THE FLUTTER OF MONOPLANES, BIPLANES AND TAIL UNITS.

FRAZER AND DUNCAN.

1931

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The Flutter of Monoplanes, Biplanes and Tail Units

(A SEQUEL TO R. & M. 1155)

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JANUARY 1931

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PREFACE.

The present report is the second in the Monograph series of the Aeronautical Research Committee to be devoted to the subject of flutter. The first, R. & M. 1155, appeared in 1928 and was entitled "The Flutter of Aeroplane Wings"; it contained the essentials of a tolerably general theory of flutter, but the problem discussed in detail was the prevention of the wing flutter of monoplanes. As the outcome of this earlier investigation, a list of recommendations was drawn up for the guidance of designers.

Since the publication of R. & M. 1155, research on wing flutter has been continued, and the subject of tail flutter has also received attention. The progress made is already recorded in separate reports issued from time to time in the R. & M. series; and these are, with slight modifications, now brought together under one cover. It is hoped that this compilation, which includes the recommendations regarding the design both of tail units in general and of the wings of biplanes, will be found convenient by designers.

To relieve the somewhat severely technical general text, an elementary introduction has been added. This consists mainly of an account, expressed in more popular language and illustrated by the aid of cinematography, of some simple experiments which are intended to show what actually happens in the various kinds of flutter, and what problems arise for solution. Some of the experiments were shown before the Royal Aeronautical Society early in 1929, and others have since been demonstrated elsewhere, but no connected description has hitherto been given.

Taken together, the present monograph and R. & M. 1155 provide a complete account of the work on wing flutter and tail flutter carried out at the National Physical Laboratory in recent years. A study of the very complex subject of airscrew flutter has been in progress for some time, but this will form the theme of a separate report.

The writers wish to express their thanks to all who have aided in the work. They are greatly indebted to the Superintendent of the Aerodynamics Department of the National Physical Laboratory and to the members of the recent Flutter Sub-Committee of the Aeronautical Research Committee for their help and encouragement. Special mention should also be made of the skilful assistance rendered in constructional work by the members of the Aerodynamics Workshop Staff. The more detailed acknowledgments are made in the text.

R. A. F.
W. J. D.

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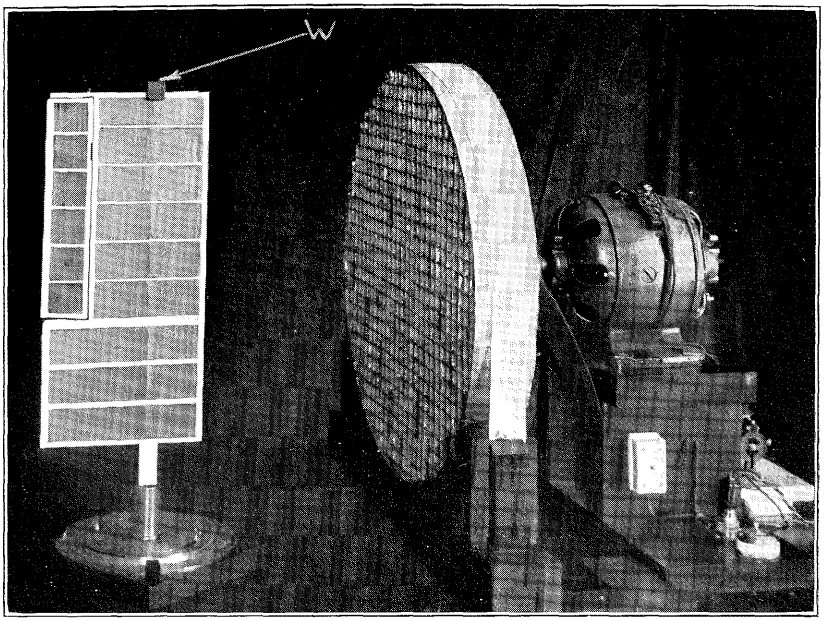

←——Direction of Air Current.

FIG. 1.—Air Blower and Flexible Model Wing.

(Note that aileron is freely hinged and that wing can bend and twist. W is a detachable lead weight.)

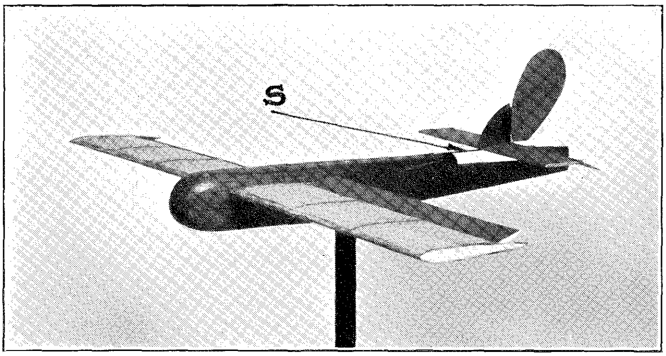

FIG. 2.—Model Monoplane for Demonstration of Wing Flutter and Tail Flutter.

(Tail unit is mounted to rock on a spring steel bridge S; the ailerons, elevators and rudder are freely hinged.)

THE FLUTTER OF MONOPLANES, BIPLANES AND TAIL UNITS.

INTRODUCTORY OUTLINE OF THE SUBJECT.

By R. A. Frazer, B.A., B.Sc., and W. J. Duncan, D.Sc., A.M.I.Mech.E.

§ 1. *Preliminary.*—The present monograph is a continuation of R. & M. 1155, "The Flutter of Aeroplane Wings".* To assist readers who are not very familiar with the earlier work, we shall begin with an elementary review of the principles of the subject. The exposition of these principles will be approached by a description of some simple experiments on models.

§ 2. *Wing Flutter and Tail Flutter of a Model Monoplane.*—The scientific study of flutter is most conveniently carried out with the aid of a wind tunnel; but for rough qualitative experiments a simpler and more portable apparatus has advantages. Fig. 1 shows an arrangement which serves well for demonstrations with small models. A wooden airscrew, driven by a half horse-power electric motor, is mounted in front of a circular honeycomb, 2 feet in diameter and provided with $\frac{7}{8}$-in. cells. The object of this honeycomb is to smooth the flow in the airscrew slip-stream; although not nearly so uniform and free from turbulence as that in a wind tunnel, the resulting air stream is sufficiently even for the purpose in view.

A description of the models in the strictly logical order would begin with those of the simplest nature from the theoretical point of view. But we shall better understand the meaning of the simpler models if we begin by studying the behaviour of a complete model aeroplane, showing flutter both of wings and tail. A view of this model is shown in Fig. 2.

Its fuselage is of solid wood, and carries a single flexible cross-spar supporting the wing ribs. The wings are covered with silk fabric, and are provided with freely hinged ailerons extending (for simplicity) along the whole span. Clearly the construction is such that each wing can bend and twist, and each aileron swing on its hinges, under air load; and, in fact, ample variety of movement has been allowed to render wing flutter a possibility.

In order to provide opportunity for tail flutter, appropriate freedom must also be offered to the various organs of the tail unit. This can be done in several ways, but in the present instance the rudder and the elevators are hinged freely, and the whole tail unit

* Ref. 1. *See* the list of numbered references given on p. 175.

is mounted on a strip of watch spring (marked S in Fig. 2) so that it can rock about a longitudinal axis in the fuselage. In an actual aeroplane this rocking of the tail can occur owing to the torsional flexibility of the fuselage itself.

Let us now study the behaviour of this model when placed in a wind. At low wind speeds the " bumps " or irregularities of the air stream set up small occasional vibrations of the wings and tail. These vibrations, however, show no tendency to grow with time; on the contrary, the vibration set up by any particular impulse very rapidly dies away. At such wind speeds the model is said to be completely stable. As the speed is increased, a critical stage is reached when either the wing vibrations or the tail vibrations caused by the casual disturbances are no longer damped; and, at a very slightly higher speed, the particular parts so affected are set into a vibration which grows to a large amplitude.* There is no general rule as to which parts of an aeroplane will begin to show this *flutter* first. In fact, with this model, we can bring the *critical speed* for tail flutter either below, or above, that for wing flutter merely by slight changes of the mass distribution. For definiteness, let us suppose that matters have been so adjusted that wing flutter begins first, and let us now examine this wing flutter in greater detail.

In the first place we find that the flutter involves bending and twisting of the wings, and swinging of the ailerons on their hinges; since there are here three distinct kinds of motion the flutter is said to be *ternary*. Secondly, although wing and aileron do not reach their extreme positions at the same instant (i.e. they are moving out of phase), yet they are both vibrating at the same rate. A further striking fact is that the amplitude of the motion is largest at the wing tips and decreases continuously towards the roots.

Now the wings of the model are actually cantilever beams, and it is known that any such beam can vibrate in a number of quite distinct ways, usually described as *modes*; moreover, the frequencies of vibration in the several modes are widely different. In the *fundamental mode*, whose frequency is the lowest, the amplitude of the vibration is greatest at the free end and decreases continuously towards the support; whereas in the higher modes there are one or more *nodes* where the amplitude is zero. Thus we learn that the flutter of the wings in the present instance is in the fundamental mode. Experience and theory are in agreement that with any normally designed cantilever wing, flutter always occurs first in the fundamental mode (*see* § 13 and Chapter II).

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* The amplitude, of course, cannot grow indefinitely without fracture of the parts; but in many cases (e.g. with the models here described) it becomes limited before failure occurs. The reason is that as soon as the vibration becomes fairly large, the various forces acting cease to be proportional to the displacements and the velocities. The motion then changes character and assumes a constant amplitude.

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(a) Ternary. (a) *contd.* (a) *contd.* (b) Rudder-fuselage.

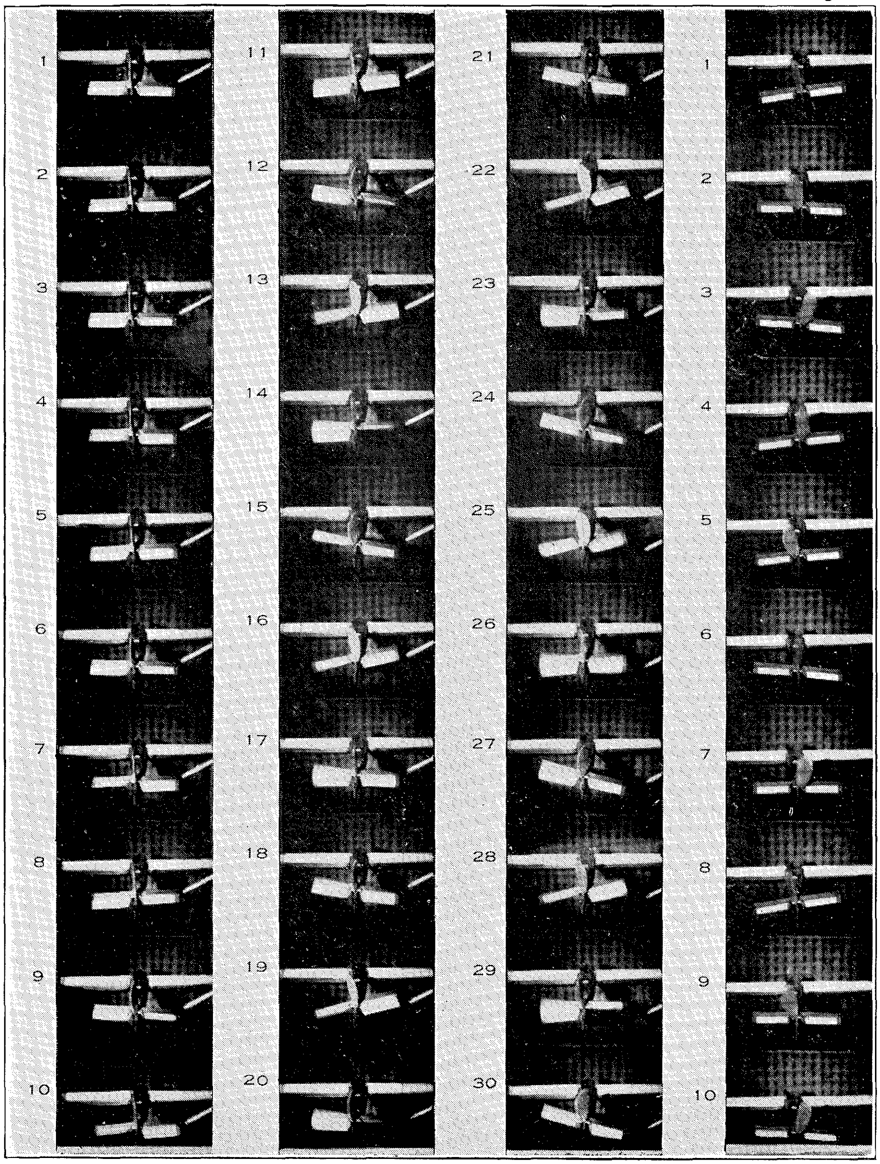

FIG. 3.—Some Types of Tail Flutter.

(Rear view of the model monoplane shown in Fig. 2.)

For Series (a) tail held stationary by rod before exposures commenced. No. 1 shows release, and later pictures illustrate growth of flutter, which involves rocking of tail and swinging of rudder and elevators on their hinges. Note scissors-like motion of elevators (*see* Nos. 13 and 14).

Series (b) shows fully developed rudder-fuselage flutter with fixed elevators.

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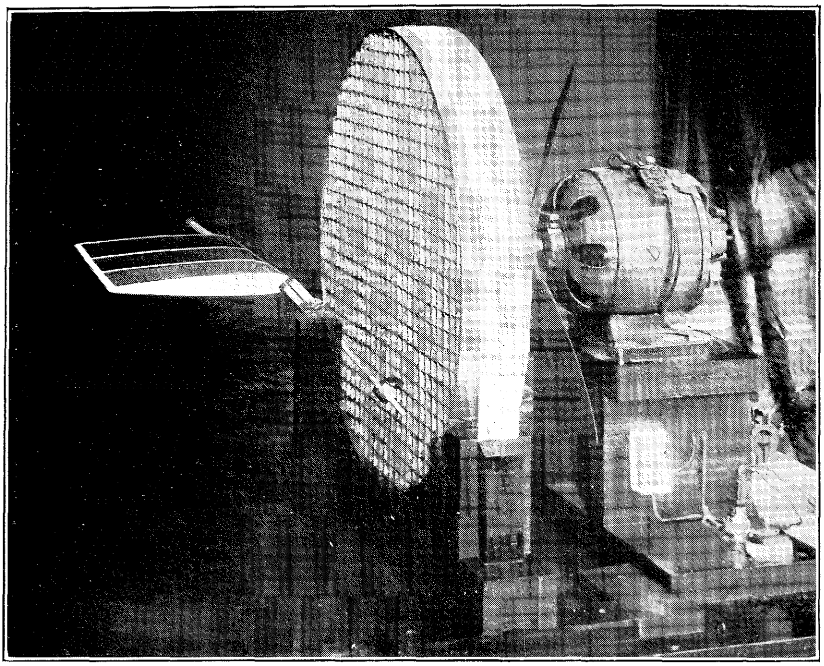
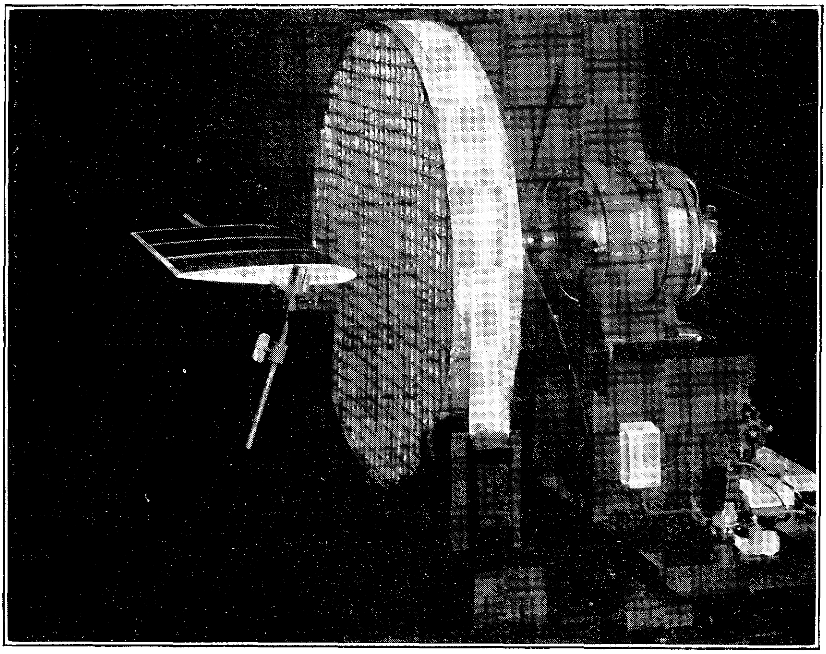


Wing A with Pitching Axis near Leading Edge.



Wing B with Pitching Axis at Mid-Chord.

FIG. 4.—Model Wings of Symmetrical Section Mounted for Pitching Oscillation Tests.

(Note the adjustable bob weights and horizontal attitude of the wings.)

Let us now proceed a stage further, and increase the speed well above the critical value at which wing flutter began. As the speed rises the flutter does not disappear as would be the case if the vibrations were due to resonance ; on the contrary, it gains such violence that any really large increase of speed above the critical value would prove destructive to the wing.

Assuming next that the model has been adjusted so that its tail flutters before the wings, we find that the vibrations of the tail involve three distinct types of motion, namely, rocking of the whole tail about the axis of the fuselage, swinging of the rudder on its hinges, and oscillation of the elevators on their hinges. The growth of a *ternary* tail flutter (i.e. one involving all three of these motions) is illustrated in Fig. 3 (*a*), which is reproduced from a cinematograph film. The wind speed was rather above the critical value, but the tail was held at rest by means of a rod until the exposures began. The tip of this rod appears on the right in the photographs. We see that the tail began to oscillate as soon as the rod was withdrawn, and that the motion grew rapidly until it became really violent (*see* last photograph of Series 3 (*a*)). An important feature of the flutter here shown is that the two elevators move in opposition. This scissors-like action could, of course, occur on an actual aeroplane, but would then be resisted by the elasticity of the connection between the elevators. Such a connection was not represented on the model.

The group of photographs, Fig. 3 (*b*), illustrate the simpler kind of flutter which resulted when the elevators were definitely fixed to the tailplane. Since only two motions are now involved, namely, swinging of the rudder and rocking of the tail, the flutter is said to be of the *binary rudder-fuselage* type. The pictures show that the rudder reached its extreme position approximately at the moment when the tailplane was in its mean (i.e. horizontal) position ; thus the two oscillations were by no means in step, and, in fact, happened to be here almost in quadrature.

§ 3. *Models with One Degree of Freedom.*—The experiments already described will help to indicate some of the main facts which a theory of flutter must explain. In order to study the part played in flutter by particular factors, we next turn to models of a much simpler type. Now one of the features of a wing which greatly influences its behaviour in a wind is the fore-and-aft position of the spar, or spars, in the chord. An illustration is provided by the pair of models (A) and (B), shown in Fig. 4.

Each model consists of a wing of symmetrical section, mounted on a horizontal axis of rotation parallel to the span, and an adjustable bob-weight carried below the axis imparts static stability. Here, the only wing motion allowed is " pitching " against the gravitational constraint ; the analogue for an actual aeroplane wing is twisting against torsional elastic stiffness. The two models are identical, except that for (A) the axis of rotation is very near to the leading edge, whereas for (B) it is close to mid-chord. For the tests described below, the bob weight was in each case adjusted so that in still air the wing was horizontal.

In the case of model (A) the influence of the wind is purely stabilising, as it provides both additional restoring moment and damping. Any motion resulting from an initial disturbance rapidly decays; indeed—if the wind speed be high enough—the wing returns to rest in the horizontal attitude without oscillation (i.e. it becomes dead beat). These effects are illustrated in the series of pictures, Fig. 5. Here and elsewhere, the wind speed V in feet per second, and the approximate frequency F of the oscillations in cycles per second, are indicated in the photographs. The oscillations of the wing in still air, following a rather large displacement, are shown in series (*a*). For series (*b*) the wind speed is 10 ft. per second and the damping has increased; whilst for series (*c*) at 25 ft. per second the motion is almost dead beat.

The behaviour of model (B) is very different. Certainly at low speeds (*see* Fig. 6, series (*a*) and (*b*)) it behaves like wing (A), but the return to equilibrium becomes more and more sluggish as the speed is raised, until a stage is reached (*see* Fig. 6, series (*c*)) when the wing *diverges* from its horizontal attitude even without artificial disturbance (note that the rod seen in (*a*) and (*b*) is absent from (*c*)). In the condition (*c*) it is just as impossible to keep the wing horizontal without some additional support as it is to balance a needle on its point. In fact, the wing is now statically unstable, the restoring effect of the bob weight having been more than neutralised by an upsetting aerodynamical action which increases as the square of the wind speed. The instability of the wing is, however, more complicated than that of a vertical needle; for, when the divergence has developed to such an extent that stalling incidence is exceeded, loss of damping ensues, and thereafter the wing executes an oscillation of an extremely large amplitude (*see* end of series (*c*)). Under some conditions this motion may even develop into a continuous rotation.

§ 4. *Model with Two Degrees of Freedom.*—The models whose behaviour has just been described were only allowed a single degree of freedom, namely, rotation about one axis. Next in order of complexity is the model illustrated in Fig. 7. Its appearance is rather unconventional; the aileron is absent and the very flexible single spar is placed far forward in the chord. The two essential degrees of freedom are here bending and twisting of the spar. At low wind speeds this wing is completely stable, but above a certain critical speed a very violent flutter occurs (*see* Fig. 8 (*a*), where the wing is seen almost end on). This experiment shows that the presence of an aileron is not essential for the development of wing flutter. In view of the two degrees of freedom involved, the flutter is said to be of the binary *flexural-torsional* type.

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(a) Still air. (b) Low speed. (b) *contd.* (c) High speed.

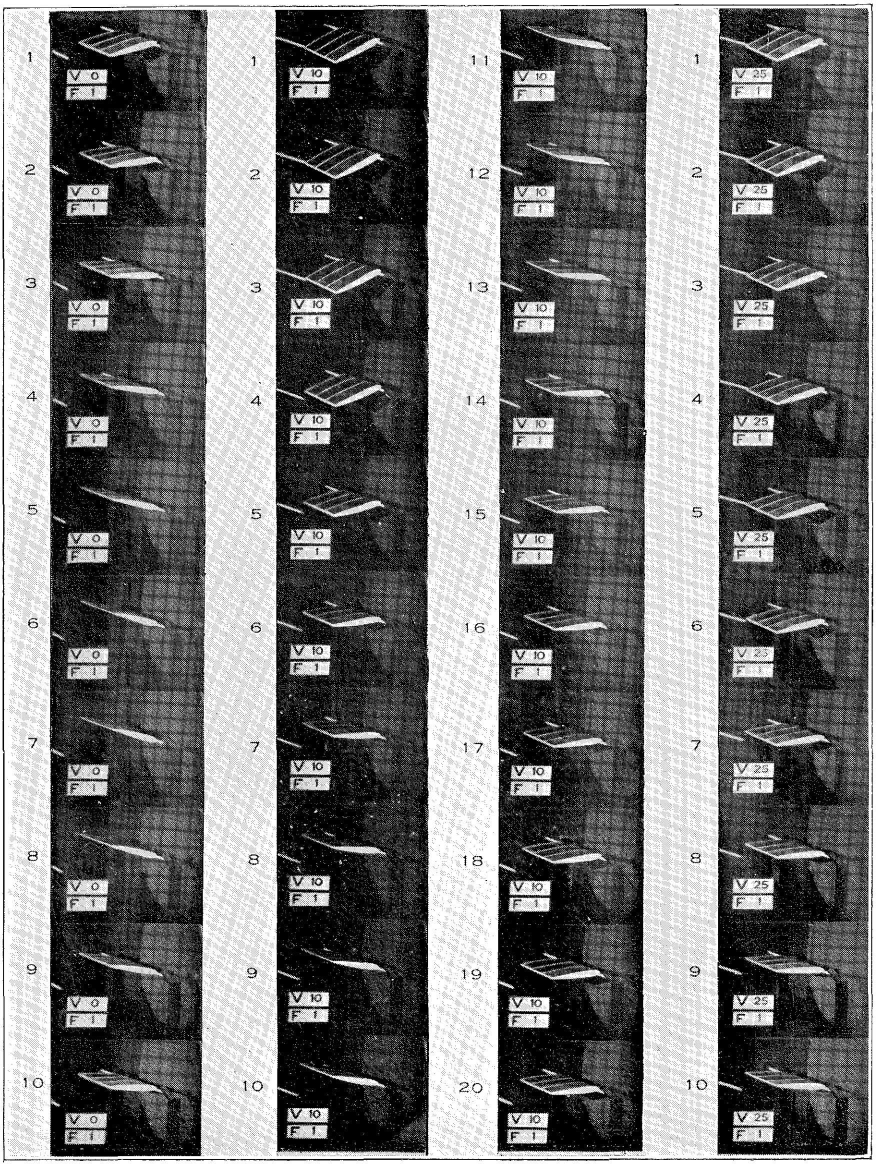

FIG. 5.—Damped Pitching Oscillations of Model A.
(Axis of rotation near leading edge.)
Wing displaced by rod and then released. Motion is damped at all air speeds, and almost dead beat in Series (c). The air speed V and approximate frequency of oscillation F are indicated in the photographs.

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(a) Still air. (b) Low speed. (c) Divergence speed (c) *contd.*

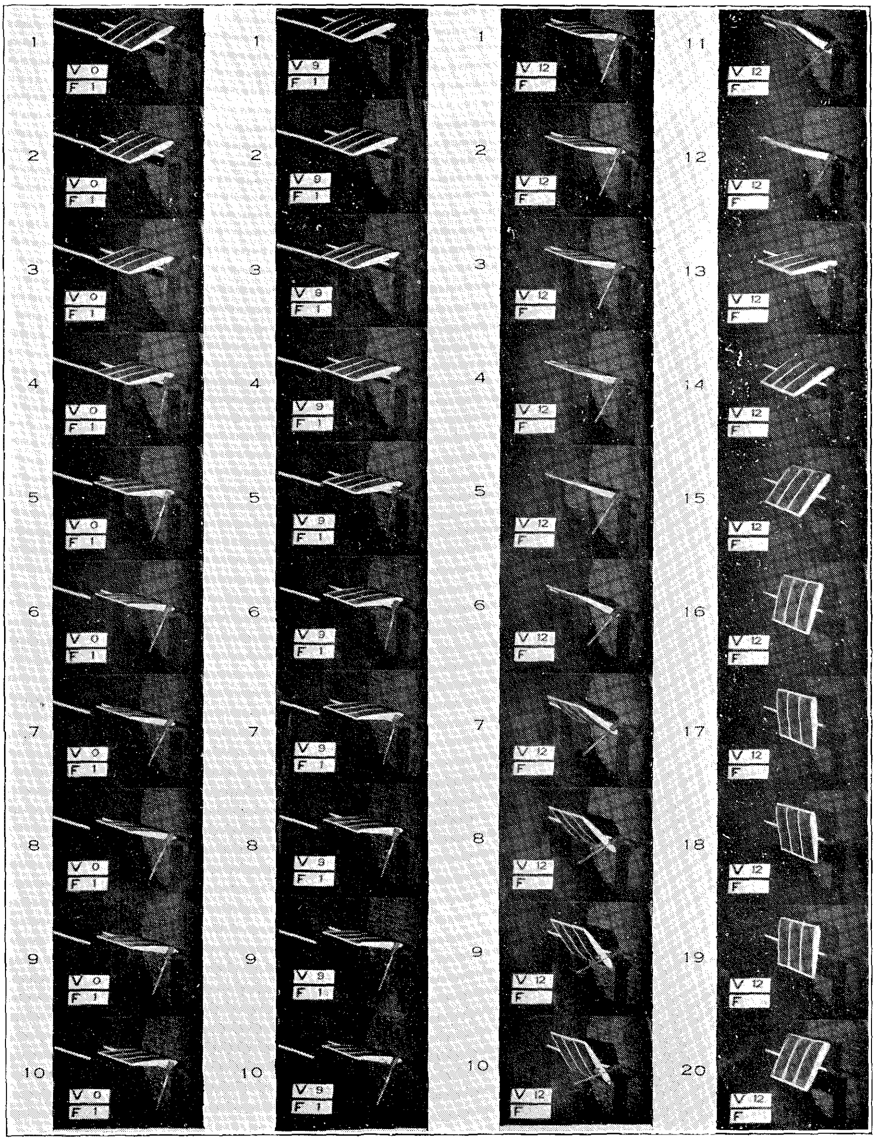

FIG. 6.—Pitching Oscillations and Divergence of Model B.
(Axis of rotation at mid-chord.)

In still air and at low speeds the motion following disturbance is a damped oscillation. At the divergence speed (Series c) the wing leaves its horizontal position even without initial displacement by the rod, and ultimately acquires an oscillation of very large amplitude.

§ 5. *Model with Three Degrees of Freedom.*—A rather more elaborate wing, fitted with an aileron, is shown in Fig. 1. The model may be used to demonstrate on a somewhat larger scale the ternary type of wing flutter already described in connection with the small complete monoplane (*see* §2). Its chief value, however, is to teach us that mass distribution has a very important influence upon the critical flutter speed.

