These equations lead to a quadratic equation in the frequency ratio Ω, whose solution for the lowest frequency, provided ϵ is constant along the beam, is

$$\Omega\equiv\frac{\omega^2}{\omega_b^2}=\frac{1-\gamma}{2(1-\epsilon)}\left[1-\sqrt{1-\frac{4\gamma(1-\epsilon)}{(1-\gamma)^2}}\right] \qquad \text{(C2)}$$

where

Ω frequency ratio, $(\omega/\omega_b)^2$
γ uncoupled frequency ratio, $(\omega_t/\omega_b)^2$
ϵ coupling coefficient, $(r/r_g)^2$

This quadratic has been plotted in figure 4 for values of ϵ ranging from 0 to 1 and values of $\gamma=(\omega_t/\omega_b)^2$ from 1 to 100.

APPENDIX D

EXACT SOLUTION FOR COUPLED BENDING-TORSION VIBRATIONS OF UNIFORM CANTILEVER BEAM

The differential equations for the equilibrium of an element of a beam vibrating in coupled bending-torsion vibrations can be put in the following dimensionless form:

$$\left.\begin{aligned} \frac{d^4Y_1}{dx^4}&=\frac{ml^4}{B}\omega^2Y_1+\frac{ml^4}{B}\omega^2Y_2 \\ \frac{d^2Y_2}{dx^2}&=-\epsilon\frac{Il^2}{C}\omega^2Y_1-\frac{Il^2}{C}\omega^2Y_2 \end{aligned}\right\} \qquad \text{(D1)}$$

where

$$Y_1\equiv y/r$$

$$Y_2\equiv\theta$$

$$x\equiv\frac{\text{distance from root}}{l}$$

$$\epsilon=(r/r_g)^2$$

Now

$$\omega_b^2=\frac{c_4B}{ml^4}$$

$$\omega_t^2=c_5\frac{C}{Il^2}$$

where

$$c_4=12.36$$

$$c_5=2.467$$

Equations (D1) become

$$\left.\begin{aligned} \frac{d^4Y_1}{dx^4}&=c_4\Omega(Y_1+Y_2) \\ \frac{d^2Y_2}{dx^2}&=-\epsilon\frac{c_5\Omega}{\gamma}Y_1-\frac{c_5\Omega}{\gamma}Y_2 \end{aligned}\right\} \qquad \text{(D2)}$$

where

$$\Omega\equiv(\omega/\omega_b)^2$$

$$\gamma\equiv(\omega_t/\omega_b)^2$$

Let

$$\left.\begin{aligned} \frac{dY_1}{dx}&=Y_3 \\ \frac{dY_3}{dx}&=Y_4 \\ \frac{dY_4}{dx}&=Y_5 \\ \frac{dY_2}{dx}&=Y_6 \\ \text{Then}\quad \frac{dY_5}{dx}&=c_4\Omega(Y_1+Y_2) \\ \frac{dY_6}{dx}&=-\frac{c_5\Omega}{\gamma}(\epsilon Y_1+Y_2) \end{aligned}\right\} \qquad \text{(D3)}$$

Equation (D3) can be written as the single matrix equation

$$\frac{d}{dx}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c_4\Omega & c_4\Omega & 0 & 0 & 0 & 0 \\ \frac{-\epsilon c_5\Omega}{\gamma} & \frac{-c_5\Omega}{\gamma} & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} \qquad \text{(D4)}$$

or

$$\frac{dY}{dx}=AY \tag{D4a}$$

where Y and A are the matrices indicated.

The solution to the matrix equation (D4) is given by

$$Y=e^{Ax}Y_0 \tag{D5}$$

where Y_0 is a column of arbitrary constants.

From the boundary conditions

$$\text{at}\quad x=0 \qquad Y_1=Y_2=Y_3=0$$

$$x=1 \qquad Y_4=Y_5=Y_6=0$$

$$Y_0=Y(0)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ Y_4(0) \\ Y_5(0) \\ Y_6(0) \end{bmatrix}$$

If then Ω_{ij} is an element of the matrizant e^A, the boundary conditions give

$$\begin{vmatrix} \Omega_{44} & \Omega_{45} & \Omega_{46} \\ \Omega_{54} & \Omega_{55} & \Omega_{56} \\ \Omega_{64} & \Omega_{65} & \Omega_{66} \end{vmatrix}=0 \tag{D6}$$

Equation (D6) is the frequency equation. It has an infinite number of roots for ω.

