A GENERALISED TYPE OF JOUKOWSKI AEROFOIL.

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Summary.—The present report gives a short account of a generalised type of Joukowski aerofoil which avoids the difficulty of extreme thinness near the trailing edge associated with the ordinary Joukowski aerofoils. Calculations have been made for three different aerofoils of this type which might form a suitable basis for an experimental test of the theory. Details are also given of a fourth aerofoil which should have a constant centre of pressure according to Munk's theory of thin aerofoils.

1. *Introduction.*—The calculation of the characteristics of an aerofoil in two dimensional motion by the method initiated by Joukowski depends on the possibility of deriving the aerofoil shape from a circle by means of a suitable conformal transformation. The general form of this transformation may be taken to be :—

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where ζ and z are the complex variables

$$\zeta = \xi + i\,\eta$$
$$z = x + i\,y$$

and the coefficients a_1, a_2, etc., may also be complex quantities. An aerofoil shape in the ζ plane can be derived by means of the formula from any circle in the z plane which encloses within its periphery all the poles of $\frac{dz}{d\zeta}$. It will be noticed that the form of the transformation is such that the region at infinity is unaltered.

The Joukowski series of aerofoils is obtained from the simple transformation

$$\zeta = z + \frac{c^2}{z} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and the circle in the z plane is chosen to pass through the pole $z = -c$ and to enclose the pole $z = c$. The aerofoil then has a cusp at the trailing edge corresponding to the point $z = -c$ of the circle, and the circulation is chosen so that the flow leaves the trailing edge smoothly. This implies that the rear stagnation point on the circle must occur at the point $z = -c$. A full account of the Joukowski aerofoils has been given by M. Glauert (*Journal of the Royal Aeronautical Society*, July 1923).

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2. *Generalised Joukowski Aerofoils.*—The Joukowski aerofoils are not suitable for use owing to the cusp at the trailing edge, but a method of obtaining more suitable aerofoil shapes has been suggested by Karman and Trefftz (*Zeitschrift fur Flugtechnik und Motorluftschiffahrt*, 1918). The Joukowski transformation (2) can be written in the alternative form

$$\frac{\zeta - 2c}{\zeta + 2c} = \left(\frac{z - c}{z + c}\right)^2$$

and may be generalised as

$$\frac{\zeta - nc}{\zeta + nc} = \left(\frac{z - c}{z + c}\right)^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This generalised transformation leads to aerofoil shapes in which the upper and lower surfaces meet at an acute angle $(2-n)\,\pi$. Thus by choosing n slightly less than 2, a generalised form of Joukowski aerofoil can be obtained which does not suffer from the defect of being too thin near the trailing edge. The present report summarises the theoretical results for aerofoils of this type and gives the shapes of a few typical aerofoils which may be suitable for experimental investigation.

Diagrams of the circle in the z plane and of the aerofoil in the ζ plane are shown in Fig. 1, where corresponding points are indicated by the same letters. In the z plane A and B are the poles of the transformation ($z = \pm c$) and C is the centre of a circle of radius a passing through the point B and enclosing the point A. To determine the circle there are two arbitrary parameters, the radius a and the angle β between the lines B A and B C. The transformation formula also contains the arbitrary parameter n, while the value of c merely determines the scale of the figure. It is, therefore, possible to obtain a triply infinite series of aerofoils by use of the transformation (3).

In the ζ plane the points A_1 and B_1 are $\zeta = \pm nc$, corresponding to the points A and B of the z plane. Now let P be any point of the circle and P_1 the corresponding point of the aerofoil. Denote the angles A P B and $A_1 P_1 B_1$ by ϕ and ϕ_1 respectively. Then the geometrical interpretation of the conformal transformation (3) is

$$\left.\begin{aligned} \frac{A_1 P_1}{B_1 P_1} &= \left(\frac{A\,P}{B\,P}\right)^n \\ \phi_1 &= n\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The first relation shows that P_1 lies on one of the family of coaxial circles having A_1 and B_1 as limiting points, and the second relation shows that P_1 also lies on one of the family of coaxial circle passing through A_1 and B_1. This geometrical property can be used to obtain the form of the aerofoil, but unless it is possible to work on a very large scale this method is not sufficiently accurate, and it is preferable to use an analytical method.