A detachable lead weight, marked W in Fig. 1, is placed on the wing tip, and the wind speed is raised until flutter begins (*see* Fig. 8 (*b*)). On removal of the weight flutter stops and only reappears at a higher wind speed (*see* Fig. 8 (*c*)). In series (*b*) the torsional component of the motion happens to be extremely small, and the flutter approximates to the binary *flexural-aileron* type; but in series (*c*) the torsion is quite distinct.

The effect of mass distribution is strikingly illustrated by another experiment in which the aileron is loaded with lead at the leading edge until its centre of gravity lies upon the hinge axis. With this modification the wing becomes completely stable for all speeds obtainable with the blower. The importance of dynamical balance of the ailerons as a preventive of flutter is, of course, well known.

§ 6. *Introduction to the "Semi-Rigid" Theory.*—Any real wing is elastic, and its oscillations obey very complicated laws. A theory of flutter which paid strict attention to all these laws, would be too intricate to be useful. As a first approximation it has been found convenient to substitute for the real elastic wing a "semi-rigid" counterpart, representing only the more important of the deformations. This simplification has been justified both by experiments and by a more exact theory (*see* § 13 and Chapter II).

The fact that an elastic wing can be distorted to almost any shape by a suitable loading leads to great mathematical difficulties. However, experiments show that when a wing oscillates in a wind, its modes of deformation do not change greatly for a wide range of the air speed. Accordingly, the assumption is made that these modes actually are invariable. More precisely, the important degrees of freedom are first ascertained, and the mode of deformation appropriate to each such degree of freedom is then treated as invariable. For instance, in the case of a cantilever wing without aileron, the two important degrees of freedom are obviously bending and twisting; and the invariability of the modes means that the bending displacement at any particular point in the span is always a definite fraction of the bending displacement at the wing tip—and that the twist at any section is always a definite fraction of that at the tip.

Fig. 9 shows the skeleton of a simple semi-rigid wing. Its spars are two stiff rods, and they are mounted on small spindles so that they can oscillate vertically under the elastic constraint of springs. For simplicity, and in order to reduce friction, the ribs are loosely threaded over the spars.* On application of a load, the model in general displaces both in flexure and in twist ; but, no matter where the load is applied, the "mode" both in flexure and in twist is invariable, since in each case the displacement is directly proportional to the distance from the wing root.

The simple model just described provides only a very crude imitation of the modes of deflection of a real wing, since, in fact, bending always involves some curvature of the spars. However, we can generalise the construction of the wing mechanism so that any desired distribution of deflection is provided. The possibility, rather than the practicability, of such constructions is the point of importance for the development of the argument.

In any semi-rigid wing having two degrees of freedom the displacements of all points of the wing are completely determined when the displacement of any one section is known. Thus, in the study of the motion of the wing it is enough to attend merely to the motion of the selected *reference section* (e.g. the wing tip). The angular displacement of the reference section from the equilibrium position provides an unambiguous measure of torsion or twist; but to define flexure we must fix on some convenient *reference centre*† situated in the reference section, and adopt its displacement as the measure of flexure. The choice of this reference centre is arbitrary, but, as will be seen shortly, particular selections lead to simplifications in the analysis.

The measures of flexure and twist are referred to as the *dynamical co-ordinates* of the wing, and are usually expressed in non-dimensional form (e.g. as angles). The flexural co-ordinate is conveniently defined as the displacement of the reference centre, divided by the span of the wing (root to tip).

§ 7. *Statics of a Semi-Rigid Wing.*—A semi-rigid wing has the important simple property that a load applied at any point of the wing is exactly equivalent to a certain load applied in the reference section. We shall, therefore, examine the effects of loads applied in the reference section.

It is obvious from Fig. 10 that a single vertical load applied at random in the reference section produces twist as well as bending. There is, however, one particular point of application (*see* Fig. 10 (*b*)) for which the twist is absent, and this point is called the *flexural centre* (sometimes the *elastic centrum*) of the chosen reference section. When a normal load is applied at the flexural centre the pure flexural

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* On account of the looseness of the connections between the ribs and spars, the arrangement shown is not strictly a semi-rigid mechanism. The construction of a true mechanism was viewed as unnecessary for the purpose of illustration.

† The *reference axis* used in R. & M. 1155 is the axis through the reference centre and parallel to the span.

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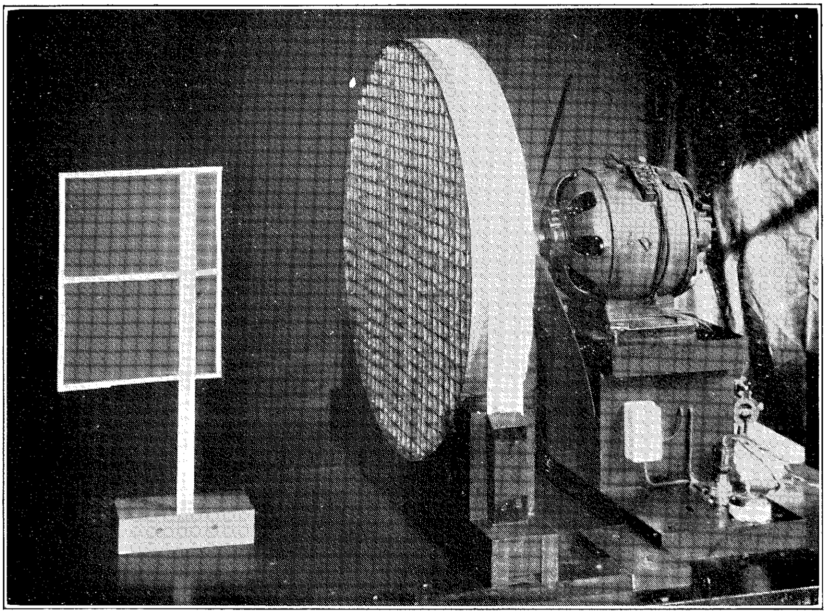

FIG. 7.—Model Wing to Show Flexural-Torsional Flutter.
(The wing has two degrees of freedom : it can bend and twist, but has no aileron.)

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(a) Flexural-torsional. (b) Ternary, with loaded wing tip. (c) Ternary, with load removed.

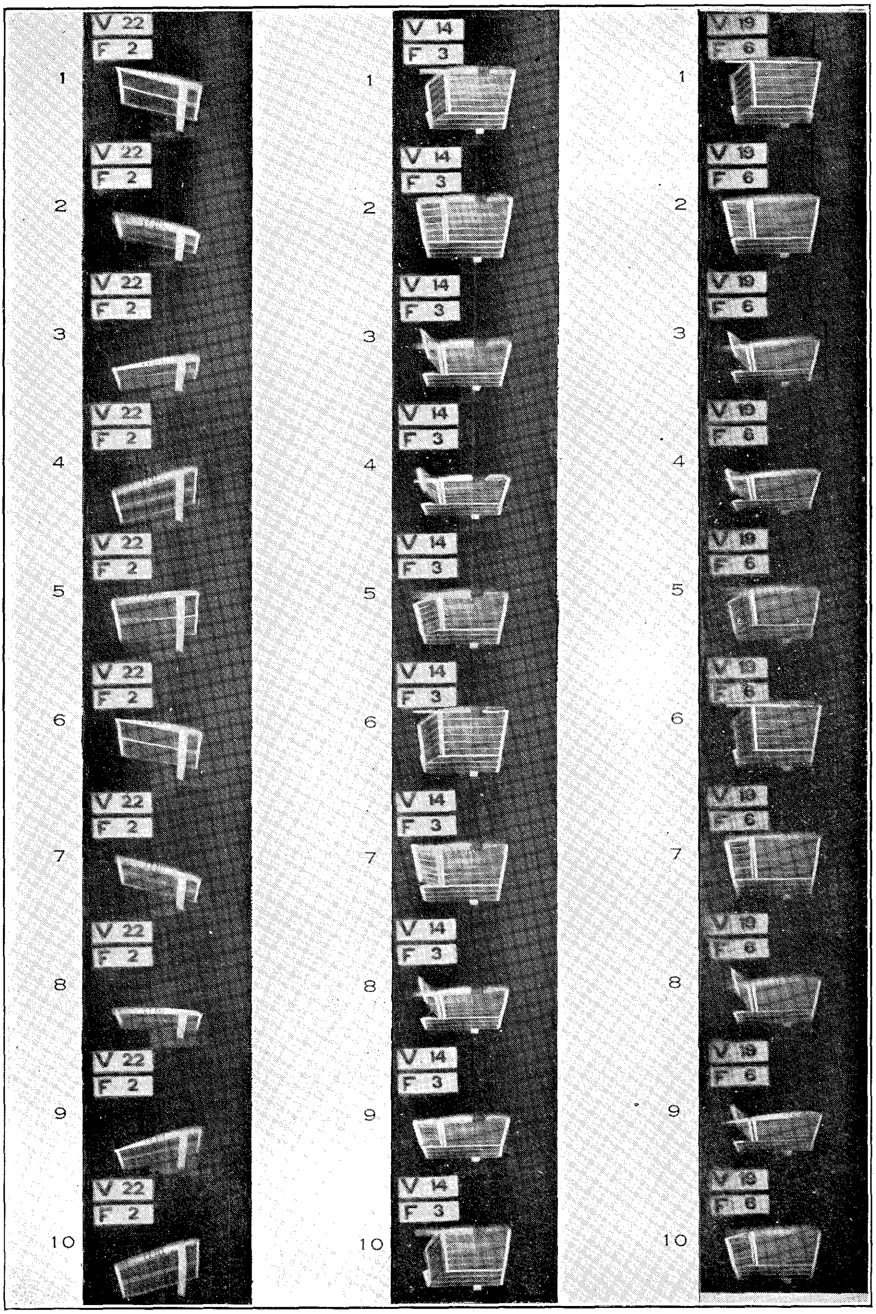

FIG. 8.—Some Types of Wing Flutter.
(Wings seen nearly in end view.)

Series (a) shows flexural-torsional flutter of wing Fig. 7. Series (b) and (c) show ternary flutter of wing Fig. 1. In (b) the wing tip carries a lead weight; in (c) this weight is absent, and the critical speed is raised. The amount of bending and twisting at each stage can be judged by use of the fixed number cards as a datum.

displacement produced is in direct proportion to the load. The *direct flexural stiffness* is defined as the moment of the load about the wing root divided by the angular flexural displacement (*see* end of § 6). A reciprocal property of the flexural centre is that it does not move when a pure twisting couple is applied to the reference section.

In statical problems it is obviously an advantage to adopt the flexural centre as the reference centre, because then flexural moments produce only flexural displacements, and, reciprocally, twisting moments produce only twisting displacements. When the reference centre is arbitrary and a pure twisting couple acts, the reference centre is necessarily displaced, since the flexural centre remains stationary. In this case, therefore, a pure twisting couple produces bending as well as twisting. In the mathematical theory this *coupling* effect is allowed for by the introduction of a *cross-stiffness*—which may be defined as the flexural moment required to keep the reference centre stationary when unit twist is produced by a pure twisting couple. Reciprocally, it is the measure of the twisting couple required to prevent twist when unit flexural displacement is produced by a load applied at the reference centre.

The *direct torsional stiffness* appropriate to a general reference centre is defined as the twisting moment per radian of twist when displacement of the reference centre is prevented. The value of this stiffness varies with the choice of the reference centre, and is a minimum when the flexural centre is adopted.

§ 8. *Oscillations of a Semi-Rigid Wing in Still Air.*—As already explained in § 6, the displacements of all points of a semi-rigid wing are completely known when the values of the dynamical co-ordinates are specified. Similarly, in the motion of such a wing the velocities and the accelerations of all points are determined by the corresponding time rates of change of the dynamical co-ordinates. For brevity, these time rates are referred to as the *velocities* and the *accelerations* of the wing.

As a preliminary to the study of oscillations in a wind, we shall consider the oscillations of a semi-rigid wing in still air.

(a) *Pure flexural oscillations.*—Dealing firstly with the simple model shown in Fig. 9, we suppose that the spars are firmly interlocked at their outer ends. Then twist is prevented entirely and the wing can only oscillate in "roll." This motion is simple harmonic, and the governing factors are the elastic stiffness due to the constraining springs and the ordinary moment of inertia about the axis of rotation.

In the case of the generalised semi-rigid wing with flexural curvature, the pure flexural oscillations are a direct generalisation of the rolling oscillations of the simpler model; and the controlling factors now are the direct flexural stiffness (*see* § 7) and the *flexural moment of inertia.* This moment of inertia is the measure of the moment of the flexural force acting at the reference section required to produce unit flexural acceleration. In general, it differs in value from the ordinary moment of inertia.

(b) *Pure torsional oscillations.*—Next, suppose the reference centre to be held by a pin. Then the oscillations are of the pure torsional type, and are governed by the direct torsional stiffness and the *torsional moment of inertia.*

(c) *Coupled oscillations.*—Lastly, let the wing be quite free from artificial constraints such as locking bars or pins. Then, if the flexural centre is displaced and released, the wing acquires an oscillation in which twist is combined with flexure, despite the fact that elastic coupling is absent. This effect is due to inertial coupling—that is, to a *product of inertia.* Imagine, for instance, the wing to have merely a flexural acceleration of unit magnitude. Then an effective inertial force acts on each mass of the wing; and the aggregate of these forces produces not only a flexural moment (measured by the flexural moment of inertia), but also a twisting moment (measured by the product of inertia).

If the reference section is chosen at a special position, known as the *principal centre of inertia,* the inertial coupling will be absent, but elastic coupling will then in general be introduced. The mass distribution of the wing can, however, be adjusted to make the principal centre of inertia coincident with the flexural centre. The wing is then said to be *dynamically balanced.* In this case both the pure flexural oscillation and the pure torsional oscillation are possible free motions of the wing; and a general initial disturbance results in these two *constituent* oscillations occurring at the same time. When the principal centre of inertia and the flexural centre are not coincident, there are again two constituent oscillations* of different frequencies; but each of these will be of a hybrid nature, involving both flexure and torsion.

In the discussion we have so far implicitly assumed that damping forces are absent. However, these forces have a great influence on the oscillations in a wind, and are by no means negligible even for still air. When damping is taken into account, the most general motion of the wing can still be built up by the superposition of constituents. As it is important to grasp the exact meaning of this term " constituent " a rather fuller explanation will now be given. A constituent is a possible free motion of the wing and, in general, involves movements in all the degrees of freedom. Its essential characteristics are that all these movements have a common single frequency and a common rate of damping; but the movements are, as

* In general, the number of constituent oscillations is equal to the number of degrees of freedom.

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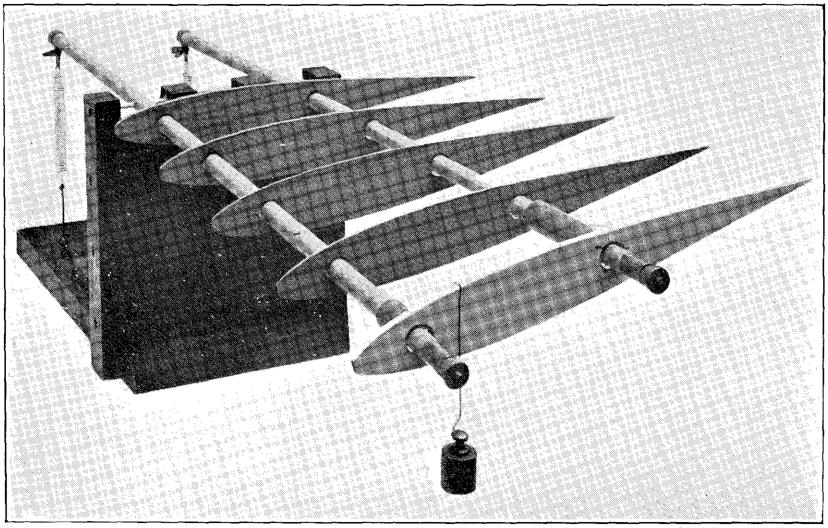
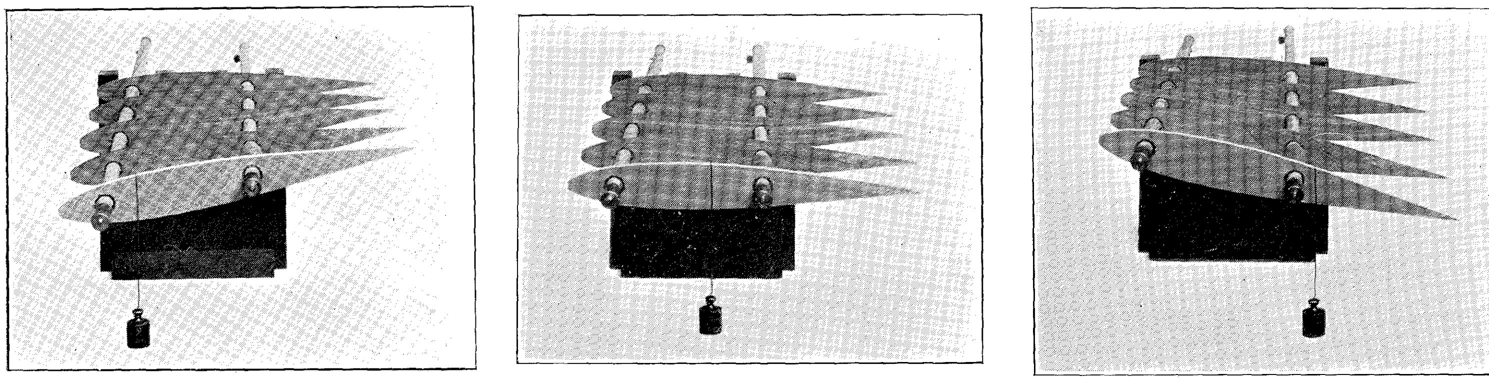


FIG. 9.—Skeleton of a Simple Semi-Rigid Wing Deflected under Load.
(Note hinge spindle of front spar just visible at root of wing, and helical springs to provide elastic stiffnesses.)

[To face page 9.]



(*a*) Negative twist due to load forward of flexural centre.

(*b*) No twist due to load at flexural centre.

(*c*) Positive twist due to load behind flexural centre.

FIG. 10.—Determination of Flexural Centre of Semi-Rigid Wing.

a rule, out of phase and have different amplitudes. The phase differences and the ratios of the amplitudes, as well as the frequency and the rate of damping, are all definite for any one constituent, and have values dependent on the dynamical constants of the wing, but quite independent of the initial conditions of the motion. The motion of the wing which follows any given initial disturbance can always be built up directly by the superposition of the constituents, their magnitudes and epochs* being suitably chosen.

§ 9. *Oscillations in a Wind.*—When the wing is placed in an air current, various further direct moments and couplings (compound moments) are incurred due to the wind action—some proportional to the displacements of the wing and analogous to the elastic moments, others to the velocities of the wing and of the nature of dampings. These moments are listed in full in Table I.

The aerodynamical moments gain importance as the wind speed increases, and may profoundly modify the nature of the constituents. As an example, let us consider the wing shown in Fig. 7. This wing is, of course, elastic, but as already indicated (*see* § 6) its behaviour approximates closely to that of a semi-rigid wing having two degrees of freedom—namely, flexure and torsion. Denote the two constituent oscillations by (A) and (B): then each of these has, at any given wind speed, a characteristic frequency and rate of damping. In still air, both constituents are slightly damped, and the two rates of damping increase with the wind speed up to a certain limit. When the wind speed exceeds this limit, the damping of (A) begins to decrease, whilst that of (B) continues to increase. At the critical speed for flutter (A) loses its damping entirely and becomes simple harmonic, whereas (B) is so heavily damped as to be practically negligible after a very short interval of time following the initial disturbance. A slight increase of speed above the critical value results in the constituent (A) becoming definitely unstable; it is this unstable constituent which gives rise to the flutter shown in Fig. 8(*a*).

§ 10. *Principles of Flutter Prevention.*—Flutter of the flexural-torsional type just described cannot occur if both of the constituent oscillations are positively damped at all wind speeds. Now if a pure flexural oscillation of the wing were actually realisable, it would certainly be more and more heavily damped as the wind speed increased; and the same would normally be true of a pure torsional oscillation. Thus, if matters could be arranged so that these were actually the two constituent oscillations of the wing at all speeds, flutter would be prevented. This would require all couplings to be absent, but fortunately it is equally effective to eliminate only the

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* The "epoch" measures the extent to which a constituent as a whole is out of step with another.

group of compound torsional moments—in other words, to prevent flexural movements from producing any twisting moments. In this case the pure flexural oscillation is obviously a possible free motion of the wing, and is, therefore, one of the constituents. To find the frequency and damping factor of the other, imagine a purely torsional initial disturbance given to the wing; then, since the compound flexural moments are not all absent, flexural movements will be induced. However, these induced movements cannot react in any way on the torsional motion, so that the latter continues as though it were strictly independent. Hence, although the second constituent must involve both flexure and torsion, its frequency and damping factor are those characteristic of the pure torsional motion.

Let us now suppose that a wing is constructed with its flexural centre F and its principal centre of inertia P coincident, and that the measurement of flexure is referred to this common centre. In this case a flexural displacement produces no elastic twisting moment; moreover, when the wing is exposed to wind action, such a displacement produces no change of aerodynamical twisting moment for the small angles of incidence which correspond to normal or high speed flight. Again, since F and P are coincident, a flexural acceleration produces no twisting moment. The only possible coupling remaining is that due to flexural velocity. Such a velocity virtually induces a change of incidence at each section of the wing, and consequently a change of the aerodynamical forces. In the aggregate these forces produce not only a flexural moment opposing the wing movement (i.e. a *direct flexural damping* moment, actually proportional to the air speed), but also a twisting moment (i.e. a *compound torsional damping* moment, also proportional to the air speed). Thus, the compound torsional damping measures the aerodynamical twisting moment per unit flexural velocity. This coupling only vanishes if the reference centre is at a particular position known as the *centre of independence* I (*see* Table 1 (*b*)). If, therefore, the wing is so constructed that the centres F and P are coincident with I, flexural movements can produce no twisting moment, whatever be the wind speed, and flutter will accordingly be prevented.

The position of the centre of independence must vary to some extent with the design of the wing. The somewhat scanty experimental evidence on this question indicates that for a square tipped wing of thin section it lies at about 0·3 chord from the leading edge.

The case of flexural-torsional flutter has here been discussed at length, since it lends itself to a particularly easy theoretical treatment. However, flutter involving aileron motion is actually more important, because—when the wing has this extra degree of freedom —the critical flutter speed is greatly reduced unless special precautions are taken. In the prevention of ternary wing flutter, the

TABLE 1 (*a*).

COEFFICIENTS FOR THE FLEXURAL AND TORSIONAL GROUPS OF MOMENTS.

	Flexural Moments.		Torsional Moments.	
	Direct.	Compound.	Compound.	Direct.
Inertia ..	Flexural moment of inertia.	Flexural-torsional product of inertia.	Flexural-torsional product of inertia.	Torsional moment of inertia.
Wind ..	Direct flexural damping.	Compound flexural damping.	Compound torsional damping.	Direct torsional damping.
	Direct flexural wind stiffness.*	Compound flexural wind stiffness.	Compound torsional wind stiffness.*	Direct torsional wind stiffness.
Elasticity ..	Direct flexural elastic stiffness.	Elastic cross-stiffness.	Elastic cross-stiffness.	Direct torsional elastic stiffness.

* Negligible, except for large angles of incidence.

TABLE 1 (*b*).

POSITIONS OF REFERENCE CENTRE FOR VANISHING OF PARTICULAR COEFFICIENTS.

Vanishing Coefficients.	*Position of Reference Centre.*
Flexural-torsional product of inertia.	Principal centre of inertia. (P.)
Compound torsional damping.	Centre of independence. (I.)
Elastic cross-stiffness.	Flexural centre. (F.)
Direct torsional wind stiffness.	" Quarter-chord " from leading edge. (Q.)

general aim is again to secure simple and definitely damped constituents by a suppression of couplings. The most objectionable of the couplings are the aileron products of inertia, and these can be suppressed by a suitable concentration of the mass forward of the hinge. When these products of inertia are absent the aileron is said to be *dynamically balanced* (or *mass balanced*).

The preceding arguments have been restricted to conditions of a very simple type which suffice to prevent flutter. The more thorough study of the stability is too complex to be made intelligible without mathematics, but is greatly facilitated by the use of certain graphical methods (*see* R. & M. 1155, Chapters III and VIII). We may broadly summarise the results of the mathematical investigation in the statement that—with certain exceptions which are unimportant in practice—flutter will only be *absolutely prevented* (i.e. avoided at all speeds, however high) when a properly selected group of couplings are sufficiently small. Thus, the more thorough analysis confirms the simple argument, but shows that the couplings must not exceed certain tolerances, which can be calculated.

Dimensional theory shows that if all the elastic stiffnesses are increased in the same proportion, then the critical speed will vary as the square root of a typical stiffness. Thus, if the measures for absolute flutter prevention should prove inconvenient in any particular design, a proportional increase of all stiffnesses is always available as a means of raising the critical speed beyond the highest possible speed of flight. The caution should, however, be added that an increase of only a single stiffness is not necessarily advantageous (*see*, for example, § 88).

§ 11. *Divergent Instability*.—The experiments described in § 3 show that flutter is not the only kind of instability to which wings are subject. The departure of a wing from its position of equilibrium may be in the nature of a continuous *divergence* instead of a growing oscillation as in flutter. As already indicated in § 3, a divergence is the result of an instability of the ordinary static type, and its initiation corresponds to the disappearance of an effective stiffness.

Suppose a cantilever wing to be constructed with the centres F, P and I coincident, so that, as shown in § 10, flutter cannot occur. Let us trace the influence of increasing wind speed on the two constituents of the wing motion :—

(a) *Pure flexural constituent*.—The rate of damping of this constituent becomes greater as the wind speed rises ; at a sufficiently high speed the oscillation would disappear, and the motion would become dead beat.

(b) *Torsional constituent*.—The second constituent involves flexure as well as torsion, yet the torsional motion is uninfluenced by the movement in flexure. Consequently, the

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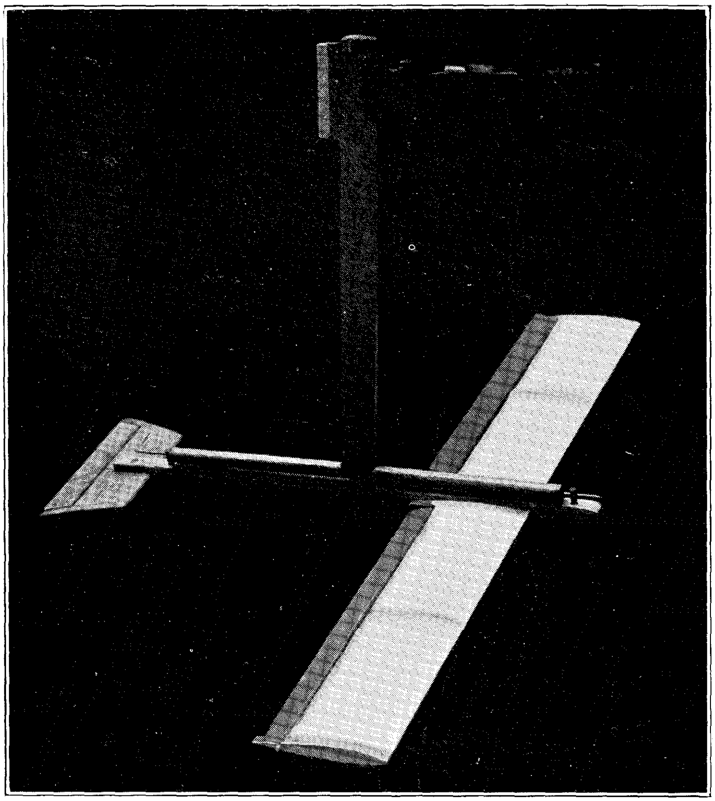

FIG. 11.—Model to Show Effect of Mobility of Fuselage.
(Model mounted on pivots to provide freedom in roll. Clamping of body to fixed support reduces critical flutter speed.)

torsional motion would not be affected if the flexural centre were actually held by a pin. Under this condition the cantilever wing would behave very much like the models (A) and (B) described in § 3. Thus, if the flexural centre, which now defines the axis of torsional motion, lies well forward in the chord, the motion would be similar to that of model (A) and therefore, stable at all wind speeds ; if on the contrary, it lies far aft, a divergence would eventually occur as with model (B). To be more precise, divergence would, or would not, ultimately occur according as the flexural centre lay aft or forward of a point Q, situated usually at about one-quarter of the chord from the leading edge. For brevity, this will be referred to as the *quarter-chord position* (*see* Table 1 (*b*)). When the wing is freed by removal of the pin holding the flexural centre, divergence will occur at the same wind speed as before, but will now involve a flexural movement due to the increase of lift associated with the twist.

From this discussion we see that for *complete* stability the common centre F, P, I, would have to lie at, or forward of, Q. Unfortunately, although the positions of F and P are readily controllable, those of I and Q are not. As already stated in § 10, experiment shows that in some cases I lies at about 0·3 chord from the leading edge, and, therefore, aft of Q. Thus, although a coincidence of F, P and I will obviate flutter, yet a divergence must be expected. However, the speed at which this divergence occurs will be very high when I lies not far aft of Q.

§ 12. *Influence of the Mobility of the Fuselage on Wing Flutter.*— Passing from the first principles of our subject, we may now briefly consider the questions dealt with in the main text.