In order to determine the elements Ω_{ij}, e^A must be evaluated. Use will be made of Sylvester's theorem (reference 13).

The λ matrix of A is

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 \\ c_4\Omega & c_4\Omega & 0 & 0 & -\lambda & 0 \\ -\frac{c_5\Omega}{\gamma}\epsilon & -\frac{c_5\Omega}{\gamma} & 0 & 0 & 0 & -\lambda \end{bmatrix}.$$

The characteristic equation $\Delta(\lambda)=0$ is

$$\lambda^6+\frac{c_5\Omega}{\gamma}\lambda^4-c_4\Omega\lambda^2-(1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma}=0 \tag{D7}$$

Equation (D7) is a cubic equation in λ^2. Let the roots be

$$\lambda_1, -\lambda_1,\ \lambda_2, -\lambda_2,\ \lambda_3, -\lambda_3$$

Then by the confluent form of Sylvester's theorem,

$$e^A=\sum_{i=1}^{r}\frac{1}{(\alpha_i-1)!}\frac{d^{\alpha_i-1}}{d\lambda^{\alpha_i-1}}\left[\frac{e^{\lambda}F(\lambda)}{\prod\limits_{k\neq i}(\lambda-\lambda_k)^{\alpha_k}}\right]_{\lambda=\lambda_i} \tag{D8}$$

where $F(\lambda)$ is the adjoint matrix, r is the number of distinct roots, and α_i is the multiplicity of the i^{th} root.

If the roots are all distinct, this relation becomes

$$e^A=\sum_{i=1}^{3}\frac{e^{\lambda_i}F(\lambda_i)-e^{-\lambda_i}F(-\lambda_i)}{2\lambda_i\prod\limits_{j\neq i}(\lambda_i-\lambda_j)(\lambda_i+\lambda_j)} \tag{D9}$$

where the adjoint matrix $F(\lambda)$ is given by

$$F(\lambda)=-\begin{bmatrix} \lambda^5+\frac{c_5\Omega}{\gamma}\lambda^3 & c_4\Omega\lambda & \lambda^4+\frac{c_5\Omega}{\gamma}\lambda^2 & \lambda^3+\frac{c_5\Omega}{\gamma}\lambda & \lambda^2+c_5\frac{\Omega}{\gamma} & c_4\Omega \\ -\epsilon c_5\frac{\Omega}{\gamma}\lambda^3 & \lambda^5-c_4\Omega\lambda & -\epsilon\frac{c_5\Omega}{\gamma}\lambda^2 & -\epsilon\frac{c_5\Omega}{\gamma}\lambda & -\epsilon\frac{c_5\Omega}{\gamma} & \lambda^4-c_4\Omega \\ c_4\Omega\lambda^2+(1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma} & c_4\Omega\lambda^2 & \lambda^5+\frac{c_5\Omega}{\gamma}\lambda^3 & \lambda^4+\frac{c_5\Omega}{\gamma}\lambda^2 & \lambda^3+\frac{c_5\Omega}{\gamma}\lambda & c_4\Omega\lambda \\ c_4\Omega\lambda^3+(1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma}\lambda & c_4\Omega\lambda^3 & c_4\Omega\lambda^2+(1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma} & \lambda^5+\frac{c_5\Omega}{\gamma}\lambda^3 & \lambda^4+\frac{c_5\Omega}{\gamma}\lambda^2 & c_4\Omega\lambda^2 \\ c_4\Omega\lambda^4+(1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma}\lambda^2 & c_4\Omega\lambda^4 & c_4\Omega\lambda^3+(1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma}\lambda & c_4\Omega\lambda^2+(1-\epsilon)\frac{c_4c_5}{\gamma}\Omega^2 & \lambda^5+\frac{c_5\Omega}{\gamma}\lambda^3 & c_4\Omega\lambda^3 \\ -\frac{\epsilon c_5\Omega}{\gamma}\lambda^4 & \frac{c_5\Omega}{\gamma}\lambda^4+(1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma} & -\epsilon\frac{c_5\Omega}{\gamma}\lambda^3 & -\epsilon\frac{c_5\Omega}{\gamma}\lambda^2 & -\epsilon\frac{c_5\Omega}{\gamma}\lambda & \lambda^5-c_4\Omega\lambda \end{bmatrix} \tag{D10}$$

