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The Buckling of a Flat Rectangular Plate under Axial Compression and its Behaviour after Buckling

By

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Summary.—(*a*) *Purpose of Investigation.*—To rehearse the theory of buckling of a flat rectangular plate with particular reference to the effect of restraint at the edges of the plate against movement in the plane of the plate.

(*b*) *Range of Investigation.*—The effects of various types and degrees of edge restraint on the behaviour of the plate are considered with regard both to the value of the critical stress and to the behaviour after buckling. An attempt is made to assess the relative importance of the several factors, which influence the buckling of the plate, in respect of the probable range of variation of these factors in practical structures.

(*c*) *Conclusions.*—The buckling stress of a flat rectangular plate may be determined chiefly by the degree of restraint applied to its edges against rotation; but restraint against movement of the edges in the plane of the plate may also have considerable influence. In their effects on the behaviour of the plate after buckling both types of restraint are equally important; in particular lateral restraint may occasionally cause the plate to behave in an apparently anomalous manner.

Analysis of the behaviour of a plate under any specified edge conditions is straightforward, and exact or very nearly exact solutions are often accessible; but the computations necessary are usually extremely tedious.

(*d*) *Further Developments.*—In relation to practical structures the present review of plate buckling is largely qualitative. A fully quantitative survey of the effect of torsional restraint on the buckling stress is in preparation, and this survey is expressed in terms of practical conditions in which the torsional restraint is afforded by stringers; in the present paper an attempt is made similarly to present a quantitative assessment of the effect of lateral restraint on the buckling stress in terms of practical conditions. Complete quantitative examination of the behaviour of a plate after buckling as it may be affected by both types of edge restraint would involve a prohibitive amount of computation; but there is clearly a need for a limited investigation to establish the behaviour in one or two typical cases. Further needs are to extend the type of analysis here envisaged to cases of shear loading and to the case of the slightly curved plate.

Notation

E = Young's modulus.

σ = Poisson's ratio.

E' = $E/(1-\sigma^2)$.

a, b, t = length, width and thickness of plate.

x, y, z = co-ordinates parallel to length, width and thickness of plate, measured in the mid-surface from one corner.

u, v, w = displacements of the mid-surface of the plate, measured parallel to x, y and z.

e_x, e_y = average compressive strains parallel to x and y.

ϱ = $-e_y/e_x$ before buckling begins.

A_r = area of section of cross members at spacing a $[\varrho = \sigma/\{1 + (A_r/at)(1-\sigma^2)\}]$

U_d, U_w = strain energies of distortion and warping.

W = amplitude, $\psi(x, y)$ = form } of buckled form of plate, so that $w = W\psi(x, y)$.

e = general parameter representing a state of strain.

A, B, k = coefficients in expression of U_d in terms of W, e_x and e_y.

dS = element of plate surface.

r, s; m, n = positive integers: r and s are sometimes restricted to odd values, when m and n are restricted to even values.

A_{rs} = coefficients in Fourier expansion of $w = \sum_r \sum_s A_{rs} \sin(r\pi x/a) \sin(s\pi y/b)$.

C_{rs} = $(\pi rs/2a)\, A_{rs}$.

θ = $(r\pi x/a)$ (§10) or $(\pi x/a)$ (§12).

ϕ = $(\pi y/b)$.

$D_r = \sum_s C_{rs}$, $D_s = \sum_r C_{rs}$ } Σ is sometimes restricted to summation over odd values, when Σ' denotes summation over even values.

U_{mn}, V_{mn} = coefficients in Fourier expansions of u and v, e.g., $u = (a/\pi) \sum_m \sum_n (1/m)\, U_{mn} \sin m\theta \cos n\phi$ (§12).

X_m, Y_m = coefficients in quasi-Fourier expansions of v: $v = \frac{1}{2}(2\phi - \pi) \sum_m X_m \cos m\theta - (\pi/2) \sum_m Y_m \cos m\theta$ (§12).

$U_0\ U_1 \ldots U_s \ldots$; $X\ V_1 \ldots V_s \ldots$; $V_1'\ V_2' \ldots V_s' \ldots$ = restricted forms of U_{mn}, V_{mn} and X_m, when $n = 0$ or 2 only (§10).

μ_r or μ, ν_s or ν = coefficients of torsional restraint at edges along length and width of the plate respectively.

e_0 = $\pi^2 t^2/3(1-\sigma^2)\, b^2$.

e_c = critical value of e_x for first buckle in mode ψ.

e_b = least value of e_c for any function ψ.

f_0 = Ee_0.

f_0' = $(1-\rho\sigma)\, f_0/(1-\sigma^2)$ or $E'(1-\rho\sigma)\, e_0$.

Notation—continued

$f_c = E'(1 - \rho\sigma)\ e_c$.

$f_b = E'(1 - \rho\sigma)\ e_b$.

$f_e = E'(1 - \rho\sigma)\ e_x$.

f_a = average stress in plate = (End load/bt).

$f_2 = E'(1 - \rho\sigma)\ C_2^2$ }
$f_3 = E'(1 - \rho\sigma)\ C_3^2$ } (§14).

η = slope of $f_a - f_e$ diagram after buckling, i.e., $f_a = \eta f_e +$ constant.

$c = rb/a$ or $mb/2a$.

$\lambda = a/r$, half-wavelength of buckled form down length of plate.

$H = \sum_{n=1}^{n=\infty} 2c^4/(n^2 + c^2)^2 = \frac{1}{2}\{\pi c \coth \pi c + (\pi c/\sinh \pi c)^2 - 2\}$ coefficient of lateral restraint against waving of the plate edges.

H_m = value of H with $c = mb/2a$.

α, β, β' = parameters in formula for e_c (equation (11) §5).

$\chi(r, s) = (s^2 + c^2)^2 - \{4/(1 - \sigma^2)\ e_0\}\{(e_x + \sigma e_y)\ c^2 + (e_y + \sigma e_x) s^2\}$

$S_r = (16\mu_r/\pi^2)\ \sum_s\{s^2/\chi(r, s)\}$.

$S_s = (16\nu_s/\pi^2)\ \sum_r\{c^2/\chi(r, s)\}$.

$\xi_{rs}\ \zeta_{rs}$ = coefficients in formula for e_c (§4).

J = constant of integration = 1, $\frac{1}{2}$ or $\frac{1}{4}$.

J_{q+r} = coefficient of lateral restraint against waving (§13).

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1. *Introduction.*—The literature on the buckling of plates under compression and on their behaviour after buckling is now fairly extensive and a critical review of published data would be of considerable value. In the present note, however, a comprehensive survey of past work will not be attempted; instead the general theory will be rehearsed upon a fundamental basis in the hope that this rehearsal may provide a background against which the value of existing information may be assessed and the need for further investigation judged. The present report contains no essentially new results; the only real element of novelty lies in the attempt to present established results in logical sequence. Nevertheless this presentation inevitably suggests developments, some major and others minor, and a few of these have been explored a little further than the stages to which they had been carried in previous investigations.

The need for a greater degree of precision in estimation of the behaviour of plates arises out of the present trend towards thick wing coverings stiffened by light stringers closely spaced. In such construction the plates may carry a considerable proportion, or even the greater part, of the total load, and inaccuracies of estimation of plate loads, which previously have been of little moment, are now worth correction.

2. *The Strain Energy of a Deformed Plate.*—The strain energy of a deformed plate may be divided into two parts, the energy of warping, associated with systems of stress varying from positive to negative through the plate thickness t, and the energy of distortion associated with systems of stress symmetrical about the mid-surface of the plate. If u, v are the displacements of the point (x, y) in the mid-surface of a flat plate parallel to mutually perpendicular axes Ox and Oy in this surface and if w is the normal displacement, the strain energy of warping

$$U_w = \tfrac{1}{24} E't^3 \int \{ (w_{xx} + w_{yy})^2 + 2(1-\sigma)(w_{xy}{}^2 - w_{xx}w_{yy}) \} \, dS \qquad (1)$$

and the strain energy of distortion

$$U_d = \tfrac{1}{4}E't \int [(1+\sigma)(u_x + \tfrac{1}{2}w_x{}^2 + v_y + \tfrac{1}{2}w_y{}^2)^2 + (1-\sigma)\{ (u_x + \tfrac{1}{2}w_x{}^2 - v_y - \tfrac{1}{2}w_y{}^2)^3 + (u_y + v_x + w_xw_y)^2 \}] \, dS \qquad (2)$$

where w_x represents $\delta w/\delta x$, w_{xx} represents $\delta^2 w/\delta x^2$, etc., $E' = E/(1-\sigma^2)$ and the integrations are taken over the plate surface. The second order terms $w_x{}^2$, $w_y{}^2$ and w_xw_y being included the form for U_d is appropriate to large deflections w. For a rectangular plate having its edges held in the original plane $\int (w_{xx}w_{yy} - w_{xy}{}^2)dS$ is zero and the integral for U_w reduces to the first term only.

3. *Ignoration of the Buckled Form.*—Suppose that $w = W\psi(x, y)$ where ψ is some specified function and suppose that u and v are subject to some boundary conditions representing a specified state of (overall) strain of magnitude e. Then clearly $U_w = Ae_cW^2$ and $U_d = A(e^2 + eW^2 + kw^4)$ where A, e_c and k are constants derived from the function ψ; a constant multiplying the term eW^2 is avoided by appropriate division of w in the form $W\psi(x,y)$. Then the load system corresponding to the strain system e in the deflected mode ψ is proportional to $e + \frac{1}{2}W^2$, and the mode ψ is in equilibrium at amplitude W if $W^2 = -(1/2k)(e + e_c)$, so that the load system is proportional to $(1 - 1/4k)\, e - e_c/4k$, provided that $e + e_c$ is negative, otherwise to e simply.

Thus e_c is the buckling strain $(-e)$ in the mode ψ and $1 - 1/4k$ is the "stiffness after buckling", that is $1/4k$ is the fraction by which the ratio of rate of increase of load to rate of increase of strain is reduced by buckling in the mode ψ. The true buckling strain e_b is of course the least value of e_c for any possible form of ψ.

The effect of restraint against rotation of the edges of the plate, if it be unaffected by the strain e, is merely to add further energy terms in W^2 and hence merely to increase e_c; this effect is discussed quantitatively in §6 below. More often in practice the restraint at the plate edges afforded by stringers will be itself dependent on e, and the two elements, plate and stringer, must be considered together. A detailed analysis of this problem is in preparation. Excepting the discussion in §6 the effect of restraint against rotation of the plate edges will henceforward be ignored; or its effect may often be included by suitable redefinition of W and by adjustment of the values of e_c and (if necessary) k. Yet this enlargement of the scope of §§8–14 is scarcely worth while, because one chief effect of restraint against rotation at the plate edges is to modify e_c very differently for different forms of ψ, so that forms appropriate to one degree of restraint may be entirely inappropriate to another.

4. *The Critical Strain e_c.*—If the rectangular plate be subjected to uniform compressive strains e_x and e_y parallel to its edges

$$e_c = \frac{t^2}{12} \iint (w_{xx} + w_{yy})^2 \, dx\, dy \Big/ \{ (1 - \varrho\sigma) \iint w_x{}^2 dxdy + (\sigma - \varrho) \iint w_y{}^2 dxdy \} \qquad (3)$$

where the type of strain system is represented by the ratio $-\varrho$ of e_y to e_x and where e_c is the critical value of e_x. For any assigned form of buckling $w = W\psi(x, y)$, e_c is a function of ψ only, and the problem of initial instability is to choose ψ so that $e_c = e_b$, its minimum value.

The edges of the plate being held in the original plane, the most general form for w is $\sum_r \sum_s A_{rs} \sin (r\pi x/a) \sin (s\pi y/b)$ where a and b are the length and width of the plate, r and s are integers and A_{rs} are arbitrary coefficients. All the integrands in (3) are perfect squares, so that the integrals (between limits which in effect are 0 and π) involve only the squares $A_{rs}{}^2$ and no cross products $A_{mn} A_{rs}$. Thus the critical strain

$$e_c = \sum_r \sum_s \xi_{rs} A_{rs}{}^2 / \sum_r \sum_s \zeta_{rs} A_{rs}{}^2, \text{ where } \xi_{rs} \text{ and } \zeta_{rs} \text{ are constants.}$$

The least value e_b of e_c is therefore ξ_{rs}/ζ_{rs} for *one* single pair of integers r and s, and the buckled form is sinusoidal in both sections parallel to Ox and Oy. When the edges of the plate are restrained against rotation the expression for the energy includes terms of the forms $(\sum_s A_{rs})^2$ and $(\sum_r A_{rs})^2$ and buckled form is no longer sinusoidal (cf. §6). When the plate is subjected to shear the integral $\iint w_x w_y \, dxdy$ is involved and, the integrand not being a perfect square, cross products of the type $A_{mn} A_{rs}$ are introduced, so that the buckled form is not sinusoidal. It is therefore convenient for reference later to retain the summation and to write $\pi rs A_{rs}/2a = C_{rs}$, $\pi^2 t^2/3(1 - \sigma^2)b^2 = e_0$ and $rb/a = c$, when

$$\frac{e_c}{e_0} = \frac{1-\sigma^2}{4} \sum_r \sum_s \left\{ \left(\frac{s}{c} + \frac{c}{s}\right)^2 C_{rs}{}^2 \right\} \Big/ \sum_r \sum_s \left\{ (1 - \varrho\sigma) \frac{C_{rs}{}^2}{s^2} + (\sigma - \varrho) \frac{C_{rs}{}^2}{c^2} \right\} \quad \ldots \quad (4)$$

and

$$\frac{e_b}{e_0} = \tfrac{1}{4} (1 - \sigma^2) (s^2 + c^2)^2 / \{ (1 - \varrho\sigma) c^2 + (\sigma - \varrho) s^2 \} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

the values of s and c ($= rb/a$) in (5) being chosen so as to render e_b a minimum.