To understand correctly the meaning of Chapter I, we have to remember that flutter theory, in its ordinary form, gains great simplification from the assumption that an aeroplane as a whole does not oscillate in response to flutter of its wings. In other words, the fuselage is usually supposed to provide an absolutely rigid and immobile support for the wings (*see*, for example, § 5 of R. & M. 1155). This assumption was natural, if not almost necessary, in the early stages of the investigation, but an experiment will now be described which shows that it is not always justifiable. The model shown in Fig. 11 is a rough representation of a monoplane, and is suspended on two pivots carried by a fixed wooden support. When not otherwise constrained, the model can " roll " freely on its pivots ; but the " fuselage " can be made immobile at will by means of a clamp or a grip with the fingers. When the model is placed in an air current and the body is held fixed, the wings begin to flutter at a moderate wind speed ; and it is important to note that the flutter here happens to be of the antisymmetrical type in which the port wing moves down when the starboard one moves up, and *vice versa*. However, when the body is released, this flutter at once stops, and only reappears at a much higher wind speed. In the present instance, therefore, the mobility of the fuselage has a definite stabilising influence upon the wing oscillations.

Chapter I is devoted to the mathematics of these questions, but the main conclusions can be summarised here. The most general small oscillation of the wings and body of a monoplane in straight flight is shown to be compounded of oscillations of only two types. In the first, called the *longitudinal-symmetrical* type, the port and starboard wings move equally and in phase, and the oscillations of the aeroplane as a whole are "longitudinal" (i.e. they involve longitudinal and normal translations, together with pitching). In this case the true critical flutter speed differs only very slightly from that calculated on the assumption that the fuselage is fixed. The second type of oscillation is called *lateral-antisymmetrical*. Here the wings move equally but in opposition, and the oscillations of the aeroplane as a whole are "lateral" (i.e. are compounded of a lateral translation, or side slip, and of rolling and yawing). The critical flutter speed for this type is shown mathematically to be decidedly higher than that corresponding to an immobile fuselage. This result accords with the experiment already described.

It is not difficult to give a broad explanation of these effects. In the symmetrical motion the alternating bending moments at the roots of the port and starboard wings act upon the fuselage in opposite senses and neutralise one another. Moreover, although the vertical reactions at the wing roots, as also the torsional moments, reinforce one another, yet the mass and the pitching moment of inertia of the fuselage are both large, so that the impressed normal and pitching motions are small. Thus, on all accounts, the response of the fuselage to the wing motions is trifling, and the critical flutter speed for the wings is almost unaffected. On the other hand, when the wings flutter in opposition, the bending moments at the wing roots reinforce one another ; and, since the fuselage has a relatively small moment of inertia in roll, its impressed rolling oscillation will be large. Thus, a marked change of the critical flutter speed is to be expected.

§ 13. *Flutter of a Wing Supported by Elastic Spars.*—In the "semi-rigid" theory of flutter, a cantilever wing without aileron is treated as though it were a mechanism having definite modes of distortion in flexure and twist. We know, however, that a real wing is elastic, and can be distorted in an arbitrary manner. Thus, since the semi-rigid theory cannot pretend to exactitude, any verification of its conclusions provided by an application of elastic theory is to be welcomed. Chapter II supplies such a verification.

Let us now consider a cantilever wing supported by two elastic spars ; and firstly, let us examine the oscillations in still air. For simplicity these will be assumed undamped. The exact way in which such a wing oscillates depends, of course, upon the nature of the initial disturbance, but the oscillations can always be analysed into a set of quite definite simple constituents. The frequency and the modes of flexural and torsional deformation are fixed characteristics for each such constituent; and we find that the constituents occur in pairs whose frequencies are in a constant ratio. In Chapter II the term " mode " is, for brevity, used in a somewhat extended sense to denote such a pair of constituents. For the purpose of classification let us confine attention to, say, the slower constituent of each mode: then the modes can be arranged uniquely in ascending order of frequency. The first of the series is the *fundamental*, and the remainder, whose frequencies are higher, are known as the *overtones*.

Next, suppose the wing to be exposed to an air current whose speed is continuously increased from zero. Then, although the constituents of the original fundamental mode undergo continuous modification, yet the pair may still with propriety be identified at any given wind speed as the "fundamental mode." Similar considerations apply to the overtones.

The mathematical analysis of Chapter II shows that the first flutter of the wing necessarily occurs in the fundamental mode, and that the forms of the stability criteria are the same as for a semi-rigid wing. In particular, when the design is such that the centres F, P, I, are coincident in every section of the wing, flutter cannot occur (cp. § 10). Thus, some of the most important deductions from the semi-rigid theory are strictly valid for the type of elastic wing considered.

§ 14. *Wing Flutter of Biplanes.*—A mathematical treatment of the wing flutter of biplanes was not attempted in R. & M. 1155, but certain provisional conclusions regarding flutter prevention were stated in § 67 of that report. This subject has since been studied both experimentally and theoretically in some detail. The introductory section of Chapter III provides a full summary of the conclusions, so that a brief reference to certain points will here suffice.

The theory is based on an extension of the notion of semi-rigidity to the deformations of the biplane structure. The dynamical system actually considered comprises the upper and the lower starboard (or port) half planes with their ailerons, and in the most general form of the theory given the freedoms selected correspond to:—

(*a*) Flexural, torsional and aileron co-ordinates for the separate upper and lower wing extensions.

(*b*) A flexural and a torsional co-ordinate defining the displacements of the outermost incidence truss.

Since the ailerons are interconnected by a strut or wire, there are just seven effective degrees of freedom.

As pointed out in § 67 of R. & M. 1155, if there were actually a node at, or close to, the outermost incidence truss, the mere extension of the ailerons far into the bay would be an effective preventive of flutter. However, flutter tests on a model biplane indicate that the flexural motion at the truss is by no means negligible with staying of a normal stiffness in relation to the spars. Thus the measure cited, although certainly desirable, cannot by itself be regarded as sufficient.

Both experiment and theory confirm that dynamical balance of the ailerons is a measure of the utmost importance. Its effectiveness will, however, be vitiated in a biplane with an extended upper wing, unless the interaileron strut is placed close to the outermost interplane struts.

The design recommendations regarding flutter prevention on biplanes are listed in full on p. 73.

§ 15. *Flutter of Tail Units.*—Instances of tail flutter have long been known, and as early as 1916 a theory of elevator flutter was given by Bairstow and Fage.* Some recent occurrences have led to a more complete investigation of tail flutter. Chapter IV contains an outline of the theory—which is again based on the principle of semi-rigidity—together with an account of some qualitative wind tunnel experiments. A quantitative investigation of a particular instance of tail flutter is described in Chapter V.

Tail flutter can be either of the *symmetrical* or the *antisymmetrical* type. In symmetrical tail flutter the port and starboard tailplanes oscillate in phase, there is no resultant twisting moment on the fuselage,† and the rudder remains at rest. The type of motion is closely analogous to ordinary wing flutter, the elevators playing the part of the ailerons. Thus, the rules for the avoidance of wing flutter, when suitably interpreted, are applicable. In antisymmetrical tail flutter the port and starboard tailplanes oscillate in opposition; consequently, a resultant twisting moment is produced, causing torsion of the fuselage, and associated with this is swinging of the rudder on its hinge. The scissors-like motion of the elevators is subject to an elastic constraint due to their interconnection. Flutter of this kind is illustrated in Fig. 3 (*a*).

The most general type of antisymmetrical tail flutter is very complex, but two simple cases are important in practice. The first is binary *rudder-fuselage* flutter, in which fuselage twist is accompanied by rudder oscillation and the other motions are negligibly small (*see* Fig. 3 (*b*)). It could be completely avoided by recourse to a rudder symmetrically disposed about the torsional axis of the fuselage, since with this arrangement all couplings between the rudder and fuselage motions are eliminated. However, this measure is not

* R. & M. 276 (Ref. 2).
† Cp. remarks on symmetrical wing flutter p. 14.

generally practicable, and the undesirable couplings can only be avoided by special attention to the rudder design. When the product of inertia is zero, torsional acceleration of the fuselage produces no rudder hinge moment, and the rudder is then said to be *dynamically balanced*. The precise condition for dynamical balance is stated on p. 144, and is not in general met when the centre of gravity of the rudder is on the hinge axis. The aerodynamical coupling (compound rudder damping coefficient) could be made to vanish by use of a suitable horn balance, but this measure is usually of minor importance.

The second of the simple cases mentioned is *elevator-fuselage* flutter, in which the significant motions are fuselage twist and opposed oscillation of the elevators. The most effective safeguards are the provision of a very stiff interconnection of the elevators (a device originally recommended by Bairstow and Fage*) and dynamical balance of the elevators.

Throughout the discussion of tail flutter the assumption is made that the aeroplane as a whole does not oscillate in response to the flutter of the tail. This assumption is, of course, not strictly true, and the mobility of the wings in tail flutter could be taken into account by methods similar to those adopted for the mobility of the fuselage in wing flutter (*see* § 12). Since, however, the moment of inertia of the wings about the longitudinal axis of the aeroplane is very large compared to that of the tail unit and rear fuselage, the " mobility effect " in antisymmetrical tail flutter is almost certainly negligible in practice. On the other hand, an appreciable pitching oscillation of the aeroplane might accompany symmetrical tail flutter. The influence of this motion on the critical flutter speed has not been investigated.

The complete list of suggestions regarding the design of tail units is given on p. 143.

§ 16. *Investigation of a Particular Instance of Tail Flutter.*—During a test flight of a certain aeroplane violent oscillations of the tail occurred, resulting in fracture of the sternpost and loss of control. The evidence suggested that flutter, predominantly of the rudder-fuselage type, had developed. In order to test this explanation of the accident, the critical speed for rudder-fuselage flutter was calculated and compared with the observed speed of flight. The calculation was based on aerodynamical coefficients derived from wind tunnel experiments on a scale model of the aeroplane; whilst the necessary inertial and elastic constants were found by direct measurement where possible, or by calculation from the drawings. The agreement between the calculated and observed speeds was sufficiently good to confirm the theory of the accident. A detailed account of the investigation is given in Chapter V.

**loc. cit.*

CHAPTER I.

WING FLUTTER AS INFLUENCED BY THE MOBILITY OF THE FUSELAGE.*

By R. A. FRAZER, B.A., B.Sc., and W. J. DUNCAN, D.Sc., A.M.I.Mech.E.

PREAMBLE AND SUMMARY.

§17. *Object of the Investigation.*—Hitherto, wing flutter has been investigated on the simplifying assumption that the fuselage is not set in oscillation by the flutter of the wings.† Strictly, however, the fuselage and wings form a single dynamical system, all parts of which must participate to some extent in any vibration. Thus the critical speed at which instability develops in actual flight will be influenced by the mobility of the fuselage, and the object of the present investigation is to find the importance of the effect.

§ 18. *Summary and Conclusions.*—The theory is restricted to the case of a monoplane in rectilinear flight at small angles of incidence.‡ The fuselage and ailerons are treated as rigid bodies, while, in accordance with the usual theory of wing flutter, the wings proper are assumed to be semi-rigid, i.e., the displacements of all points of a wing relative to the fuselage are assumed to be uniquely specified by the flexural and torsional co-ordinates.

It is shown in Part I that the most general oscillation of the complete wing-body system is compounded of oscillations of merely two types, called " longitudinal-symmetrical " and " lateral-anti-symmetrical," respectively. The characteristics of these motions will now be described.

In the longitudinal-symmetrical oscillations, the port and starboard wings move equally and in phase, and the ailerons are consequently subject to an elastic constraint provided by the cables. The motion of the aeroplane as a whole is of the " longitudinal " type, i.e., it involves merely longitudinal and normal translations, together with pitching. Equations of motion are obtained in Part II and applied to a particular monoplane. It is found that the critical flutter speed is almost identical with that calculated on the assumption of an immobile fuselage. Thus it may be concluded that the influence of the mobility of the fuselage upon the critical speeds is negligible in this type of motion.

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* Originally issued as R. & M. 1207 (Ref. 3).
† *See* R. & M. 1155 (Ref. 1).
‡ Biplanes are briefly considered in §§ 22 and 35.

On the other hand, in the lateral-antisymmetrical oscillations, the port and starboard wings move equally but in opposition, and the ailerons are virtually free from elastic control. The oscillations of the aeroplane as a whole are of the " lateral " type, being compounded of a lateral rectilinear motion, and of rolling and yawing. Equations of motion are obtained in Part III and applied to the monoplane considered in Part II. In this case the mobility of the fuselage is found to have an important stabilising influence.

Approximate theories of the two types of flutter are worked out in Part IV, and shown to be in good agreement with the more exact theories. In the longitudinal-symmetrical motion it is assumed that the only important oscillation of the fuselage is that normal to the flight path, whereas in the lateral-antisymmetrical motion the important oscillation is in roll. The approximate theory is applied to find the influence of variation of the longitudinal moment of inertia of the fuselage upon the stability, and it is shown that the stabilising effect of fuselage mobility is greatest when this moment of inertia is small.

Some experimental confirmation of the theoretical results has been obtained.

PART I.

RESOLUTION THEOREM FOR THE OSCILLATIONS OF THE SYSTEM.

§ 19. *The Dynamical Equations.*—It is assumed that the port and starboard wings are equal in all respects, or, more precisely, that they are exact mirror images in the longitudinal plane of symmetry of the monoplane. The wing and aileron motions relative to the fuselage will be described by the co-ordinates ϕ, ξ, θ; these co-ordinates have the usual significance,* but the subscripts s and p will be used to indicate quantities appropriate to the starboard and port wings, respectively. Corresponding co-ordinates and moments will be deemed to have the same sign in a symmetrical motion.

The fuselage with its appendages will be treated as a rigid body, and its motions will be specified by means of the component linear velocities (u, v, w) of the centre of mass of the complete machine† and the component angular velocities (p, q, r). Certain gravitational moments are determined by the Eulerian angles, but these angles are supposed expressed in terms of the time integrals of the angular velocities in the manner usual in aeroplane stability theory.

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* Throughout, the notation used is an extension of that adopted in the monograph R. & M. 1155.

† Strictly, the point of the fuselage which coincides with the centre of mass when the wings are undisturbed.

In the present section merely the *forms* of the dynamical equations are required, and detailed expressions for the coefficients are irrelevant; moreover, the dynamical equations are extremely unwieldy when written at length. Hence, for convenience, a condensed notation will be used; thus, typically:—

$$A_1(\phi) \equiv A_1\ddot{\phi} + B_1\dot{\phi} + C_1\phi.$$

In certain cases a time integral may appear, or one or more of the terms may be absent, but the essential point is that the operators A_1 ( ) etc., have unique meanings. In the determinant of motion the element corresponding to $A_1(\phi)$ is

$$A_1\lambda^2 + B_1\lambda + C_1,$$

and for conciseness this will be contracted to A_1 merely.

The following principles are used in the construction of the dynamical equations:—

(1) The velocities *u*, *w*, and *q* have no influence upon the lateral force, and upon the rolling and yawing moments. Consideration of symmetry establishes the truth of this as regards first order derivatives, which are alone retained.

(2) Similarly, the velocities *v*, *p*, and *r* have no influence upon the longitudinal and normal forces and upon the pitching moment.

(3) The velocities *u*, *w*, and *q* produce equal moments on the port and starboard wings.

(4) The velocities *v*, *p*, and *r* produce equal and opposite moments on the port and starboard wings.

(5) Direct elastic couplings of the ailerons is provided by the control cables, but the two wings have no other direct coupling. The elastic coupling of the ailerons is represented in equation (4) by the term

$$D_o(\xi_p) \equiv \tfrac{1}{2}h_\xi\,\xi_p \qquad \ldots \qquad (1)$$

and a similar term in equation (10). It should be noted that*

$$D_2(\xi) = D_2\ddot{\xi} + E_2\dot{\xi} + (\tfrac{1}{2}h_\xi - H_\xi)\,\xi. \qquad \ldots \qquad (2)$$

The factor $\frac{1}{2}$ is introduced in (1) and (2) in order to retain the definition of the elastic stiffness h_ξ as given in R. & M. 1155 (*see* present text, equation (18)). Inertia of the aileron controls is neglected; if this simplification were invalid, a further co-ordinate indicating the position of the control column would be required (*see* § 21).

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* It is here assumed that the gravitational stiffness of the aileron is negligible, as is certainly true when the plane of the wings is horizontal.

(6) Equal displacements (or velocities, or accelerations) of the port and starboard wings have equal effects upon the longitudinal and normal forces and upon the pitching moment, but they have equal and opposite effects upon the lateral force and upon the rolling and yawing moments.

In accordance with these principles the equations of motion can be written in the abbreviated notation :—

Flexural Moments on Starboard Wing.

$$A_1(\phi_s) + D_1(\xi_s) + G_1(\theta_s) + U_1(u) + W_1(w) + Q_1(q) + V_1(v) + P_1(p) + R_1(r) = 0. \quad \ldots \quad (3)$$

Hinge Moments on Starboard Aileron.

$$A_2(\phi_s) + D_2(\xi_s) + G_2(\theta_s) + U_2(u) + W_2(w) + Q_2(q) + D_o(\xi_p) + V_2(v) + P_2(p) + R_2(r) = 0. \quad \ldots \quad (4)$$

Torsional Moments on Starboard Wing.

$$A_3(\phi_s) + D_3(\xi_s) + G_3(\theta_s) + U_3(u) + W_3(w) + Q_3(q) + V_3(v) + P_3(p) + R_3(r) = 0. \quad \ldots \quad (5)$$

Longitudinal Forces on Complete Machine.

$$A_4(\phi_s) + D_4(\xi_s) + G_4(\theta_s) + U_4(u) + W_4(w) + Q_4(q) + A_4(\phi_p) + D_4(\xi_p) + G_4(\theta_p) = 0. \quad \ldots \quad (6)$$

Normal Forces on Complete Machine.

$$A_5(\phi_s) + D_5(\xi_s) + G_5(\theta_s) + U_5(u) + W_5(w) + Q_5(q) + A_5(\phi_p) + D_5(\xi_p) + G_5(\theta_p) = 0. \quad \ldots \quad (7)$$

Pitching Moments on Complete Machine.

$$A_6(\phi_s) + D_6(\xi_s) + G_6(\theta_s) + U_6(u) + W_6(w) + Q_6(q) + A_6(\phi_p) + D_6(\xi_p) + G_6(\theta_p) = 0. \quad \ldots \quad (8)$$

Flexural Moments on Port Wing.

$$U_1(u) + W_1(w) + Q_1(q) + A_1(\phi_p) + D_1(\xi_p) + G_1(\theta_p) - V_1(v) - P_1(p) - R_1(r) = 0. \quad \ldots \quad (9)$$

Hinge Moments on Port Aileron.

$$D_o(\xi_s) + U_2(u) + W_2(w) + Q_2(q) + A_2(\phi_p) + D_2(\xi_p) + G_2(\theta_p) - V_2(v) - P_2(p) - R_2(r) = 0. \quad \ldots \quad (10)$$

Torsional Moments on Port Wing.

$$U_3(u) + W_3(w) + Q_3(q) + A_3(\phi_p) + D_3(\xi_p) + G_3(\theta_p) - V_3(v) - P_3(p) - R_3(r) = 0. \quad \ldots \quad (11)$$

Lateral Forces on Complete Machine.

$$A_7(\phi_s) + D_7(\xi_s) + G_7(\theta_s) - A_7(\phi_p) - D_7(\xi_p) - G_7(\theta_p) + V_7(v) + P_7(p) + R_7(r) = 0. \quad \ldots \quad (12)$$

Rolling Moments on Complete Machine.

$$A_8(\phi_s) + D_8(\xi_s) + G_8(\theta_s) - A_8(\phi_p) - D_8(\xi_p) - G_8(\theta_p) + V_8(v) + P_8(p) + R_8(r) = 0. \quad .. \quad .. \quad (13)$$

Yawing Moments on Complete Machine.

$$A_9(\phi_s) + D_9(\xi_s) + G_9(\theta_s) - A_9(\phi_p) - D_9(\xi_p) - G_9(\theta_p) + V_9(v) + P_9(p) + R_9(r) = 0. \quad .. \quad .. \quad (14)$$

§ 20. *Resolution of the Determinant of Motion.*—The determinant of motion corresponding to the equations (3) to (14) is written down in the Appendix*, where it is shown to be resoluble into the following factors :—

$$\Delta_1(\lambda) \equiv \begin{vmatrix} A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 \\ A_2 & D_2+D_0 & G_2 & U_2 & W_2 & Q_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 \\ 2A_4 & 2D_4 & 2G_4 & U_4 & W_4 & Q_4 \\ 2A_5 & 2D_5 & 2G_5 & U_5 & W_5 & Q_5 \\ 2A_6 & 2D_6 & 2G_6 & U_6 & W_6 & Q_6 \end{vmatrix} \quad .. \quad (15)$$

and

$$\Delta_2(\lambda) \equiv \begin{vmatrix} A_1 & D_1 & G_1 & V_1 & P_1 & R_1 \\ A_2 & D_2-D_0 & G_2 & V_2 & P_2 & R_2 \\ A_3 & D_3 & G_3 & V_3 & P_3 & R_3 \\ 2A_7 & 2D_7 & 2G_7 & V_7 & P_7 & R_7 \\ 2A_8 & 2D_8 & 2G_8 & V_8 & P_8 & R_8 \\ 2A_9 & 2D_9 & 2G_9 & V_9 & P_9 & R_9 \end{vmatrix} \quad .. \quad (16)$$

§ 20(*a*). *Longitudinal-Symmetrical Motion.*— $\Delta_1(\lambda)$ is the determinant of the equations (3) to (8), subject to the conditions

$$\left.\begin{array}{l} v = p = r = 0, \\ \text{and} \\ \phi_s = \phi_p\,;\ \xi_s = \xi_p\,;\ \theta_s = \theta_p. \end{array}\right\} \quad .. \quad (17)$$

Thus $\Delta_1(\lambda)$ corresponds to the motion in which the wing oscillations are symmetrical and the body oscillations are of the longitudinal type. These oscillations are described as " longitudinal-symmetrical."

From (1) and (2)

$$D_2(\xi) + D_o(\xi) \equiv D_2\ddot{\xi} + E_2\dot{\xi} + (h_\xi - H_\xi)\,\xi, \quad (18)$$

so that in motion of the " longitudinal-symmetrical " type the ailerons are subject to an elastic control, as is otherwise obvious.

§ 20(*b*). *Lateral-Antisymmetrical Motion.*—The determinant $\Delta_2(\lambda)$ corresponds to the equations of motion (3) to (5) and (12) to (14), subject to the conditions

$$\left.\begin{array}{l} u = w = q = 0, \\ \text{and} \\ \phi_p = -\phi_s\,;\ \xi_p = -\xi_s\,;\ \theta_p = -\theta_s. \end{array}\right\} \quad .. \quad (19)$$

---

* *See* p. 49.

Thus, $\Delta_2(\lambda)$ is the determinant of the motion in which the wing oscillations are antisymmetrical and the body oscillations are of the lateral type. These oscillations are described as "lateral-antisymmetrical."

From (1) and (2)

$$D_2(\xi) - D_o(\xi) \equiv D_2\ddot{\xi} + E_2\dot{\xi} - H_\xi \xi \quad .. \quad (20)$$

so that in motion of the "lateral-antisymmetrical" type there is effectively no elastic constraint of the ailerons.*

§ 21. *Influence of the Inertia of the Aileron Controls.*—It can be shown without difficulty that when the inertia of the aileron controls is taken into account, resolution of the general motion into the two types of oscillation still takes place. The "longitudinal-symmetrical" oscillations are quite unaltered, but those of the "lateral-antisymmetrical" type have an extra degree of freedom. At a critical flutter speed for motion of the latter type the effect of the inertia of the controls is the same as if the ailerons had a small negative elastic stiffness, or, alternatively, no elastic stiffness and a slightly augmented moment of inertia.

§ 22. *Extension of the Argument to an Unstaggered Biplane with Equal Upper and Lower Planes.*—It is pointed out in Chap. III that, when the fuselage is supposed immobile, possible types of motion of the wings of an equal unstaggered biplane are where—

(1) The upper and lower wings oscillate equally and in the same phase.

(2) The upper and lower wings oscillate equally but in opposition.

When the restriction as to the immobility of the fuselage is removed, possible oscillations will evidently be such that—

(1) All four wings move equally and in phase, while the body motions are longitudinal.

(2) Upper and lower wings on the same side move equally and in phase but in opposition to those on the other side, while the body motions are lateral.

(3) Upper and lower wings on the same side move equally and in opposition, while body motion is absent.

It appears that the sum of the numbers of degrees of freedom in the three motions specified is equal to the number of degrees of freedom of the whole system. The three motions thus exhaust the possibilities of the system.

* In the discussion it is assumed that direct elastic couplings between φ_S and φ_P, and between θ_S and θ_P, are absent. When such couplings are present, the resolution theorem remains valid, but the effective value of the flexural (or torsional) stiffness differs in the symmetrical and antisymmetrical motions. This is precisely analogous to the effect of the elastic coupling of the ailerons.

PART II.

THEORY OF THE LONGITUDINAL-SYMMETRICAL OSCILLATIONS.

§23. *The Dynamical Equations.*—For simplicity, wing torsion will be neglected. Thus, the investigation will be limited to flutter of the so-called flexural-aileron type. The dynamical equations will first be obtained and then applied to a particular monoplane.

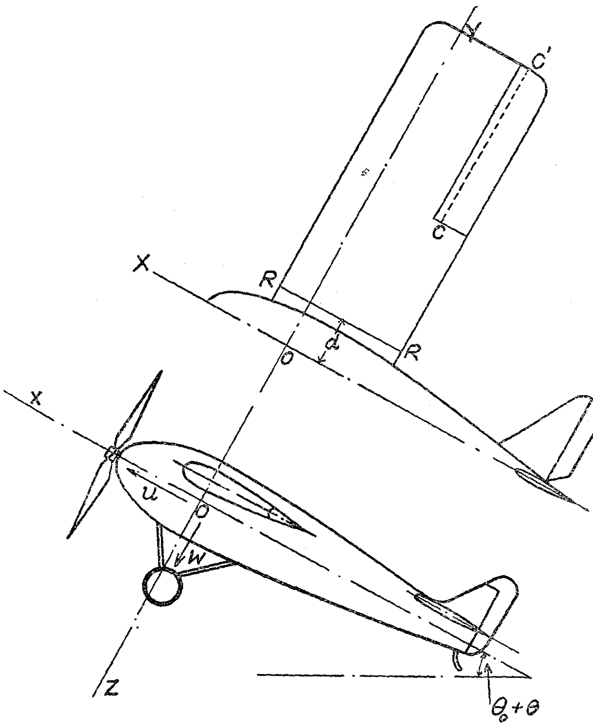

FIG. 12.—Diagram of Monoplane.

The motion will be referred to axes which move with the fuselage (*see* Fig. 12). The origin O is the point of the fuselage which coincides with the C.G. of the complete machine when the wings are undisturbed, while OX and OZ are the perpendicular reference axes in the plane of symmetry. Their exact location need not at present be specified, but the angle between OX and the wing chord will be presumed small.

At any instant after disturbance, the inclination of OX to the horizontal is $\theta_0 + \theta$, and the components of the velocity of O parallel to OX and OZ are $u_0 + u$, $w_0 + w$, respectively. The angle ϕ measures the equal flexural displacements of the two wings, and ξ

the equal angular displacements of the ailerons. For simplicity, the equations will be stated for the case where flexural curvature is negligible, but they can readily be extended to a general law of flexure.

The components of acceleration α, β, for the various points of the system are specified below :—

(*a*) *Points of Fuselage.*—

$$\alpha = \dot{u} + \dot{q}\, z + q\, w_o \qquad (21)$$
$$\beta = \dot{w} - \dot{q}\, x - q\, u_o. \qquad (22)$$

(*b*) *Points of Starboard Wing.*—

$$\alpha = \dot{u} + \dot{q}\, z + q\, w_o \qquad (23)$$
$$\beta = \dot{w} - \dot{q}\, x - q\, u_o + \ddot{\phi}\,(y - d). \qquad (24)$$

Here d is the distance from the plane of symmetry to the wing root R R′ (*see* Fig. 1).

(*c*) *Points of Starboard Aileron.*—

$$\alpha = \dot{u} + \dot{q}\, z + q\, w_o \qquad (25)$$
$$\beta = \dot{w} - \dot{q}\, x - q\, u_o + \ddot{\phi}\,(y - d) - \ddot{\xi}\, x_a. \qquad (26)$$

where the co-ordinate x_a is the distance of a point of the aileron measured *forward* of the hinge.

Let X = Component of aerodynamical force* on complete machine parallel to OX per unit mass.

Z = Component of aerodynamical force* on complete machine parallel to OZ per unit mass.

M = Aerodynamical pitching moment on complete machine about OY.

M = Total Mass of the Machine.

g = Acceleration due to gravity.

B = Moment of inertia of the complete machine about OY.

L = Flexural moment on the starboard wing.

H = Hinge moment on the starboard aileron.

l_ϕ = Flexural elastic stiffness of starboard wing.

h_ξ = Elastic stiffness of starboard aileron control.

$\sum_t$ denote summation over the whole machine.

$\sum_w$ denote summation over starboard wing with aileron.

$\sum_a$ denote summation over starboard aileron.

δ denote increment from equilibrium value.

* This is assumed to include airscrew thrust.