RAF 30. $n = 1.08$ $n = 1.95$

In the z plane, let C P make angle θ with the x axis and let A P $= r$, B P $= s$. Also write

$$k = \frac{a}{c} \text{ and } \lambda = \frac{s}{r} \quad \text{.........................(5)}$$

Then P is the point

$$z = a(e^{i}\beta + e^{i}\theta) - c$$

and the vectors A P, B P are respectively

$$\text{A P} = a(e^{i}\beta + e^{i}\theta) - 2c$$
$$\text{B P} = a(e^{i}\beta + e^{i}\theta)$$

After a little reduction it can now be shown that

$$\left.\begin{aligned} \tan\phi &= \frac{b+t}{k(1+bt) - (1-bt)} \\ \lambda &= \frac{k(1+bt)}{(b+t)} \sin\phi \end{aligned}\right\} \quad \text{......................(6)}$$

$$\text{where } b = \tan \tfrac{1}{2}\beta$$
$$t = \tan \tfrac{1}{2}\theta$$

By means of these formulæ it is possible to calculate the values of λ and ϕ corresponding to any point P of the circle.

Turning next to the ζ plane, we have in the first place from (4)

$$\left.\begin{aligned} \mu &= \frac{s_1}{r_1} = \lambda^n \\ \phi_1 &= n\phi \end{aligned}\right\} \quad \text{...........................(7)}$$

Also denoting the angle A_1 B_1 P_1 by ψ, we obtain for the co-ordinates of P_1

$$\xi = s_1 \cos\psi - nc = nc + r_1 \cos(\phi_1 + \psi)$$
$$\eta = s_1 \sin\psi = r_1 \sin(\phi_1 + \psi)$$

from which we deduce the final forms

$$\left.\begin{aligned} \frac{\xi}{nc} &= \frac{\mu^2 - 1}{\mu^2 - 2\mu\cos\phi_1 + 1} \\ \frac{\eta}{nc} &= \frac{2\mu\sin\phi_1}{\mu^2 - 2\mu\cos\phi_1 + 1} \end{aligned}\right\} \quad \text{.......................(8)}$$

In this way any desired number of points on the aerofoil can be calculated to determine the shape.

One further point is worthy of attention. The point B of the circle transforms into the trailing edge B_1 of the aerofoil at $\zeta = -nc$. Also the point Q of the circle transforms into the leading edge Q_1 of the aerofoil. Now Q is given by $\theta = -\beta$, and for this point $\phi = o$

$$\text{and } \lambda_o = \frac{\text{BQ}}{\text{AQ}} = \frac{a\cos\beta}{a\cos\beta - c} = \frac{k\cos\beta}{k\cos\beta - 1}$$

Then $\mu_o = \lambda_o{}^n$

$$\text{and } \xi_o = \frac{\mu_o + 1}{\mu_o - 1} nc.$$

Thus the full chord $B_1 Q_1$ of the aerofoil is

$$nc(1 + \xi_o) = \frac{2\mu_o\, nc}{\mu_o - 1} \quad \text{.........................(9)}$$

which is slightly greater than $2\, n\, c$.

3. *Lift and Pitching Moment.*—It has been shown quite generally by Mises (*Zeitschrift fur Flugtechnik und Motorluftschiffahrt*, 1917) that the lift of the aerofoil at incidence α is

$$L = 4\,\pi\, a\, \rho\, V^2 \sin\,(\alpha + \beta) \quad \text{.................. (10)}$$

and that the pitching moment round the centre C is

$$M_C = 2\,\pi\, b^2\, \rho\, V^2 \sin 2\,(\alpha + \gamma)$$

where the co-efficient a_1 of the general transformation (1) is

$$a_1 = b^2\, e^{2i\gamma}$$

In the case of the generalised Joukowski transformation (3), we find on expansion

$$a_1 = \frac{1}{3}\,(n^2 - 1)\, c^2$$

and so it follows that

$$M_C = \frac{2}{3}\,(n^2 - 1)\,\pi\, c^2 \rho V^2 \sin 2\alpha \quad \text{.............. (11)}$$