From equations (D9) and (D10), the elements Ω_{ij} are seen to be given by

$$
\left.
\begin{aligned}
\Omega_{44} &= -\sum_{i=1}^{3} \frac{\lambda_i^4 + \frac{c_5\Omega}{\gamma}\lambda_i^2}{\prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \cosh \lambda_i \\
\Omega_{45} &= -\sum_{i=1}^{3} \frac{\lambda_i^4 + \frac{c_5\Omega}{\gamma}\lambda_i^2}{\lambda_i \prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \sinh \lambda_i \\
\Omega_{46} &= -\sum_{i=1}^{3} \frac{c_4\Omega\lambda_i^2}{\lambda_i \prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \sinh \lambda_i \\
\Omega_{54} &= -\sum_{i=1}^{3} \frac{c_4\Omega\lambda_i^2 + \frac{c_4c_5\Omega^2}{\gamma}(1-\epsilon)}{\lambda_i \prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \sinh \lambda_i \\
\Omega_{55} &= \Omega_{44} \\
\Omega_{56} &= -\sum_{i=1}^{3} \frac{c_4\Omega\lambda_i^2}{\prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \cosh \lambda_i \\
\Omega_{64} &= -\sum_{i=1}^{3} \frac{-\epsilon c_5 \frac{\Omega}{\gamma}\lambda_i}{\prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \sinh \lambda_i \\
\Omega_{65} &= -\sum_{i=1}^{3} \frac{-\epsilon c_5 \frac{\Omega}{\gamma}\lambda_i}{\lambda_i \prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \cosh \lambda_i \\
\Omega_{66} &= -\sum_{i=1}^{3} \frac{\lambda_i^4 - c_4\Omega}{\prod\limits_{j\neq i}(\lambda_i^2-\lambda_j^2)} \cosh \lambda_i
\end{aligned}
\right\} \qquad \text{(D11)}
$$

The value of the determinant in equation (D6) must be plotted against the frequency; the value of the frequency for which this determinant becomes zero is thereby obtained. This procedure involves first solving the cubic equation (D7) for each assumed value of frequency parameter and then calculating the elements of the determinant from equations (D11). The process is evidently long and laborious.

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TABLE I—STATION NUMBERS

$n=1$

	j \ k	1
M	1	$\frac{2}{3}$
N		$\frac{5}{12}$
P		$\frac{3}{20}$
Q		$\frac{7}{180}$
M'		$\frac{2}{5}$
N'		$\frac{18}{45}$
P'		$\frac{71}{630}$
Q'		$\frac{31}{1008}$

TABLE II—STATION NUMBERS

$n=2$

	j \ k	1	2
M	1	$\frac{11}{12}$	$\frac{5}{12}$
N		$\frac{8}{15}$	$\frac{8}{15}$
P		0.183333	0.025000
Q		.046032	.029365
M'		.536364	.627273
N'		.367100	.851948
P'		.137933	.057955
Q'		.036616	.069733
M	2	$-\frac{13}{48}$	$\frac{29}{48}$
N		$-\frac{31}{240}$	$\frac{239}{240}$
P		−0.037500	0.143750
Q		−.008135	.181448
M'		−.060795	.448674
N'		−.034875	.758085
P'		−.011252	.118462
Q'		−.002014	.150415

TABLE III—STATION NUMBERS

$n=3$

	$j \backslash k$	1	2	3
M	1	0.950000	0.450000	−0.050000
N		.545833	.587500	−.120833
P		.186310	.032143	−.005357
Q		.046577	.038244	−.011756
M'		.596268	.533205	−.097426
N'		.399646	.708205	−.239994
P'		.148013	.042560	−.012798
Q'		.038884	.050843	−.028318
M	2	−0.525000	0.725000	0.475000
N		−.241667	1.175000	1.091667
P		−.068452	.160714	.031548
Q		−.014583	.202083	.068750
M'		−.149356	.602896	.625418
N'		−.083406	.994860	1.475153
P'		−.026378	.143948	.057937
Q'		−.006034	.181698	.127659
M	3	0.235185	−0.153704	0.568519
N		.106019	−.231944	1.513426
P		.029563	−.023677	.139749
Q		.006222	−.028963	.316408
M'		.040630	−.072928	.445812
N'		.022325	−.111744	1.200133
P'		.006972	−.012081	.118007
Q'		.001579	−.014830	.267865