The variables u, w, θ, ϕ, ξ will be adopted as generalised co-ordinates, and the aerodynamical terms expressed in terms of derivatives, as usual. Thus, typically

$$\delta X = uX_u + wX_w + qX_q + \dot{\phi}X_{\dot{\phi}} + \phi X_{\phi} + \dot{\xi}X_{\dot{\xi}} + \xi X_{\xi}. \quad (27)$$

On application of the formulae (21) .. (26), the equations of motion can be derived in the form :—

(1) *Longitudinal Forces on Whole Machine.*—

$$\dot{u} + q\,w_o = \delta X - \theta g \cos \theta_o. \quad .. \quad .. \quad (28)$$

(2) *Normal Forces on Whole Machine.*—

$$\dot{w} - q\,u_0 + 2\ddot{\phi}\,\frac{\sum_w m\,(y-d)}{M} - 2\ddot{\xi}\,\frac{\sum_a mx_a}{M} = \delta Z - \theta g \sin \theta_o. \quad (29)$$

(3) *Pitching Moments.*—

$$\dot{q}\,B - 2\ddot{\phi}\sum_w mx\,(y-d) + 2\ddot{\xi}\sum_a mxx_a$$
$$= \delta M - 2\phi\, g \sin \theta_o \sum_w m(y-d) + 2\xi g \sin \theta_o \sum_a mx_a. \quad (30)$$

(4) *Flexural Moments on Starboard Wing.*—

$$\dot{w}\sum_w m(y-d) - \dot{q}\sum_w mx(y-d) - qu_o\sum_w m(y-d) + \ddot{\phi}\sum_w m(y-d)^2$$
$$- \ddot{\xi}\sum_a mx_a(y-d) = -\phi l_{\phi} + \delta L - \theta g \sin \theta_o \sum_w m(y-d). \quad (31)$$

(5) *Hinge Moments on Starboard Aileron.*—

$$-\dot{w}\sum_a mx_a + \dot{q}\sum_a mxx_a + q\,u_o\sum_a mx_a - \ddot{\phi}\sum_a mx_a\,(y-d)$$
$$+ \ddot{\xi}\sum_a mx^2_a = -\xi h_{\xi} + \delta H + \theta g \sin \theta_o \sum_a mx_a. \quad .. \quad (32)$$

The determinant of motion corresponding to the foregoing equations is (33), and is an octic in λ. The three-row determinant of conventional longitudinal stability theory, and the two-row determinant* of conventional flutter theory, constitute the two minors in the principal diagonal. Couplings between the two types of oscillation are represented by the remaining terms.

§24. *Application to a Particular Monoplane in a Vertical Dive.*— (*a*) *General Data.*—The case of a vertical dive, with engine off, has been selected for simplicity. It is assumed that the machine has been so rigged that in the dive the lift and pitching couple are both zero, and that it can therefore gather speed without appreciable change of incidence. The axis OX will be chosen to coincide with the vertical, prior to the disturbance ; thus $\theta_o = -\pi/2$, and $w_o = 0$. Under the stated conditions the derivatives Z_u, M_u, L_u, H_u may be taken to vanish. The determinant of motion now resolves into the product of $\lambda - X_u$ (which denotes a subsidence), and its minor, which is a determinant of the fourth order and of the seventh degree in λ. In the sequel, attention will be confined to this minor.

The data assumed for the machine are listed in Table 2.

* The negative sign associated with the flexural-aileron product of inertia is due to the fact that the co-ordinate x_a is here measured forward of the hinge.

(4453)

$$
\left|\begin{array}{c|cccc}
-X_u , & -X_w , & \lambda(w_o-X_q)+g\cos\theta_o , & -\lambda X_\phi - X_\phi , & -\lambda X_\xi - X_\xi \\
\hdashline
-Z_u , & \lambda - Z_w , & -\lambda(u_o+Z_q)+g\sin\theta_o , & \frac{2\lambda^2}{M}\sum\limits_w m(y-d)-\lambda Z_\phi - Z_\phi , & -\frac{2\lambda^2}{M}\sum\limits_a m x_a - \lambda Z_\xi - Z_\xi \\
-M_u , & -M_w , & \lambda^2 B - \lambda M_q , & \begin{array}{l} -2\lambda^2\sum\limits_w mx(y-d)-\lambda M_\phi \\ -M_\phi+2g\sin\theta_o\sum\limits_w m(y-d), \end{array} & \begin{array}{l} 2\lambda^2\sum\limits_a mxx_a-\lambda M_\xi - M_\xi \\ -2g\sin\theta_o\sum\limits_a mx_a \end{array} \\
-L_u , & \lambda\sum\limits_w m(y-d)-L_w , & \begin{array}{l} -\lambda^2\sum\limits_w mx(y-d)-\lambda\{u_o\sum\limits_w m(y-d) \\ +L_q\}+g\sin\theta_o\sum\limits_w m(y-d) , \end{array} & \begin{array}{l} \lambda^2\sum\limits_w m(y-d)^2-\lambda L_\phi \\ +l_\phi - L_\phi , \end{array} & \begin{array}{l} -\lambda^2\sum\limits_a mx_a(y-d) \\ -\lambda L_\xi - L_\xi \end{array} \\
-H_u , & -\lambda\sum\limits_a mx_a - H_w , & \begin{array}{l} \lambda^2\sum\limits_a mxx_a+\lambda\{u_o\sum\limits_a mx_a - H_q\} \\ -g\sin\theta_o\sum\limits_a mx_a , \end{array} & \begin{array}{l} -\lambda^2\sum\limits_a mx_a(y-d)-\lambda H_\phi \\ -H_\phi , \end{array} & \begin{array}{l} \lambda^2\sum\limits_a mx_a^2-\lambda H_\xi + h_\xi \\ -H_\xi \end{array}
\end{array}\right|
\qquad \ldots \quad \ldots \quad \ldots \quad (33)
$$

B 3

TABLE 2.

Particulars of Monoplane.

Item.	Specification.
Type of Machine	Cantilever Monoplane.
Overall Span	35 ft.
Wing Chord	5 ft.
Span of Fixed Centre Section ..	5 ft.
Span of each Wing	15 ft.
Aileron Span	7 ft.
Aileron Chord	1·4 ft.
Aileron Hinge	0·14 ft. from nose.
C.G. of Machine	0·35 chord aft of leading edge.
Distance C.G. from Wing Chord ..	0.
Distance C.G. to C.P. of Tail Unit ..	15 ft.
Area of Tail Unit	31·5 ft.^{2}
Mass of Machine	2,600 lb. (80·7 slugs.).
Radius Gyration about C.G. ..	0·8 wing chord.
Wing Weight	2 lb. per ft.^{2}
$\frac{2}{M}\sum_w m\,(y-d)$	0·583 ft.
$-\frac{2}{M}\sum_a m\,x_a$	0·00421 ft.
B	1,300 slug. ft.^{2}
$-\sum_w m\,x\,(y-d)$	0.
$\sum_a mxx_a$	0·3 slug. ft.^{2}
$-\sum_a mx_a$	0·17 slug. ft.
$\sum_w m\,(y-d)^2$	250 slug. ft.^{2}
$-\sum_a m\,x_a\,(y-d)$	2 slug. ft.^{2}
$\sum_a m\,x_a{}^2$	0·175 slug. ft.^{2}
$l\phi$	3 $\times$ 10^6 lb. ft. per radian.
$h\xi$	867 lb. ft. per radian.

(*b*) *Aerodynamical Derivatives.*—The wing and aileron are similar as regards plan form to the model wing of 27 in. span used in the derivative measurements of R. & M. 1155. The purely flexural-aileron group of derivatives has been derived from these measurements, aerodynamical scale effect and the influence of wing section being neglected. As regards the ordinary longitudinal derivatives, these have been calculated in the orthodox manner. The remainder have either been estimated by strip theory, or deduced from the results given in R. & M. 1155.

TABLE 3.

Values of Aerodynamical Derivatives.

(Slug-foot-second units).

Derivative.	Value.	Derivative.	Value.	Derivative.	Value.	Derivative.	Value.
Z_w	$-0\cdot01u_0$	M_w	$-1\cdot0u_0$	L_w	$-0\cdot239u_0$	H_w	$-0\cdot00325u_0$
Z_q	$-0\cdot028u_0$	M_q	$-33\cdot6u_0$	L_q	$-12\cdot5u_0$	H_q	$-0\cdot05u_0$
$Z_{\dot\phi}$	$-0\cdot048u_0$	$M_{\dot\phi}$	0*	$L_{\dot\phi}$	$-13\cdot2u_0$	$H_{\dot\phi}$	$-0\cdot045u_0$
Z_ϕ	0	M_ϕ	0	L_ϕ	0	H_ϕ	0
$Z_{\dot\xi}$	$-0\cdot001u_0$	$M_{\dot\xi}$	$-0\cdot2u_0$	$L_{\dot\xi}$	$-0\cdot48u_0$	$H_{\dot\xi}$	$-0\cdot02u_0$
Z_ξ	$-0\cdot001u_0^2$	M_ξ	$-0\cdot13u_0^2$	L_ξ	$-0\cdot5u_0^2$	H_ξ	$-0\cdot008u_0^2$

On substitution of the numerical data, the determinant under consideration assumes the form (34), where the figures for the last two rows have been doubled in order to exhibit the symmetry of the inertial couplings.

* This derivative is zero since the transverse axis OY approximately coincides with the " axis of independence."

Determinant for Longitudinal Oscillations and Flutter of a Monoplane.

(Case of a Vertical Dive).

$$\left|\begin{array}{cc|cc} \lambda+0\cdot 01u_0 , & -0\cdot 972u_0\lambda-32\cdot 2 , & 0\cdot 583\lambda^2+0\cdot 048u_0\lambda , & 0\cdot 00421\lambda^2+0\cdot 001u_0\lambda+0\cdot 001u_0{}^2 \\ 1\cdot 0u_0 , & 1{,}300\lambda^2+33\cdot 6u_0\lambda , & -1513\cdot 4 , & 0\cdot 6\lambda^2+0\cdot 2\,u_0\lambda+0\cdot 13u_0{}^2-10\cdot 95 \\ \hline 47\lambda+4\cdot 78u_0 , & -22\cdot 0u_0\lambda-1513\cdot 4 , & 500\lambda^2+26\cdot 4u_0\lambda+6\times 10^6 , & 4\lambda^2+0\cdot 96u_0\lambda+1\cdot 0u_0{}^2 \\ 0\cdot 34\lambda+0\cdot 0065u_0 , & 0\cdot 6\lambda^2-0\cdot 24u_0\lambda-10\cdot 948 , & 4\lambda^2+0\cdot 09u_0\lambda , & 0\cdot 35\,\lambda^2+0\cdot 04u_0\lambda+1{,}734+0\cdot 016u_0{}^2 \end{array}\right| \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (34)$$

(*c*) *Critical Flutter Speed when the Machine as a Whole does not Oscillate.*—The relevant determinant in this case is the minor enclosed within dotted lines in (34). By the usual methods it is found that the lowest critical speed is 245 ft. per second, and the critical frequency about 1,055 cycles per minute. It may be noted that if the elastic control of the ailerons be supposed absent, the lowest critical speed rises to 310 ft. per sec.

Solution of the determinant for the longitudinal oscillations of the aeroplane (with the wings supposed held rigid) at 245 ft./sec. gives

$$\lambda = -0\cdot 102, \ -0\cdot 196 \text{ and } -4\cdot 34 \pm 6\cdot 376\sqrt{-1}.$$

Thus the machine itself is longitudinally stable at this speed.

(*d*) *True Critical Flutter Speed for the Complete System.*—The true critical flutter speed is found from the complete determinant (34). It is clearly inconvenient to use test functions in the present instance, and the critical speed was found by plotting the real part of the relevant root of (34) on a base of wind speed. The roots were obtained very readily by Bairstow's method of successive approximation to a quadratic factor,* the first trial factor being indicated by the roots for the pure flexural-aileron motion. In this way the critical speed was found to be 245·5 ft./sec. Thus the mobility of the fuselage raises the critical speed by merely 0·5 ft./sec.; the difference is practically negligible.

The ratios of the amplitudes of the oscillations were found by methods similar to those given in R. & M. 1155. The final results are that the amplitude of the normal motion of the C.G. of the machine in space at the critical speed is one twenty-sixth of the amplitude of the motion of the wing tip, measured relative to the fuselage: and that the amplitudes of the pitching motion of the machine and of the angular motion of the wings in flexure are in the ratio 1 : 120.

§25. *Conclusion.*—In the present case the influence of the mobility of the fuselage may be considered as practically negligible. More generally, it appears probable that for the type of motion considered, the critical flutter speed calculated on the assumption that the fuselage is immobile would be a sufficiently reliable guide for practical purposes.

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* Refs. 4 and 5.

PART III.

THEORY OF THE LATERAL-ANTISYMMETRICAL OSCILLATIONS.

§ 26. *The Dynamical Equations.*—Wing torsion will be neglected as in Part II and the same system of reference axes will be adopted (*see* Fig. 13).

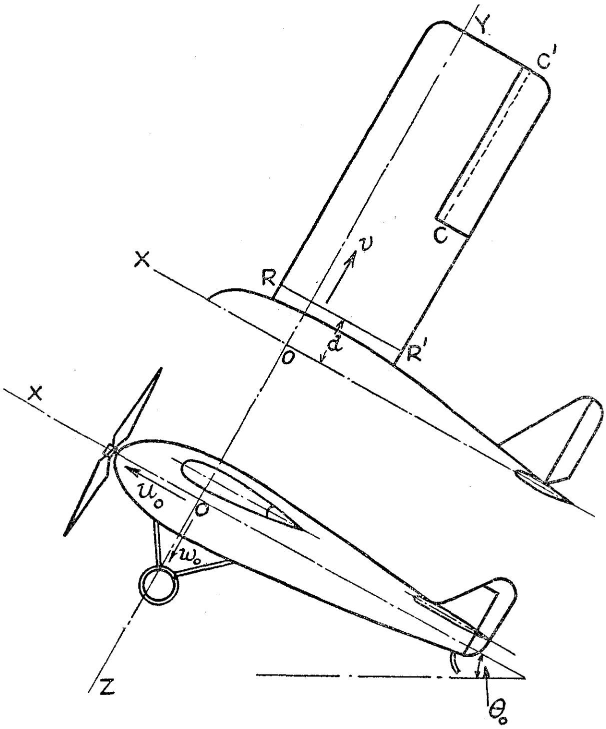

FIG. 13.—Diagram of Monoplane.

At any instant the linear velocity of the origin O is (u_0, v, w_0) and the angular velocity is (p, 0, r), while in the steady undisturbed motion v, p and r, vanish. The pitching velocity q will be zero and u and w will be constant, since there are no couplings between the corresponding motions and the "lateral-antisymmetrical oscillations."

When squares and products of small quantities are neglected, the components of acceleration (α, β, γ) for the various points of the system are :—

(*a*) *Points of Fuselage.*—

$$\alpha = -y\dot{r} \quad \ldots \quad (35)$$

$$\beta = \dot{v} + x\dot{r} - z\dot{p} - pw_0 + ru_0 \quad \ldots \quad (36)$$

$$\gamma = y\dot{p}. \quad \ldots \quad (37)$$

(*b*) *Points of Starboard Wing.*—It is assumed that the angle between OX and the wing chord is small; the influence of this angle and of the dihedral angle on the acceleration is neglected.

$$\alpha = -y\dot{r} \qquad (38)$$

$$\beta = \dot{v} + x\dot{r} - z\dot{p} - pw_o + ru_o \qquad (39)$$

$$\gamma = y\dot{p} + (y-d)\ddot{\phi}. \qquad (40)$$

(*c*) *Points of Starboard Aileron.*—

$$\alpha = -yr \qquad (41)$$

$$\beta = \dot{v} + x\dot{r} - zp - pw_o + ru_o \qquad (42)$$

$$\gamma = y\dot{p} + (y-d)\ddot{\phi} - x_a\ddot{\xi}. \qquad (43)$$

Let Y = Component of aerodynamical force parallel to OY per unit mass of the complete machine.

L = Aerodynamical rolling moment on complete machine.*
N = Aerodynamical yawing moment on complete machine.
A = Moment of inertia of complete machine about OX.
C = Moment of inertia of complete machine about OZ.
$-$E = $-\sum_t mzx$ = rolling-yawing product of inertia of complete machine.
χ = angular displacement of fuselage in roll.†

In the undisturbed position of the machine the wing span OY is assumed horizontal, and the longitudinal axis OX is supposed inclined at θ_0 to the horizontal. The variables v, p, r, ϕ, ξ will be adopted as generalised co-ordinates, and the aerodynamical terms expressed by means of derivatives, as usual. Thus, typically,

$$\delta Y = vY_v + pY_p + rY_r + \dot{\phi}\,Y_{\dot{\phi}} + \phi\,Y_{\phi} + \dot{\xi}\,Y_{\dot{\xi}} + \xi\,Y_{\xi}. \qquad (44)$$

In all cases the combined effect of the two wings is just double the effect of the starboard wing alone. Hence, by the aid of the formulae (35)–(43), it can be shown that the equations of motion are as follows:—

(1) *Lateral Forces per Unit Mass of Whole Machine,*—

$$\dot{v} - pw_o + ru_o = \delta Y + \chi g \cos\theta_o. \qquad (45)$$

(2) *Rolling Moments on Whole Machine.*—

$$\dot{p}A - \dot{r}E + \ddot{\phi}\,2\sum_w my\,(y-d) - \ddot{\xi}\,2\sum_a mx_a y = \delta\,\mathbf{L}. \qquad (46)$$

---

* The heavy capital **L** is used, since L is required for the flexural moment.

† This angle is usually denoted by φ, but the latter symbol has been adopted for the flexural co-ordinate.

(3) *Yawing Moments on Whole Machine.*—

$$-\dot{p}\mathrm{E} + \dot{r}\mathrm{C} = \delta\,\mathbf{N}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (47)$$

(4) *Flexural Moments on Starboard Wing.*—

$$\dot{p}\sum_w my\,(y-d) + \ddot{\phi}\sum_w m\,(y-d)^2$$
$$-\ddot{\xi}\sum_w mx_a\,(y-d) = -\phi\, l_\phi + \delta\,\mathrm{L}. \qquad \ldots \qquad (48)$$

(5) *Hinge Moments on Starboard Aileron.*—

$$-\dot{p}\sum_a mx_a y - \ddot{\phi}\sum_a mx_a\,(y-d)$$
$$+\ddot{\xi}\sum_a mx_a^2 = \delta\mathrm{H} - \xi g \sin\theta_o \sum_a mx_a. \qquad \ldots \qquad (49)$$

The angle χ is eliminated from equation (45) by means of the relation

$$\dot{\chi} = p + r\tan\theta_o. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (50)$$

Since p and r are proportional to $e^{\lambda t}$ it follows that

$$\chi g\cos\theta_o = \frac{g\,(p\cos\theta_o + r\sin\theta_o)}{\lambda}. \qquad \ldots \qquad (51)$$

The determinant of motion can accordingly be written :—

$$\Delta(\lambda) \equiv \begin{vmatrix} \lambda - \mathrm{Y}_v, & -w_o - \mathrm{Y}_p - \dfrac{g\cos\theta_o}{\lambda}, & u_o - \mathrm{Y}_r - \dfrac{g\sin\theta_o}{\lambda}, & -\lambda\mathrm{Y}_{\dot\phi} - \mathrm{Y}_{\phi} & , \quad -\lambda\mathrm{Y}_{\dot\xi} - \mathrm{Y}_{\xi} \\ -\mathbf{L}_v & , \mathrm{A}\lambda - \mathbf{L}_p & , -\mathrm{E}\lambda - \mathbf{L}_r & , \lambda^2 2\sum_w my(y-d) - \lambda\mathbf{L}_{\dot\phi} - \mathbf{L}_{\phi} & , -\lambda^2 2\sum_a mx_a y - \lambda\mathbf{L}_{\dot\xi} - \mathbf{L}_{\xi} \\ -\mathbf{N}_v & , -\mathrm{E}\lambda - \mathbf{N}_p & , \mathrm{C}\lambda - \mathbf{N}_r & , -\lambda\mathbf{N}_{\dot\phi} - \mathbf{N}_{\phi} & , -\lambda\mathbf{N}_{\dot\xi} - \mathbf{N}_{\xi} \\ -\mathrm{L}_v & , \lambda\sum_w my(y-d) - \mathrm{L}_p & , -\mathrm{L}_r & , \lambda^2\sum_w m(y-d)^2 - \lambda\mathrm{L}_{\dot\phi} + l_{\phi} - \mathrm{L}_{\phi} & , -\lambda^2\sum_a mx_a(y-d) - \lambda\mathrm{L}_{\dot\xi} - \mathrm{L}_{\xi} \\ -\mathrm{H}_v & , -\lambda\sum_a mx_a y - \mathrm{H}_p & , -\mathrm{H}_r & , -\lambda^2\sum_a mx_a(y-d) - \lambda\mathrm{H}_{\dot\phi} - \mathrm{H}_{\phi} & , \lambda^2\sum_a mx_a^2 - \lambda\mathrm{H}_{\dot\xi} - \mathrm{H}_{\xi} + g\sin\theta_o\sum_a mx_a \end{vmatrix} = 0 \quad \ldots \ldots \ldots \quad (52)$$

It will be seen that the third order determinant of conventional lateral stability theory and the second order determinant of conventional flexural-aileron flutter theory constitute the two minors in the principal diagonal of Δ (λ). The remaining terms represent couplings between the two types of oscillation.

§ 27. *Application to a Particular Monoplane in a Vertical Dive.*—(*a*) *General Data.*—The mathematical analysis will now be applied to the monoplane whose longitudinal-symmetrical oscillations are investigated in Part II. As before, the case of a vertical dive with engine off will be selected, and it will be assumed that the machine has been so rigged that in the dive the lift and pitching couple are both zero, so that it can gather speed without appreciable change of incidence. The axis OX will be chosen to coincide with the vertical prior to the disturbance ; thus $\theta_o = -\pi/2$, and $w_o = 0$.

The data assumed for the machine are listed in Table 4 and the values of the aerodynamical derivatives are given in Table 5. The purely flexural-aileron group of derivatives has been derived from the values as given in R. & M. 1155* for a model wing of 27 in. span, while the ordinary lateral derivatives have been estimated with the help of measurements made on the Bristol Fighter. As regards the derivatives in the remaining coupling terms, these have been estimated by strip theory, or deduced from the known derivatives.

TABLE 4.

Particulars of Monoplane.

(*Supplementary to Table* 2.)

Item.	Specification.
Dihedral Angle	3°
A	950 slug ft.^{2}
C	2,000 slug ft.^{2}
E	0
$\sum_{w} my\,(y-d)$	310 slug ft.^{2}
$-\sum_{a} mx_a y$	2·425 slug ft.^{2}

* Ref. 1.

TABLE 5.

Values of Aerodynamical Derivatives. (Slug-foot-second units.)

Derivative.	Value.	Derivative.	Value.	Derivative.	Value.	Derivative.	Value.	Derivative.	Value.
Y_v	$-0\cdot 001u_o$	$\mathbf{L}_v$	$-0\cdot 3u_o$	$\mathbf{N}_v$	0	L_v	$-0\cdot 1u_o$	H_v	0
Y_p	$-0\cdot 35u_o$	$\mathbf{L}_p$	$-50\cdot 0u_o$	$\mathbf{N}_p$	0	L_p	$-17\cdot 5u_o$	H_p	$-0\cdot 055u_o$
Y_r	0	$\mathbf{L}_r$	0	$\mathbf{N}_r$	$-4\cdot 0u$	L_r	0	H_r	0
$Y_{\dot\phi}$	$-0\cdot 26u_o$	$\mathbf{L}_{\dot\phi}$	$-35\cdot 0u_o$	$\mathbf{N}_{\dot\phi}$	0	$L_{\dot\phi}$	$-13\cdot 2u_o$	$H_{\dot\phi}$	$-0\cdot 045u_o$
Y_ϕ	0	$\mathbf{L}_\phi$	0	$\mathbf{N}_\phi$	0	L_ϕ	0	H_ϕ	0
$Y_{\dot\xi}$	$-0\cdot 002u_o$	$\mathbf{L}_{\dot\xi}$	$-1\cdot 2u_o$	$\mathbf{N}_{\dot\xi}$	0	$L_{\dot\xi}$	$-0\cdot 48u_o$	$H_{\dot\xi}$	$-0\cdot 02u_o$
Y_ξ	$-0\cdot 002u_o^2$	$\mathbf{L}_\xi$	$-1\cdot 2u_o^2$	$\mathbf{N}_\xi$	0	L_ξ	$-0\cdot 5u_o^2$	H_ξ	$-0\cdot 008u_o^2$

It will be seen from Table 5 that the derivatives $\mathbf{N}_p$ and $\mathbf{L}_r$ have been assumed to vanish. The validity of those assumptions under the condition of no lift is supported by the experimental results for a Bristol Fighter model given in R. & M. 932.* Since $\mathbf{L}_r$ vanishes, it is clearly legitimate to assume that L_r and H_r vanish also. The magnitude and sign of Y_r depend on the distribution of the vertical surfaces of the tail and fuselage, and in the present case, for simplicity, the derivative has been assumed zero. Again, since $\mathbf{N}_p$ is zero, $\mathbf{N}_{\dot{\phi}}$ must vanish. The derivatives $\mathbf{N}_{\dot{\xi}}$, $\mathbf{N}_{\xi}$ and H_v are certainly very small, and have been taken to vanish. Finally, all the ϕ derivatives are zero, since the incidence is small.

When the data from Tables 2, 4, and 5 are substituted in the determinant of motion (52), the latter becomes merely the product of the term $(C\lambda - \mathbf{N}_r)$, which represents a subsidence in yaw, and its minor. The latter is alone of interest from the present point of view, and it is written at length in equation (53), where the last two rows have been doubled so as to preserve the symmetry of the inertial coupling terms. The very small gravitational stiffness of the ailerons, represented by the term $g \sin \theta_o \sum_a mx_a$ in (52), has been neglected.

(b) *Stability of the System.*—When the fuselage is supposed held rigid, the lowest critical flutter speed for elastically free ailerons is found to be at 310 ft./sec., but the determinantal equation (53) shows that when the fuselage is free there is no flutter up to a speed of 800 ft./sec. It has not been considered necessary to carry the calculations to higher speeds.

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* R. & M. 932. Section 2 (Ref. 6).

$$\begin{vmatrix} \lambda + 0\cdot001u_o, & 0\cdot35u_o, & 0\cdot26u_o\,\lambda \quad , & +0\cdot002u_o\,\lambda + 0\cdot002u_o^2 \\ 0\cdot3u_o \quad , & 950\lambda + 50\cdot0u_o, & 620\,\lambda^2 + 35\cdot0u_o\,\lambda \quad , & 4\cdot85\,\lambda^2 + 1\cdot2u_o\,\lambda + 1\cdot2u_o^2 \\ 0\cdot2u_o \quad , & 620\lambda + 35\cdot0u_o, & 500\,\lambda^2 + 26\cdot4u_o\,\lambda + 6 \times 10^6, & 4\,\lambda^2 + 0\cdot96u_o\,\lambda + 1\cdot0u_o^2 \\ 0 \quad , & 4\cdot85\lambda + 0\cdot11u_o, & 4\,\lambda^2 + 0\cdot09u_o\,\lambda \quad , & 0\cdot35\,\lambda^2 + 0\cdot04u_o\,\lambda + 0\cdot016u_o^2 \end{vmatrix} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (53)$$

§ 28. *Conclusions.*—In the present instance the stabilising effect of the mobility of the fuselage is very great. When torsional wing motion is taken into account the effect is less, but still large (*see* § 33; also §34 for a general explanation of the stabilising effect).

PART IV.

APPROXIMATE THEORIES OF THE OSCILLATIONS.

§ 29. *Preliminary.*—Attention will be confined to the vibrations of comparatively high frequency, which, when unstable, constitute wing flutter. Then, in the longitudinal-symmetrical motion, the pitching of the fuselage must be very small on account of its large moment of inertia about the transverse axis; this conclusion is confirmed by the detailed calculation mentioned at the end of § 24. Evidently also the axial oscillation (oscillation in u) is negligible from the present point of view, and only the normal oscillation of the fuselage is of any importance. Similarly, in the lateral antisymmetrical motion the yawing and side slipping oscillations will be very small, and only the rolling of the fuselage important. The theories of the two types of oscillation will now be worked out on the basis of the foregoing assumptions, and will be checked by application to the monoplane already investigated by the more exact theories.

Longitudinal-Symmetrical Oscillations.

§ 30(a). *The Dynamical Equations.*—The wing motions are specified by means of the co-ordinates ϕ, ξ, θ as usual, while the co-ordinate z measures the displacement of the fuselage in a direction normal to the flight path. Since the angle of incidence is supposed small, the displacement z may be treated as normal to the wings.