To proceed further we shall regard α, β, $(k-1)$, $(2-n)$ and $1/\mu_0$ as small quantities whose squares may be neglected. The chord of the aerofoil then is

$$4c\left(\tfrac{1}{2}n + \frac{1}{\mu_0}\right).$$

The lift co-efficient of the aerofoil becomes

$$k_L = \pi\left\{k + \tfrac{1}{2}\,(2 - n) - \frac{1}{\mu_0}\right\}(\alpha + \beta) \quad \text{...........(12)}$$

and the pitching moment coefficient round the leading edge Q_1 is found to be

$$k_m = -\frac{\pi}{2}\,\beta - \tfrac{1}{4}\left\{1 + \frac{5}{6}\,(2 - n) + \frac{3}{\mu_0}\right\} k_L \quad \text{........(13)}$$

4. *Typical Aerofoils.*—Calculations have been made in the method described above for three aerofoil sections, defined by the following values of the arbitrary parameters.

Aerofoil.	k	n	β
A - - - - - -	1·050	1·950	6°
B - - - - - -	1·025	1·975	3°
C - - - - - -	1·050	1·950	0°

The general shape of these aerofoils is shown in Figs. 2 and 3, and numerical data are given in Table 1. A and B are aerofoils of different thickness and mean camber, and C is a symmetrical tailplane section.

The aerodynamical characteristics for two dimensional motion can be deduced from equations (12) and (13). In equation (12) the angle of incidence is measured from the base-line, while it is more customary to use the tangent to the lower surface. This small correction has been made in the case of aerofoils A and B in the following table of the theoretical characteristics of the aerofoils.

Aerofoil.	A	B	C
Angle of zero lift	$-6^\circ\cdot7$	$-3^\circ\cdot4$	0°
$\frac{dk_L}{da}$ two dimensions	$0\cdot059$	$0\cdot057$	$0\cdot059$
$\frac{dk_L}{da}$ aspect ratio 6	$0\cdot043$	$0\cdot042$	$0\cdot043$
k_m at $k_L = 0$	$-0\cdot082$	$-0\cdot041$	0
$\frac{dk_m}{dk_L}$	$-0\cdot262$	$-0\cdot256$	$-0\cdot262$

Details are also given in Table 1 and Fig. 3 of an aerofoil D. This aerofoil has the same thickness at every point as the symmetrical section C, but its central axis is taken as the curve

$$y = 0\cdot05 \ x(1-x)\ (7-8x)$$

According to the theory of Munk given in report 142 of the National Advisory Committee for Aeronautics this Aerofoil D should have a practically constant position of the centre of pressure and a no-lift angle of $-0^\circ\cdot7$ relative to the base line given in the figure.

TABLE I.

x	Aerofoil A.		Aerofoil B.	
	y_1	y_2	y_1	y_2
0	0	0	0	0
·05	·0330	−·0120	·0163	−·0067
·10	·0495	−·0115	·0252	−·0064
·15	·0630	−·0095	·0321	−·0056
·20	·0730	−·0065	·0379	−·0043
·25	·0810	−·0035	·0421	−·0029
·30	·0870	0	·0450	−·0013
·35	·0910	·0030	·0470	+·0002
·40	·0930	·0055	·0480	·0019
·45	·0930	·0080	·0480	·0034
·50	·0920	·0010	·0472	·0048
·55	·0895	·0130	·0456	·0059
·60	·0850	·0140	·0431	·0067
·65	·0790	·0150	·0400	·0071
·70	·0715	·0155	·0364	·0074
·75	·0630	·0150	·0318	·0072
·80	·0530	·0140	·0266	·0069
·85	·0410	·0120	·0207	·0059
·90	·0285	·0090	·0142	·0043
·95	·0145	·0050	·0074	·0024
1·00	0	0	0	0