TABLE IV—STATION NUMBERS

$n=4$

	$j \backslash k$	1	2	3	4
M	1	1.022222	0.429630	−0.061852	0.022222
N		.576455	.557937	−.127249	.076455
P		.194478	.029597	−.006581	.002612
Q		.048240	.035167	−.014547	.008359
M'		.623188	.511862	−.082891	.042276
N'		.413738	.676680	−.203719	.146954
P'		.152256	.039616	−.010651	.005795
Q'		.039818	.047267	−.023551	.018630
M	2	−0.647917	0.747917	0.518750	−0.085417
N		−.292857	1.207143	1.207143	−.292857
P		−.081920	.163021	.041295	−.009598
Q		−.017295	.204828	.090642	−.030688
M'		−.211987	.667412	.544025	−.112648
N'		−.116662	1.091462	1.289193	−.390585
P'		−.036502	.153469	.044508	−.015000
Q'		−.008281	.198310	.097745	−.048203
M	3	0.522222	−0.255556	0.633333	0.522222
N		.229365	−.381746	1.673810	1.729365
P		.062798	−.037401	.148512	.037202
Q		.013040	−.045624	.335791	.118397
M'		.122052	−.166738	.582158	.643846
N'		.065879	−.252823	1.545802	2.164827
P'		.020304	−.026235	.140622	.060554
Q'		.004551	−.032114	.318707	.194016
M	4	−0.221701	0.094850	−0.105961	0.543924
N		−.096544	.140724	−.267841	1.997206
P		−.026267	.013391	−.017322	.136803
Q		−.005428	.016301	−.038574	.446729
M'		−.035456	.042628	−.064723	.438962
N'		−.019023	.064205	−.164169	1.622066
P'		−.005836	.006481	−.010669	.117037
Q'		−.001303	.007917	−.024222	.382752

TABLE V—STATION NUMBERS

$n=5$

	$j \backslash k$	1	2	3	4	5
M	1	1.097991	0.408755	−0.040898	0.019866	−0.013120
N		.608222	.527493	−.100112	.069159	−.058445
P		.202887	.026908	−.005074	.002776	−.001627
Q		.049943	.031910	−.011210	.008927	−.006836
M'		.649902	.492141	−.070298	.034939	−.024007
N'		.427616	.647530	−.172488	.121519	−.107689
P'		.156411	.036908	−.008903	.004824	−.003375
Q'		.040729	.043977	−.019678	.015509	−.014228
M	2	−0.839550	0.799339	0.493783	−0.089550	0.049339
N		−.373049	1.282044	1.145470	−.310549	.219544
P		−.103119	.169599	.037952	−.011949	.006007
Q		−.021583	.212792	.083245	−.038398	.025243
M'		−.255330	.699256	.523828	−.099723	.058134
N'		−.139170	1.138472	1.219101	−.345551	.260447
P'		−.043239	.157833	.041704	−.013166	.008078
Q'		−.009758	.198610	.091532	−.042299	.034055
M	3	0.782798	−0.313591	0.651687	0.575298	−0.126091
N		.329315	−.465823	1.718204	1.923065	−.559573
P		.089079	−.044602	.150458	.049347	−.014691
Q		.018334	−.054326	.340126	.157843	−.061693
M'		.197103	−.228783	.633812	.573549	−.137821
N'		.105126	−.344915	1.674943	1.916360	−.616234
P'		.032115	−.034985	.148520	.048503	−.018600
Q'		.007151	−.042759	.335798	.156076	−.078385
M	4	−0.548214	0.187897	−0.159325	0.576786	0.562897
N		−.233780	.276637	−.400446	2.109970	2.432587
P		−.062665	.025479	−.024868	.140460	.042295
Q		−.012809	.030950	−.055302	.458397	.177006
M'		−.117990	.117132	−.136452	.557359	.664560
N'		−.062435	.175188	−.344108	2.041363	2.901646
P'		−.018958	.017186	−.021868	.137234	.063512
Q'		−.004201	.020954	−.048664	.447980	.267043
M	5	0.214238	−0.069026	0.050904	−0.080137	0.525349
N		.090928	−.101352	.127410	−.283759	2.458336
P		.024289	−.009225	.007667	−.013645	.134478
Q		.004952	−.011196	.017031	−.044065	.573757
M'		.033722	−.031711	.032307	−.056459	.432107
N'		.017789	−.047311	.081140	−.200039	2.030260
P'		.005389	−.004593	.005002	−.009675	.116059
Q'		.001192	−.005596	.011120	−.031249	.495663