5. *Initial Instability.*—The formula (5) is of course essentially symmetrical in r and s, and a and b. In the general case for which a/b and ϱ may take any values, the formula (5) may be applied to show that either $s = 1$ and $c \doteqdot 1$ or $r = 1$ and $sa/b \doteqdot 1$. For practical purposes attention may, however, be confined to cases in which $a/b >$ about $1 \cdot 5$ and $\sigma > \rho > 0$, that is to plates loaded in the direction of their lengths and merely restrained against lateral expansion; this restriction has been anticipated in the definition of e_0. In such cases $s = 1$ and

$$e_b/e_0 = \tfrac{1}{4} (1 - \sigma^2) (1 + c^2)^2 / \{ (\sigma - \varrho) + (1 - \varrho\sigma) c^2 \};$$

if a/b is large, c may take any value and the least value of e_b is

$$e_b/e_0 = (1 + \sigma) (1 - \sigma)^2 (1 + \varrho) / (1 - \varrho\sigma)^2$$

for which $\quad c^2 = (1 - 2\sigma + 2\varrho - \varrho\sigma) / (1 - \varrho\sigma) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$

but when a/b is not large, the restriction that r must be integral may increase e_b/e_0 appreciably. The buckling stress $f_b = E' (e_x + \sigma e_y) = E e_b (1 - \varrho\sigma) / (1 - \sigma^2)$ so that, writing $f_0 = E e_0$,

$$f_b/f_0 = (1 - \sigma) (1 + \varrho) / (1 - \varrho\sigma) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

If the lateral restraint represented by ϱ is due to lateral members of the same material as the plate of section A_r spaced at distance a apart, the average value of

$$\varrho = \sigma / \{ 1 + (A_r/at) (1 - \sigma^2) \}.$$

Values of e_b/e_0, c and f_b/f_0 for a series of values of A_r/at, assuming $\sigma = 0 \cdot 3$, are given in Table 1 below.

TABLE 1

Effect of Lateral Restraint on the Buckling under Compression of a Flat Rectangular Plate Simply Supported at all Four Edges

A_r/at		infinity	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0
ϱ (average)		0	0·065	0·106	0·157	0·206	0·244	0·269	0·300 ($=\sigma$)
e_b/e_0		0·637	0·705	0·752	0·812	0·873	0·923	0·957	1
f_b/f_0		0·700	0·760	0·800	0·850	0·900	0·940	0·967	1
$c = rb/a$		0·632	0·721	0·775	0·837	0·894	0·938	0·966	1
Values of f_b/f_0 when $a/b =$	1·6	0·700	0·767	0·819	0·894	0·985	1·012	1·029	1·051
	1·8	0·703	0·780	0·842	0·902	0·935	0·964	0·985	0·011
	2·0	0·710	0·797	0·833	0·870	0·909	0·943	0·968	1·000
	2·2	0·718	0·788	0·812	0·854	0·900	0·941	0·970	1·009
	2·4	0·722	0·768	0·803	0·850	0·904	0·951	0·986	1·034
	2·6	0·710	0·762	0·800	0·854	0·916	0·972	0·996	1·021
	2·8	0·704	0·760	0·803	0·864	0·924	0·955	0·976	1·005
	3·0	0·701	0·762	0·809	0·869	0·909	0·943	0·968	1·000

The diagonal lines divide the lower part of Table 1 into three regions, in which $r = 1$ at the top left, $r = 2$ in centre and $r = 3$ at bottom right ; at values of $a/b > 3$, the variation of f_b/f_0 with a/b is inappreciable. In certain cases the a/b ratios may be somewhat indefinite ; in that case for low values of a/b variations of f_b/f_0 up to 5 or even 10 per cent. above the true minimum must be expected.

Lateral restraint reduces the buckling stress and increases the wavelength of the buckle in the direction of loading ; the latter effect may markedly affect the behaviour of the plate after buckling (*see* §14).

6. *Effect of Restraint against Rotation of the Edges of the Plate.*—If the edges of the plate are restrained against rotation it may be assumed that the torque per unit length of edge is

$$\frac{E't^3}{3b} \sum_r \mu_r \left(\sum_s \frac{s\pi}{b} A_{rs} \right) \sin \frac{r\pi x}{a} \qquad \text{at } y = 0,$$

where the rotation

$$w_y = \sum_r \left(\sum_s \frac{s\pi}{b} A_{rs} \right) \sin \frac{r\pi x}{a}$$

and

$$\frac{E't^3}{3b} \sum_r \mu_r' \left(\sum_s (-1)^s \frac{s\pi}{b} A_{rs} \right) \sin \frac{r\pi x}{a} \qquad \text{at } y = b,$$

where the rotation

$$w_y = \sum_r \left(\sum_s (-1)^s \frac{s\pi}{b} A_{rs} \right) \sin \frac{r\pi x}{a} \quad .$$

and similarly with different coefficients ν_s and ν_s' for restraint at the edges $x = 0$ and $x = a$. The only restriction introduced by these forms is the assumption that a sinusoidal distribution of torque on the restraining element produces a similar sinusoidal distribution of rotation ; this assumption is valid for a stringer or rib of constant section, and in general it is likely to be seriously at fault only in quite abnormal cases. It will be assumed further that $\mu' = \mu$ and that ν and ν' are zero.

In that case the total energy of the plate and edge restraints is the sum of U_w (equation (1)), U_d (equation (2)) and the term $(2Eabt\, e_0/\pi^2c^2)\mu\{(\Sigma C_s)^2 + (\Sigma' C_s)^2\}$*, where Σ denotes summation over all odd values of s and Σ' summation over all even values. The total energy of deformation divided by $\frac{1}{2}Eabt\, e_0$ is then

$$\tfrac{1}{4}(\Sigma + \Sigma')\left[\left(\frac{s}{c} + \frac{c}{s}\right)^2 C_s^{\,2} + \frac{4}{(1-\sigma^2)e_0}\left\{(e_x^{\,2} + e_y^{\,2} + 2\sigma e_x e_y - (e_x + \sigma e_y)\frac{C_s^{\,2}}{s^2} - (e_y + \sigma e_x)\frac{C_s^{\,2}}{c^2}\right\}\right] + \frac{4\mu}{\pi^2c^2}\{(\Sigma C_s)^2 + (\Sigma' C_s)^2\} \quad .. \quad (8)$$

Then by differentiation with respect to C_s for s odd

$$[(s^2 + c^2)^2 - \{4/(1-\sigma^2)e_0\}\{(e_x + \sigma e_y)c^2 + (e_y + \sigma e_x)s^2\}]\,C_s = -(16\mu s^2/\pi^2)(\Sigma C_s) \quad .. \quad (9)$$

or $C_s = 0$ and similarly for s even. The relations (9), which express the relative amplitudes of the several components C_s, are consistent only if $C_s = 0$, unless

$$\frac{1}{\mu} = -\frac{16}{\pi^2}\sum_s \frac{s^2}{[(s^2+c^2)^2 - \{4/(1-\sigma^2)e_0\}\{(e_x + \sigma e_y)c^2 + (e_y + \sigma e_x)s^2\}]}, \quad .. \quad (10)$$

so that this relation expresses the condition for buckling; the similar condition for even values of s leads to a greater value of the critical strain.† The series in (10) is of the form $\Sigma s^2/(s^2 + \alpha^2)\ (s^2 + \beta^2)$ and may be summed, so that (10) becomes

$$\mu = \pi(\alpha^2 - \beta^2)/4\{\beta \tanh(\pi\beta/2) - \alpha\tanh(\pi\alpha/2)\}\ \ddagger \quad .. \quad .. \quad (11)$$

where $\alpha^2 + \beta^2 = 2c^2 - 4(e_y + \sigma e_x)/(1-\sigma^2)\, e_0$

$\alpha^2\beta^2 = c^4 - 4c^2(e_x + \sigma e_y)/(1-\sigma^2)\, e_0$

and $\alpha^2 - \beta^2 = 4\{(e_y + \sigma e_x)^2/(1-\sigma^2)^2 e_0^{\,2} + c^2(e_x - e_y)/(1+\sigma)\, e_0\}^{1/2}$

the positive root of $(\alpha^2 - \beta^2)^2$ is taken, so that $\alpha^2 > \beta^2$.

When e_x and e_y are both small, $-\mu \rightarrow \frac{1}{2}(1 + \cosh \pi c)/(1 + \sinh \pi c/\pi c)$ and when c is small, $-\mu \rightarrow \frac{1}{2}$; the torsional stiffness $\frac{1}{2}(E't^3/3b)$ is that of a wide beam of length b and thickness t simply supported at its ends under equal and opposite moments applied to these ends. $-\mu$ is a comparative measure of the torsional stiffness of the plate edge to this manner of twisting in r-half waves. The torsional stiffness restraining a single stringer between two similar plates is represented by -2μ, which tends to unity at low loads and over long wavelengths; it is for this reason that the factor $E't^3/3b$ is introduced into the definition of μ.

If $e_x > e_y$, $\alpha^2 - \beta^2 > 4(e_y + \sigma e_x)/(1-\sigma^2)\, e_0$ and $\alpha^2 > c^2$, so that α is always real; but from the formula for $\alpha^2\beta^2$ it is clear that if $e_x + \sigma e_y > (1-\sigma^2)\, c^2 e_0/4$, β is imaginary. In that case the first term in the denominator of (11) becomes $-\beta' \tan(\pi\beta'/2)$, where $\beta' = i\beta$. If $\beta' = 1$, $\mu = 0$ and the formulae for $\alpha^2 + \beta^2$ and $\alpha^2\beta^2$ imply the condition $(e_x + \sigma e_y)c^2 + (e_y + \sigma e_x) = \frac{1}{4}(1 + c^2)^2(1 - \sigma^2)\, e_0$ as in §5. If $\mu \rightarrow$ infinity, equation (11) requires that

$$\beta' \tan(\pi\beta'/2) + \alpha\tanh(\pi\alpha/2) = 0.$$

Since $\alpha^2 > \beta^2$ and $\tanh(\pi\alpha/2)$ differs little from unity the lowest root of this equation lies between $\beta' = 1$ and $3/2$. By trial of a series of values of β' in this range, the values in Table 2 have been computed.

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* The suffix r in C_{rs} is dropped as it is unnecessary here.

† This higher value actually represents the case of a plate of width $b/2$ restrained against rotation at one edge and simply supported at the other.

‡ This condition was derived by a different method and in a slightly different form by Timoskenko, 1940.

TABLE 2.

Effect of Lateral Restraint on Buckling under Compression of a Flat Rectangular Plate Simply Supported along its Loaded Edges and Clamped along its other Pair of Edges.

A_r/at	infinity	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0
ϱ (average)	0	0·065	0·106	0·157	0·206	0·244	0·269	0·300 ($=\sigma$)
e_b/e_0	1·343	1·418	1·469	1·534	1·602	1·658	1·695	1·743
f_b/f_0	1·476	1·527	1·563	1·606	1·651	1·689	1·712	1·743
$c = b/\lambda$	1·310	1·352	1·385	1·420	1·452	1·476	1·491	1·510

Lateral restraint reduces f_b for a plate with clamped edges by about half the proportionate reduction caused by the same degree of restraint applied to a plate with simply supported edges, and the effect of lateral restraint in increasing the wavelength (λ) of buckle is also less marked. In the case of the plate with clamped edges the wavelength λ is never greater than 0·76 times the panel width ; as a result with values of $a/b > 1{\cdot}6$, the restriction of r to integral values has only slight effect on the value of f_b, although the number of half-waves in the buckled form may be altered.

When the loaded edges of the plate are also restrained against rotation equation (9) is modified by the addition of the term $(-\,16\,\nu r^2/\pi^2)\,(\sum_r C_{rs})$ to the right-hand side and the suffix r in C_{rs} suppressed in equation (9) must be replaced. Writing $\sum_s C_{rs} = D_r$, $\sum_r C_{rs} = D_s$ and $(s^2 + c^2)^2 - \{4/(1-\sigma^2)e_0\}\,\{(e_x + \sigma e_y)c^2 + (e_y + \sigma e_x)s^2\} = \chi(r,\, s)$ the condition for instability is the condition for compatibility of the equations

$$\left.\begin{aligned} D_r &= -\frac{16\nu}{\pi^2}\,\frac{c^2}{1+S_r}\sum_s\{D_s/\chi(r,s)\} \\ \text{and}\qquad D_s &= -\frac{16\mu}{n^2}\,\frac{\pi^2}{1+S_s}\sum_r\{D_r/\chi(r,s)\} \end{aligned}\right\} \quad \cdots \quad (12)$$

where

$$S_r = \frac{16\mu}{\pi^2}\sum_s\left\{\frac{s^2}{\chi\,(r,\,s)}\right\}$$

and

$$S_s = \frac{16\nu}{\pi^2}\sum_r\left\{\frac{c^2}{\chi\,(r,\,s)}\right\}$$

which are equivalent to the known series summed above in equation (11). Unfortunately, no means have yet been found to eliminate D_r and D_s from equations (12). For the special case in which all four edges of the plate are completely clamped approximate solutions based on limited expansions of the deflection w are available (Timoskenko, 1940, §64), and these overestimates are supplemented by underestimates found by another method (Taylor, 1933). When μ and ν are finite, approximate solutions could be found from equations (12) using a limited number of coefficients C_{11}, C_{12}, C_{21}, etc. ; but the work would be extremely laborious. When μ and/or ν are large, the expansion of the deflection w in sine series may be only slowly convergent and a different mode of expansion may be preferable.