Radius of curvature at leading edge :—
Aerofoil A 0·0043
Aerofoil B 0·0012

TABLE I.—*continued.*

x	Aerofoil C.		Aerofoil D.	
	y_1	y_2	y_1	y_2
0	0	0	0	0
·05	·0237	—·0237	·0394	—·0080
·10	·0328	—·0328	·0607	—·0049
·15	·0390	—·0390	·0760	—·0020
·20	·0428	—·0428	·0860	+·0004
·25	·0451	—·0451	·0920	·0018
·30	·0467	—·0467	·0950	·0016
·35	·0470	—·0470	·0948	·0008
·40	·0463	—·0463	·0919	—·0007
·45	·0450	—·0450	·0870	—·0030
·50	·0429	—·0429	·0804	—·0054
·55	·0403	—·0403	·0725	—·0081
·60	·0372	—·0372	·0636	—·0108
·65	·0335	—·0335	·0540	—·0130
·70	·0292	—·0292	·0439	—·0145
·75	·0245	—·0245	·0338	—·0152
·80	·0196	—·0196	·0244	—·0148
·85	·0147	—·0147	·0160	—·0134
·90	·0098	—·0098	·0089	—·0107
·95	·0049	—·0049	·0035	—·0063
1·00	0	0	0	0

Radius of curvature at leading edge is 0·0054 for both aerofoils.

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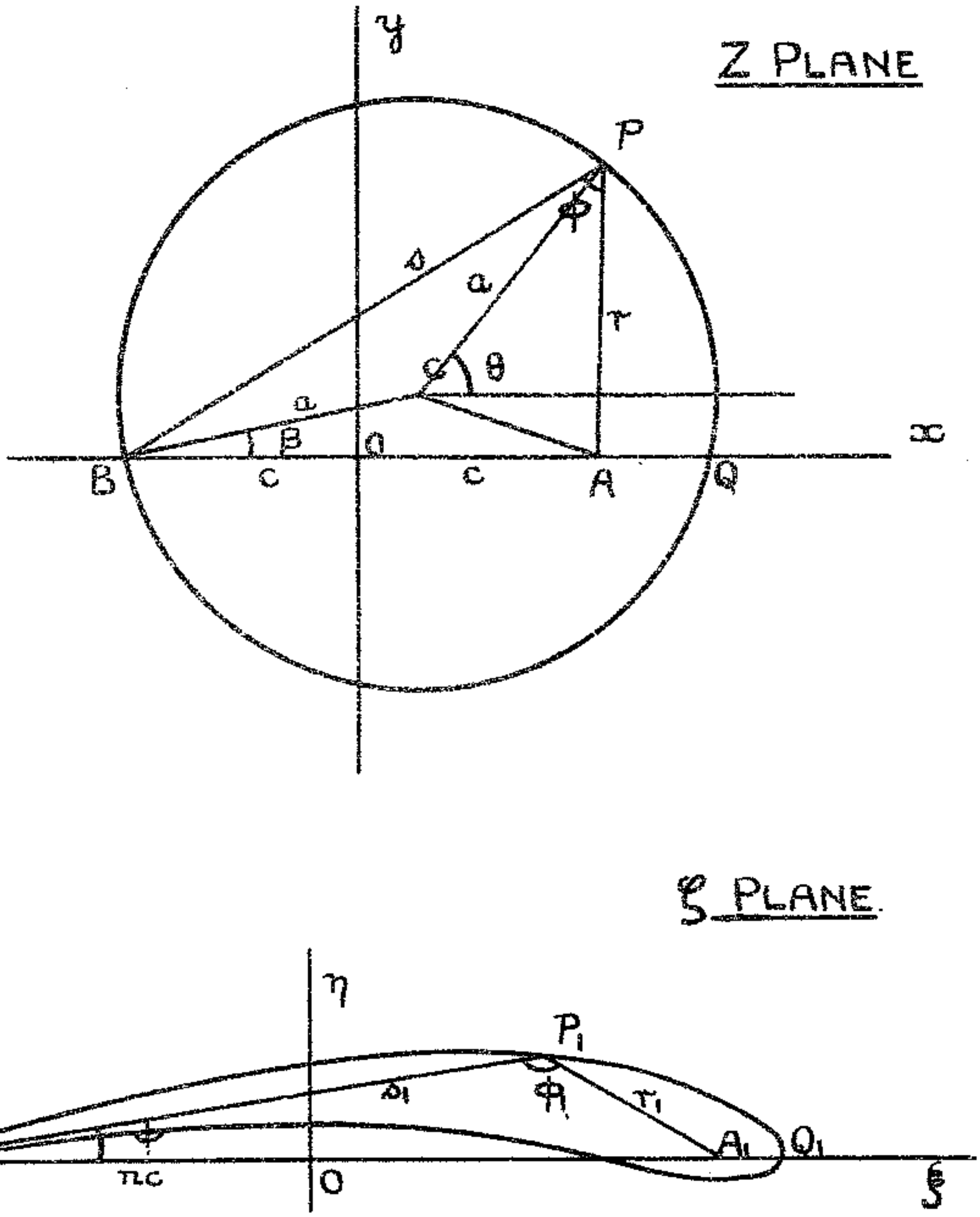
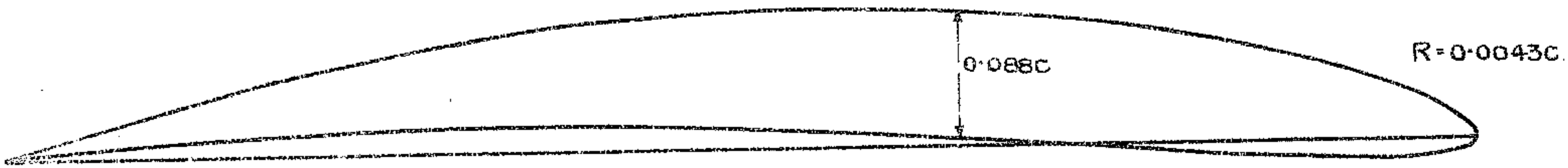
FIG 1