TABLE VI—STATION NUMBERS

$n=6$

	$j \backslash k$	1	2	3	4	5	6
M	1	1.172073	0.391101	−0.062371	0.013323	−0.010149	0.008879
N		.638800	.501856	−.078978	.046300	−.045644	.048522
P		.210893	.024685	−.003894	.001823	−.001498	.001143
Q		.051551	.029221	−.008595	.005860	−.006322	.005949
M'		.676394	.474177	−.059129	.026582	−.018685	.015649
N'		.441269	.621067	−.144759	.092370	−.083901	.085903
P'		.160476	.034473	−.007337	.003631	−.002691	.002241
Q'		.041616	.041021	−.016206	.011674	−.011350	.011694
M	2	−1.066598	0.853106	0.468124	−0.070505	0.044513	−0.035782
N		−.466718	1.360105	1.081893	−.244062	.199943	−.195451
P		−.127634	.176358	.034411	−.009203	.006452	−.004561
Q		−.026505	.220969	.075401	−.029565	.027216	−.023740
M'		−.303948	.731991	.503742	−.085145	.049793	−.038661
N'		−.164215	1.186681	1.169252	−.294736	.223322	−.212149
P'		−.050692	.162263	.035896	−.011103	.007043	−.005503
Q'		−.011384	.203987	.085309	−.035665	.029701	−.028708
M	3	1.150584	−0.404200	0.693794	0.546418	−0.133366	0.069627
N		.489124	−.597296	1.822457	1.822457	−.597296	.489124
P		.130870	−.055956	.156259	.045293	−.018484	.011225
Q		.026719	−.068060	.352909	.144808	−.077925	.058410
M'		.267118	−.275585	.662165	.553477	−.126314	.083246
N'		.141177	−.413819	1.745286	1.846431	−.564890	.456509
P'		.042839	−.041308	.152473	.045980	−.017100	.011713
Q'		.009490	−.050434	.344556	.146996	−.072063	.061097
M	4	−0.930965	0.273028	−0.194854	0.592473	0.624591	−0.171416
N		−.390902	.399897	−.488124	2.163786	2.720209	−.933437
P		−.103635	.036020	−.029594	.142229	.056698	−.020561
Q		−.021011	.043690	−.065761	.464052	.238215	−.106938
M'		−.210538	.182630	−.190822	.598464	.604424	−.166643
N'		−.110263	.271857	−.454522	2.185427	2.628772	−.912352
P'		−.033225	.026154	−.028221	.143453	.053359	−.022768
Q'		−.007320	.031846	−.062753	.468015	.224100	−.118729
M	5	0.581796	−0.156399	0.092907	−0.111954	0.537351	0.599157
N		.242612	−.228221	.231501	−.394888	2.509279	3.194001
P		.063996	−.020218	.013474	−.018261	.134578	.046991
Q		.012925	−.024496	.029896	−.058920	.574036	.243745
M'		.120308	−.097119	.081568	−.110987	.535339	.685021
N'		.062735	−.144101	.204039	−.391731	2.499695	3.678582
P'		.018841	−.013674	.012212	−.018226	.133968	.066470
Q'		.004140	−.016634	.027119	−.058815	.571521	.345986
M	6	−0.209220	0.054246	−0.030239	0.031561	−0.064035	0.510543
N		−.086982	.079042	−.075223	.110987	−.291427	2.902746
P		−.022893	.009657	−.004323	.004969	−.011243	.132560
Q		−.004616	.008425	−.009587	.016022	−.047575	.698254
M'		−.033141	.025961	−.020586	.024431	−.049677	.425817
N'		−.017248	.038471	−.051417	.085977	−.225999	2.427820
P'		−.005173	.003631	−.003042	.003877	−.008676	.115150
Q'		−.001135	.004415	−.006753	.012502	−.036708	.606979