7. *Initial Buckling of a Practical Plate.*—The factors which influence the stress at which a plate first buckles under compression may be summarized as follows, roughly in order of importance :—

(*a*) Restraint against rotation of the edges parallel to the load.
(*b*) Restraint against lateral expansion.
(*c*) Restraint against rotation of the loaded edges.
(*d*) Aspect ratio a/b.
(*e*) Variations of plate thickness.
(*f*) Presence of initial irregularities.
(*g*) Elastic failure of material.

On account of (*a*) the buckling stress may vary over the range 1 to 1·743; with closed section stringers a value near the upper limit is probable, but with open section stringers any value in this range (or even in exceptional cases below the lower limit) may be appropriate. Quantitative estimate of this ratio for open section stringers necessitates detailed consideration of stringer deformation and this forms the subject of a separate report.

On account of (*b*) the buckling stress may vary over the range 0·7 to 1·0 for a plate with simply supported edges or over the range 1·476 to 1·743 for a plate with clamped edges; but with practical values of the ratio A_r/at values below 0·85 in the first case or 1·60 in the second are unlikely. (The effect of lateral restraint on the behaviour after buckling is more important; this effect is discussed below.)

So long as $a/b > 3$, the influence of the factors (*c*) and (*d*) is probably negligible, although on account of (*c*) the effective value of a/b may be reduced, so that the value of r may be altered. For values of a/b between 1·6 and 3·0, (*d*) may cause an increase of buckling stress up to 10 per cent.; but only in exceptional cases will the increase exceed 5 per cent. On account of (*c*), the correction due to (*d*) is difficult to assess quantitatively for any given plate.

On account of (*e*) variations of buckling stress of the order of $\pm$ 5 per cent. cannot be regarded as abnormal; but with care this variation might be halved. The initial buckling stress, defined as the point of intersection of the load-strain lines before and after buckling, may be 5, 10 or even 15 per cent. below its correct value because the buckled form develops from the initial irregularities.* The influence of (*f*) may be still more important in combination with (*g*) in cases in which the buckling stress is comparable with the proportional limit of the material. However, the effect of (*g*) is as yet entirely unexplored.

From this survey of the factors affecting the buckling stress it is concluded that reasonably accurate allowances for the effects of (*a*) and (*b*) having been made, variations of at least $\pm$5 per cent. and quite probably $\pm$10 per cent. must be expected on account of factors (*c*), (*d*) and (*e*). Still greater discrepancies between theory and experiment may be ascribable to (*g*), or to (*g*) and (*f*) in combination (as in the normal "strut curve"), but no theoretical estimate of the effect of elastic failure is yet available.

8. *Behaviour after Buckling.*—In §3, in discussion of the behaviour of the plate after buckling, it was tacitly assumed that the state of *strain* represented by the ratio of e_y to e_x was maintained constant both before and after buckling. Later, in §5, by representation of the ratio $\varrho = -e_y/e_x$ in terms of the ratio A_r/at, this assumption was in effect abandoned and replaced by a condition relating the lateral strain to the mean lateral load. This change of condition necessitates a slight revision of the argument of §3. It is, however, convenient to retain the symbol ϱ to represent the degree of lateral restraint, that is to denote the ratio of $-e_y$ to e_x *before buckling begins.*

The total strain energy of the plate is proportional to

$$e_x^2 + e_y^2 + 2\sigma e_x e_y - (e_x + \sigma e_y) AW^2 - (e_y + \sigma e_x) BW^2 + kW^4 + \{(A + \sigma B) - \varrho(B + \sigma A)\} e_c W^2 \quad \ldots (13)$$

where $w = W\psi(x, y)$ as in §3. ψ is now a completely arbitrary function merely consistent with the boundary conditions imposed on w, and the coefficients A and B depend upon ψ only. The last term of (13) represents U_w and the coefficient $\{(A + \sigma B) - \varrho(B + \sigma A)\} e_c$ is also a function of ψ only; in writing the coefficient in this form simplification later is anticipated. The strains e_x and e_y are the *mean* compressive strains lengthwise and laterally; in addition (see §10) the strains u_x, v_y and $u_y + v_x$ in the formula (2) for U_d may include terms representing variations of strain over the plate. These varying strains all result from the deflection w and their magnitudes are all proportional to W^2; the energies associated with these strains are thus proportional to W^4 and these energies are included in the term with coefficient k.

* A note on the effect of initial irregularities and of elastic failure on estimation of the buckling stress is in preparation.

By differentiation of the total energy (13) with respect to e_x, e_y and W^2 in succession, the load in the direction x is proportional to

$$e_x + \sigma e_y - (A + \sigma B)W^2/2, \qquad (14)$$

the load in the direction y is proportional to

$$e_y + \sigma e_x - (B + \sigma A)W^2/2 \qquad (15)$$

and the condition for equilibrium of the buckled form ψ at amplitude W is

$$2kW^2 = (A + \sigma B)(e_x - e_c) + (B + \sigma A)(e_y + \varrho e_c) . \qquad (16)$$

If the lateral load in the direction y is throughout proportional to the strain e_y, equation (15) leads to the condition

$$(\sigma/\varrho)e_y + \sigma e_x = (B + \sigma A)W^2/2 \qquad (17)$$

because when $W = 0$, $e_y = -\varrho e_x$ by the definition of ϱ.

Then equations (16) and (17) together define both e_y and W^2 in terms of e_x, and by elimination of e_y

$$\{4k - (\varrho/\sigma)(B + \sigma A)^2\} W^2/2 = \{(A + \sigma B) - \varrho(B + \sigma A)\}(e_x - e_c), \quad \text{or } W = 0 \qquad (18)$$

Also by eliminating e_y between (14) and (17), the end load in direction x is proportional to $e_x(1 - \varrho\sigma) - \{(A + \sigma B) - \varrho(B + \sigma A)\} W^2/2$ and then, by substitution for W^2 from equation (18), the end load P is

$$P = E'bt\,[\,e_x(1 - \varrho\sigma) - (e_x - e_c)\{(A + \sigma B) - \varrho(B + \sigma A)\}^2/\{4k - (\varrho/\sigma)(B + \sigma A)^2\}\,] \qquad (19)$$

When $\varrho = 0$, so that e_y is zero throughout, equation (19) reduces to the form given in §3 with $A + \sigma B = 1$, as defined in that section; but if $\varrho \neq 0$ the state of strain (represented by the ratio of e_y to e_x) varies as the buckle develops and equation (19) differs in form from that given in §3.

Before buckling begins $f_e = E'(e_x + \sigma e_y) = Ee_x(1 - \varrho\sigma)/(1 - \sigma^2)$ represents the longitudinal stress in the plate, and f_c, the critical value of f_e at which buckling would begin in the mode ψ, is defined by the formula*

$$\frac{f_c}{f_0} = \frac{1 - \varrho\sigma}{1 - \sigma^2}\frac{e_c}{e_0} = \frac{1 - \varrho\sigma}{4} \sum_r \sum_s \left(\frac{s}{c} + \frac{c}{s}\right)^2 C_{rs}^{\;2} \Big/ \sum_r \sum_s \left\{(1 - \varrho\sigma)\frac{C_{rs}^{\;2}}{s^2} + (\sigma - \varrho)\frac{C_{rs}^{\;2}}{c^2}\right\} \qquad (20)$$

where the value of e_c/e_0 is taken from equation (4), $f_0 = Ee_0 = \pi^2Et^2/3(1 - \sigma^2)b^2$, $c = rb/a$ and $C_{rs} = \pi rs\, A_{rs}/2a$ as in §4.

* In a later section (§13) interaction between buckled forms characterized by different values of r is discussed, and it is there demonstrated that simultaneous development of two such modes is normally unstable, except possibly in respect of modification of one r mode by the introduction of its harmonics $2r$, $3r$, etc. In all normal cases the basis of the buckled form is characterized by $s = 1$; but in this case harmonics may be introduced from the beginning of buckling by the effect of restraint of the plate edges against rotation (*cf.* §6). The possible modification of the r-mode by introduction of harmonics is unlikely to be an important effect, and therefore in equation (20) and elsewhere it is usually permissible to omit the r-summation and the suffix r from C_{rs} and to regard $c = rb/a$ as a single constant. On the other hand interaction between modes characterized by different values of s is an essential feature of the behaviour after buckling, even when the plate edges are simply supported, and the forms of equation (20) and subsequent formulae have been chosen to facilitate application to such cases.

It is permissible to rewrite equation (19) in the form

$$f_a = (P/bt) = f_e - (f_e - f_c)\{(A + \sigma B) - \varrho(B + \sigma A)\}^2/(1 - \varrho\sigma)\{4k - (\varrho/\sigma)(B + \sigma A)^2\} \quad .. \quad (21)$$

where $f_e = E'(1 - \varrho\sigma)\, e_x$ and $f_c = E'(1 - \varrho\sigma)\, e_c$; that is f_e is the "apparent" value of the edge stress, the product of the edge strain and the effective modulus $E'\,(1 - \varrho\sigma)$ prior to buckling. Actually the stress in the edge of the plate after buckling varies down the length of the edge; its maximum possible value is

$$E'(e_x + \sigma e_y) \text{ or}$$

$$\{(A + \sigma B)f_e - \varrho(B + \sigma A)f_a\}/\{(A + \sigma B) - \varrho(B + \sigma A)\} \quad .. \quad .. \quad .. \quad .. \quad (22)$$

which is greater than f_e and its mean value is

$$E'\,[e_x + \sigma\{e_y - (B + \sigma A)\,W^2/2\}] \text{ or}$$

$$f_e - (f_e - f_a)(\sigma - \varrho)(B + \sigma A)/\{(A + \sigma B) - \varrho(B + \sigma A)\} \quad .. \quad .. \quad .. \quad (23)$$

which is less than f_e. The form (21) is convenient, but in view of these variations of actual edge stress the form (19) is to be regarded as the substantive form (see Appendix II).

9. *Forms for the Coefficients A and B.*—Although throughout §8 the coefficients A and B have been left unspecified, their forms are apparent in equation (20), that is $A = \sum_r\sum_s C_{rs}{}^2/s^2$ and $B = \sum_r\sum_s C_{rs}{}^2/c^2$. The amplitude of buckle being represented by W, the C coefficients are merely ratios and it is permissible to specify $C_{r1} = 1$ or $\sum_s C_{rs} = 1$ or otherwise according to convenience; but of course this choice affects the value of k. In subsequent sections §§10 and 11 in which attention is confined to one coefficient, C_{r1}, its value will be taken as unity, when $A = 1$ and $B = 1/c^2$. In §§12, 13 and 14, in which more than one C coefficient is involved, attention is concentrated on *variation* of buckled form so that it becomes convenient to use the C—coefficients to represent absolute magnitudes, that is in effect to put $W = 1$ in §8. Nevertheless the chief conclusion of §8 stands, that so long as any particular buckled form, represented by constant ratios of the C's remains unchanged, the $f_a - f_e$ relation after buckling is a straight line.

10. *Computation of the Value of the Coefficient k for the Double Sinusoidal Wave Form.*—The coefficient k is the coefficient of W^4 in the integral

$$\frac{2U_d}{Eabt} = \frac{1}{2ab}\iint\Big[(1 + \sigma)\left(u_x + v_y + \tfrac{1}{2}w_x{}^2 + \tfrac{1}{2}w_y{}^2\right)^2$$
$$+ (1 - \sigma)\left\{\left(u_x - v_y + \tfrac{1}{2}w_x{}^2 - \tfrac{1}{2}w_y{}^2\right)^2 + (u_y + v_x + w_x w_y)^2\right\}\Big]dxdy \quad .. \quad (24)$$

when $w = W\,(2a/\pi r)\sin(r\pi x/a)\sin(\pi y/b)$, and u and v are chosen in conformity with the boundary conditions so as to render the value of k a minimum. Then writing $(r\pi x/a) = \theta$, $(\pi y/b) = \phi$ and $(rb/a) = c$

$$\left.\begin{aligned} (w/W) &= (2a/r\pi)\sin\theta\sin\phi \\ (w_x/W)^2 &= (1 + \cos 2\theta)(1 - \cos 2\phi) \\ (w_y/W)^2 &= (1/c^2)(1 - \cos 2\theta)(1 + \cos 2\phi) \\ \text{and}\quad w_x w_y/W^2 &= (1/c)\sin 2\theta\sin 2\phi\,. \end{aligned}\right\} \quad (25)$$

The bending stresses are

$$\tfrac{1}{2}E't\,(w_{xx} + \sigma w_{yy}) = -E'W\,(\pi t/b)\,(c + \sigma)\sin\theta\sin\phi$$

and $$\tfrac{1}{2}E't\,(w_{yy} + \sigma w_{xx}) = -E'W\,(\pi t/b)\,(1 + c\sigma)\sin\theta\sin\phi\,.$$

Forms for the displacements u and v consistent with these forms for $w_x{}^2$, etc., are

$$\left.\begin{aligned} (2r\pi/a)u &= -e_x\,2\theta + (U_0 + U_1\cos 2\phi + U_2\cos 4\phi + \ldots)\sin 2\theta \\ \text{and}\quad (2\pi/b)v &= (-e_y + X\cos 2\theta)(2\phi - \pi) + (V_1' + V_1\cos 2\theta)\sin 2\phi \\ &\quad + (V_2' + V_2\cos 2\theta)\sin 4\phi + \ldots \end{aligned}\right\} \quad (26)$$

The form for u is based upon the assumption that the edges of the plate at $x = 0$ and $x = a$ are held straight; but the form for v includes the term with coefficient X which represents waving of the edges at $y = 0$ and $y = b$ in the plane of the plate; this waving may be restricted or prevented.* Direct interaction of u_x with w_x^2 etc., is confined to terms of u and v in 2θ and 2ϕ; but terms in 4ϕ, 6ϕ etc., interact with the term X and hence indirectly with w_x^2, etc.