AEROFOIL A.



AEROFOIL B.

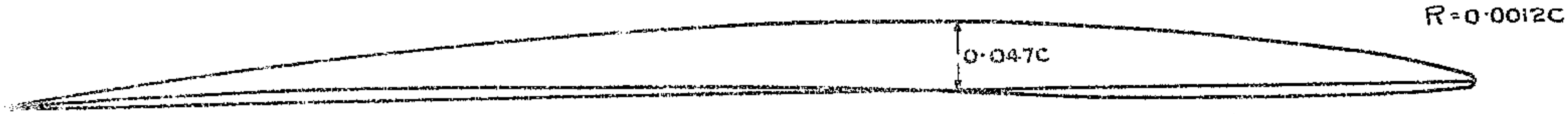


FIG. 2.

AEROFOIL C

AEROFOIL D

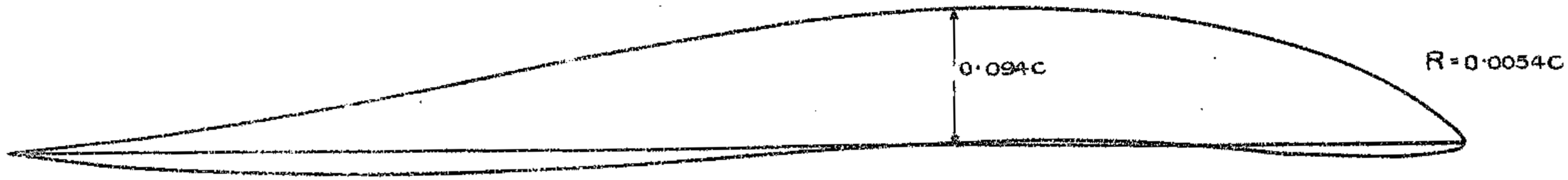


FIG 3.