TABLE VII—STATION NUMBERS

$n=7$

	j \ k	1	2	3	4	5	6	7
M	1	1.243487	0.376396	−0.026266	0.009112	−0.005396	0.006025	−0.006513
N		.667840	.480602	−.063889	.031590	−.026481	.033195	−.042160
P		.218415	.022882	−.003069	.001211	−.000853	.000925	−.000963
Q		.053049	.027042	−.006769	.008890	−.003599	.004829	−.005357
M'		.702228	.458303	−.050122	.019989	−.012820	.011370	−.011099
N'		.454474	.597757	−.122429	.069352	−.057533	.062529	−.072060
P'		.164382	.032358	−.006088	.002679	−.001831	.001689	−.001614
Q'		.042465	.038456	−.013441	.008611	−.007724	.008815	−.010037
M	2	−1.321299	0.905487	0.446479	−0.055674	0.029812	−0.027896	0.028701
N		−.570270	1.435730	1.028397	−.192270	.133730	−.153603	.185730
P		−.154452	.182773	.031489	−.007052	.004233	−.004239	.003780
Q		−.031847	.228718	.068932	−.022639	.017853	−.022136	.023463
M'		−.357636	.764868	.485193	−.071706	.038132	−.031029	.028988
N'		−.191650	1.234950	1.123273	−.247832	.170920	−.170555	.188216
P'		−.058807	.166641	.036328	−.009168	.005346	−.004565	.004199
Q'		−.013146	.209297	.079621	−.029438	.022542	−.023823	.026110
M	3	1.672922	−0.511106	0.737737	0.516672	−0.104856	0.081487	−0.077078
N		.701415	−.751768	1.931045	1.718603	−.468955	.448232	−.498585
P		.185835	−.069049	.162183	.040990	−.014225	.012162	−.010056
Q		.037666	−.063877	.366024	.130958	−.059956	.063490	−.062408
M'		.355047	−.329159	.692136	.532099	−.108454	.073511	−.063669
N'		.186093	−.492459	1.819566	1.771836	−.484684	.403671	−.413268
P'		.056122	−.048434	.156613	.042913	−.014528	.010624	−.009166
Q'		.012374	−.059075	.353730	.137128	−.061218	.055439	−.056987
M	4	−1.605312	0.409927	−0.250374	0.629063	0.592219	−0.182666	0.144688
N		−.664780	.597643	−.625274	2.291470	2.574726	−1.002357	.935220
P		−.174510	.052757	−.037056	.147488	.051939	−.026098	.018551
Q		−.035122	.063907	−.082279	.480975	.218352	−.136173	.115110
M'		−.317567	.247432	−.216700	.623577	.584350	−.157315	.115706
N'		−.164912	.366953	−.543417	2.273022	2.538692	−.861833	.750618
P'		−.049381	.034761	−.083166	.147041	.050504	−.021773	.016464
Q'		−.010326	.042284	−.073708	.479557	.212061	−.113560	.102355
M	5	1.129029	−0.264373	0.134585	−0.136596	0.551252	0.668960	−0.220971
N		.464325	−.384008	.334325	−.480675	2.570992	3.589325	−1.425675
P		.121269	−.033334	.019010	−.021700	.136202	.063551	−.027154
Q		.024312	−.040333	.042147	−.069981	.590356	.330701	−.168426
M'		.234183	−.168109	.122950	−.143020	.568583	.634247	−.197770
N'		.120971	−.248408	.306705	−.503685	2.649577	3.397336	−1.281188
P'		.036084	−.023166	.017983	−.022914	.139081	.057851	−.027322
Q'		.007887	−.028149	.039907	−.073907	.593022	.300912	−.160818
M	6	−0.617807	0.137401	−0.064103	0.054068	−0.034474	0.507772	0.632193
N		−.252961	.199169	−.158887	.189539	−.383331	2.883613	4.007039
P		−.065852	.017129	−.008878	.008211	−.014243	.130010	.051386
Q		−.013170	.020712	−.019672	.026453	−.060230	.684756	.318021
M'		−.125078	.086131	−.058376	.057414	−.092005	.516450	.704705
N'		−.064447	.127049	−.145336	.201491	−.417397	2.931074	4.491403
P'		−.019183	.011759	−.008393	.008826	−.015450	.131144	.069349
Q'		−.004186	.014281	−.018616	.028441	−.065330	.690659	.430362
M	7	0.205449	−0.044613	0.020059	−0.015852	0.021392	−0.053078	0.498306
N		.083943	−.064608	.049675	−.055505	.096798	−.295079	3.334065
P		.021819	−.005533	.002756	−.002373	.003481	−.009551	.130932
Q		.004358	−.006689	.006106	−.007644	.014713	−.049981	.820708
M'		.033023	−.022307	.014642	−.013579	.018939	−.044219	.420133
N'		.016993	−.032878	.036425	−.047602	.085721	−.245636	2.816717
P'		.005053	−.003033	.002091	−.002060	.008074	−.007854	.114318
Q'		.001102	−.003682	.004637	−.006638	.012992	−.041094	.716958