From the formulae (26) for u and v

$$\left.\begin{aligned} u_x &= -e_x + (U_0 + U_1 \cos 2\phi + U_2 \cos 4\phi + \ldots) \cos 2\theta \\ v_y &= -e_y + V_1' \cos 2\phi + 2V_2' \cos 4\phi + \ldots \\ &\quad + (X + V_1 \cos 2\phi + 2V_2 \cos 4\phi + \ldots) \cos 2\theta \\ u_y + v_x &= - \{ cX (2\phi - \pi) + (cV_1 + U_1/c) \sin 2\phi \\ &\quad + (cV_2 + 2U_2/c) \sin 4\phi \ldots \} \sin 2\theta \\ &= - \sum_{s=1}^{s=\infty} \{ cV_s + (sU_s/c) - (2cX/s) \} \sin 2s\phi \sin 2\theta \end{aligned}\right\} \quad (27)$$

by expansion of $2\phi - \pi$ in Fourier series.

The integrand in the expression (24) for U_d is the sum of three squares and in each term integration eliminates all cross products between terms with different values of r or s (in the form $\sin r_1 \theta \sin r_2\theta$, for instance) or between sine and cosine terms. Hence, writing

$$\begin{aligned} u_x + v_y + \tfrac{1}{2} w_x^2 + \tfrac{1}{2} w_y^2 = &- e_x - e_y + \frac{1 + c^2}{2c^2} W^2 + \left(V_1' + \frac{1 - c^2}{2c^2} W^2\right) \cos 2\phi \\ &+ 2V_2' \cos 4\phi + \ldots \\ &+ \left\{\left(U_0 + X - \frac{1 - c^2}{2c^2} W^2\right) + \left(U_1 + V_1 - \frac{1 + c^2}{2c^2} W^2\right) \cos 2\phi \right. \\ &\left. + (U_2 + 2V_2) \cos 4\phi + \ldots \right\} \cos 2\theta \end{aligned}$$

and similarly for $u_x - v_y + \frac{1}{2} w_x^2 - \frac{1}{2} w_y^2$ and $u_y + v_x + w_x w_y$, the energy U_d is the product of $\frac{1}{2}Eabt$ and

$$\begin{aligned} &\tfrac{1}{2} (1 + \sigma) \left\{ \left(e_x + e_y - \frac{1 + c^2}{2c^2} W^2\right)^2 + \tfrac{1}{2} \left(V_1' + \frac{1 - c^2}{2c^2} W^2\right)^2 + \tfrac{1}{2} (2V_2')^2 + \ldots \right. \\ &\left. + \tfrac{1}{2} \left(U_0 + X - \frac{1 - c^2}{2c^2} W^2\right)^2 + \tfrac{1}{4} \left(U_1 + V_1 - \frac{1 + c^2}{2c^2} W^2\right)^2 + \tfrac{1}{4} (U_2 + 2V_2)^2 + \ldots \right\} \\ &+ \tfrac{1}{2} (1 - \sigma) \left\{ \left(e_x - e_y + \frac{1 - c^2}{2c^2} W^2\right)^2 + \tfrac{1}{2} \left(V_1' + \frac{1 + c^2}{2c^2} W^2\right)^2 + \tfrac{1}{2} (2V_2')^2 + \ldots \right. \\ &\left. + \tfrac{1}{2} \left(U_0 - X + \frac{1 + c^2}{2c^2} W^2\right)^2 + \tfrac{1}{4} \left(U_1 - V_1 + \frac{1 - c^2}{2c^2} W^2\right)^2 + \tfrac{1}{4} (U_2 - 2V_2)^2 + \ldots \right\} \\ &+ \tfrac{1}{8} (1 - \sigma) \{ (cV_1 + U_1/c - 2cX - W^2/c)^2 + \ldots + (cV_s + sU_s/c - 2cX/s)^2 + \ldots \} \end{aligned} \quad (28)$$

* Other restrictions on u and v may be imposed; for instance if the plate edges at $y = 0$ and $y = b$ are rigidly attached to stringers of heavy section, variation of u_x with θ at $\phi = 0$ and π may be prevented, so that $U_0 + U_1 + U_2 + \ldots = 0$. Restrictions of this type, as also the restraint against waving of the plate edges (term in X) may always be treated by inclusion of the energies in the restraining elements; but by the introduction of additional parameters to represent the several possible types of edge restraint the analysis is rendered more cumbersome.

The terms in (28) involving e_x and e_y correspond to the form (13) with $A = 1$ and $B = 1/c^2$ as stated in § 9; therefore k is the coefficient of W^4 in (28) when $U_0\ U_1 \ldots V_1 \ \ldots \ V_1' \ldots$ etc. are chosen so as to render k a minimum. Clearly V_2', $V_3' \ldots$ etc. are all zero, and U_s and V_s for $s > 1$ are zero if $X = 0$.

It may be shown that

$$\left.\begin{aligned} U_s &= 2c^2\,(s^2 - c^2\sigma)X/(s^2 + c^2)^2 \\ \text{and} \qquad sV_s &= 2c^2\,(c^2 - s^2\sigma)X/(s^2 + c^2)^2 \end{aligned}\right\} \text{ for } s > 1 \qquad (29)$$

and these formulae hold also for $s = 1$, if $U_1 - W^2/2$ and $V_1 - W^2/2c^2$ are substituted for U_1 and V_1 respectively.

Also $$U_0 + \sigma X = -\,(c^2 - \sigma)\,W^2/2c^2 \qquad (30)$$

and $$V_1' = -\,(1 - c^2\sigma)\,W^2/2c^2. \qquad (31)$$

Finally, if X may take any value,

$$\sigma U_0 + X\,\{\,1 + (1 - \sigma^2)H\,\} = (1 - c^2\sigma)\,W^2/2c^2 \qquad (32)$$

so that $$U_0 = -\,(W^2/2c^2)\,\{\,c^2 + (c^2 - \sigma)H\,\}\,/(H + 1) \text{ and } X = (W^2/2c^2)/(H + 1)$$

where $$H = \sum_{s=1}^{s=\infty} \frac{2c^4}{(s^2 + c^2)^2} = \left(\frac{\pi c}{2}\coth \pi c + \frac{\pi^2 c^2}{2 \sinh^2 \pi c} - 1\right). \qquad (33)$$

Values of H and of $1/(H + 1)$ are plotted against values of c in Fig. 1; for large values of c (> 1), $1/(H + 1) \doteqdot 2/\pi c$.

By substitution in (28) after some simplification

$$k = \frac{1 + 2\sigma c^2 + c^4}{4c^4} + (1 - \sigma^2)\left\{\frac{1}{8} + \frac{1}{2}\left(\frac{1}{2c^2} - \frac{X}{W^2}\right)^2 + \frac{H}{2}\left(\frac{X}{W^2}\right)^2\right\} \qquad (34)$$

If the edges of the plate at $y = 0$ and b are held straight, $X = 0$ and

$$k = \{\,2\,(1 + 2\sigma c^2 + c^4) + (1 - \sigma^2)\,(1 + c^4)\,\}\,/8c^4 \qquad (35)$$

and, if they are free, $X = W^2/2c^2\,(H + 1)$

and $$k = \left\{\,2\,(1 + 2\sigma c^2 + c^4) + (1 - \sigma^2)\left(c^4 + \frac{H}{H + 1}\right)\right\}/8c^4 \qquad (36)$$

(35) may be derived from (36) by assuming H to become infinite.

11. *The Stiffness of the Plate after Buckling*.—Writing equation (21) in the form

$$f_a = \eta f_e + (1 - \eta)\,f_c \qquad (37)$$

the ratio of the stiffness after buckling to the stiffness before buckling is

$$\eta = \left[\frac{1 - \sigma^2}{1 - \varrho\sigma}\right]\left[\frac{2\,(\sigma - \varrho) + \sigma\,(1 - \varrho\sigma)\,\{\,c^4 + H/(H + 1)\,\}}{2\,(\sigma - \varrho)\,(1 + \sigma c^2)^2 + \sigma\,(1 - \sigma^2)\,\{\,3c^4 + H/(H + 1)\,\}}\right] \qquad (38)$$

where H is given by equation (33) when the unloaded edges are free to wave or where $H \to \infty$, when the unloaded edges are held straight.

Values of η computed from formula (38) for these two limiting cases, edges free to wave and edges held straight, are given in Tables 3 and 4 respectively. In these tables the values of ϱ and c $(= rb/a)$ correspond to the values used in Table 1, so that the values of η in Tables 3 and 4 represent lower and upper limits to the stiffnesses immediately after buckling of the plates which buckle in the modes and at the critical stresses defined by Table 1. Even within this restricted field the value of η ranges from $0 \cdot 376$ to $0 \cdot 845$. The chief cause of this wide variation is the factor c, that is the wavelength $\lambda = a/r$. For instance, with the edges held straight, if $c = 1$ in all cases, η would vary only from $0 \cdot 500$ when $\varrho = \sigma$ (plate free to expand laterally) through a maximum value of $0 \cdot 521$ and then to $0 \cdot 518$ when $\varrho = 0$ (plate completely restrained laterally) ; but in fact lateral restraint, by increasing the wavelength λ, so much reduces c that the complete variation is much greater.

In the absence of lateral restraint, $\varrho = \sigma$ and formula (38) reduces to

$$\eta = \{ c^4 + H/(H+1) \} / \{ 3c^4 + H/(H+1) \} \qquad \ldots \quad \ldots \quad \ldots \quad (39)$$

When c has its maximum possible " natural " value $\sqrt{2}$*, corresponding to two half-waves in a panel for which $a/b = \sqrt{2}$, $\eta = 0 \cdot 385$ when the unloaded edges are held straight and $0 \cdot 363$ when the unloaded edges are free to wave. If a wave length still shorter than $b/\sqrt{2}$ could be forced, the limiting value $\eta = 1/3$ would still more nearly approached ; this limit corresponds to neglect of all strain energy except that of compression parallel to the load (Cox 1933, Ref. 2), and by reference to the expression (24) it may be shown that such complete elimination of all shear and transverse stresses can never actually be realized, whatever the wave form of the buckle. Yet in respect of the value of η the effect of this unattainable limit may be very nearly approached. If the plate is long, c differs little from unity and η is about 1/2 if the unloaded edges are held straight or about $0 \cdot 41$ if these edges are free to wave.

At the other extreme, if a long wavelength is by any means imposed, c may be small and very high values of η may result, particularly if the plate edges are held straight. This possibility appears worth exploration as a practical artifice to improve the performance of plates after buckling (*see* § 15).

It has already been remarked that lateral restraint, apart from its indirect effect, due to depression of the value of c, only slightly increases γ. This is further illustrated in Tables 3 and 4, where the values in any row vary only slowly so long as r is constant. As might be expected, the effect of freedom or restraint of the unloaded edges to wave in the plane of the plate is less when the plate is restrained laterally than when it is free to expand.

Perhaps the chief conclusion to be drawn from this preliminary examination of the behaviour of flat plates after buckling, is that a close estimate of the slope of the $f_a - f_c$ line requires a much more detailed specification of the conditions than has hitherto been thought necessary. In many practical cases this detailed knowledge of the loading conditions (cf. Ref. 4) will be lacking. In such cases, apart from the effect of the conditions imposed on the unloaded edges, of which the limits are represented by Tables 3 and 4, the most important factor is the wavelength ; if the wavelength may be guessed the value of η may be estimated within narrow limits. On the other hand, when the value of ϱ is uncertain (as usually it will be), either of two wavelengths may appear equally probable ; this uncertainty is illustrated by a test on a practical panel in Ref. 4, Fig. 9.

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* That is, a wavelength occurring naturally in primary buckling.