The equations of small motion will be written at length as follows:—

Equation of Flexural Moments on Wings.

$$A_1 \ddot{\phi} + B_1 \dot{\phi} + C_1 \phi + D_1 \ddot{\xi} + E_1 \dot{\xi} + F_1 \xi + G_1 \ddot{\theta} + J_1 \dot{\theta} + K_1 \theta + P_1 \ddot{z} + Q_1 \dot{z} = 0. \quad (54)$$

Equation of Aileron Hinge Moments.

$$A_2 \ddot{\phi} + B_2 \dot{\phi} + C_2 \phi + D_2 \ddot{\xi} + E_2 \dot{\xi} + F_2 \xi + G_2 \ddot{\theta} + J_2 \dot{\theta} + K_2 \theta + P_2 \ddot{z} + Q_2 \dot{z} = 0. \quad (55)$$

Equation of Torsional Moments on Wings.

$$A_3 \ddot{\phi} + B_3 \dot{\phi} + C_3 \phi + D_3 \ddot{\xi} + E_3 \dot{\xi} + F_3 \xi + G_3 \ddot{\theta} + J_3 \dot{\theta} + K_3 \theta + P_3 \ddot{z} + Q_3 \dot{z} = 0. \quad (56)$$

Equation of Normal Forces on Complete Machine.

$$A_4 \ddot{\phi} + B_4 \dot{\phi} + C_4 \phi + D_4 \ddot{\xi} + E_4 \dot{\xi} + F_4 \xi + G_4 \ddot{\theta} + J_4 \dot{\theta} + K_4 \theta + P_4 \ddot{z} + Q_4 \dot{z} = 0 \ldots \quad (57)$$

Each term in the equations (54) to (56) is *twice* the measure of the corresponding moment on a single wing or aileron. Subject to this understanding the inertial coupling coefficients can be equated in pairs :—

$$P_1 \equiv A_4, \qquad P_2 \equiv D_4, \qquad P_3 \equiv G_4 \quad .. \quad (58)$$

§ 30(*b*). *Values of the Coefficients.*—Every one of the coefficients of ϕ, ξ and θ and of their derivatives in the equations (54) to (56) is twice the corresponding coefficient of ordinary flutter theory, and the values of all these will be presumed known. The remaining coefficients will now be examined.

In conformity with Appendix I of R. & M. 1155, it will be assumed that the normal displacement of a point on the wing relative to the fuselage is given by the expression.

$$z_w = \phi f(y) - \theta x F(y). \quad .. \quad .. \quad .. \quad (59)$$

As in the Appendix cited, the inertial constants will be evaluated by Appell's method. When the normal motion of the fuselage is taken into account, the expression for the "kinetic energy of the accelerations" is

$$T_a = \tfrac{1}{2} \sum_f m\ddot{z}^2 + \sum_{w-a} m \{\ddot{\phi} f(y) - \ddot{\theta} x F(y) + \ddot{z}\}^2$$
$$+ \sum_a m \left\{ \begin{array}{l} \ddot{\phi} \left[s + \dfrac{y-s}{s_a} \{ s - f(s - s_a) \} \right] \\ + \ddot{\theta} \left[-x + \dfrac{a(y-s)}{s_a} \{ 1 - F(s - s_a) \} \right] - \ddot{\xi} x_a + \ddot{z} \end{array} \right\}^2, \quad (60)$$

where s = semi-span ;
s_a = aileron span ;
$\sum\limits_f$ denotes summation over the fuselage with its appendages.

The inertial term in the equation of normal forces is

$$\frac{\partial T_a}{\partial \ddot{z}} = 2\ddot{\phi} \left\{ \sum_{w-a} m f(y) + \sum_a m \left[s + \frac{y-s}{s_a} \{ s - f(s-s) \} \right] \right\}$$
$$-2\ddot{\xi} \sum_a m x_a + 2\ddot{\theta} \left\{ - \sum_{w-a} m x F(y) + \sum_a \left[-x \frac{a(y-s)}{s_a} \{ 1 - F(s - s_a) \} \right] \right\}$$
$$+ \ddot{z} M. \quad .. \quad (61)$$

Hence, the expressions for the new inertial coefficients in the dynamical equations are

$$A_4 = P_1 = 2\sum_{w-a} mf(y) + 2\sum_a m\left[s + \frac{y-s}{s_a}\{s - f(s-s_a)\}\right]; \quad (62)$$

$$D_4 = P_2 = -2\sum_a mx_a, \quad \ldots \quad (63)$$

$$G_4 = P_3 = -2\sum_{w-a} mx\,F(y)$$
$$+ 2\sum_a m\left[-x + \frac{a(y-s)}{s_a}\{1 - F(s-s_a)\}\right] \quad (64)$$

$$P_4 = M. \quad \ldots \quad (65)$$

With regard to the aerodynamical derivatives $Q_1 \ldots Q_4$, it is clear that $\dot{z}$ is equivalent to w. Hence

$$\left.\begin{aligned} Q_1 &= -L_w, \\ Q_2 &= -H_w, \\ Q_3 &= -M_w, \\ Q_4 &= -MZ_w. \end{aligned}\right\} \quad \ldots \quad (66)^*$$

These derivatives and those which occur in equation (57) can be estimated by strip theory or analogous methods.

§ 30(*c*). *Application to a Particular Monoplane.*—The approximate theory will now be applied to the monoplane investigated by the " exact " theory in §24. As previously, attention will be confined to flexural-aileron flutter and the wing flexure will, for simplicity, be assumed " linear."

The data are given in Table 6.

TABLE 6.

(*Slug-foot-second units.*)

Coefficient.	Value.	Coefficient.	Value.	Coefficient.	Value.
A_1	500	A_2	4·0	A_4	47·05
B_1	26·4u	B_2	0·09u	B_4	3·87u
C_1	6×10^6	C_2	0	C_4	0
D_1	4·0	D_2	0·35	D_4	0·34
E_1	0·96u	E_2	0·04u	E_4	0·0807u
F_1	1·0u^2	F_2	1,734 +0·016u^2	F_4	0·0807u^2
P_1	47·05	P_2	0·34	P_4	80·7
Q_1	4·78u	Q_2	0·0065u	Q_4	0·807

* L and M here signify twice the flexural and twice the torsional moments, respectively, on the starboard wing. The factor M appears in Q_4 since Z_w is conventionally measured per unit mass.

On the basis of the foregoing data the lowest critical flutter speed is found to be 245·0 ft./sec., which is the same as for a fixed fuselage. The " exact " theory (*see* § 24) gives the critical speed as 245·5 ft./sec., so that the approximate method appears to be amply accurate for practical purposes.

Lateral-Antisymmetrical Oscillations.

§31(*a*). *The Dynamical Equations.*—The displacement of the fuselage in roll is denoted by χ, and is taken as positive when the displacement of the starboard wing tip is downwards. It will be assumed for simplicity that the wing displacements due to χ are normal to the wing, i.e., that the axis of rotation lies at least approximately in the plane of the wings. Evidently a small dihedral angle is negligible from the present point of view.

The equations of small oscillation are :—

Equation of Flexural Moments on Wings.—

$$A_1\ddot{\phi} + B_1\dot{\phi} + C_1\phi + D_1\ddot{\xi} + E_1\dot{\xi} + F_1\xi + G_1\ddot{\theta} + J_1\dot{\theta} + K_1\theta + S_1\ddot{} + T_1\dot{\chi} = 0 \ldots \quad (67)$$

Equation of Aileron Hinge Moments.—

$$A_2\ddot{\phi} + B_2\dot{\phi} + C_2\phi + D_2\ddot{\xi} + E_2\dot{\xi} + F_2\xi + G_2\ddot{\theta} + J_2\dot{\theta} + K_2\theta + S_2\ddot{\chi} + T_2\chi = 0 \ldots \quad (68)$$

Equation of Torsional Moments on Wings :—

$$A_3\ddot{\phi} + B_3\dot{\phi} + C_3\phi + D_3\ddot{\xi} + E_3\dot{\xi} + F_3\xi + G_3\ddot{\theta} + J_3\dot{\theta} + K_3\theta + S_3\chi + T_3\chi = 0 \ldots \quad (69)$$

Equation of Rolling Moments on Whole Machine :—

$$A_5\ddot{\phi} + B_5\dot{\phi} + C_5\phi + D_5\ddot{\xi} + E_5\dot{\xi} + F_5\xi + G_5\ddot{\theta} + J_5\dot{\theta} + K_5\theta + S_5\ddot{\chi} + T_5\dot{\chi} = 0 \ldots \quad (70)$$

§31(*b*). *Values of the Co-efficients.*—Appell's method will again be used in the evaluation of the inertial constants. Let A_f be the moment of inertia of the fuselage about the longitudinal axis. Then, clearly, the expression for the " kinetic energy of the accelerations " is

$$T_a = \tfrac{1}{2} A_f \ddot{\chi}^2 + \sum_{w-a} m \{\ddot{\phi} f(y) - \ddot{\theta} x F(y) + y \ddot{\chi}\}^2$$

$$+ \sum_a m \left\{ \begin{array}{l} \ddot{\phi} \left[s + \dfrac{y-s}{s} \{ s - f(s - s_a) \} \right] \\ + \ddot{\theta} \left[-x + \dfrac{a(y-s)}{s_a} \{ 1 - F(s - s_a) \} \right] - \ddot{\xi} x_a + y \ddot{\chi} \end{array} \right\}^2 . \quad (71)$$

The inertial term in the equation of rolling moments is

$$\frac{\partial T_a}{\partial \ddot{\chi}} = 2\ddot{\phi}\left\{\sum_{w-a} myf(y) + \sum_a my\left[s + \frac{y-s}{s_a}\{s - f(s-s_a)\}\right]\right\}$$
$$-2\ddot{\xi}\sum_a mx_a y + 2\ddot{\theta}\left\{-\sum_{w-a} mxy\,F(y)\right.$$
$$\left.+\sum_a my\left[-x + \frac{a(y-s)}{s_a}\{1-F(s-s_a)\}\right]\right\} + A\ddot{\chi}. \quad (72)$$

Thus, the expressions for the new inertial coefficients are

$$A_5 = S_1 = 2\sum_{w-a} myf(y) + 2\sum_a my\left[s + \frac{y-s}{s_a}\{s - f(s-s_a)\}\right], \quad (73)$$

$$D_5 = S_2 = -2\sum_a mx_a y, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

$$G_5 = S_3 = -2\sum_{w-a} mxy\,F(y)$$
$$+2\sum_a my\left[-x + \frac{a(y-s)}{s_a}\{1 - F(s-s_a)\}\right], \quad (75)$$

$$S_5 = A. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (76)$$

Evidently $\dot{\chi}$ is equivalent to p. Hence

$$\left.\begin{aligned} T_1 &= -L_p \\ T_2 &= -H_p \\ T_3 &= -M_p \\ T_5 &= -\mathbf{L}_p. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (77)$$

§ 31 (*c*). *Application to a Particular Monoplane.*—The theory will be applied to the case investigated by the "exact" theory in §27.

The numerical data are given in Table 7.

TABLE 7.

(*Slug-foot-second units.*)

Coefficient.	Value.	Coefficient.	Value.	Coefficient.	Value.
A_1	500	A_2	4·0	A_5	620
B_1	26·4u	B_2	0·09u	B_5	35·0u
C_1	6×10^6	C_2	0	C_5	0
D_1	4·0	D_2	0·35	D_5	4·85
E_1	0·96u	E_2	0·04u	E_5	1·2u
F_1	1·0u^2	F_2	0·016u^2	F_5	1·2u^2
S_1	620	S_2	4·85	S_5	950
T_1	35·0u	T_2	0·11u	T_5	50·0u

* As before, L & M are double the flexural and torsional moments on the starboard wing. $\mathbf{L}$ is the rolling moment on the complete machine.

The critical flutter speed with fixed fuselage and with ailerons free from elastic control is 310 ft./sec. A calculation based on the data of Table 7 shows that when the fuselage is free there is no critical flutter speed up to 800 ft./sec. This result is in agreement with the "exact" theory (*see* § 27). The stability has not been examined at higher speeds.

A comparison of the frequencies and damping factors of the oscillations of the system as given by the "exact" and approximate theories constitutes a much more severe test of the latter. The oscillation chosen for the comparison is the least damped of the high frequency oscillations of the system. Frequencies and damping factors at two flight speeds are given in Table 8.

TABLE 8.

Flight Speed (ft./sec.).	Damping Factor.		Frequency (cycles/sec.).	
	"Exact" Theory.	Approximate Theory.	"Exact" Theory.	Approximate Theory.
400	3·5036	3·5033	40·105	40·106
600	4·3740	4·3730	39·733	39·733

It will be seen that the agreement of the results of the two theories is everywhere excellent.

§ 32. *Influence of the Moment of Inertia of the Fuselage.*—The calculations for the monoplane have been repeated with the moment of inertia A, increased from 950 to 1,500 and to 2,000, all the remaining data being unchanged. Since all the products of inertia are unaltered, the increase in A is entirely attributable to an increase in the moment of inertia of the fuselage about the longitudinal axis.

In Fig. 14 the cube root of the test function T_4 is plotted against flight speed, and it will be seen that the increase of A has resulted in the development of flutter at comparatively low speeds. The critical speeds are given in Table 9.

TABLE 9.

Value of moment of inertia A (slug ft.^{2}).	Critical Flutter Speed (ft./sec.).
950	Above 800, if any.
1,500	440
2,000	360
∞	310

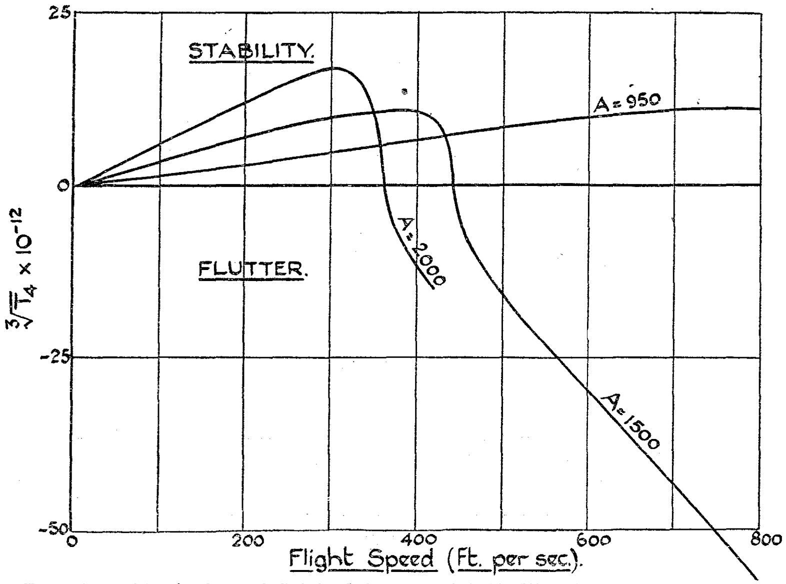

FIG. 14.—Variation of Critical Speed with Rolling Moment of Inertia.

The increase of A from 950 to 1,500 corresponds, of course, to a very large increase of A_f, since A_f is initially only a small fraction of A.

§ 33. *Lateral - Antisymmetrical Oscillations with Wing Torsion Included.*—The results of an application of equations (67) to (70) to the case where wing torsion is included, will now be quoted. The complete data, which relate to the same monoplane as before, are listed in Table 10

TABLE 10.

Coeff.	Value.	Coeff.	Value.	Coeff.	Value.	Coeff.	Value.
A_1	500	A_2	$4 \cdot 0$	A_3	$10 \cdot 0$	A_5	620
B_1	$26 \cdot 4u$	B_2	$0 \cdot 09u$	B_3	0	B_5	$35 \cdot 0u$
C_1	6×10^6	C_2	0	C_3	0	C_5	0
D_1	$4 \cdot 0$	D_2	$0 \cdot 35$	D_3	$1 \cdot 2$	D_5	$4 \cdot 85$
E_1	$0 \cdot 96u$	E_2	$0 \cdot 04u$	E_3	$0 \cdot 2u$	E_5	$1 \cdot 2u$
F_1	$1 \cdot 0u^2$	F_2	$0 \cdot 016u^2$	F_3	$0 \cdot 1u^2$	F_5	$1 \cdot 2u^2$
G_1	$10 \cdot 0$	G_2	$1 \cdot 2$	G_3	$8 \cdot 0$	G_5	$10 \cdot 5$
J_1	$5 \cdot 6u$	J_2	$0 \cdot 08u$	J_3	$0 \cdot 56u$	J_5	$7 \cdot 0u$
K_1	$2 \cdot 4u^2$	K_2	$0 \cdot 009u^2$	K_3	$3 \times 10^5 - 0 \cdot 10u^2$	K_5	$3 \cdot 0u^2$
S_1	620	S_2	$4 \cdot 85$	S_3	$10 \cdot 5$	S_5	950
T_1	$35 \cdot 0u$	T_2	$0 \cdot 11u$	T_3	0	T_5	$50u$

When the fuselage is supposed fixed, the (ternary) critical flutter speed is 311 ft. per sec., which happens to be almost identical with the flexural-aileron critical speed of 310 ft. per sec.

When, however, the fuselage is free to roll, the critical flutter speed rises to 485 ft. per sec. Thus the mobility of the fuselage has a decided stabilising effect on the ternary wing motion, but the effect is not so marked as for the flexural-aileron motion.

§ 34. *General Explanation of the Stabilising Influence of the Freedom of the Fuselage in Roll.*—A broad understanding of the effect can be gained by the consideration of a very simple dynamical system (*see* Fig. 15).

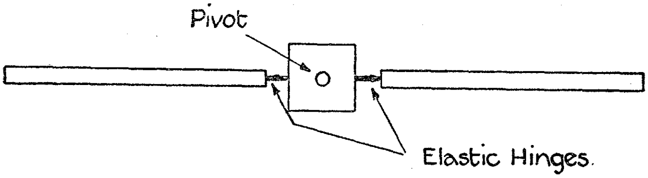

FIG. 15.—Simple Dynamical System Illustrating Mobility Effect.

The block shown is pivoted, but can be rigidly clamped if desired. A pair of equal massive bars are attached to it through elastic hinges. Suppose, firstly, that the block is clamped: then the frequency of the system is that corresponding to the elastically hinged massive bar, and is low. On the other hand, let the block be free to rotate on its pivot, and let impulses be given to the system, so as to set up an antisymmetrical oscillation. Clearly, since the moment of inertia of the block is much less than that of the bars, the block will oscillate with a large amplitude while the bars will remain almost at rest. Thus the frequency will correspond roughly to that of the block constrained by the elastic hinges, and will be high; so that, as far as concerns the frequency, release of the block is equivalent to a large increase of the flexural stiffness of the hinges. By analogy, the mobility of the fuselage in roll is virtually equivalent to a large increase in the flexural stiffness of the wings in an antisymmetrical oscillation, and there is a concomitant rise of the critical flutter speed.

The foregoing rough argument can easily be put in a mathematical form. Obviously the equations (67) and (70) are applicable, merely the co-ordinates ϕ and χ being retained. Then, for still air, the dynamical equations become

$$A_1 \ddot{\phi} + C_1 \phi + S_1 \ddot{\chi} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (78)$$

$$A_5 \ddot{\phi} \qquad\qquad + S_5 \ddot{\chi} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (79)$$

where S_1 ($\equiv A_5$) denotes the product of inertia. Let f and f_o be the frequencies when the block is free and clamped, respectively. Then it readily follows that—

$$(f/f_o)^2 = A_1 S_5 / (A_1 S_5 - S_1^2). \qquad \ldots \quad \ldots \quad (80)$$

Thus f is always greater than f_o. When the block is small, S_1 and A_1 do not differ greatly from S_5, and f greatly exceeds f_o.

§ 35. *Experimental Evidence of the Stabilising Influence of the Freedom of the Fuselage in Roll.*—In order to provide a qualitative demonstration a small model monoplane with flexible wings was constructed, and the fuselage was pivoted to a fixed stand about a longitudinal axis, but could be clamped to the stand when desired. When the fuselage was clamped, antisymmetrical wing flutter developed, but immediately ceased on release of the fuselage (*see* § 12).

Further evidence is provided by experiments carried out at the Royal Aircraft Establishment and described in R. & M. 1197.* A half model biplane was mounted on a pivoted base-board, which could be fixed by wedges. No flutter could be obtained when the base was free, and the flutter which did occur when the base was fixed immediately died out on withdrawal of the wedges.

The writers wish to acknowledge their indebtedness to the Superintendent of the Aerodynamics Department of the National Physical Laboratory for advice regarding the estimation of the aerodynamical derivatives: and to Miss S. W. Skan, who has carried out the numerical calculations.

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* Ref. 7.

APPENDIX TO CHAPTER I.

Resolution of the Determinant of Motion.

The determinant of motion corresponding to equations (3) to (14) of the text can be written in condensed notation as follows :—

$$\Delta(\lambda) \equiv \begin{vmatrix} A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 & 0 & 0 & 0 & V_1 & P_1 & R_1 \\ A_2 & D_2 & G_2 & U_2 & W_2 & Q_2 & 0 & D_0 & 0 & V_2 & P_2 & R_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 & 0 & 0 & 0 & V_3 & P_3 & R_3 \\ A_4 & D_4 & G_4 & U_4 & W_4 & Q_4 & A_4 & D_4 & G_4 & 0 & 0 & 0 \\ A_5 & D_5 & G_5 & U_5 & W_5 & Q_5 & A_5 & D_5 & G_5 & 0 & 0 & 0 \\ A_6 & D_6 & G_6 & U_6 & W_6 & Q_6 & A_6 & D_6 & G_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_1 & W_1 & Q_1 & A_1 & D_1 & G_1 & -V_1 & -P_1 & -R_1 \\ 0 & D_0 & 0 & U_2 & W_2 & Q_2 & A_2 & D_2 & G_2 & -V_2 & -P_2 & -R_2 \\ 0 & 0 & 0 & U_3 & W_3 & Q_3 & A_3 & D_3 & G_3 & -V_3 & -P_3 & -R_3 \\ A_7 & D_7 & G_7 & 0 & 0 & 0 & -A_7 & -D_7 & -G_7 & V_7 & P_7 & R_7 \\ A_8 & D_8 & G_8 & 0 & 0 & 0 & -A_8 & -D_8 & -G_8 & V_8 & P_8 & R_8 \\ A_9 & D_9 & G_9 & 0 & 0 & 0 & -A_9 & -D_9 & -G_9 & V_9 & P_9 & R_9 \end{vmatrix} = 0. \quad (1)$$

On addition of the 7th, 8th and 9th columns to the 1st, 2nd and 3rd columns, respectively, this becomes

$$\begin{vmatrix} A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 & 0 & 0 & 0 & V_1 & P_1 & R_1 \\ A_2 & D_2+D_0 & G_2 & U_2 & W_2 & Q_2 & 0 & D_0 & 0 & V_2 & P_2 & R_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 & 0 & 0 & 0 & V_3 & P_3 & R_3 \\ 2A_4 & 2D_4 & 2G_4 & U_4 & W_4 & Q_4 & A_4 & D_4 & G_4 & 0 & 0 & 0 \\ 2A_5 & 2D_5 & 2G_5 & U_5 & W_5 & Q_5 & A_5 & D_5 & G_5 & 0 & 0 & 0 \\ 2A_6 & 2D_6 & 2G_6 & U_6 & W_6 & Q_6 & A_6 & D_6 & G_6 & 0 & 0 & 0 \\ A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 & A_1 & D_1 & G_1 & -V_1 & -P_1 & -R_1 \\ A_2 & D_2+D_0 & G_2 & U_2 & W_2 & Q_2 & A_2 & D_2 & G_2 & -V_2 & -P_2 & -R_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 & A_3 & D_3 & G_3 & -V_3 & -P_3 & -R_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_7 & -D_7 & -G_7 & V_7 & P_7 & R_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_8 & -D_8 & -G_8 & V_8 & P_8 & R_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_9 & -D_9 & -G_9 & V_9 & P_9 & R_9 \end{vmatrix}$$

Next, subtract the 1st, 2nd and 3rd rows from the 7th, 8th and 9th rows, respectively. Then

$$\Delta(\lambda) \equiv \begin{vmatrix} A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 & 0 & 0 & 0 & V_1 & P_1 & R_1 \\ A_2 & D_2+D_0 & G_2 & U_2 & W_2 & Q_2 & 0 & D_0 & 0 & V_2 & P_2 & R_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 & 0 & 0 & 0 & V_3 & P_3 & R_3 \\ 2A_4 & 2D_4 & 2G_4 & U_4 & W_4 & Q_4 & A_4 & D_4 & G_4 & 0 & 0 & 0 \\ 2A_5 & 2D_5 & 2G_5 & U_5 & W_5 & Q_5 & A_5 & D_5 & G_5 & 0 & 0 & 0 \\ 2A_6 & 2D_6 & 2G_6 & U_6 & W_6 & Q_6 & A_6 & D_6 & G_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1 & D_1 & G_1 & -2V_1 & -2P_1 & -2R_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2 & D_2-D_0 & G_2 & -2V_2 & -2P_2 & -2R_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_3 & D_3 & G_3 & -2V_3 & -2P_3 & -2R_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_7 & -D_7 & -G_7 & V_7 & P_7 & R_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_8 & -D_8 & -G_8 & V_8 & P_8 & R_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -A_9 & -D_9 & -G_9 & V_9 & P_9 & R_9 \end{vmatrix} = 0. \quad (2)$$

Hence $\Delta(\lambda)$ is the product of

$$\Delta_1(\lambda) \equiv \quad \cdots \begin{vmatrix} A_1 & D_1 & G_1 & U_1 & W_1 & Q_1 \\ A_2 & D_2+D_0 & G_2 & U_2 & W_2 & Q_2 \\ A_3 & D_3 & G_3 & U_3 & W_3 & Q_3 \\ 2A_4 & 2D_4 & 2G_4 & U_4 & W_4 & Q_4 \\ 2A_5 & 2D_5 & 2G_5 & U_5 & W_5 & Q_5 \\ 2A_6 & 2D_6 & 2G_6 & U_6 & W_6 & Q_6 \end{vmatrix}, \quad \cdots \quad \cdots \quad (3)$$

and

$$\Delta_2(\lambda) \equiv \quad \cdots \begin{vmatrix} A_1 & D_1 & G_1 & V_1 & P_1 & R_1 \\ A_2 & D_2-D_0 & G_2 & V_2 & P_2 & R_2 \\ A_3 & D_3 & G_3 & V_3 & P_3 & R_3 \\ 2A_7 & 2D_7 & 2G_7 & V_7 & P_7 & R_7 \\ 2A_8 & 2D_8 & 2G_8 & V_8 & P_8 & R_8 \\ 2A_9 & 2D_9 & 2G_9 & V_9 & P_9 & R_9 \end{vmatrix}. \quad \cdots \quad \cdots \quad (4)$$

CHAPTER II.

CONDITIONS FOR THE PREVENTION OF FLEXURAL-TORSIONAL FLUTTER OF AN ELASTIC WING.*

BY R. A. FRAZER, B.A., B.Sc., AND W. J. DUNCAN, D.Sc., A.M.I.Mech.E.

§36. *Preamble.*—The theoretical discussion of wing flutter given in R. & M. 1155† is based on the assumption that the wing can be treated as " semi-rigid "—in the sense that the flexural and torsional displacements at any section are supposed to be determined by the corresponding displacements at the reference section (e.g., the wing tip). The practical validity of the assumption is well supported by the good agreement obtained in comparisons between experimentally observed critical flutter speeds and those predicted by the " semi-rigid " theory.‡ Nevertheless, an analysis of the problem based on a direct application of elastic theory—if mathematically feasible—would be useful as affording an independent test of the conclusions already drawn from the simpler theory.

The oscillations of an elastic cantilever wing without aileron are studied mathematically in a paper by S. B. Gates§, but simple conditions for stability are not stated. In the present chapter a similar analysis is adopted, and a general method for the discussion of the stability is developed. Further, the treatment is extended to stayed wings of a certain type.

§37. *The Dynamical Equations.*—The wing will be supposed thin and composed of rigid strips perpendicular to the span, the strips being carried by the two parallel elastic spars but otherwise independent (*see* Fig. 16). Thus, the loading of the spars at any section will be determined by the inertial reactions of the corresponding strip and by the aerodynamical forces acting over the surface of the strip.

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* Originally issued as R. & M. 1217 (Ref. 8).

† Ref. 1.

‡ *See* R. & M. 1155, Chap. VII.

§ Ref. 9.

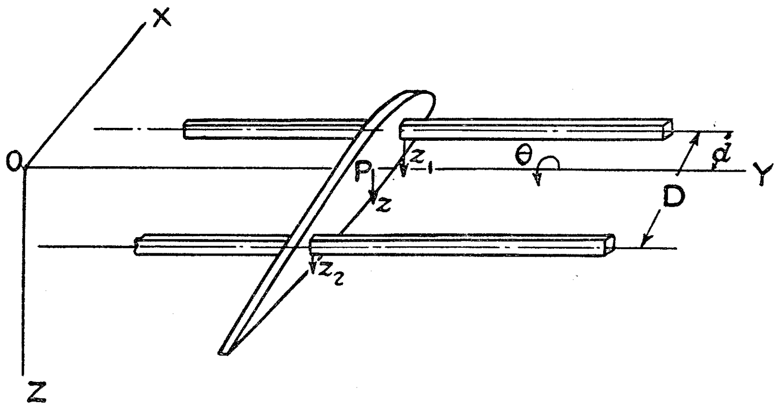

FIG. 16.—Diagram showing Spars of Elastic Wing.

In the diagram the origin O lies at the root of the wing, and the rectangular axes OX, OY, OZ are chosen, respectively, parallel to the chord, to the span, and to the downward normal. The position of OY in the wing chord is at present left arbitrary.

Notation.

(The positive senses of co-ordinates are indicated in Fig. 16.)

x, y = co-ordinates of current point of wing.

z_1 = downward displacement of point of neutral axis of front spar.

z_2 = downward displacement of point of neutral axis of rear spar.

z = downward displacement of "reference point" P on OY.

d = distance of OY from front spar.