TABLE VIII—STATION NUMBERS

$n=8$

	j \ k	1	2	3	4	5	6	7	8
M	1	1.312192	0.364019	−0.021829	0.006490	−0.003545	0.003081	−0.003931	0.005039
N		.695399	.462793	−.052953	.022453	−.015893	.016958	−.025621	.037691
P		.225483	.021401	−.002485	.000839	−.000499	.000464	−.000622	.000684
Q		.054447	.025255	−.005476	.002694	−.002103	.002422	−.003869	.004929
M'		.727233	.444367	−.043004	.015242	−.006698	.007062	−.007522	.008334
N'		.467152	.577362	−.104816	.052799	−.038988	.038933	−.048943	.062492
P'		.168111	.030533	−.005119	.002003	−.001220	.001045	−.001145	.001228
Q'		.043271	.036245	−.011294	.006433	−.005146	.005456	−.007123	.006863
M	2	−1.600390	0.955663	0.428499	−0.045077	0.020351	−0.016189	0.019610	−0.024337
N		−.682210	1.507998	.984086	−.155335	.091123	−.089039	.127790	−.182002
P		−.183160	.188781	.029123	−.005560	.002806	−.002412	.003062	−.003289
Q		−.037524	.235968	.063696	−.017807	.011838	−.012593	.019183	−.023705
M'		−.416706	.797175	.468735	−.060778	.028715	−.021414	.021627	−.023284
N'		−.221089	1.282229	1.082551	−.209723	.128559	−.117660	.140686	−.174560
P'		−.067464	.170868	.034087	−.007611	.003953	−.003129	.003272	−.003421
Q'		−.015018	.214419	.074659	−.024428	.016664	−.016330	.020358	−.024694
M	3	2.337500	−0.630523	0.780889	0.491625	−0.062648	0.054514	−0.060500	0.071477
N		.967933	−.923581	2.036155	1.631308	−.368956	.299530	−.394067	.534419
P		.254178	−.083331	.167794	.037442	−.010886	.007975	−.009418	.009602
Q		.051181	−.101109	.378439	.119546	−.045861	.041626	−.058605	.069206
M'		.462586	−.388839	.722423	.512112	−.091428	.056607	−.052469	.054058
N'		.240602	−.579786	1.894495	1.702145	−.408100	.310711	−.341167	.406216
P'		.072148	−.036240	.160734	.040069	−.012016	.008120	−.007864	.007911
Q'		.015836	−.068533	.362854	.127977	−.050617	.042389	−.048918	.057107
M	4	−2.630070	0.593584	−0.315718	0.667195	0.558819	−0.143453	0.132430	−0.143916
N		−1.075644	.861856	−.786292	2.424356	2.424356	−.786292	.861856	−1.075644
P		−.279850	.074709	−.045646	.152881	.046981	−.020073	.020253	−.019152
Q		−.055949	.090392	−.101283	.498324	.197214	−.104712	.126011	−.138032
M'		−.466670	.329868	−.258293	.650770	.561588	−.135711	.104892	−.100193
N'		−.240468	.487559	−.646305	2.367821	2.436346	−.743106	.681473	−.750648
P'		−.071591	.045534	−.038818	.150902	.047159	−.018614	.015449	−.014572
Q'		−.015627	.055337	−.086223	.491981	.197950	−.097075	.096064	−.105178
M	5	2.192995	−0.454018	0.201514	−0.175130	0.584107	0.633389	−0.237005	0.215982
N		.890699	−.656778	.499217	−.614926	2.718847	3.393592	−1.539301	1.613222
P		.230544	−.055975	.027793	−.027134	.141097	.058226	−.034728	.028253
Q		.045912	−.067648	.061577	−.087460	.601511	.302906	−.215987	.203600
M'		.389428	−.253524	.165704	−.170613	.391124	.614158	−.191506	.155413
N'		.199636	−.373346	.412443	−.509857	2.750891	3.287044	−1.241724	1.164103
P'		.059201	−.034316	.023783	−.026822	.142374	.054969	−.027027	.022340
Q'		.012882	−.041657	.052749	−.066477	.606915	.285874	−.168026	.161236
M	6	−1.350286	0.265559	−0.108064	0.078152	−0.102878	0.520374	0.709714	−0.274441