TABLE 3

Stiffness after Buckling under Compression of a Flat Rectangular Plate Simply Supported at All Four Edges, its Unloaded Edges being Laterally Restrained but Free to Wave (Values of η from (38) with H Computed from (33))

A_r/at ϱ (average)		infinity 0	4 0·065	2 0·106	1 0·157	$\frac{1}{2}$ 0·206	$\frac{1}{4}$ 0·244	$\frac{1}{8}$ 0·269	0 0·300 (= σ)
Values of a/b	1·6	0·688	0·690	0·688	0·681	0·663	0·384	0·380	0·376
	1·8	0·731	0·730	0·737	0·425	0·417	0·410	0.402	0·392
	2·0	0·763	0·772	0·470	0·462	0·451	0·438	0·427	0·408
	2·2	0·789	0·516	0·512	0·502	0·487	0·470	0·453	0·426
	2·4	0·561	0·557	0·553	0·542	0·525	0·502	0·481	0·443
	2·6	0·598	0·596	0·591	0·580	0·562	0·535	0·395	0·386
	2·8	0·631	0·630	0·627	0·616	0·428	0·418	0·410	0·397
	3·0	0·661	0·662	0·659	0·462	0·451	0·438	0·427	0·408
a/b large		0·684	0·613	0·590	0·540	0·494	0·458	0·436	0·408

TABLE 4

Stiffness after Buckling under Compression of a Flat Rectangular Plate Simply Supported at All Four Edges, its Unloaded Edges being Laterally Restrained and Held Straight (Values of η from (38) with $H \to \infty$)

A_r/at ϱ (average)		infinity 0	4 0·065	2 0·106	1 0·157	$\frac{1}{2}$ 0·206	$\frac{1}{4}$ 0·244	$\frac{1}{8}$ 0·269	0 0·300 (= σ)
Values of a/b	1·6	0·751	0·763	0·771	0·778	0·786	0·417	0·416	0·414
	1·8	0·792	0·807	0·815	0·468	0·466	0·462	0·458	0·453
	2·0	0·822	0·837	0·521	0·519	0·516	0·512	0·507	0·500
	2·2	0·845	0·573	0·573	0·573	0·570	0·566	0·561	0·552
	2·4	0·616	0·621	0·623	0·624	0·623	0·619	0·615	0·606
	2·6	0·654	0·665	0·668	0·671	0·672	0·670	0·443	0·439
	2·8	0·694	0·703	0·707	0·712	0·482	0·477	0·474	0·495
	3·0	0·724	0·735	0·742	0·519	0·516	0·512	0·507	0·500
a/b large		0·746	0·696	0·665	0·621	0·580	0·548	0·526	0·500

12. *Effect of Variation of Wave Form on the Behaviour after Buckling.*—(*a*) *General.*—The analysis carried through in §10 for the simplest possible buckled form (25) may be repeated for any wave form, but once the simple form is abandoned the choice of buckled forms to investigate presents difficulties. Obviously the buckled form of a plate with clamped edges would be an interesting case ; but this form is by no means simple (*cf.* equations (9), μ is infinite but $\Sigma C_s = 0$ and the product $\mu \Sigma C_s$ is finite) ; it includes all C_s with s odd (the r suffix is dropped in conformity with the notation of §6), and the work involved in computing η for a wave form characterized by several C's is extremely tedious. For this reason in the present section the analysis is developed in general terms for the most general buckled form, but no attempt is made to apply the results of this analysis to specific cases. One such case is treated in the succeeding section §13, and that section demonstrates the difficulty of application of the method to even moderately complex cases. Further comments on the question of application of the results of the present section are made at the end of this section.

(*b*) *Strains Directly due to Buckling.*—The general form for the deflection of a flat rectangular plate with its edges held in the original plane is

$$w = \frac{2a}{\pi} \sum_r \sum_s \frac{1}{rs} C_{rs} \sin r\theta \sin s\phi, \text{ where } \theta = \pi x/a \text{ and } \phi = \pi y/b \qquad .. \quad (40)$$

The coefficients C_{rs} are to be regarded as undetermined. It may be shown that

$$\left.\begin{aligned}
w_x^2 &= \sum_m \sum_n \Bigg[\sum_r \sum_s \frac{C_{rs}}{s} \Bigg\{ \frac{1}{n+s} (C_{m+r,n+s} + C_{m-r,\,n+s}) \\
&\qquad - \frac{1}{n-s} (C_{m+r,n-s} + C_{m-r,n-s}) \Bigg\} \Bigg] \cos m\theta \cos n\phi \\
w_y^2 &= \frac{a^2}{b^2} \sum_m \sum_n \Bigg[\sum_r \sum_s \frac{C_{rs}}{r} \Bigg\{ \frac{1}{m+r} (C_{m+r\,n+s} + C_{m+r,n-s}) \\
&\qquad - \frac{1}{m-r} (C_{m-r,n+s} + C_{m-r,n-s}) \Bigg\} \Bigg] \cos m\theta \cos n\phi \\
\text{and} \quad w_x w_y &= -\frac{a}{b} \sum_m \sum_n \Bigg[\sum_r \sum_s \frac{C_{rs}}{r} \Bigg\{ \frac{1}{n+s} (C_{m+r,n+s} - C_{m-r,n+s}) \\
&\qquad + \frac{1}{n-s} (C_{m+r,n-s} - C_{m-r,n-s}) \Bigg\} \Bigg] \sin m\theta \sin n\phi
\end{aligned}\right\} \qquad .. \quad (41)$$

Both the $m - n$ and the $r - s$ summations are taken over the whole range from 1 to infinity and the signs of the suffixes in $C_{m-r,n-s}$ etc. are disregarded so that $C_{\pm r,\pm s} \equiv C_{r,s}$; it is for convenience in this respect that the coefficients C_{rs} are divided by rs, so that $(1/rs) \sin r\theta \sin s\phi$ shall be unaltered by change of sign of either r or s. In the $r - s$ summation r and s in taking *all* values repeat coefficients C_{rs} otherwise obtained in forms $C_{m+r,n+s}$ etc. ; in fact each product $C_{rs} C_{m\pm r,n\pm s}$ occurs twice, except when it is a perfect square, when it occurs once only. The coefficients in w_x^2 and w_y^2 are merely repeated, but those in $w_x w_y$ have two different parts. The complete coefficients are thus :—

	in w_x^2	in w_y^2	in $- w_x w_y$	
of $C_{rs} C_{m+r,n+s}$	$2/s\,(n + s)$	$2/r\,(m + r)$	$1/r\,(n + s) + 1/(m + r)s$	
of $C_{rs} C_{m+r,n-s}$	$-2/s\,(n - s)$	$2/r\,(m + r)$	$1/r\,(n - s) - 1/(m + r)s$	
of $C_{rs} C_{m-r,n+s}$	$2/s\,(n + s)$	$-2/r\,(m - r)$	$-1/r\,(n + s) + 1/(m - r)s$	
of $C_{rs} C_{m-r,n-s}$	$-2/s\,(n - s)$	$-2/r\,(m - r)$	$-1/r\,(n - s) - 1/(m - r)s$	.. (41a)

Actually only the first line is needed, because the others follow by substituting $-r$ for r and/or $-s$ for s.

If shear is absent and if the plate is symmetrically loaded, s is restricted to odd values and therefore n to even values; or if there is only one value of r or s, the corresponding m or n is even, that is 0 and $2r$ or $2s$. These special cases are illustrated in §10.

(c) *Strains due Indirectly to Buckling.*—Disregarding the mean strains imposed (*cf.* §3), it may be assumed that the displacements in the plane of the plate are

$$u = (a/\pi) \sum_m \sum_n (1/m)\, U_{mn} \sin m\theta \cos n\phi$$

and

$$v = (b/\pi) \left\{ \sum_m \sum_n (1/n)\, V_{mn} \cos m\theta \sin n\phi + \tfrac{1}{2}(2\phi - \pi) \sum_m X_m \cos m\theta - (\pi/2) \sum_m Y_m \cos m\theta \right\} \quad \ldots \quad (42)$$

As in §3, these forms are based on the assumptions that the edges of the plate at $x = 0$ and $x = a$ are held straight but that its edges at $y = 0$ and $y = b$ may be free to wave; if these unloaded edges are also held straight the terms X_m and Y_m should be omitted.

Then
$$u_x = \sum_m \sum_n U_{mn} \cos m\theta \cos n\phi,$$
$$v_y = \sum_m \sum_n V_{mn} \cos m\theta \cos n\phi + \sum_m X_m \cos m\theta$$

and
$$u_y + v_x = -\sum_m \sum_n \left(\frac{n\,a}{mb} U_{mn} + \frac{mb}{n\,a} V_{mn}\right) \sin m\theta \sin n\phi - \left(\phi - \frac{\pi}{2}\right) \sum_m \frac{mb}{a} X_m \sin m\theta + \frac{\pi}{2} \sum_m \frac{mb}{a} Y_m \sin m\theta \quad \ldots \quad (43)$$

or
$$u_y + v_x = -\sum_m \sum_n \left(\frac{n\,a}{mb} U_{mn} + \frac{mb}{n\,a} V_{mn} - 2\,\frac{mb}{n\,a} X_m\right) \sin m\theta \sin n\phi$$

by expansion of $\phi - \dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ in series of sines, the term X_m becoming Y_m when n is odd. (The differences between the formulae (43) and (27) are due to the slightly varied specifications for u and v in (42) in comparison with (26)).

(d) *Strain Energy of Distortion.*—The strain energy of distortion U_d is the sum of integrals of the squares of the expressions (*cf.* equation (24))

$$u_x + v_y + \tfrac{1}{2}w_x^2 + \tfrac{1}{2}w_y^2 = \sum_m \sum_n \left(U + V + X + \tfrac{1}{2} W_{xx} + \frac{a^2}{2b^2} W_{yy}\right)_{mn} \cos m\theta \cos n\phi$$
$$u_x - v_y + \tfrac{1}{2}w_x^2 - \tfrac{1}{2}w_y^2 = \sum_m \sum_n \left(U - V - X + \tfrac{1}{2} W_{xx} - \frac{a^2}{2b^2} W_{yy}\right)_{mn} \cos m\theta \cos n\phi \quad \ldots \quad (44)$$
$$u_y + v_x + w_x w_y = \sum_m \sum_n \left(\frac{n\,a}{mb} U + \frac{mb}{n\,a} V - 2\,\frac{mb}{n\,a} X + \frac{a}{b} W_{xy}\right)_{mn} \sin m\theta \sin n\phi$$

Where W_{xx}, W_{yy} and W_{xy} represent the expressions within the square brackets in the formulae (41), and the suffixes m and n outside a bracket refer to all the terms within the bracket except X to which only m applies. In the first two formulae of (44) X appears only when n is zero; in the third this term is X_m for all even values of n and Y_m for all odd values.

By virtue of the orthogonal properties of the trigonometrical functions, substitution of those forms for $u_x + v_y + \frac{1}{2}w_x^2 + \frac{1}{2}w_y^2$ etc., in (24) yields the constant k of §8 as the sum for all possible pairs of m and n of the expressions

$$\tfrac{1}{2}J\left\{ (1+\sigma)\left(U + V + X + \tfrac{1}{2}W_{xx} + \frac{a^2}{2b^2}W_{yy}\right)^2 + (1-\sigma)\left(U - V - X + \tfrac{1}{2}W_{xx} - \frac{a^2}{2b^2}W_{yy}\right)^2 + (1-\sigma)\left(\frac{na}{mb}U + \frac{mb}{na}V - 2\frac{mb}{na}X + \frac{a}{b}W_{xy}\right)^2\right\} \quad \ldots \quad (45)$$

where $J = \frac{1}{4}$ if neither m nor n is zero
$= \frac{1}{2}$ if either m or n is zero
$= 1$ if both m and n are zero
(these factors result from integration)

and where X is omitted or replaced by Y according to the schedule above.

Since our present purpose excludes consideration of shear and may reasonably be restricted to symmetrical cases, from now on it will be assumed then s is odd and therefore that n is even; on that basis the coefficients Y_m are no longer needed.

Provided that U_{mn} and V_{mn} are not restricted by boundary conditions their values may be chosen so as to render the expression (45) a minimum and it is proved in Appendix I that the minimum value of $(1+\sigma)(U + V + A)^2 + (1-\sigma)(U - V + B)^2 + (1-\sigma)(pU + V/p + C)^2$ is

$$\tfrac{1}{2}(1-\sigma^2)\left\{\left(p + \frac{1}{p}\right)A + \left(p - \frac{1}{p}\right)B - 2C\right\}^2 \Big/ \left(p + \frac{1}{p}\right)^2$$

or

$$\tfrac{1}{2}(1-\sigma^2)\left\{p(A+B) + \frac{(A-B)}{p} - 2C\right\}^2 \Big/ \left(p + \frac{1}{p}\right)^2 \quad \ldots \quad (46)$$

and p in this application is na/mb. When neither m nor n is zero, $A + B = W_{xx}$, $A - B = (a^2/b^2)W_{yy}$ (X disappears from the first two terms of (45)) and $C = (a/b)W_{xy} - 2(mb/na)X$ so that the minimum value of (45) for these terms is

$$\tfrac{1}{16}(1-\sigma^2)\frac{a^2}{b^2}\sum_m\sum_n\left\{\frac{n}{m}W_{xx} + \frac{m}{n}W_{yy} - 2W_{xy} + 4\frac{mb^2}{na^2}X\right\}^2 \Big/ \left(\frac{na}{mb} + \frac{mb}{na}\right)^2 \quad \ldots \quad (47)$$

excluding m or n zero.

When $m = 0$ and $n \neq 0$, the last term of (45) disappears, because $\sin m\theta$ is zero, and for the same reason u contributes nothing* to the expression (45). X is to be omitted from the first two terms of this expression, but in any case it is clear that the minimum value is independent of X and of W_{yy} also. In fact this minimum value is

$$\tfrac{1}{8}(1-\sigma^2)\sum_n W_{xx}^2 \qquad \text{for } m = 0 \quad \ldots \quad (48)$$

which is twice the limit to which (47) reduces when $m \to 0$; the factor 2 results from integration.

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* A contribution from u to these terms would be made if the edges at $x = 0$ and $x = a$ were not held straight.

When $m \neq 0$ and $n = 0$, the last term in (45) disappears, and V disappears from the first two terms, but the term X representing v displacements remains. The minimum value of these terms is then

$$\tfrac{1}{8}(1-\sigma^2)\sum_m \{2X + (a/b)^2 W_{yy}\}^2 \quad \text{for } n = 0 \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

When both m and n are zero, the last term of (45) and U and V in the first two terms all disappear, whilst X_0 (which would represent e_y) is taken as zero. The value of this term is therefore

$$\tfrac{1}{8}(1+\sigma)\{W_{xx} + (a/b)^2 W_{yy}\}^2 + \tfrac{1}{8}(1-\sigma)\{W_{xx} - (a/b)^2 W_{yy}\}^2$$

or

$$\tfrac{1}{4}\{W_{xx}^2 + 2\sigma(a/b)^2 W_{xx}W_{yy} + (a/b)^4 W_{yy}^2\} \quad \text{for } m = n = 0 \quad \ldots \quad (50)$$

which is invariable.