D = distance between spars.

b = thickness of wing at any point.

s = span of wing measured from OX.

h = distance of stay attachments (if any) from OX.

θ = angular (torsional) displacement of section.

R_1, R_2 = flexural rigidities of front and rear spars, respectively.

$C [\equiv R_1 + R_2]$ = total flexural rigidity.

$m [\equiv d^2R_1 + (D - d)^2R_2]$ = torsional stiffness of wing due to differential bending of spars.

n = sum of torsional rigidities of spars.

F_1, F_2 = shearing forces in front and rear spars.

M_1, M_2 = bending moments in front and rear spars.

T = sum of torsional moments in spars.

η = density of material at current point of wing.

V = wind speed.

In the construction of the equations of motion aerodynamical forces will, for simplicity, at first be omitted. Consider, then, the forces acting on a typical strip of width δy. The downward forces on the strip at the front and rear spars are, respectively, $\frac{dF_1}{dy}\delta y$ and $\frac{dF_2}{dy}\delta y$: in addition, there will be torsional couples amounting in the aggregate to $\frac{dT}{dy}\delta y$. Again, the normal displacement of a current point of the strip at distance x from P is $z - x\theta$, so that the effective inertial force at this position is $-(\ddot{z} - x\ddot{\theta})\, b\eta\, \delta x\, \delta y$. Thus the two dynamical equations required are

$$\int b\eta(\ddot{z} - x\ddot{\theta})dx - \frac{dF_1}{dy} - \frac{dF_2}{dy} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (81a)$$

$$\int b\eta x(x\ddot{\theta} - \ddot{z})dx + d\frac{dF_1}{dy} - (D - d)\frac{dF_2}{dy} - \frac{dT}{dy} = 0, \qquad \ldots \quad (81b)$$

where the integrations extend over the chord.

Now

$$F_1 = -\frac{d}{dy}\left(R_1\frac{d^2 z_1}{dy^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82a)$$

$$F_2 = -\frac{d}{dy}\left(R_2\frac{d^2 z_2}{dy^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82b)$$

$$T = n\frac{d\theta}{dy}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82c)$$

Further

$$z_1 = z - d\theta, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83a)$$

$$z_2 = z + (D - d)\,\theta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83b)$$

Hence, on elimination of the variables z_1 and z_2, equations (81) become

$$\int b\eta(\ddot{z} - x\,\ddot{\theta})\,dx + \frac{d^2}{dy^2}\left\{C\frac{d^2 z}{dy^2} + (R_2\overline{D - d} - R_1 d)\frac{d^2\theta}{dy^2}\right\} = 0, \quad (84a)$$

$$\int b\eta x(x\ddot{\theta} - \ddot{z})\,dx - \frac{d}{dy}\left(n\frac{d\theta}{dy}\right) + \frac{d^2}{dy^2}\left\{(R_2\overline{D - d} - R_1 d)\frac{d^2 z}{dy^2}\right\}$$
$$+ \frac{d^2}{dy^2}\left(m\frac{d^2\theta}{dy^2}\right) = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (84b)$$

where C and m are as defined in the list of symbols. The further treatment will be restricted to the case where the ratio R_2/R_1 is constant along the span. On this assumption the position of the axis OY—hitherto arbitrary—may be chosen so that

$$R_2(D - d) - R_1 d = 0. \quad \ldots \quad \ldots \quad \ldots \quad (85)$$

The term involving $\frac{d^2\theta}{dy^2}$ in (84a), and that involving $\frac{d^2 z}{dy^2}$ in (84b), then vanish, and the dynamical equations reduce to the simple forms

$$A_1\ddot{z} + \frac{d^2}{dy^2}\left(C\,\frac{d^2z}{dy^2}\right) + G_1\ddot{\theta} = 0, \qquad \ldots \qquad (86a)$$

$$A_3\ddot{z} + G_3\ddot{\theta} - \frac{d}{dy}\left(n\,\frac{d\theta}{dy}\right) + \frac{d^2}{dy^2}\left(m\,\frac{d^2\theta}{dy^2}\right) = 0, \qquad \ldots \qquad (86b)$$

where the inertial coefficients A_1, G_1, etc., are defined as follows :—

$$A_1 \equiv \int b\eta\, dx, \qquad \ldots \qquad (87a)$$

$$G_1 \equiv A_3 \equiv -\int b\eta\, x\, dx, \qquad \ldots \qquad (87b)$$

$$G_3 \equiv \int b\eta\, x^2\, dx. \qquad \ldots \qquad (87c)$$

The equations (86) are only appropriate to oscillation of the wing *in vacuo*, so that the additional terms representing the wind actions must now be introduced. It will be assumed that for any given wind speed and angle of incidence the aerodynamical forces on a strip due to the motion are determined by the displacement and velocity of that strip: in other words, as far as concerns small motions about the position of equilibrium, the strips will be treated as aerodynamically independent. As in the treatment of the semi-rigid wing the aerodynamical actions will be represented by means of " damping derivatives " varying directly as V, and of " stiffness derivatives " varying directly as V^2. If, as will be supposed, the angle of incidence is small, the two stiffness derivatives corresponding to a pure flexural displacement (i.e., to a displacement in z only) can be taken to vanish. The dynamical equations, when generalized to include the aerodynamical terms, will accordingly be of the following forms :—

$$A_1\ddot{z} + B_1V\dot{z} + \frac{d^2}{dy^2}\left(C\,\frac{d^2z}{dy^2}\right) + G_1\ddot{\theta} + J_1V\dot{\theta} + K_1V^2\theta = 0, \qquad (88a)$$

$$A_3\ddot{z} + B_3V\dot{z} + G_3\ddot{\theta} + J_3V\dot{\theta} + K_3V^2\theta - \frac{d}{dy}\left(n\,\frac{d\theta}{dy}\right) + \frac{d^2}{dy^2}\left(m\,\frac{d^2\theta}{dy^2}\right)$$
$$= 0. \qquad \ldots \qquad (88b)$$

It may be noted that these equations are valid even when the several coefficients are variable along the span. The sole restriction hitherto imposed is that the ratio R_2/R_1 of the flexural rigidities of the two spars is constant along the span.

§38. *Conditions of Support.*—Attention will be restricted to the case where the spars are encastré at their roots. The support conditions are then as follows :—

(*a*) *Wing Root.*—When $y = o$ the displacement and the slope of each spar vanish. Hence

$$z_1 = z_2 = \frac{dz_1}{dy} = \frac{dz_2}{dy} = 0,$$

or

$$z = \theta = \frac{dz}{dy} = \frac{d\theta}{dy} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (89)$$

(*b*) *Wing Tip.*—At the free end ($y = s$), since the bending moments on both spars vanish

$$R_1 \frac{d^2 z_1}{dy^2} = R_2 \frac{d^2 z_2}{dy^2} = 0,$$

or

$$\frac{d^2 z}{dy^2} = \frac{d^2 \theta}{dy^2} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (90)$$

Further, the shearing stresses in the spars must together be equivalent to zero force and zero couple. Hence

$$F_1 + F_2 = 0,$$

and

$$T - d\,F_1 + (D - d)\,F_2 = 0,$$

which reduce to

$$\frac{d}{dy}\left(C \frac{d^2 z}{dy^2} \right) = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (91)$$

and

$$n \frac{d\theta}{dy} - \frac{d}{dy}\left(m \frac{d^2\theta}{dy^2} \right) = 0. \qquad \ldots \quad \ldots \quad (92)$$

(*c*) *Stay Attachments.*—It remains to consider the further conditions to be imposed when the wing is externally braced. The stays will be supposed fixed to the spars at the section $y = h$; and, for simplicity, the points of attachment will be assumed to lie upon the neutral axes, so that the bending moments will be continuous. In order to distinguish displacements of the outer and the inner segments of the wing, accented and unaccented symbols will be used to refer to the "overhang" and to the "bay," respectively. Since the displacements, slopes and bending moments are continuous at the stay attachments ($y = h$), it readily follows that

$$z' - z = \theta' - \theta = 0, \ldots \qquad \ldots \quad \ldots \quad \ldots \quad (93)$$

$$\frac{dz'}{dy} - \frac{dz}{dy} = \frac{d\theta'}{dy} - \frac{d\theta}{dy} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (94)$$

$$\frac{d^2 z'}{dy^2} - \frac{d^2 z}{dy^2} = \frac{d^2\theta'}{dy^2} - \frac{d^2\theta}{dy^2} = 0. \qquad \ldots \quad \ldots \quad (95)$$

Two further conditions are obtained from a consideration of the change of shearing force at the bracing points. Let P, P′ be points separated by, but indefinitely close to, the front stay attachment, and Q, Q′ be points similarly situated on the rear spar. Then

$$F_P' = F_P + \sigma_1 z_1, \quad \ldots \quad (96a)$$

$$F_Q' = F_Q + \sigma_2 z_2, \quad \ldots \quad (96b)$$

where σ_1, σ_2 are the stiffnesses of the respective stays, expressed as normal restoring force per unit normal displacement of the point of attachment.* On substitution from equations (82) and (83) the foregoing relations become

$$\frac{d}{dy}\left[R_1 \frac{d^2}{dy^2}\left\{ (z' - z) - (\theta' - \theta)d \right\} \right] + \sigma_1(z - \theta d) = 0, \quad \ldots \quad (97a)$$

$$\frac{d}{dy}\left[R_2 \frac{d^2}{dy^2}\left\{ (z' - z) + (\theta' - \theta)(D - d) \right\} \right] + \sigma_2\left\{ z + \theta (D - d) \right\} = 0. \quad \ldots \quad (97b)$$

§39. *Solution of Dynamical Equations for Uniform Wing.*—The detailed solution of the equations of motion will only be attempted for the case where the dynamical coefficients are constant along the span, although some deductions regarding the stability when this restriction is not introduced will be given in §43. In order to render the problem tractable it will also be necessary to assume that each stay stiffness is proportional to the flexural rigidity of the corresponding spar (*see* Appendix 1, page 63). Thus

$$\frac{\sigma_1}{R_1} = \frac{\sigma_2}{R_2} = c \text{ (say)}. \quad \ldots \quad (98)$$

The physical interpretation of this restriction is that a concentrated load which produces no torsion at any particular wing section will produce no torsion at any other. Finally, the torsional rigidity n of the spars themselves will be neglected, and the stiffness in torsion will accordingly be attributed entirely to differential bending action. As pointed out by Gates (*loc. cit.*), the inclusion of the torsional rigidity complicates the analysis immensely.

In accordance with these simplifications the dynamical equations (88) become†

$$A_1\ddot{z} + B_1V\dot{z} + C\frac{d^4z}{dy^4} + G_1\ddot{\theta} + J_1V\dot{\theta} + K_1V^2\theta = 0, \quad \ldots \quad (99a)$$

$$A_3\ddot{z} + B_3V\dot{z} + G_3\ddot{\theta} + J_3V\dot{\theta} + K_3V^2\theta + m\frac{d^4\theta}{dy^4} = 0, \quad \ldots \quad (99b)$$

where all the coefficients are now independent of y.

---

* The effect of the compression of the spars within the bay due to the staying is neglected.

† Equations (99) differ from those given by Gates, who adopts for z the normal displacement of the C.G. of the section.

To solve the equations (99), assume

$$z = Ae^{y\varkappa\sqrt{V} + t\lambda V}, \quad \ldots \quad \ldots \text{(100a)}$$

$$\theta = \rho\, Ae^{y\varkappa\sqrt{V} + t\lambda V}. \quad \ldots \quad \ldots \text{(100b)}$$

Then the differential equations will be satisfied provided that

$$\Delta(\lambda, \varkappa) \equiv \begin{vmatrix} A_1\lambda^2 + B_1\lambda + C\varkappa^4, & G_1\lambda^2 + J_1\lambda + K_1 \\ A_3\lambda^2 + B_3\lambda & , G_3\lambda^2 + J_3\lambda + K_3 + m\varkappa^4 \end{vmatrix}$$

$$= 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(101)}$$

and

$$\rho = -\frac{A_1\lambda^2 + B_1\lambda + C\varkappa^4}{G_1\lambda^2 + J_1\lambda + K_1} \ldots \quad \ldots \quad \ldots \text{(102)}$$

The period and the damping factor of the motion represented by (100) are determined by λ, while $\varkappa$ only affects the distribution of the displacements along the span. Hence all the terms (100) corresponding to any one value of λ must be combined linearly in such a manner that the conditions of support are satisfied at all instants.

The relation (101), considered as an equation in $\varkappa$, has eight roots of the form $\pm\varkappa_1$, $\pm i\varkappa_1$, $\pm\varkappa_2$, $\pm i\varkappa_2$. However, from (102), it appears that there are only two distinct values of ρ, which will be designated ρ_1 and ρ_2 respectively. For conciseness write

$$\mu = \varkappa\sqrt{V}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(103)}$$

Then the complete expressions for the displacements within the " bay " corresponding to a given λ are

$$ze^{-\lambda Vt} = A_1e^{\mu_1 y} + A_2e^{-\mu_1 y} + A_3e^{i\mu_1 y} + A_4e^{-i\mu_1 y}$$
$$+ A_5e^{\mu_2 y} + A_6e^{-\mu_2 y} + A_7e^{i\mu_2 y} + A_8e^{-i\mu_2 y}, \ldots \text{(104a)}$$

$$\theta e^{-\lambda Vt} = \rho_1\left(A_1e^{\mu_1 y} + A_2e^{-\mu_1 y} + A_3e^{i\mu_1 y} + A_4e^{-i\mu_1 y}\right)$$
$$+ \rho_2\left(A_5e^{\mu_2 y} + A_6e^{-\mu_2 y} + A_7e^{i\mu_2 y} + A_8e^{-i\mu_2 y}\right). \text{(104b)}$$

Similar expressions with accented coefficients A refer to the " overhang."

It is shown in Appendix 1 that all the conditions of support will be satisfied if all the coefficients A and A′ involving the suffices 5—8 are assumed zero, and μ_1 is taken to be any root of a certain " modal equation " (*see* Appendix 1, equation (20))

$$1 + \cosh\mu s\cos\mu s + \frac{c}{2\mu^3}F(\mu s, h/s) = 0. \quad \ldots \quad \ldots \text{(105)}$$

Further, it is shown that the value of μ_1^4 is necessarily real and positive.

§40. *Discussion of the Stability.*—Let the distinct values of μ^4, appropriate to the roots of the modal equation (105), be in ascending order of magnitude μ_0^4, μ_1^4,----μ_r^4, etc. Then the values of λ corresponding to any particular root μ_r of (105) will be given by the determinantal equation (*see* equations (101) and (103))

$$\Delta_r(\lambda) \equiv \begin{vmatrix} A_1\lambda^2 + B_1\lambda + X, & G_1\lambda^2 + J_1\lambda + K_1 \\ A_3\lambda^2 + B_3\lambda & , G_3\lambda^2 + J_3\lambda + Y \end{vmatrix} = 0, \qquad \ldots (106)$$

where

$$X = \frac{C\mu_r^4}{V^2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (107a)$$

and

$$Y = K_3 + \frac{m\mu_r^4}{V^2}. \qquad \ldots \quad \ldots \quad \ldots (107b)$$

The discussion of the stability is now formally identical with that given in R. & M. 1155 (Chapters III and VIII) for flexural-torsional motion of a semi-rigid wing. It is there shown that if X and Y be regarded as current co-ordinates in a plane, the test function for stability corresponding to (106), when equated to zero, leads to a certain " test conic," and that the critical wind speeds for flutter correspond to the points of intersection of this conic with the " stiffness line." In the present case the equation to the " stiffness line " is obtained by the elimination of V from equations (107). Clearly this line passes through the fixed " stiffness point " whose co-ordinates are $(0, K_3)$, and has the slope m/C. It appears, therefore, that both the test conic and the stiffness line are the same for all wind speeds and values of μ. Consider then one of the intersections of the conic with the stiffness line, and let the abscissa be X_c. From (107*a*) it follows that

$$V_{rc}^2 = \frac{C\mu_r^4}{X_c}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (108)$$

where V_{rc} is the critical speed corresponding to the selected root μ_r. As with the corresponding diagram for a semi-rigid wing, only real intersections of the stiffness line which lie to the right of the stiffness point correspond to real critical speeds: this is clear since μ^4 is necessarily real and positive.

The equation (108) shows that the critical flutter speeds are directly proportional to the squares of the real roots of the modal equation. Moreover, if the wing is stable at low wind speeds, as is normally the case, the first flutter is necessarily in the " fundamental " mode; hence, if flutter in the " fundamental " mode is prevented, it will not occur in any higher mode. The term " fundamental " is here used to designate that mode which corresponds to the numerically smallest root of the modal equation. In §41 it will be shown that the gravest oscillation *in vacuo* occurs in the fundamental mode.

As regards divergences, the inception of this type of instability is indicated by the occurrence of a zero real root of the determinantal equation (106). Thus the critical divergence speed V_{rd} will be given by the condition $XY = 0$; and since X cannot vanish at any finite speed it follows that

$$V_{rd} = \mu_r^2 \sqrt{\frac{m}{-K_3}} \quad \ldots \quad (109)$$

This equation immediately leads to the conclusions that the earliest divergence occurs in the fundamental mode, and that the divergence speed is proportional to the square root of the torsional wing stiffness m. A result equivalent to (109) was obtained by H. Bolas in a paper dealing with the divergent instabilities of cantilever tail planes.*

The influence of change of span upon the critical flutter speed (or divergence speed) of a cantilever wing having prescribed spar sections, etc., is easily deduced from equation (105). For an unstayed wing c is zero, and (105) determines the permissible values of the product μs, so that the root μ_r varies inversely as s. Hence the critical speed varies inversely as the square of the span, since, by (108), V_{rc} varies as μ_r^2. More generally, for a series of stayed wings of different spans, the foregoing deductions will hold good provided that h/s is constant—which implies that the stays are attached at a constant fraction of the span—and that

$$c = \sigma/R = \text{const.}/s^3.$$

For then (105) is, as before, merely an equation in the product μs. The condition $\sigma/R = \text{const}/s^3$ ensures that the staying is of a constant proportional effectiveness in relation to the spars.

§41. *Frequencies in the Several Modes.*—The appropriate form of solution for motion *in vacuo* is

$$z = Ae^{\mu y + ipt}, \quad \ldots \quad (110a)$$

$$\theta = \rho Ae^{\mu y + ipt}, \quad \ldots \quad (110b)$$

and the determinantal equation corresponding to (101) is

$$\begin{vmatrix} -A_1p^2 + C\mu^4, & -G_1p^2 \\ -A_3p^2 & , -G_3p^2 + m\mu^4 \end{vmatrix} = 0,$$

or

$$\left(\frac{p^2}{\mu^4}\right)^2 (A_1G_3 - A_3G_1) - \left(\frac{p^2}{\mu^4}\right)(mA_1 + CG_3) + mC = 0. \quad (111)$$

Hence the frequencies of oscillation *in vacuo* are directly proportional to the squares of the roots of the modal equation, and the lowest root μ_0 corresponds to the gravest oscillation.

* Ref. 10.

When the wing oscillates in a wind, the frequencies cannot in general be expressed by simple formulae. However, it can be shown by the usual method* that at a critical flutter speed the frequency of the simple harmonic oscillation is given by

$$p_{rc}^2 = \frac{(B_1K_3 - B_3K_1)V_{rc}^2 + \mu_r^4\,(mB_1 + CJ_3)}{(A_1J_3 - A_3J_1) + (B_1G_3 - B_3G_1)} \quad \ldots \quad \ldots \quad (112)$$

Since by equation (108) V_{rc}^2 is proportional to $\mu_r{}^4$, it follows that the critical frequencies in the several modes are directly proportional to the squares of the corresponding roots of the modal equation.

§42. *Conditions for the Prevention of Flutter.*—A detailed discussion of the sufficient conditions for the avoidance of flexural-torsional flutter of a semi-rigid wing is given in R. & M. 1155, Chap. VIII. In view of the exact parallelism between the stability criteria for the elastic and the semi-rigid wings, the known results for the semi-rigid wing can be interpreted immediately for the case of the elastic wing. Without a detailed recapitulation of all the conditions it may be remarked that, theoretically, no flutter is possible when the design is such that the coefficients A_3 and B_3 vanish.† The former coefficient will be zero if the C.G. of each strip of the wing lies on the "flexural axis" OY as defined by equation (85): and the coefficient B_3 will also be zero if the aerodynamical torsional moment on each strip due to flexural velocity vanishes. In the terminology of R. & M. 1155, these conditions require coincidence of the principal axis of inertia and the flexural axis with the axis of independence. It may be added that when a wing conforms to these requirements the motions in flexure and in torsion are effectively independent.

From equation (109) it will be seen that no divergence could occur if the coefficient K_3 were positive. This would actually be the case if the flexural axis OY were situated forward of about one-quarter of the chord from the leading edge.

§43. *Sufficient Conditions for Stability of a Non-Uniform Wing.*—The treatment has hitherto been limited to the case where all the dynamical coefficients are constants and the torsional rigidity n of the individual spars is neglected. When these simplifications are abandoned the mathematical analysis in general becomes intractable. Nevertheless, the stability admits discussion for the important special case where the coefficients A_3 and B_3 in the dynamical equations (88) are zero for every section. The remaining coefficients may be variable, but it must be remembered that in the reduction of the dynamical equations to the form (88) the ratio R_1/R_2 of the flexural rigidities of the spars has been assumed constant along the span. For simplicity only the unstayed wing will be considered, although the analysis can readily be extended to the case of the stayed wing, provided that the relation (98) is satisfied.

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* *See* R. & M. 1155, p. 36.

† As is usual in flutter theory, it is assumed that the two direct damping coefficients B_1 and J_3 are positive.

When $A_3 = B_3 = 0$ the dynamical equations are

$$A_1\ddot{z} + B_1V\dot{z} + \frac{d^2}{dy^2}\left(C\frac{d^2z}{dy^2}\right) + J_1V\dot{\theta} + K_1V^2\theta = 0, \quad .. \quad (113a)$$

$$G_3\ddot{\theta} + J_3V\dot{\theta} + K_3V^2\theta - \frac{d}{dy}\left(n\frac{d\theta}{dy}\right) + \frac{d^2}{dy^2}\left(m\frac{d^2\theta}{dy^2}\right) = 0. \quad (113b)$$

The independent torsional motion represented by (113*b*) admits discussion by aid of the equation of energy. Multiply by $\dot{\theta}$ and integrate over the span ; then

$$\frac{d}{dt}\int_0^s\left\{\tfrac{1}{2}G_3\dot{\theta}^2 + \tfrac{1}{2}K_3V^2\theta^2\right\}dy + \int_0^s J_3V\dot{\theta}^2dy + I = 0, \quad .. \quad (114)$$

where

$$I = \int_0^s \dot{\theta}\left\{\frac{d^2}{dy^2}\left(m\frac{d^2\theta}{dy^2}\right) - \frac{d}{dy}\left(n\frac{d\theta}{dy}\right)\right\}dy.$$

On successive integration by parts

$$I = \left[\dot{\theta}\left\{\frac{d}{dy}\left(m\frac{d^2\theta}{dy^2}\right) - n\frac{d\theta}{dy}\right\} - m\frac{d\dot{\theta}}{dy}\frac{d^2\theta}{dy^2}\right]_0^s$$
$$+ \frac{d}{dt}\int_0^s\left\{n\left(\frac{d\theta}{dy}\right)^2 + m\left(\frac{d^2\theta}{dy^2}\right)^2\right\}dy.$$

On account of the conditions of support (*see* equations (89), (90) and (92)) the expression in square brackets vanishes, and equation (114) becomes

$$\frac{d}{dt}\int_0^s\left\{\tfrac{1}{2}G_3\dot{\theta}^2 + \tfrac{1}{2}K_3V^2\theta^2 + n\left(\frac{d\theta}{dy}\right)^2 + m\left(\frac{d^2\theta}{dy^2}\right)^2\right\}dy$$
$$+ \int_0^s J_3V\dot{\theta}^2dy = 0. \quad .. \quad .. \quad .. \quad .. \quad (115)$$

It follows that if J_3 is positive for all values of y, the first integral in (115) necessarily diminishes as time increases. Hence, if in addition K_3 is positive at every section, the motion must decay, and there can be neither oscillatory nor divergent torsional instability.

Next suppose that K_3 is not everywhere positive, whilst J_3 is still positive at all sections. Since the term in (115) involving K_3 is proportional to V^2, whereas that involving J_3 is proportional to V, it is clear that at sufficiently low wind speeds the influence of K_3 is negligible, and the motion is completely stable. If now on a continuous increase of speed an oscillatory instability could ultimately develop, then an intermediate critical speed would exist at which the predominant motion was simple harmonic. At that speed the first integral in (115) would tend to become periodic, and its rate of change with time would not be consistently one-signed. Such a reversal of sign is impossible, since the second integral is always positive. The general conclusion is that no unstable torsional oscillation can occur provided that J_3 is positive for all sections.

The torsional motion just discussed is not by itself a possible free motion of the wing, since—owing to the coupling terms in (113*a*)—any torsional motion necessarily induces a flexural motion having the same frequencies and damping factors. The complete flexural motion consists of this effectively forced component (equivalent to the particular integral of (113*a*)), superposed on the free flexural motion which would occur if θ were zero. This free motion corresponds to the complementary function of (113*a*). Clearly the forced component due to the torsion will be stable under the conditions already examined, so that it remains to consider the pure flexural motion corresponding to the equation

$$A_1\ddot{z} + B_1 V\dot{z} + \frac{d^2}{dy^2}\left(C\frac{d^2z}{dy^2}\right) = 0. \quad \ldots \quad \ldots \quad (116)$$

Multiply by $\dot{z}$ and integrate over the span. Then, on integration by parts and introduction of the conditions of support

$$\frac{d}{dt}\int_0^s\left\{\tfrac{1}{2}A_1\dot{z}^2 + C\left(\frac{d^2z}{dy^2}\right)^2\right\}dy + \int_0^s B_1 V\dot{z}^2 dy = 0. \quad \ldots \quad (117)$$

The form of this equation shows at once that the pure flexural motion is completely stable provided that B_1 is positive at all sections.

The general conclusions from the foregoing argument will now be stated. In the investigation it is supposed that the ratio of the flexural rigidities of the two parallel spars is constant for all sections, and that the stiffnesses of the stays, if any, are in proportion to the flexural rigidities of the corresponding spars: under these conditions the locus of the flexural centres will be rectilinear and parallel to the span. Apart from these restrictions the flexural and torsional rigidities of the spars may vary in any manner. Then, no flutter can occur in any mode provided that both the centre of independence* and the centre of mass at every section lie upon the straight flexural locus. If, in addition, the aerodynamical stiffness coefficient K_3 could be positive at every section, divergence would also be avoided.

§ 44. *General Conclusions.*—The present investigation is believed to provide strong support for the principal deductions regarding stability of the flexural-torsional motion of a wing, drawn in R. & M. 1155 from the "semi-rigid" theory. Important additional conclusions are that the earliest flexural-torsional flutter of a monoplane wing whose mass distribution is approximately uniform will occur in the "fundamental" mode, and that if flutter in the fundamental mode has been prevented, it will not occur in any higher mode.

A recent re-examination of the problem discussed in § 39 has led to a demonstration of the principle of the "invariability of the modes," which is one of the fundamental assumptions of the "semi-rigid" theory. The proof will be found in Appendix 2.

* This point may be defined as the intersection with the chord of the line of action of the aerodynamical normal force produced by normal or flexural velocity of the wing.

APPENDIX 1 TO CHAPTER II.