N		−.546005	.383370	−.267117	.273370	−.466005	2.952258	4.523995	−2.046630
P		−.140865	.032367	−.014611	.011568	−.016907	.131525	.070024	−.034431
Q		−.027983	.039093	−.032351	.037251	−.071468	.692634	.434540	−.248035
M'		−.263362	.163432	−.096083	.084195	−.115917	.543778	.662638	−.230699
N'		−.134575	.240192	−.243600	.294907	−.525024	3.081687	4.216400	−1.725906
P'		−.039810	.021885	−.013810	.012656	−.019020	.135367	.062213	−.032195
Q'		−.008646	.026552	−.030611	.040765	−.080395	.712714	.385904	−.232314
M	7	0.654484	−0.124395	0.048081	−0.031856	0.034801	−0.066829	0.484484	0.662747
N		.263837	−.179348	.118705	−.111252	.157170	−.370670	3.239551	4.867795
P		.067914	−.015052	.006430	−.004626	.005472	−.011570	.126311	.055545
Q		.013468	−.018172	.014232	−.014589	.023114	−.060517	.791745	.399381
M'		.130945	−.079057	.045388	−.036054	.041862	−.077799	.500211	.723527
N'		.066783	−.116059	.112606	−.126105	.189085	−.431248	3.341811	5.337321
P'		.019727	−.010524	.006331	−.005328	.006593	−.013324	.128649	.072134
Q'		.004280	−.012763	.014029	−.017155	.027846	−.069680	.806281	.519815
M	8	−0.202414	0.037821	−0.014268	0.009104	−0.009307	0.015388	−0.045167	0.487926
N		−.081470	.054494	−.035205	.031774	−.041992	.085181	−.296574	3.754721
P		−.020947	.004560	−.001898	.001312	−.001443	.002573	−.008294	.129520
Q		−.004150	.005504	−.004201	.004221	−.006093	.013452	−.051711	.941457
M'		−.033107	.019722	−.011093	.008527	−.009326	.015058	−.039785	.414996
N'		−.016869	.028937	−.027507	.029609	−.042089	.083310	− 260993	3.198326
P'		−.004979	.002618	−.001541	.001252	−.001450	.002493	−.007173	.113558
Q'		−.001080	.003174	−.003414	.004029	−.006122	.013034	−.044710	.825790

TABLE IX—COMPARISON OF RESULTS

Number of stations	Torsion			Bending			Coupled		
	$\omega_1\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_2\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_3\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_1\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_1\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3\sqrt{\frac{m_0 l^4}{B_0}}$
Station-Function method									
1	1.549	----------	----------	3.493	----------	----------	3.46	----------	----------
2	1.571	4.526	----------	3.516	21.71	----------	3.48	19.7	----------
3	1.571	4.689	7.502	3.516	22.04	60.20	3.48	20.6	48.2
Weighted influence coefficients									
2	1.575	5.39	----------	3.56	15.63	----------	----------	----------	----------
4	1.571	4.73	----------	3.52	22.80	----------	----------	----------	----------
Exact theoretical value									
	1.571	4.712	7.854	3.516	22.04	61.70	3.49	20.6	49.1

TABLE X—STATIONS REQUIRED FOR SATISFACTORY ACCURACY

Method	Torsion			Bending		
	$\omega_1\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_2\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_3\sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_1\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2\sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3\sqrt{\frac{m_0 l^4}{B_0}}$
Station Functions	1	3	4	1	2	3
Weighted influence coefficients	2	4	----------	3	6	----------