(*e*) *Check on the Formula* (47) *to* (50) *by Application to the Simple Case Treated in* §10.—Using formulae (41) and taking $C_{r1} = 1$, all other C's zero, the coefficients of W^4 in W_{xx} etc., are

	$m=0, n=0$	$m=0, n=2$	$m=2r, n=0$	$m=2r, n=2$
W_{xx}/W^2	1	-1	1	-1
W_{yy}/W^2	$1/r^2$	$1/r^2$	$-1/r^2$	$-1/r^2$
W_{xy}/W^2	—	—	—	$-1/r$

Then for the case $m = 0$, $n = 0$, formula (50) gives $\frac{1}{4}\left(1 + \frac{2\sigma}{c^2} + \frac{1}{c^4}\right)$ where $c = rb/a$ as in §10; for the case $m = 2r$, $n = 0$, formula (49) gives $\frac{1}{2}(1-\sigma^2)\{(X/W^2) - (1/2c^2)\}^2$; for the case $m = 0$, $n = 2$, formula (48) gives $(1-\sigma^2)/8$. For the cases $m = 2r$, $n = 2$, 4, 6 etc., formula (47) includes W_{xx}, W_{yy} and W_{xy} only for the case $m = 2r$, $n = 2$ and for that case $\frac{n}{m}W_{xx} + \frac{m}{n}W_{yy} - 2W_{xy}$ is identically zero. Therefore formula (47) reduces to $(1-\sigma^2)\sum_{n\ even}\left\{\left(\frac{2c}{n}\right)^2\left(\frac{X}{W^2}\right)^2\Big/\left(\frac{2c}{n}+\frac{n}{2c}\right)^2\right\}$ or $(1-\sigma^2)\sum_s \frac{c^4}{(s^2+c^2)^2}\left(\frac{X}{W^2}\right)^2$ where the s summation is now over all integral values. It will be seen that the sum of these four terms corresponds to formula (34).

(*f*) *Effect of Freedom of Unloaded Edges to Wave.*—In general X_m (with a specific value of m) occurs once in (49) and in a single n—array in (47); in fact the terms including X_m are

$$\tfrac{1}{16}(1-\sigma^2)\left[2\{2X + (a/b)^2 W_{yy}\}_{m0}^2 + \frac{a^2}{b^2}\sum_n\left\{R + 4\frac{mb^2}{na^2}X_m\right\}^2\Big/\left(\frac{na}{mb}+\frac{mb}{na}\right)^2\right] \quad \ldots \quad (51)$$

where R stands for $\left(\frac{n}{m}W_{xx} + \frac{m}{n}W_{yy} - 2W_{xy}\right)_{mn}$.

Omitting the factor $\frac{1}{16}(1-\sigma^2)$ and the suffix m from X_m and expanding, the expression (51) may be written

$$8X^2 + 8X\left(\frac{a}{b}\right)^2(W_{yy})_0 + 2\left(\frac{a}{b}\right)^4(W_{yy})_0^2$$
$$+ \sum_n\left(\frac{a^2}{b^2}R^2 + 8R\frac{m}{n}X + 16\frac{m^2b^2}{n^2a^2}X^2\right)\Big/\left(\frac{mb}{na}+\frac{mb}{na}\right)^2 \quad \ldots \quad (52)$$

the summation with respect to n being over all even values from 2 upwards.

But
$$\sum_{n \text{ even}} 2\,\frac{m^2b^2}{n^2a^2} \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2 = \sum_{\text{all } s} 2c^4/(s^2 + c^2)^2$$

where
$$c = \frac{mb}{2a} \left(\text{or } \frac{rb}{a} \text{ if } m = 2r\right)$$

and this sum has been previously defined in (33), § 10, as H. Therefore (52) may be rewritten as

$$8\,(H+1)\,X^2 + 8X\left[(a/b)^2\,(W_{yy})_0 + \sum_{n=2}\left\{\frac{m}{n}\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_n \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2\right\}\right]$$
$$+ 2\,(a/b)^4\,(W_{yy})_0{}^2 + \sum_{n=2}\left\{\frac{a^2}{b^2}\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_n^2 \Big/ \left(\frac{n\,a}{mb} + \frac{n\,a}{mb}\right)^2\right\} \quad \ldots \quad (53)$$

But $(a/b)^2\,(W_{yy})_0$ is the limit of $\frac{m}{n}\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_n \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2$ when $n \to 0$, so that the expression may be slightly simplified in the form

$$8\,(H+1)\,X^2 + 8X \sum_{n=0}\left\{\frac{m}{n}\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_n \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2\right\}$$
$$+ (a/b)^4\,(W_{yy})_0{}^2 + \frac{a^2}{b^2}\sum_{n=0}\left\{\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_n^2 \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2\right\} \quad \ldots \quad (54)$$

By completing the square in terms containing X, the least value of the sum of terms in (47) and (49) when X may take any value is

$$\tfrac{1}{16}\,(1-\sigma^2)\sum_m\left[\frac{a^4}{b^4}\,(W_{yy})_{m0}{}^2 + \frac{a^2}{b^2}\sum_{n=0}\left\{\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_{mn}^2 \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2\right\}\right.$$
$$\left. - \frac{2}{H+1}\left\{\sum_{n=0}\frac{m}{n}\left(\frac{n}{m}\,W_{xx} + \frac{m}{n}\,W_{yy} - 2W_{xy}\right)_{mn} \Big/ \left(\frac{n\,a}{mb} + \frac{mb}{n\,a}\right)^2\right\}^2\right] \quad \ldots \quad (55)$$

the n summations being over even values including zero and H being given by (33) with $c = mb/2a$.

If the restraint against waving of the unloaded edges of the plate is afforded by edge members which resist bending, the energy associated with this flexure is (cf. formulae (42)) proportional to X^2. Thus inclusion of this energy would have the effect of increasing the value of H, and for infinitely stiff edge members, $H \to \infty$ (cf. § 10). Inclusion of a parameter to represent the flexural resistance of edge members is thus straightforward and would not render the formula (55) unduly cumbersome. On the other hand in practice any considerable restraint against waving of the unloaded edges is more likely to result from conditions of continuity laterally of adjacent plates ; consideration of this type of restraint would necessitate revision of (42), because the X and Y terms are appropriate only in the range $0 < \phi < \pi$. This development for a flat panel composed of several plates in parallel is straightforward, but the results would be cumbersome. Moreover in practice the effect of lateral curvature is likely to be so predominantly important in determining the degree of restraint against waving that the analysis for a flat panel might be misleading rather than helpful. Accordingly as in § 10, attention will be paid mainly to the two limiting cases, edges entirely free to wave, and edges held straight ; intermediate conditions will be discussed only in general terms. On that basis the constant k is made up of the three terms (48), (50) and (55) with a value of H between the limits of (33) and infinity.

(g) *Specification of k in terms of C's.*—From (41), when $m = 0$ and $n = 0$, $W_{xx} = \sum_r \sum_s C_{rs}^2/s^2$, $W_{yy} = \sum_r \sum_s C_{rs}^2/r^2$ and W_{xy} disappears. Formula (50) may then be written in the form

$$\tfrac{1}{4} \sum_r \sum_s \sum_R \sum_S \left\{ \frac{1}{s^2 S^2} + \frac{\sigma a^2}{b^2}\left(\frac{1}{s^2 R^2} + \frac{1}{S^2 r^2}\right) + \frac{a^4}{b^4 r^2 R^2} \right\} C_{rs}\, C_{RS} \qquad \ldots \quad (56)$$

where the summations are taken over full ranges, so that terms for which either $r \neq R$ or $s \neq S$ are repeated, whilst terms in C_{rs}^4 appear once only. Similarly using (41), formula (48) becomes

$$\tfrac{1}{8}(1 - \sigma^2) \sum_n \left\{ \sum_r \sum_s \frac{C_{rs}}{s}\left(\frac{C_{r,\,n+s}}{n + s} - \frac{C_{r,\,n-s}}{n - s}\right) \right\}^2 \qquad \ldots \quad (57)$$

where summation with respect to n excludes $n = 0$, and the rs summations are taken over full ranges. Similarly again, formula (49) with $X = 0$ becomes

$$\tfrac{1}{8}(1 - \sigma^2)\frac{a^4}{b^4} \sum_m \left\{ \sum_r \sum_s \frac{C_{rs}}{r}\left(\frac{C_{m+r,\,s}}{m + r} - \frac{C_{m-r,\,s}}{m - r}\right) \right\}^2 \qquad \ldots \quad (58)$$

The complete coefficient of $C_{rs}\, C_{m+r,\,m+s}$ in

$$\left(\frac{n}{m} W_{xx} + \frac{m}{n} W_{yy} - 2W_{xy}\right)$$

is

$$\frac{2n}{ms\,(n + s)} + \frac{2m}{nr\,(m + r)} - \frac{2}{r\,(n + s)} - \frac{2}{(m + r)\,s}$$

(by using (41a)), which reduces to $2(nr - ms)^2/mnrs\,(m + r)\,(n + s)$; the coefficients of $C_{rs}\, C_{m-r,n+s}$, etc., are similar with $-r$ for r, etc. If the cases $m = 0$ and $n = 0$ are excluded, the only possible square is $C_{rs}\, C_{m-r,\,n-s}$ with $m = 2r$ and $n = 2s$; but the coefficient of this term is identically zero. Yet it is still convenient to write

$$\left(\frac{n}{m} W_{xx} + \frac{m}{n} W_{yy} - 2W_{xy}\right) \quad \text{as} \quad \sum \sum_s \{ (nr - ms)^2/mnrs\,(m + r)\,(n + s) \}\, C_{rs}\, C_{m+r,\,n+s}$$

and let r and s range over all values so that each term is repeated. Using this form in (55), the complete expression for the constant k is the product of $\frac{1}{16}(1 - \sigma^2)$ and

$$\frac{4}{1 - \sigma^2} \sum_r \sum_s \sum_R \sum_S \left\{ \frac{1}{s^2 S^2} + \frac{\sigma a^2}{b^2}\left(\frac{1}{s^2 R^2} + \frac{1}{S^2 r^2}\right) + \frac{a^4}{b^4 r^2 R^2} \right\} C_{rs}^2\, C_{RS}^2$$
$$+ 2 \sum_n \left\{ \sum_r \sum_s \frac{C_{rs}}{s}\left(\frac{C_{r,\,n+s}}{n + s} - \frac{C_{r,\,n-s}}{n - s}\right) \right\}^2 + \frac{2a^4}{b^4} \sum_m \left\{ \sum_r \sum_s \frac{C_{rs}}{r}\left(\frac{C_{m+r,\,s}}{m + r} - \frac{C_{m-r,\,s}}{m - r}\right) \right\}^2$$
$$+ \frac{a^2}{b^2} \sum_m \sum_n \left\{ \sum_r \sum_s \frac{(nr - ms)^2}{mnrs\,(m + r)\,(n + s)}\, C_{rs}\, C_{m+r,\,n+s} \right\}^2 \Big/ \left(\frac{na}{mb} + \frac{mb}{na}\right)^2$$
$$- \sum_m \frac{2}{H + 1} \left\{ \sum_{n=0} \sum_r \sum_s \frac{(nr - ms)^2}{n^2 rs\,(m + r)\,(n + s)}\, C_{rs}\, C_{m+r,\,n+s} \Big/ \left(\frac{na}{mb} + \frac{mb}{na}\right)^2 \right\}^2 \qquad \ldots \quad (59)$$

where

$$H = \frac{\pi c}{2} \coth \pi c + \frac{\pi^2 c^2}{2 \sinh^2 \pi c} - 1,$$

and $c = mb/2a$, or $H \to$ infinity.

All the mn summations exclude zero, except the n-summation in the last term.* All the rs summations are over *all* values, so that products which are not perfect squares are repeated. In the fourth and fifth terms the rs summation includes both positive and negative values, in the manner indicated at length in the second and third terms.

(*h*) *Comments on the Application of Formula* (59).—When C_{r1} alone is different from zero, the value of k from (59) is

$$\tfrac{1}{16}\,(1-\sigma^2)\left\{\frac{4}{(1-\sigma^2)}\left(1+\frac{2\sigma}{c^2}+\frac{1}{c^4}\right)+2+\frac{2}{c^4}+0-\frac{2}{(H+1)\,c^4}\right\}$$

where $c = rb/a$, which agrees with (36). Unfortunately the application of (59) to more general cases is difficult because the quadruple summations are extremely awkward. The abnormal complexity of this stage of the analysis so seriously increases the risk of casual error that at least three independent checks on any one expansion proves desirable. Obviously on this basis extensive application of (59) is virtually impossible. Expansion of (59) must be regarded as a task comparable to the preparation of mathematical tables and should be subjected to the checks and comparisons usually applied in that undertaking. Indeed the logical course would undoubtedly be to prepare tables of expansion of (59) in terms of typical sub-sets of C's, e.g. C_{rs} with r constant and s odd integers from unity upwards.

In § 13 below the formula (59) is applied to the group $C_{r1}\,C_{q1}$, as a basis for discussion of change of wavelength after buckling in § 14 ; § 13 illustrates the complexity of expansions of (59).