THE MODAL EQUATION FOR A STAYED WING.*

In the present Appendix it is proposed to obtain and reduce the condition of compatibility of the equations 104(a) and 104(b) of the text with the conditions of support. On substitution of the expressions (104) in the equations (89)–(97), the following relations are obtained :—

(i) *Conditions at Root—*

$$\sum_1^4 A + \sum_5^8 A = 0, \qquad (1)$$

$$\varrho_1 \sum_1^4 A + \varrho_2 \sum_5^8 A = 0. \qquad (2)$$

$$\sum_1^4 \mu A + \sum_5^8 \mu A = 0, \qquad (3)$$

$$\varrho_1 \sum_1^4 \mu A + \varrho_2 \sum_5^8 \mu A = 0. \qquad (4)$$

ii) *Conditions at Stay Attachments—*

$$\sum_1^4 (A - A')e^{\mu h} + \sum_5^8 (A - A')e^{\mu h} = 0, \qquad (5)$$

$$\varrho_1 \sum_1^4 (A - A')e^{\mu h} + \varrho_2 \sum_5^8 (A - A')e^{\mu h} = 0, \qquad (6)$$

$$\sum_1^4 \mu (A - A')e^{\mu h} + \sum_5^8 \mu (A - A')e^{\mu h} = 0, \qquad (7)$$

$$\varrho_1 \sum_1^4 \mu (A - A')e^{\mu h} + \varrho_2 \sum_5^8 \mu (A - A')e^{\mu h} = 0, \qquad (8)$$

$$\sum_1^4 \mu^2 (A - A')e^{\mu h} + \sum_5^8 \mu^2 (A - A')e^{\mu h} = 0, \qquad (9)$$

$$\varrho_1 \sum_1^4 \mu^2 (A - A')e^{\mu h} + \varrho_2 \sum_5^8 \mu^2 (A - A')e^{\mu h} = 0, \qquad (10)$$

$$\sum_1^4 A(1 - \varrho_1 d)\left(\mu^3 - \frac{\sigma_1}{R_1}\right)e^{\mu h} - \sum_1^4 A'(1 - \varrho_1 d)\mu^3 e^{\mu h}$$

$$+ \sum_5^8 A(1 - \varrho_2 d)\left(\mu^3 - \frac{\sigma_1}{R_1}\right)e^{\mu h} - \sum_5^8 A'(1 - \varrho_2 d)\mu^3 e^{\mu h} = 0, \qquad (11)$$

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* The simplifying conditions stated at the beginning of §39 of the text are adopted throughout.

$$\sum_1^4 A(1 + \overline{D - d\varrho_1}) \left(\mu^3 - \frac{\sigma_2}{R_2}\right) e^{\mu h} - \sum_1^4 A'(1 + \overline{D - d\varrho_1})\mu^3 e^{\mu h}$$

$$+ \sum_5^8 A(1 + \overline{D - d\varrho_2}) \left(\mu^3 - \frac{\sigma_2}{R_2}\right) e^{\mu h} - \sum_5^8 A'(1 + \overline{D - d\varrho_2})\mu^3 e^{\mu h}$$

$$= 0. \qquad (12)$$

(iii) *Conditions at Wing Tip*—

$$\sum_1^4 A'\mu^2 e^{\mu s} + \sum_5^8 A'\mu^2 e^{\mu s} = 0, \qquad (13)$$

$$\varrho_1 \sum_1^4 A'\mu^2 e^{\mu s} + \varrho_2 \sum_5^8 A'\mu^2 e^{\mu s} = 0, \qquad (14)$$

$$\sum_1^4 A'\mu^3 e^{\mu s} + \sum_5^8 A'\mu^3 e^{\mu s} = 0, \qquad (15)$$

$$\varrho_1 \sum_1^4 A'\mu^3 e^{\mu s} + \varrho_2 \sum_5^8 A'\mu^3 e^{\mu s} = 0. \qquad (16)$$

The elimination of the constants A and A′ from the equations (1) . . . (16) leads to a determinant of the 16th order. For general values of σ_1 and σ_2 the reduction of the determinant would be excessively laborious. However, when $\sigma_1/R_1 = \sigma_2/R_2$ (as has been assumed), the eliminant can be resolved into the product of two 8th order determinants which are tractable. In fact, all the conditions (1) . . . (16) can now be satisfied by the displacements corresponding merely to the single set of roots $\varkappa_1$, $-\varkappa_1$, $i\varkappa_1$, $-i\varkappa_1$, of the equation (101) of the text.* When the constants A_5, A'_5 . . . A_8, A'_8 are omitted from the equations (1) . . . (16), the latter become identical in pairs. For example, (2) is merely a repetition of (1) (the condition $\varrho_1 = 0$ is trivial); while (11) and (12) become identical when the factors $(1 - \varrho_1 d)$ and $(1 + \overline{D - d\varrho_1})$ have been extracted. Hence there are just eight equations to determine the ratios of the eight constants A_1, A'_1 . . . A_4, A'_4. The eliminant is the determinant :—

* A similar resolution occurs in the simpler case of the unstayed wing.

$$\Delta(\mu) \equiv \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mu & -\mu & i\mu & -i\mu & 0 & 0 & 0 & 0 \\ e^{\mu h} & e^{-\mu h} & e^{i\mu h} & -e^{-i\mu h} & -e^{\mu h} & -e^{-\mu h} & -e^{i\mu h} & -e^{-i\mu h} \\ \mu e^{\mu h} & -\mu e^{-\mu h} & i\mu e^{i\mu h} & -i\mu e^{-i\mu h} & -\mu e^{\mu h} & \mu e^{-\mu h} & -i\mu e^{i\mu h} & i\mu e^{-i\mu h} \\ \mu^2 e^{\mu h} & \mu^2 e^{-\mu h} & -\mu^2 e^{i\mu h} & -\mu^2 e^{-i\mu h} & -\mu^2 e^{\mu h} & -\mu^2 e^{-\mu h} & \mu^2 e^{i\mu h} & \mu^2 e^{-i\mu h} \\ (\mu^3 - c) e^{\mu h} & (-\mu^3 - c) e^{-\mu h} & (-i\mu^3 - c) e^{i\mu h} & (i\mu^3 - c) e^{-i\mu h} & -\mu^3 e^{\mu h} & \mu^3 e^{-\mu h} & i\mu^3 e^{i\mu h} & -i\mu^3 e^{-i\mu h} \\ 0 & 0 & 0 & 0 & \mu^2 e^{\mu s} & \mu^2 e^{-\mu s} & -\mu^2 e^{i\mu s} & -\mu^2 e^{-i\mu s} \\ 0 & 0 & 0 & 0 & \mu^3 e^{\mu s} & -\mu^3 e^{-\mu s} & -i\mu^3 e^{i\mu s} & i\mu^3 e^{-i\mu s} \end{vmatrix} = 0 \quad \ldots (17)$$

After much reduction this becomes

$$\frac{-\Delta(\mu)}{2^7 \mu^{12}} \equiv 1 + \cosh \mu s \cos \mu s + \frac{c}{2\mu^3} \left\{ \begin{array}{l} \cosh \mu h \sin \mu h - \cos \mu h \sinh \mu h + \cos \mu(s-h) \sinh \mu(s-h) - \cosh \mu(s-h) \sin \mu(s-h) \\ \quad + \cosh \mu h \sin \mu s \cosh \mu(s-h) - \cos \mu h \sinh \mu s \cos \mu(s-h) \end{array} \right\} \quad \ldots \quad (18)$$

$$\equiv 1 + \cosh \mu s \cos \mu s + \frac{c}{2\mu^3} \mathrm{F}(\mu s,\, h/s), \text{ say.} \quad \ldots \quad (19)$$

The trivial condition $\mu = 0$ may be discarded, and the equation required becomes

$$1 + \cosh \mu s \cos \mu s + \frac{c}{2\mu^3} F(\mu s,\ h/s) = 0. \quad \ldots \quad (20)$$

This is identical with the equation which determines the natural frequencies of a single encastré beam of length s with an elastic support at a distance h from the root.* The quantity μ would then be interpreted by the relation

$$\mu^4 = \frac{\eta p^2}{Ek^2} \quad \ldots \quad (21)$$

where η = density of the material of the beam.
E = Young's modulus for the beam.
k = radius of gyration of the section.
p = 2π times the frequency.

Since (20) determines the periods of the undamped motion of a single beam, the roots must be such that μ^4 is real and positive.† This fact is of importance in the discussion of the stability of the wing.

Two special cases of (20) may be noted. Firstly, when c is zero the equation reduces to the ordinary form for an unstayed encastré beam.‡ Secondly, when c is infinite the equation becomes

$$F(\mu s,\ h/s) = 0, \quad \ldots \quad (22)$$

and is characteristic of a beam with rigid point support at a distance h from the root. Equation (22) can be written in the form

$$\left\{1 - \cos \mu h \cosh \mu h\right\}\left\{\sin \mu(s-h) \cosh \mu(s-h) - \sinh \mu(s-h) \cos \mu(s-h)\right\}$$
$$= \left\{1 + \cos \mu(s-h) \cosh \mu(s-h)\right\}\left\{\sin \mu h \cosh \mu h - \sinh \mu h \cos \mu h\right\} \quad (23)$$

The expression (23) is identical with the period equation for this particular case as given by Relf and Cowley in their paper§ on " Some Experiments on the Vibration of Bars."

Clearly a similar analysis may be used to obtain the modal equation for other types of support of the spars. For example, the spars may be pin jointed at the root or be stayed at more than one point. However, the stiffnesses of the stays must be in proportion to the flexural rigidities of the corresponding spars.

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* This identity is readily explained, for a solution of the type

$$z = A_1 e^{\mu y} + A_2 e^{-\mu y} + A_3 e^{i\mu y} + A_4 e^{-i\mu y}$$

is appropriate to the beam problem, and the conditions of support lead precisely to the determinant (17).

† *See* Rayleigh, " Sound," vol. I, page 279.

‡ Gates (*loc. cit.*), page 104.

§ Phil. Mag., October, 1924.

APPENDIX 2 TO CHAPTER II.

THE INVARIABILITY OF THE MODES.

Attention will be confined to the case of the uniform wing whose individual spars are of negligible torsional rigidity (*see* § 39). Then, as already remarked, all the conditions of the problem will be satisfied by expressions of the type (104) with the coefficients A_5 — — — A_8 omitted.

Consider first a cantilever wing. In the present discussion it will be convenient to convert the expressions for the displacements into a " real " form. Accordingly, let

$$\lambda = \sigma + ip. \quad \ldots \quad (1)$$

Then the expression for z corresponding to (104a) can be written

$$ze^{-\sigma Vt} = Y_1 \cos pVt + Y_2 \sin p\, Vt, \quad \ldots \quad (2)$$

where $$Y_1 = A \cosh \mu y + B \sinh \mu y + C \cos \mu y + D \sin \mu y, \quad \ldots \quad (3)$$

$$Y_2 = E \cosh \mu y + F \sinh \mu y + G \cos \mu y + H \sin \mu y, \quad \ldots \quad (4)$$

and the values of the coefficients A — — — H are at present arbitrary. From equation (102) it is clear that ϱ is in general a complex number. Let

$$\varrho = \alpha + i\beta. \quad \ldots \quad (5)$$

Then equation (104b) gives

$$\theta e^{-\sigma Vt} = Y_1 (\alpha \cos pVt - \beta \sin pVt) + Y_2 (\beta \cos pVt + \alpha \sin pVt). \quad \ldots \quad (6)$$

The conditions of support are expressed by equations (89) — — — (92). Since n is zero, while C and m are constants, the equations (91) and (92) reduce to

$$\frac{d^3z}{dy^3} = \frac{d^3\theta}{dy^3} = 0. \quad \ldots \quad (7)$$

It is now clear that the conditions of support become :—

Root conditions ($y = o$)

$$Y_1 = \frac{dY_1}{dy} = 0. \quad \ldots \quad (8)$$

$$Y_2 = \frac{dY_2}{dy} = 0. \quad \ldots \quad (9)$$

Tip conditions ($y = s$)

$$\frac{d^2Y_1}{dy^2} = \frac{d^3Y_1}{dy^3} = 0. \quad \ldots \quad (10)$$

$$\frac{d^2Y_2}{dy^2} = \frac{d^3Y_2}{dy^3} = 0. \quad \ldots \quad (11)$$

On substitution from (3) in (8) and (10) :—

$$C = -A,\ D = -B, \text{ and } B = -A \frac{\cosh \mu s + \cos \mu s}{\sinh \mu s + \sin \mu s},$$

and the condition of compatibility is

$$1 + \cosh \mu s \cos \mu s = 0, \quad \ldots \quad (12)$$

which agrees with (105) when c is zero.

For convenience write A $(\cosh \mu s + \cos \mu s) = R$. Then the expression for Y_1 becomes

$$Y_1 = R\left\{\frac{\cosh \mu y - \cos \mu y}{\cosh \mu s + \cos \mu s} - \frac{\sinh \mu y - \sin \mu y}{\sinh \mu s + \sin \mu s}\right\}. \quad \ldots \quad \ldots \quad (13)$$

Since equations (9) and (11) are obtained from (8) and (10) respectively by a mere change of suffix, it follows that (9) and (11) yield (12) as condition of compatibility, and that

$$Y_2 = kY_1, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where k is a real constant whose value depends on the initial conditions of the motion. Since Y_2 is proportional to Y_1, it follows from (2) and (6) that both z and θ are proportional to Y_1. Now Y_1 is quite independent of the dynamical constants of the wing. Suppose that these constants are so chosen that the flexural and torsional motions are independent (for example, let $A_3 = B_3 = 0$), and let all damping coefficients be absent. Then the flexural displacements must be proportional to Y_1, i.e. Y_1 expresses the modes of oscillation of a single cantilever beam in still air. (Compare equation (13) with equation (9·32) of Prescott's " Applied Elasticity".) Hence the following conclusions can be drawn regarding the oscillations of a wing of the type considered :—

(1) The functions expressing the modes of displacement in flexure and in torsion are identical. In particular, the flexural and torsional *nodes* coincide.

(2) The flexural and torsional modes are independent of the values of the dynamical constants of the wing. In particular, they are independent of the wind speed.

When the wing is stayed, and the stay stiffnesses satisfy equation (98), the conclusions already drawn are valid. The direct method of solution adopted above would be cumbrous, but the argument can be cast in the following form. Assume that equations (2) and (6) are still valid, where the functions Y are again of the form (3), but the coefficients have different values for the bay and the overhang. Also assume that Y_2 is again proportional to Y_1 (equation (14)). Then the differential equations (99) will be satisfied, and the conditions of support (89) ——— (97) will also be satisfied, provided that :—

For $y = 0$,
$$Y_1 = \frac{dY_1}{dy} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

For $y = h$,
$$Y_1' = Y_1, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16a)$$

$$\frac{dY_1'}{dy} = \frac{dY_1}{dy}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16b)$$

$$\frac{d^2Y_1'}{dy^2} = \frac{d^2Y_1}{dy^2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16c)$$

$$\frac{d^3}{dy^3}(Y_1' - Y_1) + cY_1 = 0. \quad \ldots \quad \ldots \quad (16d)$$

For $y = s$,
$$\frac{d^2Y_1'}{dy^2} = \frac{d^3Y_1'}{dy^3} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In the above equations Y_1 refers to the bay and Y_1' to the overhang.

It will be seen that these equations are quite independent of the dynamical constants of the system,* and that, in fact, they determine the modes of oscillation of a simple beam, encastré at the root and provided with an elastic support. The condition of compatibility of the equations (15) to (17) can be reduced to the " modal equation " (20) of Appendix I.

A simple explanation of the fact that the earliest instability necessarily occurs in the fundamental mode can now be given. Since the flexural and torsional nodes coincide, it follows that when the wing oscillates in one of the higher modes, there are virtually a number of *short* wings vibrating together. Now shortening the wing increases both the flexural and torsional stiffness and, therefore, raises the critical speed (*see* also end of § 40).

* Excepting, of course, the relative stay stiffness coefficient c.

CHAPTER III.

THE WING FLUTTER OF BIPLANES.*

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PREAMBLE AND SUMMARY, WITH LIST OF DESIGN RECOMMENDATIONS.

§ 45. *Preliminary.*—The theoretical treatment of the wing flutter of biplanes adopted in this chapter is based upon, and forms a natural extension of, the theory of the wing flutter of monoplanes developed in R. & M. 1155.† A recapitulation of that theory cannot be attempted here, and the reader must be supposed acquainted with the Monograph, or at least with the non-mathematical summary contained in R. & M. 1177.‡ Nevertheless it will be useful to restate the following essential points :—

(1) The wing-aileron system is treated as semi-rigid, i.e. as possessing only a finite number of degrees of freedom.

(2) The general principle underlying the methods of flutter prevention advocated is the elimination, as far as is possible, of the couplings between the motions in the several degrees of freedom.

(3) Elimination of the aileron couplings is of paramount importance, since this measure is almost certainly sufficient to raise the critical flutter speeds well beyond the flying range.

§ 46. *Experimental Evidence.*—Several instances of the occurrence of wing flutter on full-scale biplanes are on record, but naturally the observations of the wing motions on these occasions were too inexact to be of much service in the formulation of a theory. However, the phenomena have now been studied in some detail by the aid of wind tunnel experiments on model biplanes. The earliest of such experiments were conducted at the Royal Aircraft Establishment upon a very accurately made model of a certain machine which had exhibited wing flutter, and are described in R. & M. 1197.§§ A large number of experiments on models were subsequently made at the National Physical Laboratory and an account of this work forms Part II of the present chapter.

All of the experiments showed that the wing motion in a flutter was most pronounced at the tips. In some cases the amplitude of the motion at the outermost interplane struts was very small, but this was by no means generally true. On the other hand, the torsional component of the motion was always found to be small at this section. It is thus evident that the occurrence of a normal or flexural motion of the outermost incidence truss must be allowed for in the theory.

* Originally issued as R. & M. 1227 (Ref. 11). † Ref. 1. ‡ Ref. 12.
§§ Ref. 7.

§ 47. *Basis of the Theoretical Treatment.*—The dynamical system comprises an upper and a lower half plane, each in general carrying an aileron, and supported at the roots by a centre section or fuselage which will be treated as rigid and incapable of any motion normal to the flight path.* The planes are interconnected by one or more pairs of suitably braced interplane struts, and the ailerons are joined by a strut or wire. In the sequel the interaileron connection will always be spoken of as a strut, but it is to be understood that a wire may be actually in use.

The wing-aileron system will be treated as semi-rigid and the simplest set of co-ordinates compatible with the essential freedoms of the system will be chosen. Evidently each overhang can be displaced both in torsion and in flexure relatively to the outermost incidence truss, while the displacement of the latter may be treated as purely "flexural." Hence the simplest system of co-ordinates will consist of a single general flexural co-ordinate which specifies the displacement of the incidence truss, and of separate flexural, torsional and aileron co-ordinates for each of the overhangs. On account of the interconnection of the ailerons, the system will have six degrees of freedom.

§ 48. *Theoretical Conclusions regarding Biplanes with Equal Overhangs.*—The case where the upper and lower wings with their ailerons are identical is of special simplicity. It can be shown (*see* §§ 54 and 59) that the most general motion of the system can be resolved into a quaternary motion in which the upper and lower wings move equally and in phase, and a binary motion in which the wings move equally, but in opposition. The stability of the wing system is determined by the stabilities of the component motions, and these will now be considered in turn.

(a) *Quaternary Motion.*—The quaternary motion in general involves movement of the interplane struts, but if the bracing could be made so stiff as to keep these struts at rest, the motion would degenerate into a ternary type. Then, if at least one half of the ailerons lie within the bay, all the aileron couplings will be very small and the only possible instability will be a flexural-torsional flutter or divergence of the overhang, which will probably only occur at an extremely high speed. However, the model experiments indicate that it is probably not practicable to stiffen the bracing to such an extent as to produce a node at the interplane struts; consequently, the mere adoption of ailerons projecting well into the bays must not be relied on for the elimination of the aileron couplings.

Suppose now that the methods of flutter prevention advocated in R. & M. 1155 have been applied to the upper and lower wings

* *See* Chap. I for an investigation of the influence of the mobility of the fuselage on wing flutter.

separately. This will imply that the centres of gravity of the ailerons lie upon, or slightly ahead of, the hinge axes, and that for each overhang the principal axis of inertia and the flexural axis coincide with the axis of independence. Then the quaternary motion will resolve into a torsional-aileron motion of the ordinary type and a certain binary motion involving merely flexure. The latter motion can be shown to be necessarily stable, and the former will have been rendered stable by the adoption of the measures already cited. It may be concluded that the adoption of the usual methods of flutter prevention will suffice to suppress the quaternary flutter. Of all such measures, mass balance of the ailerons is, in the absence of irreversible aileron controls, by far the most important.

(b) *Binary Motion.*—In the binary motion, the upper and lower wings with their ailerons move equally and in opposition, and the interplane and interaileron struts remain at rest. When the interaileron strut lies far from the interplane struts, the earliest instability is a divergence (*see* § 58). This divergence has been realised on a model biplane (*see* § 76), and is not of a violent type, but implies loss of control of the ailerons. The divergence speed rises as the interaileron strut is made to approach the interplane struts ; when it is opposite the interplane struts, the binary motion becomes of a pure flexural-torsional type, and instability will be postponed to a very high speed. Thus it appears that the interaileron strut should be as nearly as possible in the same plane as the interplane struts ; further, the aileron controls should operate in the same section.

§ 49. *Biplanes with Unequal Overhangs.*—An important type of biplane is that in which the upper overhang is long, while the lower overhang is short. Here experiment justifies a simplification of the theory, for it is found that the lower plane merely partakes of the general flexural motion, i.e., the flexural and torsional displacements of its overhang relative to the interplane struts are negligible. Thus the system can be treated as possessing only four degrees of freedom.

Suppose now that the interaileron strut is not near to the interplane struts. Then if the upper aileron remain at a fixed angle to the upper wing and the latter receive a flexural displacement, the interaileron strut will be displaced and the lower aileron will rotate. It readily follows that the lower aileron, merely in virtue of its *moment of inertia*, contributes effectively to the flexural-aileron product of inertia of the upper aileron. Thus, location of the C.G.'s of the ailerons upon the hinge axes will not suffice to eliminate the flexural aileron product of inertia, as is very desirable. This again emphasises the advantage of an interaileron strut placed as close to the interplane struts as possible. When this condition is satisfied, the usual methods of flutter prevention will be effective.

The case where both the overhangs are short is the simplest of all, for here the motion can be adequately treated as of binary flexural-aileron nature. The flexural motion is of course a general flexural motion of the upper and lower planes togather. Flutter can be avoided by the measures laid down in § 55 of R. & M. **1155** ; the most important of these is mass balance of the ailerons.

§ 50. *Miscellaneous Remarks on the Flutter of Biplanes.*—If ailerons be fitted to the upper (or lower) planes only, it is clear that the usual measures will prevent the participation of the ailerons in flutter ; thus any instability will be of a flexural-torsional character, and will probably only supervene at extremely high speeds.

The model experiments described in Part II appear to show that biplanes are relatively immune from merely flexural-torsional flutter. Thus in some instances the critical flutter speed with unbalanced ailerons was as low as about 10 ft. per second, but no flexural-torsional flutter was observed within the range of wind speeds available (maximum 75 ft. per sec.). This may probably be attributed to the fact that interplane bracing increases the general torsional stiffness of the structure in a greater proportion than the general flexural stiffness.

Theory (*see* § 66) and experiment (§ 75) are in agreement that stagger has only a slight influence on the critical speeds, provided that the wing structure is sufficiently stiff to limit the general torsional motion to a small amount. The influence of a small stagger is quite negligible, but a very large stagger may tend to introduce a torsional motion, with a consequent impairment of stability.

Experiment shows clearly that mass loading within the bays has very little influence on the critical speeds (*see* § 74). Thus, other things being equal, the fitting of wing petrol tanks is not likely to cause a serious reduction of the critical speeds.

§ 51. *Recommendations regarding the Prevention of Wing Flutter.*—The recommendations are to a large extent the same as for monoplanes, but a few novelties are added. A complete justification of all the measures is not given in the present chapter ; this is permissible since a thorough discussion of the measures applicable to monoplanes has been given in R. & M. 1155. Notes on the recommendations are given in § 52.

List of Design Recommendations.

Group I.—*General Recommendations.*

(1) All elastic stiffnesses as large as possible.

(2) Irreversibility of aileron control.

Failing (2)—

(3) Centre of gravity of aileron slightly ahead of hinge.

(4) Moment of inertia of aileron small.

(5) An appreciable part, preferably rather more than one half, of the aileron should lie inboard of the centre line of the attachments of the outermost interplane struts.

(6) Aileron heavily damped, e.g., artificially.

(7) Aileron definitely underbalanced aerodynamically.

(8) Interaileron strut not outboard of the interplane struts. (Only of secondary importance if for any reason recommendation (3) is not adopted.)

(9) Aileron controls to operate in the same section as the interaileron strut.

Group II.—*Recommendations relative to Overhangs.*

(These recommendations may be ignored unless the overhang is very long.)

(10) Balance of masses of each overhang (including corresponding portion of aileron) about its flexural axis.

(11) Flexural axis close to axis of independence.

§ 52. *Notes on the Recommendations.—Recommendation* (1).—A proportionate increase of all elastic stiffnesses raises the critical speeds. In the case of biplanes the stiffness of the staying is naturally of great importance.

Recommendation (2).—A properly designed irreversible control completely eliminates flutter involving the ailerons. All other recommendations relative to the ailerons can then be ignored.

Recommendation (3).—This recommendation is of the greatest importance and should be interpreted strictly, since partial mass balance may be of no benefit. Allowance must be made for the mass of the interaileron strut and other appendages of the aileron. Interconnection of the ailerons by a wire instead of a strut may be of assistance here on account of the smaller mass of the wire.

Recommendation (4).—All parts of the control system which move with the ailerons contribute effectively to the moment of inertia of the aileron. All such parts should therefore be as light as possible.

Recommendation (5).—This measure assists to minimise some of the aileron couplings, but it must not be considered as an effective alternative to recommendation (3).

Recommendation (6).—An artificial damping device, if employed, should be of the fluid friction or electrical type. The use of solid friction is viewed as objectionable.

Recommendation (7).—Very close approach to the condition of aerodynamical balance is considered dangerous. However, experiments described in § 72 show that an aileron hinged at about 0·2 of the chord from its leading edge may be quite satisfactory.

Recommendation (8) —The reasons for this recommendation are given in §§ 48 and 49. It is of particular importance when one of the overhangs is long and the other short, and the ailerons are mass balanced.

Recommendation (9).—This measure results in the elimination of certain couplings, and is also clearly mechanically sound.

Recommendations (10) *and* (11).—Full explanations of these recommendations are given in R. & M. 1155.

PART I.

THEORY OF THE WING FLUTTER OF BIPLANES.

§ 53. *Introductory.*—A preliminary theory of biplane flutter is given in R. & M. 1042,* where a number of simplifying assumptions are introduced. In that investigation it is postulated that the wings have flexural nodes inboard of the outermost struts and that these struts provide a kinematic link between the flexural oscillations of the wings, while, in conjunction with the bracing wires, they entirely prevent torsional oscillation. According to these assumptions the system has only two degrees of freedom—one co-ordinate determining the flexural displacements of the wings and the other determining the positions of the ailerons. This method of approach to the problem may be regarded as a legitimate first approximation, and has the decided advantage that great simplification of the analysis results ; moreover, experiment shows that the assumptions are valid to a high degree of approximation for a restricted class of biplane structures. Nevertheless, this simple theory is inadequate to deal with many important types of biplane, and it has been considered desirable to develop a theory of greater generality. The theory now presented will, it is hoped, be found adequate for most of the important classes of biplane.

Since the most general form of the theory is complicated, it has been thought advisable to begin with a restricted and simplified case and later to remove the restrictions one by one. Thus, the discussion will open with the case of an unstaggered biplane having equal upper and lower planes, and at first it will be assumed that the bracing is so stiff that the nodes lie at the outermost interplane struts.

The general mathematical treatment and notation is an extension of that adopted for monoplanes in R. & M. 1155, with which the reader is assumed to be familiar. It is postulated throughout that the fuselage provides a rigid support for the wings at their roots. The nature of the errors involved in this assumption is the subject of a special investigation† and will not be treated here.

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* Ref. 13. † See Chap. I.

SECTION A.

UNSTAGGERED BIPLANE WITH EQUAL WINGS AND THE NODE AT THE OUTERMOST INTERPLANE STRUTS.

§ 54. *Preliminary Physical Ideas.*—As already stated, it will at first be assumed that the interplane struts remain stationary during flutter. Also it will be supposed that the upper and lower wings are aerodynamically independent. Thus the only coupling of their oscillations is the mechanical one provided by the interaileron strut. Since there is no stagger, the oscillatory motion of this strut will be parallel to itself. It can thus be supposed replaced by an ideal massless strut and by a pair of massive particles at its ends, each of half its mass. These particles may be treated as appertaining to the ailerons.

Strictly speaking, a single wing-aileron system has an infinity of degrees of freedom, but experiments on monoplanes have shown that the system can usually be treated as ternary. Accordingly, the system composed by the upper and lower planes with their ailerons will be deemed to have five degrees of freedom, one degree being lost on account of the interconnection of the ailerons.

In general, the discussion of a quinary motion would be quite intractable, but a physical argument will now be advanced which shows that in the present instance (i.e. when the upper and lower wings are in all respects identical) the quinary motion can be resolved into a ternary motion of a single wing-aileron system and a binary motion in which the interaileron strut is stationary. This resolution leads to a great simplification of the analysis.

Consider the equivalent ideal system with a rigid but massless interaileron strut. Let the system be initially at rest, and, in the first place, suppose that exactly equal displacements or impulses are given to the upper and lower planes. Then, on account of the equality of the upper and lower systems, it is evident that if the ideal strut be removed, their motions will be identical at all instants. Thus the massless strut may be replaced without alteration of the motions. In other words, a ternary motion of a single wing-aileron system is also a possible motion when the interaileron strut is connected. In the second place, let the strut be connected to the ailerons, and let equal but opposite displacements or impulses be given to the upper and lower wings. Then, on account of symmetry, it is evident that the interaileron strut will remain stationary during the subsequent motion. Thus a possible motion of the system is that effectively binary motion which occurs when the interaileron strut is held stationary. These results will be proved analytically in the sequel.

§ 55. *The Equations of Motion.*—In the derivation of the equations of motion it will be assumed that the angles of inclination of the ailerons to the wings are measured in the section containing the attachment of the interaileron strut. The transformations of the equations of motion which result from change of the section in which these angles are measured are discussed in Appendix 1. (*See* page 122).

The symbols used are as follows (*see* Figs. 17 and 18) :—

ϕ_1 = flexural co-ordinate of upper wing.
ϕ_2 = ,, ,, lower ,,
θ_1 = torsional ,, upper ,,
θ_2 = ,, ,, lower ,,
ξ_1 = aileron ,, upper ,,
ξ_2 = ,, ,, lower ,,
s_0 = span of overhang.
a = distance from reference axis to aileron hinge.
d = ,, ,, aileron hinge to interaileron strut.
r = $a + d$.
T = thrust in the ideal massless strut.