13. *Interaction between Wave Forms of Different Wavelengths in the Direction of Compression.*—If C_{r1} and C_{q1} alone differ from zero, the expansion of (59) term by term is

1st term
$$\frac{4}{1-\sigma^2}\left[\left(1+\frac{2\sigma a^2}{r^2b^2}+\frac{a^4}{r^4b^4}\right)C_r^{\,4}\right.$$
$$\left.+\,2\left\{1+\frac{\sigma a^2}{b^2}\left(\frac{1}{r^2}+\frac{1}{q^2}\right)+\frac{a^4}{r^2q^2b^4}\right\}C_r^{\,2}\,C_q^{\,2}+\left(1+\frac{2\sigma a^2}{q^2b^2}+\frac{a^4}{q^4b^4}\right)C_q^{\,4}\right]$$

2nd term
$$2\,(-\,C_r^{\,2}-C_q^{\,2})^2 \qquad \text{for } n=2 \text{ only}$$

3rd term
$$2\,(a/b)^4\left\{(-\,C_r^{\,2}/r^2)^2_{m=2r}+(-\,C_q^{\,2}/q^2)^2_{m=2q}+(2C_rC_q/rq)^2_{m=q-r}+(-\,2C_rC_q/rq)^2_{m=q+r}\right\}$$

(the subscripts indicate the values of m by which the individual terms are characterized ; each of the last two terms has two equal parts in forms $(C_rC_q/rq) - (-\,C_rC_q/rq)$)

4th term
$$\left\{\frac{(2r+q-r)^2}{-\,2\,(q-r)\,rq}\,C_rC_q+\frac{(-\,2q+q-r)^2}{-\,2\,(q-r)\,qr}\,C_qC_r\right\}^2\Big/\left(\frac{2}{q-r}+\frac{q-r}{2}\,\frac{b^2}{a^2}\right)^2 \quad \text{for } m=q-r,\ n=2$$
$$+\left\{\frac{(q-r^2)}{(q+r)\,rq}\,C_rC_q\right\}^2\Big/\left(\frac{2}{q+r}+\frac{q+r}{2}\,\frac{b^2}{a^2}\right)^2 \quad \text{for } m=q+r,\ n=2$$

(the first term is written at length to demonstrate its derivation)

---

* The second and third terms of (59) are *twice* the limits of the fourth term when $m = 0$ and $n = 0$ respectively. By extending summations in this fourth term to include zero, *half* the second and third terms would be absorbed in the fourth (cf. Formula (55)) ; but since the two former terms must still be written, it is more convenient to retain the factors 2 and to limit the summation in the fourth term. In the fifth term summation including $n = 0$ enables this zero term to be absorbed completely in the general form.

5th term.—The general form for each value of m is

$$-\frac{2}{(H+1)}\frac{a^4}{b^4}\Big[\Big\{\frac{4a^2}{m^2b^2}+2-r\,(m+r)\,\frac{b^2}{a^2}\Big\}\,C_rC_{m+r}\Big/\Big(\frac{2a}{mb}+\frac{mb}{2a}\Big)^2\,r\,(m+r)\Big]^2$$

which leads to the terms

$$-\frac{2a^4}{b^4}\Big[\frac{1}{H_{2r}+1}\Big(\frac{C_r^{\,2}}{r^2}\Big)^2+\frac{1}{H_{2q}+1}\Big(\frac{C_q^{\,2}}{q^2}\Big)^2+\Big(\frac{J_{q-r}^{\,2}}{H_{q-r}+1}+\frac{J_{q+r}^{\,2}}{H_{q+r}+1}\Big)\Big(\frac{2C_qC_r}{qr}\Big)^2\Big]$$

where $$J_{q\pm r}=\Big\{\frac{4a^2}{(q\pm r)^2b^2}+2\pm qr\,\frac{b^2}{a^2}\Big\}\Big/\Big\{\frac{2a}{(q\pm r)b}+\frac{(q\pm r)b}{2a}\Big\}^2$$

and the suffixes to H denote the value of c in (33) e.g. H_{2r} implies $c=(2r)\,(b/2a)$. By regrouping the third and fourth terms may be simplified slightly; but the other three terms are not so amenable. It is therefore possible to make little progress until numerical values are assigned to the ratio a/b and σ, when reasonable values of q and r may be chosen. This is illustrated below in §14, but the following general comments may be made

Assuming that the ratio a/b is fairly large (say > 2), and that both q and r approximate to a/b, $q-r$ may be small in comparison with $q+r$. In that case the second part of the fourth term above is almost negligible in comparison with the first. In the fifth term both $J_{q\pm r}\rightarrow 1$; but H_{q-r} may be considerably less than any of the other H's. Thus the fifth term tends to reduce the coefficient of $C_r^{\,2}\,C_q^{\,2}$ in k more than it reduces the coefficients of $C_r^{\,4}$ and $C_q^{\,4}$; that this may have an important influence on change of buckled form is demonstrated in §14.

14. *Interaction between Buckles in Two and Three Half-Waves in a Plate for which* $a/b=2\cdot6$ and $\sigma=0\cdot3$.—(This plate would normally buckle in three half-waves; but two half-waves would be forced by a moderate degree of lateral restraint (cf. Table 1)). Using the expansion of (59) in §13, the value of k is the product of $\frac{1}{16}(1-\sigma^2)$ and

$$\{4/(1-\sigma^2)\}\,(4\cdot870C_2^{\,4}+6\cdot003C_2^{\,2}\,C_3^{\,2}+2\cdot015C_3^{\,4})+2\,(C_2^{\,2}+C_3^{\,2})^2$$
$$+\,(5\cdot719C_2^{\,4}+20\cdot310C_2^{\,2}\,C_3^{\,2}+1\cdot128C_3^{\,4})+(4\cdot036+0\cdot002)\,C_2^{\,2}\,C_3^{\,2}$$
$$-\,2\,(2\cdot6)^4\,[(1/1\cdot322)\,(C_2^{\,2}/4)^2+(1/1\cdot832)\,(C_3^{\,2}/9)^2$$
$$+\,\{(0\cdot937/1\cdot003)+(0\cdot982/1\cdot562)\}\,(C_2\,C_3/3)^2]$$

Thus

$$k=1\cdot656C_2^{\,4}+3\cdot113C_2^{\,2}\,C_3^{\,2}+0\cdot682\,C_3^{\,4}-(0\cdot246\,C_2^{\,4}+0\cdot903\,C_2^{\,2}\,C_3^{\,2}+0\cdot035\,C_3^{\,4})\quad\ldots\quad(60)$$

if the unloaded edges of the plate are quite free to wave or the first three terms only if these edges are held straight.

This coefficient k represents all the terms of order C^4 in $2U_d/E'abt$ (cf. formula (13) of §8). The remainder of $2U_d/E'abt$ is

$$e_x^{\,2}+e_y^{\,2}+2\sigma e_xe_y-(e_x/ab)\iint(w_x^{\,2}+\sigma w_y^{\,2})\,dxdy-(e_y/ab)\iint(w_y^{\,2}+\sigma w_x^{\,2})dxdy\quad\ldots\quad(61)$$

and

$$\frac{1}{ab}\iint w_x^{\,2}\,dxdy=C_2^{\,2}+C_3^{\,2}\ \text{and}\ \frac{1}{ab}\iint w_y^{\,2}\,dxdy=\frac{a^2}{b^2}\,(\tfrac{1}{4}\,C_2^{\,2}+\tfrac{1}{9}\,C_3^{\,2})$$

Moreover $2U_w/E'abt = \frac{1-\sigma^2}{4}\left\{\left(\frac{2b}{a}+\frac{a}{2b}\right)^2 C_2{}^2 + \left(\frac{3b}{a}+\frac{a}{3b}\right)^2 C_3{}^2\right\} e_0$ .. .. (62)

(the factor $1-\sigma^2$ is reintroduced because $e_0 = \pi^2 t^2/3\,(1-\sigma^2)b^2$). Writing (60) in the form $\frac{1}{2}\,PC_2{}^4 + QC_2{}^2\,C_3{}^2 + \frac{1}{2}\,RC_3{}^4$, where $3{\cdot}312 > P > 2{\cdot}820$, $3{\cdot}113 > Q > 2{\cdot}210$ and $1{\cdot}364 > R > 1{\cdot}294$, the total energy is proportional to

$$e_x{}^2 + e_y{}^2 + 0{\cdot}6e_xe_y - (1{\cdot}507e_x + 1{\cdot}990e_y)\,C_2{}^2 - (1{\cdot}225e_x + 1{\cdot}051e_y)\,C_3{}^2$$
$$+ (0{\cdot}974C_2{}^2 + 0{\cdot}929C_3{}^2)\,e_0 + \tfrac{1}{2}\,PC_2{}^4 + QC_2{}^2\,C_3{}^2 + \tfrac{1}{2}\,RC_3{}^4 \quad .. \quad (63)$$

Then the load (in direction x) is proportional to

$$e_x + 0{\cdot}3e_y - 0{\cdot}753\,C_2{}^2 - 0{\cdot}612\,C_3{}^2 \quad .. \quad .. \quad .. \quad .. \quad (64)$$

the lateral load (in direction y) is proportional to

$$0{\cdot}3e_x + e_y - 0{\cdot}995\,C_2{}^2 - 0{\cdot}525\,C_3{}^2 \quad .. \quad .. \quad .. \quad .. \quad (65)$$

and for equilibrium of the buckled form

$$PC_2{}^2 + QC_3{}^2 = 1{\cdot}507e_x + 1{\cdot}990e_y - 0{\cdot}974e_0 \quad \text{or } C_2 = 0 \quad .. \quad (66)$$

and

$$QC_2{}^2 + RC_3{}^2 = 1{\cdot}225e_x + 1{\cdot}051e_y - 0{\cdot}929e_0 \quad \text{or } C_3 = 0 \quad .. \quad (67)$$

If the lateral load is throughout proportional to e_y, so that, when $C_2 = C_3 = 0$, $e_y = -\varrho e_x$, the formula (65) becomes $e_y = -\varrho e_x + \frac{\varrho}{0{\cdot}3}\,(0{\cdot}995C_2{}^2 + 0{\cdot}525C_3{}^2)$. Substituting for e_y in (64) the average stress is

$$f_a = E'\,\{(1 - 0{\cdot}3\varrho)e_x - (0{\cdot}753 - 0{\cdot}995\varrho)\,C_2{}^2 - (0{\cdot}612 - 0{\cdot}525\varrho)\,C_3{}^2\}\,. \quad .. \quad .. \quad (68)$$

Substituting for e_y in (66) and (67),

$$(P - 6{\cdot}600\varrho)\,C_2{}^2 + (Q - 3{\cdot}487\varrho)\,C_3{}^2 = (1{\cdot}507 - 1{\cdot}990\varrho)e_x - 0{\cdot}974e_0 \quad \text{or } C_2 = 0 \quad .. \quad (69)$$

$$(Q - 3{\cdot}487\varrho)\,C_2{}^2 + (R - 1{\cdot}840\varrho)\,C_3{}^2 = (1{\cdot}225 - 1{\cdot}051\varrho)e_x - 0{\cdot}929e_0 \quad \text{or } C_3 = 0 \quad .. \quad (70)$$

It is convenient to write

$$E'(1 - 0{\cdot}3\varrho)e_x = f_e \text{ (cf. §8)},\; E'\,(1 - 0{\cdot}3\varrho)e_0 = f_0{}',^*$$
$$E'\,(1 - 0{\cdot}3\varrho)\,C_2{}^2 = f_2 \text{ and } E'\,(1 - 0{\cdot}3\varrho)\,C_3{}^2 = f_3.$$

Then (68) (69) and (70) become

$$f_a = f_e - \{(0{\cdot}753 - 0{\cdot}995\varrho)\,f_2 + (0{\cdot}612 - 0{\cdot}525\varrho)\,f_3\}/(1 - 0{\cdot}3\varrho) \quad .. \quad .. \quad (71)$$

$$(P - 6{\cdot}600\varrho)\,f_2 + (Q - 3{\cdot}487\varrho)\,f_3 = (1{\cdot}507 - 1{\cdot}990\varrho)\,f_e - 0{\cdot}974f_0{}' \quad \text{or } f_2 = 0 \quad .. \quad (72)$$

$$(Q - 3{\cdot}487\varrho)\,f_2 + (R - 1{\cdot}840\varrho)\,f_3 = (1{\cdot}225 - 1{\cdot}051\varrho)\,f_e - 0{\cdot}929f_0{}' \quad \text{or } f_3 = 0 \quad .. \quad (73)$$

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* $f_0{}' = \frac{1-\varrho\sigma}{1-\sigma^2}\,f_0$: cf definition of $f_0 = Ee_0$ in §5.

Taking $\varrho = 0\cdot157$ (cf. Tables 1, 3 and 4), these formulae become

$$f_a = f_e - 0\cdot627f_2 - 0\cdot556f_3 \quad \ldots \quad (74)$$

$$P'f_2 + Q'f_3 = 1\cdot195f_e - 0\cdot974f_0' \quad \text{or } f_2 = 0 \quad \ldots \quad (75)$$

and $$Q'f_2 + R'f_3 = 1\cdot060f_e - 0\cdot929f_0' \quad \text{or } f_3 = 0 \quad \ldots \quad (76)$$

where $2\cdot276 > P' > 1\cdot784$, $2\cdot565 > Q' > 1\cdot662$ and $1\cdot075 > R' > 1\cdot005$.

At low values of f_e, $f_2 = f_3 = 0$ and $f_a = f_e$; when $f_e/f_0' > 0\cdot974/1\cdot195 = 0\cdot815$, $f_2 = (1\cdot195f_e - 0\cdot974\,f_0')/P'$, $f_3 = 0$ and $f_a = (1 - 0\cdot750/P')\,f_e + (0\cdot611/P')\,f_0'$. Neither f_2 nor f_3 can of course be negative, so that for f_3 to differ from zero by the first part of (76), it is necessary that $1\cdot060f_e - 0\cdot929f_0' > (1\cdot195f_e - 0\cdot974f_0')\,Q'/P'$, or

$$f_e/f_0' > (0\cdot929 - 0\cdot974Q'/P')/(1\cdot060 - 1\cdot195Q'/P') \,. \quad \ldots \quad (77)$$

If $Q'/P' > 1$, this inequality cannot be satisfied by any value of f_e/f_0' greater than $0\cdot815$. Therefore, if the unloaded edges of the plate are held straight or severely restrained from waving, the two half-wave form of buckle is stable and entirely excludes buckling in three half-waves. Moreover, if the plate edges are entirely free, so that the lower limits of P' and Q' are appropriate, $1\cdot060 - 1\cdot195\,Q'/P'$ is again negative, and the change from two to three half-waves is still inhibited.

If the unloaded edges of the plate are moderately restrained against waving, this restraint is likely to be greater against displacement of the edge in four, five or six half-waves than against displacement in one half-wave. Therefore P' may advance towards its upper limit more rapidly than Q' (the range of variation of R' is so small that the exact value of R' is unimportant). For this reason the set of values $P' = 2\cdot2$, $Q' = 1\cdot8$ and $R' = 1\cdot05$ may be a fair estimate for a case in which the restraint against waving is moderately severe. Substituting these values in (75) and (76), solving for f_2 and f_3 and substituting in (74) leads to the relations

$$\left.\begin{aligned} f_2 &= 0\cdot7024f_e - 0\cdot6984f_0' \\ f_3 &= -0\cdot1946f_e + 0\cdot3125f_0' \\ \text{and} \quad f_a &= 0\cdot668f_e + 0\cdot264f_0' \end{aligned}\right\} \text{ for } 0\cdot995 < f_e/f_0' < 1\cdot605 \text{ because } f_2 \text{ and } f_3 \text{ must both be positive.}$$

For $0\cdot815 < f_e/f_0' < 1\cdot605$, $f_3 = 0$ and $f_a = 0\cdot660f_e + 0\cdot277f_0'$ and for $f_e/f_0' > 0\cdot995$, $f_2 = 0$ and $f_a = 0\cdot439f_e + 0\cdot492f_0'$. The inversion of the limits for f_e/f_0' in the last two formulae implies that the dual wave part of the $f_a - f_e$ diagram is actually regressive, both f_a and f_e decreasing as the change from two to three half-waves is made. This behaviour is illustrated in Fig. 2 and in Ref. 4, Fig. 9; but Fig. 2 actually shows the case with $P' = 2\cdot0$, $Q' = 1\cdot68$ and $R' = 1\cdot05$ because of the three $f_a - f_e$ lines, namely

$$f_a = 0\cdot625f_e + 0\cdot305f_0', \quad 0\cdot815 < f_e/f_0' < 1\cdot630$$

$$f_a = 0\cdot637f_e + 0\cdot287f_0', \quad 1\cdot630 > f_e/f_0' > 1\cdot035$$

$$f_a = 0\cdot439f_e + 0\cdot492f_0', \quad f_e/f_0' > 1\cdot035$$

the first two are slightly less nearly concurrent and may, therefore, be shown more clearly. However, the former case with $P' = 2\cdot20$, etc., appears to fit the experimental results of Ref. 4, Fig. 9 rather better than the case illustrated by Fig. 2. Of course in any normal test rig the regressive portion of the $f_a - f_e$ diagram will be suppressed.

15. *Conclusion.*—The chief omission from the present survey of the behaviour of flat rectangular plates under compression is examination of the effect of torsional restraint at the unloaded edges on their behaviour after buckling. Until this examination shall have been made, either by application of §12 or otherwise, no close estimate of the behaviour in typical practical cases can be attempted, and so long as this uncertainty remains unresolved, it is pointless to discuss how the several other factors mentioned in §7 may affect the post-buckling behaviour.

Nevertheless the results of §11 clearly demonstrate that the present common practice of representing the behaviour of a buckled plate merely by reference to the ratio t/b is unlikely to be sufficiently accurate if the plates are to carry any considerable portion of the total load. At the same time the need for greater precision in representing the behaviour of the plate cannot be met unless the loading conditions may be stated more precisely. This point is well illustrated in §14, where the transition from two to three half-waves (disastrous in increasing f_e) may or may not be inhibited according to the precise degree of constraint imposed on the unloaded edges of the plate.

One further point perhaps deserves comment. The behaviour of a plate after buckling is affected more by variation of the wave-length λ than by any other single factor, and if an abnormally long wave-length can be forced and *maintained* relatively high ratios of f_a to f_e result. Therefore, when thick covers are used so that the loads carried by the plates are considerable, it might be worth while to fit lateral stiffeners between the actual ribs at a spacing of (say) 1·3 times the width of the plates. These stiffeners could be relatively very light and yet be sufficient to prevent buckling except at the long wave-length. The advantage thus gained in the ratio of f_a to f_e should be considerable. In addition the buckle over a long wave-length might be expected to delay the onset of permanent deformation; but this possible gain is not certain because the curvature laterally would be slightly increased.

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(These references are incidental only: a complete list would be of considerable length.)

No.	*Authors*	*Title, etc.*
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APPENDIX I

The minimum value of $(1 + \sigma)(U + V + A)^2 + (1 - \sigma)(U - V + B)^2 + (1 - \sigma)(pU + V/p + C)^2$ *when U and V may be varied*

The expression may be written in the form

$$(1 + \sigma) X^2 + (1 - \sigma) Y^2 + (1 - \sigma) \{ \tfrac{1}{2}(p + 1/p) X + \tfrac{1}{2}(p - 1/p) Y + D \}^2$$

where $D = C - \tfrac{1}{2}(p + 1/p)A - \tfrac{1}{2}(p - 1/p) B$

Then $$2(1 + \sigma) X + (1 - \sigma)\left(p + \frac{1}{p}\right)\left\{\frac{1}{2}\left(p + \frac{1}{p}\right) X + \frac{1}{2}\left(p - \frac{1}{p}\right) Y + D\right\} = 0$$

and $$2(1 - \sigma) Y + (1 - \sigma)\left(p - \frac{1}{p}\right)\left\{\frac{1}{2}\left(p + \frac{1}{p}\right) X + \frac{1}{2}\left(p - \frac{1}{p}\right) Y + D\right\} = 0.$$

Hence $$(1 + \sigma)\left(p - \frac{1}{p}\right) X = (1 - \sigma)\left(p + \frac{1}{p}\right) Y = Q \text{ say}$$

and $$\left\{2(1 + \sigma) + \tfrac{1}{2}(1 - \sigma)\left(p + \frac{1}{p}\right)^2 + \tfrac{1}{2}(1 + \sigma)\left(p - \frac{1}{p}\right)^2\right\} Q + (1 - \sigma^2)\left(p^2 - \frac{1}{p^2}\right) D = 0$$

or $$Q = (1 - \sigma^2)\left(p - \frac{1}{p}\right) D \Big/ \left(p + \frac{1}{p}\right).$$

Then $$X = -(1 - \sigma) D \Big/ \left(p + \frac{1}{p}\right), \quad Y = (1 + \sigma)\left(p - \frac{1}{p}\right) D \Big/ \left(p + \frac{1}{p}\right)^2$$

and $$\frac{1}{2}\left(p + \frac{1}{p}\right) X + \frac{1}{2}\left(p - \frac{1}{p}\right) Y + D = 2(1 + \sigma) D \Big/ \left(p + \frac{1}{p}\right)^2$$

so that the minimum value is $2(1 - \sigma^2) D^2 \Big/ \left(p + \frac{1}{p}\right)^2$

or $$\tfrac{1}{2}(1 - \sigma^2)\left\{\left(p + \frac{1}{p}\right) A + \left(p - \frac{1}{p}\right) B - 2C\right\}^2 \Big/ \left(p + \frac{1}{p}\right)^2.$$

APPENDIX II

Stress Distribution in a Buckled Plate

The stress distribution in a buckled plate depends upon the detail of the edge conditions. The mean value of the stress down the length of the unloaded edges is always that given by equation (23); for instance, it can be shown that equation (23) is consistent with the results of §10, whether the unloaded edges are held straight or are free to wave. This mean value of the stress is always less than f_e, unless $\varrho > \sigma$, that is, unless the plate is actually compressed laterally.

On the other hand, the maximum local value of the stress down the length of the unloaded edges may be either greater or less than f_e, according to the degree of restraint imposed on movement of the plate edges laterally and longitudinally (c.f. footnote, page 12). For instance, on the basis of §10, if the edges of the plate are held straight but not restrained longitudinally, the stress is uniform down the length of the edge, and the maximum value is therefore equal to the mean value defined by equation (23); whereas, if the edges are free both to move longitudinally and to wave, the stress does vary down the length of the edge and its maximum value may or may not exceed f_e, according to the value of c (i.e. the wavelength). The value of the maximum stress quoted in equation (22) represents the effect of complete restraint against variations of *strain* along the plate edges, and this value is the highest possible value consistent with the condition $\varrho < \sigma$.

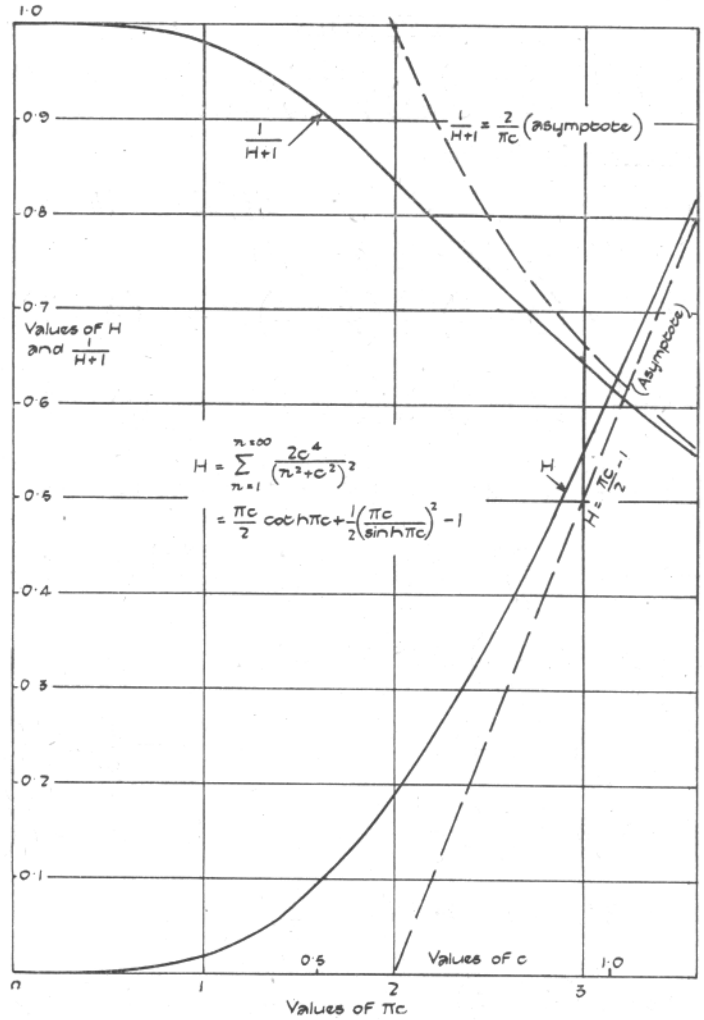

FIG. 1. Functions of Restraint against Waving of the Unloaded Edges.

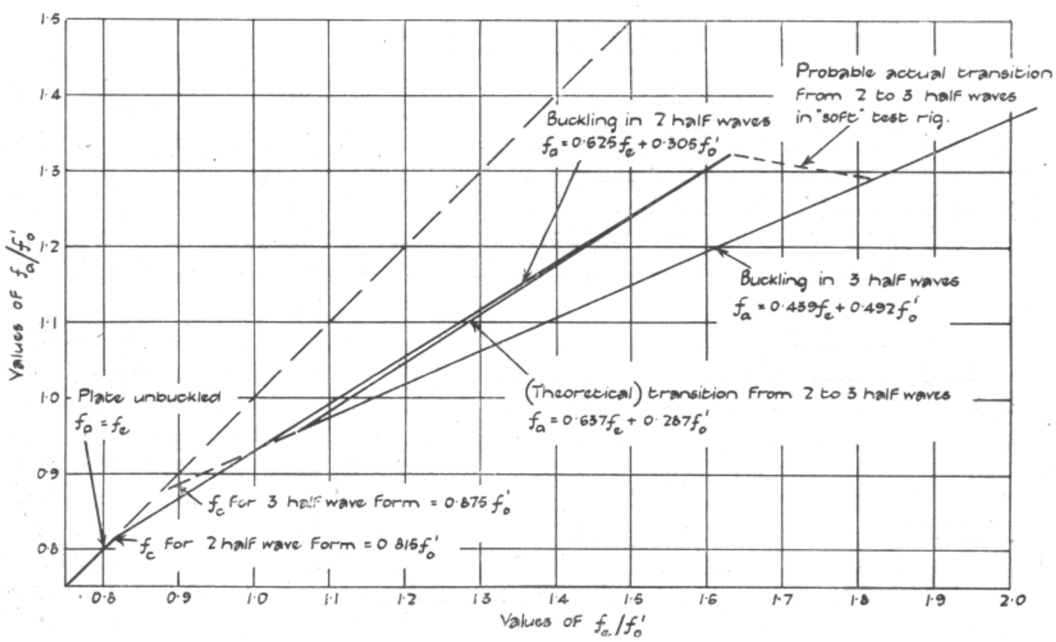

FIG. 2.—Effect of Change of Wave Form from 2 to 3 Half-waves in the Direction of Loading.

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