The flexural co-ordinate is defined as the flexural displacement of the reference centre (taken to be at the wing tip in the figures) divided by the distance of that point from the nodal line, while the torsional co-ordinate is defined as the torsional displacement measured at the reference section.

If the interaileron strut had been situated at the reference section, its displacement would have been given by the expression

$$z = s_0\phi_1 + d\,\xi_1 + r\theta_1 = s_0\phi_2 + d\,\xi_2 + r\theta_2 . \quad \ldots \quad \ldots \quad (118)$$

In general, however, the expression is

$$z = s'\phi_1 + d\,\xi_1 + r'\theta_1 = s'\phi_2 + d\,\xi_2 + r'\theta_2 , \quad \ldots \quad \ldots \quad (119)$$

where s' and r' are definite multiples of s_0 and r respectively, depending on the position of the interaileron strut and on the " laws " of flexure and torsion of the wings.

In accordance with the notation already adopted in the discussion of the wing flutter of monoplanes, the equations of motion of the free upper wing-aileron system (i.e. with the ideal massless interaileron strut disconnected) will be as follows :—

Equation of Flexural Moments.

$$A_1\ddot{\phi}_1 + B_1\dot{\phi}_1 + C_1\phi_1 + D_1\ddot{\xi}_1 + E_1\dot{\xi}_1 + F_1\xi_1 + G_1\ddot{\theta}_1 + J_1\dot{\theta}_1 + K_1\theta_1 = 0, \quad \ldots \quad (120a)$$

Equation of Aileron Hinge Moments.

$$A_2\ddot{\phi}_1 + B_2\dot{\phi}_1 + C_2\phi_1 + D_2\ddot{\xi}_1 + E_2\dot{\xi}_1 + F_2\xi_1 + G_2\ddot{\theta}_1 + J_2\dot{\theta}_1 + K_2\theta_1 = 0, \quad \ldots \quad (120b)$$

Equation of Torsional Moments.

$$A_3\ddot{\phi}_1 + B_3\dot{\phi}_1 + C_3\phi_1 + D_3\ddot{\xi}_1 + E_3\dot{\xi}_1 + F_3\xi_1 + G_3\ddot{\theta}_1 + J_3\dot{\theta}_1 + K_3\theta_1 = 0. \quad \ldots \quad (120c)$$

The equations of motion of the free lower wing-aileron system are obtained from the last equations by a mere change of the suffix of the co-ordinates from 1 to 2, for the upper and lower planes are by hypothesis identical and therefore must have the same dynamical constants.* For definiteness it should be stated that the equations of motion are in the form given by Lagrange's method, e.g. the

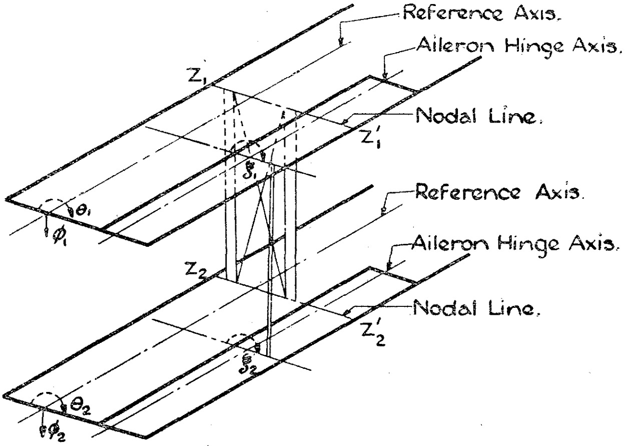

FIG. 17.—Diagram of an Unstaggered Biplane with Equal Upper and Lower Planes.

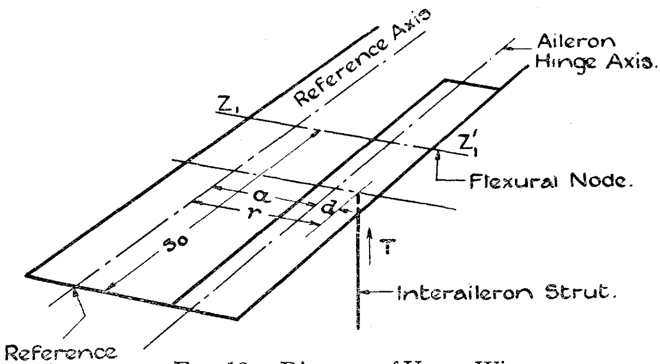

FIG. 18.—Diagram of Upper Wing.

equation of flexural moments expresses the vanishing of the virtual work of the applied forces and of the reversed mass accelerations when the flexural co-ordinate receives an arbitrary small increment. Further, the datum from which the co-ordinates are measured is the equilibrium attitude for the particular wind speed considered.

Since the equations of motion which have to be dealt with are complicated, a system of condensed notation will be adopted. Thus :—

$$\left.\begin{array}{l} A_1 \ddot{\phi}_1 + B_1 \dot{\phi}_1 + C_1 \phi_1 \equiv A_1 (\phi_1) \\ D_1 \ddot{\phi}_2 + E_1 \dot{\phi}_2 + F_1 \phi_2 \equiv D_1 (\phi_2) \\ \quad \text{etc., etc.} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (121)$$

* As already stated, aerodynamical interference is neglected.

The following further abbreviations will be used in the determinants of motion :—

$$\left.\begin{aligned} A_1\,\lambda^2 + B_1\,\lambda + C_1 &\equiv A_1\,(\lambda) \\ G_2\,\lambda^2 + J_2\,\lambda + K_2 &\equiv G_2\,(\lambda) \\ \text{etc., etc.} & \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (122)$$

Finally, the convention will be adopted that, for example,

$$\left.\begin{aligned} (x\,A_1 + y\,D_1)\,(\phi_1) &\equiv x\,A_1\,(\phi_1) + y\,D_1\,(\phi_1) \\ (l\,G_3 + m\,D_2)\,(\lambda) &\equiv l\,G_3\,(\lambda) + m\,D_2\,(\lambda). \end{aligned}\right\} \quad .. \quad .. \quad (123)$$

Consider now the equation of flexural moments of the upper wing-aileron system when the ideal interaileron strut is connected. On account of equation (119) the displacement due to a small change $\delta\phi_1$ of the flexural co-ordinate is $s'\,\delta\phi_1$ and the virtual work done against the thrust T is $s'\,T\,\delta\phi_1$. Hence the equation of flexural moments becomes, in the condensed notation,

$$A_1\,(\phi_1) + D_1\,(\xi_1) + G_1\,(\theta_1) + s'\,T = 0. \quad .. \quad .. \quad .. \quad (124a)$$

Similarly the remaining equations of motion of the upper wing become

$$A_2\,(\phi_1) + D_2\,(\xi_1) + G_2\,(\theta_1) + d\,T = 0, \quad .. \quad .. \quad .. \quad (124b)$$

$$A_3\,(\phi_1) + D_3\,(\xi_1) + G_3\,(\theta_1) + r'\,T = 0, \quad .. \quad .. \quad .. \quad (124c)$$

and the equations of motion of the lower wing-aileron system are

$$A_1\,(\phi_2) + D_1\,(\xi_2) + G_1\,(\theta_2) - s'\,T = 0, \quad .. \quad .. \quad .. \quad (125a)$$

$$A_2\,(\phi_2) + D_2\,(\xi_2) + G_2\,(\theta_2) - d\,T = 0, \quad .. \quad .. \quad .. \quad (125b)$$

$$A_3\,(\phi_2) + D_3\,(\xi_2) + G_3\,(\theta_2) - r'\,T = 0. \quad .. \quad .. \quad .. \quad (125c)$$

The final equations of motion are obtained by the elimination of T and of one of the co-ordinates (here selected as ξ_2) from the equations (124) and (125) by aid of the kinematical relation (119). Direct addition of (124a) and (125a) and subsequent elimination of ξ_2 leads to the equation

$$\begin{aligned} &(d\,A_1 + s'\,D_1)\,(\phi_1) + 2d\,D_1\,(\xi_1) + (d\,G_1 + r'\,D_1)\,(\theta_1) \\ &\quad + (d\,A_1 - s'\,D_1)\,(\phi_2) + (d\,G_1 - r'\,D_1)\,(\theta_2) = 0. \end{aligned} \quad .. \quad (126a)$$

Similarly the remaining equations taken in pairs yield

$$\begin{aligned} &(d\,A_2 + s'\,D_2)\,(\phi_1) + 2d\,D_2\,(\xi_1) + (d\,G_2 + r'\,D_2)\,(\theta_1) \\ &\quad + (d\,A_2 - s'\,D_2)\,(\phi_2) + (d\,G_2 - r'\,D_2)\,(\theta_2) = 0, \end{aligned} \quad .. \quad (126b)$$

and

$$\begin{aligned} &(d\,A_3 + s'\,D_3)\,(\phi_1) + 2d\,D_3\,(\xi_1) + (d\,G_3 + r'\,D_3)\,(\theta_1) \\ &\quad + (d\,A_3 - s'\,D_3)\,(\phi_2) + (d\,G_3 - r'\,D_3)\,(\theta_2) = 0. \end{aligned} \quad .. \quad (126c)$$

Elimination of T from (124a) and (124b) gives

$$\begin{aligned} &(s'\,A_2 - d\,A_1)\,(\phi_1) + (s'\,D_2 - d\,D_1)\,(\xi_1) \\ &\quad + (s'G_2 - d\,G_1)\,(\theta_1) = 0, \end{aligned} \quad .. \quad .. \quad .. \quad .. \quad (126d)$$

and similarly (124b) and (124c) give

$$\begin{aligned} &(r'\,A_2 - d\,A_3)\,(\phi_1) + (r'\,D_2 - d\,D_3)\,(\xi_1) \\ &\quad + (r'\,G_2 - d\,G_3)\,(\theta_1) = 0. \end{aligned} \quad .. \quad .. \quad .. \quad .. \quad (126e)$$

Equations (126) are the dynamical equations of the complete system. Hence the determinant of motion is

$$\triangle\,(\lambda) \equiv \begin{vmatrix} (d\,A_1 + s'\,D_1)\,(\lambda), & 2d\,D_1\,(\lambda), & (dG_1 + r'\,D_1)\,(\lambda), & (dA_1 - s'\,D_1)\,(\lambda), & (dG_1 - r'\,D_1)\,(\lambda) \\ (dA_2 + s'\,D_2)\,(\lambda), & 2dD_2\,(\lambda), & (dG_2 + r'\,D_2)\,(\lambda), & (dA_2 - s'\,D_2)\,(\lambda), & (dG_2 - r'\,D_2)\,(\lambda), \\ (dA_3 + s'\,D_3)\,(\lambda), & 2dD_3\,(\lambda), & (dG_3 + r'\,D_3)\,(\lambda), & (dA_3 - s'\,D_3)\,(\lambda), & (dG_3 - r'\,D_3)\,(\lambda) \\ (s'\,A_2 - dA_1)\,(\lambda), & (s'\,D_2 - dD_1)\,(\lambda), & (s'\,G_2 - dG_1)\,(\lambda), & 0 & , \quad 0 \\ (r'\,A_2 - dA_3)\,(\lambda), & (r'\,D_2 - dD_3)\,(\lambda), & (r'\,G_2 - dG_3)\,(\lambda), & 0 & , \quad 0 \end{vmatrix} .$$

On addition of the 4th and 5th columns to the 1st and 3rd columns, respectively, this reduces to

$$\begin{vmatrix} 2dA_1\,(\lambda) & , 2dD_1\,(\lambda) & , 2dG_1\,(\lambda) & , (dA_1 - s'\,D_1)\,(\lambda), & (dG_1 - r'\,D_1)\,(\lambda) \\ 2d\,A_2\,(\lambda) & , 2dD_2\,(\lambda) & , 2dG_2\,(\lambda) & , (dA_2 - s'\,D_2)\,(\lambda), & (dG_2 - r'\,D_2)\,(\lambda) \\ 2dA_3\,(\lambda) & , 2dD_3\,(\lambda) & , 2dG_3\,(\lambda) & , (dA_3 - s'\,D_3)\,(\lambda), & (dG_3 - r'\,D_3)\,(\lambda) \\ (s'\,A_2 - dA_1)\,(\lambda), & (s'\,D_2 - dD_1)\,(\lambda), & (s'\,G_2 - dG_1)\,(\lambda), & 0 & , \quad 0 \\ (r'\,A_2 - d\,A_3)\,(\lambda), & (r'\,D_2 - dD_3)\,(\lambda), & (r'\,G_2 - dG_3)\,(\lambda), & 0 & , \quad 0 \end{vmatrix} .$$

Multiply the 4th row by $2d$, add d times the 1st row and subtract s' times the 2nd row ; also multiply the 5th row by $2d$, add d times the 3rd row and subtract r' times the 2nd row.

Then

$$4d^2 \triangle(\lambda) = \begin{vmatrix} 2dA_1(\lambda), & 2dD_1(\lambda), & 2dG_1(\lambda), & (dA_1 - s'D_1)(\lambda) & , (dG_1 - r'D_1)(\lambda) \\ 2dA_2(\lambda), & 2dD_2(\lambda), & 2dG_2(\lambda), & (dA_2 - s'D_2)(\lambda) & , (dG_2 - r'D_2)(\lambda) \\ 2dA_3(\lambda), & 2dD_3(\lambda), & 2dG_3(\lambda), & (dA_3 - s'D_3)(\lambda) & , (dG_3 - r'D_3)(\lambda) \\ 0 \quad , & 0 \quad , & 0 & , d^2 A_1(\lambda) - ds'(A_2 + D_1)(\lambda) + s'^2 D_2(\lambda), & , d^2 G_1(\lambda) - d(s'G_2 + r'D_1)(\lambda) + r's'D_2(\lambda) \\ 0 \quad , & 0 \quad , & 0 & , d^2 A_3(\lambda) - d(s'D_3 + r'A_2)(\lambda) + r's'D_2(\lambda), & , d^2 G_3(\lambda) - dr'(D_3 + G_2)(\lambda) + r'^2 D_2(\lambda) \end{vmatrix} . \quad (127)$$

Hence $\triangle(\lambda)$ has the factors :—

$$\triangle_1(\lambda) \equiv \begin{vmatrix} A_1(\lambda), & D_1(\lambda), & G_1(\lambda) \\ A_2(\lambda), & D_2(\lambda), & G_2(\lambda) \\ A_3(\lambda), & D_3(\lambda), & G_3(\lambda) \end{vmatrix}, \quad \ldots \quad (128)$$

and

$$\triangle_2(\lambda) \equiv \begin{vmatrix} d^2 A_1(\lambda) - ds'(A_2 + D_1)(\lambda) + s'^2 D_2(\lambda) & , d^2 G_1(\lambda) - d(s'G_2 + r'D_1)(\lambda) + r's'D_2(\lambda) \\ d^2 A_3(\lambda) - d(s'D_3 + r'A_2)(\lambda) + r's'D_2(\lambda), & d^2 G_3(\lambda) - dr'(D_3 + G_2)(\lambda) + r'^2 D_2(\lambda) \end{vmatrix} . \quad \ldots \quad (129)$$

The determinant $\Delta_1(\lambda)$ is characteristic of the ternary motion of a single wing aileron system. It remains to interpret $\Delta_2(\lambda)$.

Evidently the equations of motion of the wing-aileron system when the point of attachment of the inter-aileron strut is held stationary are (126d) and (126e), with ξ_1 eliminated by means of the kinematical equation

$$s'\phi_1 + d\,\xi_1 + r'\theta_1 = 0 . \qquad \text{(130)}$$

They can be written concisely

$$[d^2 A_1 - ds'(A_2 + D_1) + s'^2 D_2](\phi_1) + [d^2 G_1 - d(s' G_2 + r' D_1) + r' s' D_2](\theta_1) = 0, \qquad \text{(131a)}$$

$$[d^2 A_3 - d(s' D_3 + r' A_2) + r' s' D_2](\phi_1) + [d^2 G_3 - dr'(D_3 + G_2) + r'^2 D_2](\theta_1) = 0. \qquad \text{(131b)}$$

These equations when divided by d^2 are identical with those obtained by the application of Lagrange's method. It will be seen that $\Delta_2(\lambda)$ is the determinant of motion corresponding to the equations (131), and it has thus been proved analytically that the quinary motion of the system can be resolved into a ternary motion of a single wing-aileron system and a binary motion in which the inter-aileron strut does not move. For brevity the latter type of motion will in the sequel be called simply *the binary motion.*

One important special case of the binary motion may be noted. Suppose that s' and r' are both zero, as will be at least approximately true when the interaileron strut is in the same plane as the interplane struts. Then (129) gives

$$\Delta_2(\lambda) = d^4 \begin{vmatrix} A_1(\lambda), & G_1(\lambda) \\ A_3(\lambda), & G_3(\lambda) \end{vmatrix} . \qquad \text{(132)}$$

Thus the motion is of a pure flexural-torsional type, as is otherwise obvious.

§ 56. *Stability of the System.*—From the point of view of wing flutter the important aspect of the problem is the stability of the motion. Now it has been shown that the determinant of motion has the factors $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$, which are characteristic of the ternary and binary motions respectively. Hence the stability of the system is determined by the stabilities of the two component motions. No special comment is required here upon the stability of the ternary motion, since this matter has already been discussed in R. & M. 1155; it may be recalled, however, that the optimum condition as regards stability is when the couplings of the oscillations in the several degrees of freedom have been eliminated as far as possible. The stability of the binary motion will now be examined.

It may be noted that the determinantal quartic (129) of the binary motion involves all the dynamical constants of the wing and aileron.

The expressions for the coefficients in the expanded quartic are extremely complicated and it does not appear probable that any rigorous general deductions can be made, though numerical applications in particular instances where the necessary data are available present no difficulty (*see* § 58 for numerical application to a model wing of 27 in. span and comparison with experiment).

In the absence of a rigorous general discussion, recourse must be had to simplified cases which may be expected to throw some light on the general problem. The most characteristic feature of the binary motion is the presence of coupling terms which are attributable to the fixture of the point of attachment of the interaileron strut. The influence of these couplings can be studied in the artificially simplified case in which the three component oscillations of the single wing-aileron system are uncoupled. The extent to which this ideal condition can be realised is examined subsequently.

When there is no coupling in the ternary motion

$$A_2(\lambda) = A_3(\lambda) = D_1(\lambda) = D_3(\lambda) = G_1(\lambda) = G_2(\lambda) = 0,$$

and equation (129) becomes

$$\Delta_2(\lambda) = \begin{vmatrix} d^2A_1(\lambda) + s'^2D_2(\lambda), & r's'D_2(\lambda) \\ r's'D_2(\lambda), & d^2G_3(\lambda) + r'^2D_2(\lambda) \end{vmatrix}. \quad \ldots \quad (133)$$

The peculiarity of the binary motion represented by (133) is that the couplings are symmetrical. When this is so, Theorem 1 of § 319 of Routh's " Rigid Dynamics " (Vol. II, 6th Ed.) is applicable, and it appears that *oscillatory* instability cannot occur, subject to the condition that the dissipation function of the system shall be one-signed and positive. A detailed discussion by direct algebra is given in Appendix 2 to this chapter, and it is shown that if the determinant of a binary motion be

$$\Delta(\lambda) = \begin{vmatrix} \alpha_1\lambda^2 + \beta_1\lambda + \gamma_1, & \alpha_2\lambda^2 + \beta_2\lambda + \gamma_2 \\ \alpha_2\lambda^2 + \beta_2\lambda + \gamma_2, & \delta_2\lambda^2 + \varepsilon_2\lambda + \zeta_2 \end{vmatrix}, \quad \ldots \quad (134)$$

then the system will be completely stable provided that

$$\beta_1\varepsilon_2 - \beta_2^2 > 0, \quad \ldots \quad \ldots \quad \ldots \quad (135a)$$

and

$$\gamma_1\zeta_2 - \gamma_2^2 > 0. \quad \ldots \quad \ldots \quad \ldots \quad (135b)$$

Further, no oscillatory instability can occur provided merely that the inequality (135a) be satisfied. In this case failure of (135b) implies the onset of a divergence.

Substitution of the values of β_1, ε_2, etc., from (133) in the inequalities (135) leads to the conditions

$$d^4B_1J_3 + d^2E_2(r'^2B_1 + s'^2J_3) > 0, \quad \ldots \quad \ldots \quad \ldots \quad (136a)$$

$$d^4C_1K_3 + d^2F_2(r'^2C_1 + s'^2K_3) > 0. \quad \ldots \quad \ldots \quad \ldots \quad (136b)$$

Now B_1, E_2 and J_3 are the direct dampings of the wing-aileron system; they are normally positive and will be presumed so here. Thus (136a) is satisfied and *no flutter can occur in the binary motion.* Failure of (136b) implies a binary divergence, but the only coefficient in this inequality which can become negative (in the absence of an aerodynamically overbalanced aileron) is the torsional wing stiffness K_3. Change of sign of K_3, however, implies divergence in the ternary motion, so that in the special case considered the binary motion does not become unstable before the ternary motion.

§ 57.—*Conditions favourable to the Elimination of Couplings.*— When the centre of gravity of each aileron lies upon or slightly ahead of the hinge axis,* the flexural axis of the wing coincides with the axis of independence,† and the wing is mass balanced about that axis, a considerable number of the couplings are eliminated. The effect of the position of the outermost interplane struts in relation to the aileron upon the couplings will now be examined.

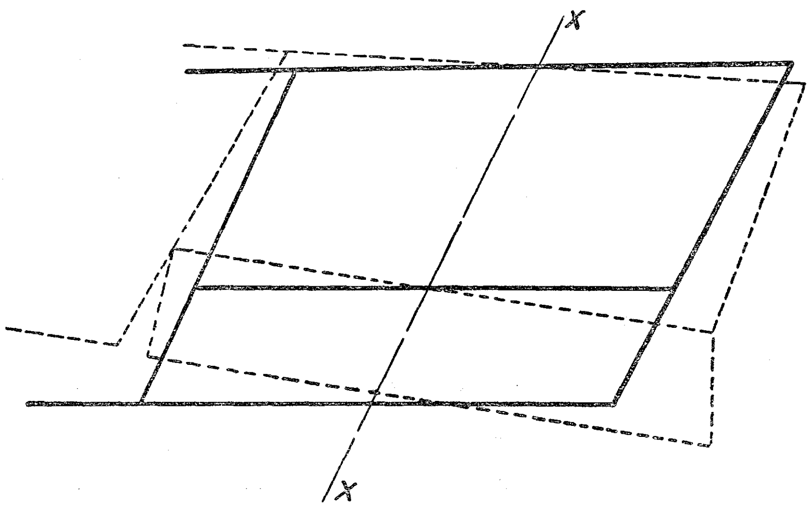

Fig. 19.—Diagrammatic Representation of Wing and Aileron Displacements.

It will be recalled that the theory is at present being developed for the case where the vibrational node coincides with the line of the attachments of the interplane struts. Suppose then that this nodal

* Due allowance must be made for the mass of the interaileron strut and other appendages.

† *See* §9 of R. & M. 1155.

line bisects the span of the aileron and suppose further for the moment that the displacements of the wing in flexure and in torsion are equal but opposite at equal distances on the two sides of the node (*see* Fig. 19). Also let the aileron control operate in the section XX and the aileron angle ξ be measured there. Then if end effects be neglected, the aerodynamical influence upon the aileron of the motions of the parts of the wing within and without the nodal line will cancel. Likewise, so far as motion of the wing about XX is concerned, the net aerodynamical effect of the aileron vanishes. Provided that the mass of the aileron is symmetrically distributed about XX, flexural or torsional accelerations of the wing will not produce moments tending to alter the angle ξ, i.e. the two aileron products of inertia will vanish. Thus, finally, all the aileron couplings vanish and the ternary motion consequently resolves into the flexural-torsional motion and an independent aileron motion which will be stable provided that the aileron is damped and not aerodynamically overbalanced. Now it has been pointed out in § 55 that, when the interaileron strut is opposite the interplane struts, the binary motion degenerates into a pure flexural-torsional motion. Thus the only possible instability of the combined wing-aileron system is in the pure flexural-torsional motion. The methods for the avoidance of such instability are stated in § 54 of R. & M. 1155, but in any case the critical speeds will be very high unless the overhangs are extremely flexible.

In the development of this argument it has been assumed that the displacements of the wing are of equal amount on the two sides of the node, but, in fact, the displacements outboard of the node will always be the greater. Hence it may be concluded that the best position for the interplane struts is rather outboard of the midspan of the aileron and that the interaileron strut and aileron control should lie as nearly as possible in the same plane as the interplane struts.

An incidental result of the foregoing argument is that all coupling terms involving the aileron can be made small, and that of the couplings in the flexural-torsional motion only J_1 $(-L_{\dot{\theta}})$ and K_1 $(-L_{\theta})$ cannot be eliminated.

§ 58. *Further Consideration of the Binary Motion.*—As already remarked, the determinantal quartic (129) for the binary motion is too complicated for general discussion. However, some light on the nature of the stability of the motion in cases where all the coupling terms are present may be obtained by a numerical application to a model wing of 27 in. span for which all the necessary data are available.*

---

* This wing was used in the derivative measurements described in R. & M. 1155.

It follows from equations (131) that the dynamical coefficients of the binary motion are given by the equations

$$A_1' = A_1 - \frac{2s'}{d} A_2 + \left(\frac{s'}{d}\right)^2 D_2, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (137a)$$

$$B_1' = B_1 - \frac{s'}{d}(B_2 + E_1) + \left(\frac{s'}{d}\right)^2 E_2, \quad \ldots \quad \ldots \quad \ldots \quad (137b)$$

$$C_1' = C_1 - \frac{s'}{d}(C_2 + F_1) + \left(\frac{s'}{d}\right)^2 F_2, \quad \ldots \quad \ldots \quad \ldots \quad (137c)$$

$$G_1' = G_1 - \frac{s'}{d} G_2 - \frac{r'}{d} D_1 + \frac{r' s'}{d^2} D_2, \quad \ldots \quad \ldots \quad \ldots \quad (137d)$$

etc.

The values of the coefficients A_1, B_1, etc., are given in Table 11, which is reproduced from R. & M. 1155. The inertial constants as tabulated refer to an aileron without attached interaileron strut, but this introduces no error since the strut does not move in the binary motion. It is assumed that the aileron angle ξ is measured at the wing tip and the calculations will first be made for the inter-aileron strut connected at that section. The influence of inboard displacement of this strut will be examined later.

Predictions of the critical flutter speeds and of the divergence speeds were made in the usual manner, on the basis of the formulæ (137) and the data of Table 11. The calculations were made for five values of d (distance of the point of attachment of the interaileron strut aft of the hinge) ranging from zero up to 0·1792 ft., which corresponds to a point almost on the trailing edge. The principal dimensions are :—

Wing Span (s) = 2·25 ft.
,, Chord = 0·75 ft.

Aileron Span = 1·042 ft.
,, Chord = 0·208 ft.
,, Hinge at 0·0208 ft. aft of the nose.
,, ,, 0·528 ,, ,, reference axis.

Experiments were carried out in order to check the theoretical predictions. The wing was mounted vertically in a 4 ft. wind tunnel as shown diagrammatically in Fig. 20. A thread was attached to the selected point of the aileron and passed over a pulley outside the tunnel. To this end a large weight was attached while the thread on the other side of the wing was wrapped round a peg. In this manner the point of attachment could be kept stationary, but the thread could be adjusted from outside the tunnel so as to maintain the aileron at zero incidence when the wing deflected under the wind load.

(4453)

TABLE 11.

Dynamical Coefficients for Model Wing of 27-in. Span.

(Units :—Slug, foot, second.)

Flexural Moments.			Aileron Hinge Moments.			Torsional Moments.		
Coefft.	Significance.	Value × 10^3	Coefft.	Significance.	Value × 10^3	Coefft.	Significance.	Value × 10^3
A_1	Inertia Moment	17·6	$A_2 = D_1$	Inertia Product	0·128	$A_3 = G_1$	Inertia Product	2·89
B_1/V	$-L_{\dot\phi}/V$	6·7 + 11·5/V	B_2/V	$-H_{\dot\phi}/V$	0·023	B_3/V	$-M_{\dot\phi}/V$	0·60
C_1/V^2	$(l_\phi - L_\phi)/V^2$	17,430/V^2	C_2/V^2	$(h_\phi - H_\phi)/V^2$	0	C_3/V^2	$(m_\phi - M_\phi)/V^2$	1,715/V^2
$D_1 = A_2$	Inertia Product	0·128	D_2	Inertia Moment	0·0095	$D_3 = G_2$	Inertia Product	0·0413
E_1/V	$-L_{\dot\xi}/V$	0·245	E_2/V	$-H_{\dot\xi}/V$	0·0104	E_3/V	$-M_{\dot\xi}/V$	0·0756
F_1/V^2	$-L_\xi/V^2$	1·89	F_2/V^2	$(h_\xi - H_\xi)/V^2$	0·027	F_3/V^2	$-M_\xi/V^2$	0·364
$G_1 = A_3$	Inertia Product	2·89	$G_2 = D_3$	Inertia Product	0·0413	G_3	Inertia Moment	0·725
J_1/V	$-L_{\dot\theta}/V$	2·10	J_2/V	$-H_{\dot\theta}/V$	0·0223	J_3/V	$-M_{\dot\theta}/V$	0·35+3·7/V
K_1/V^2	$(l_\theta - L_\theta)/V^2$	4·04+1,715/V^2	K_2/V^2	$(h_\theta - H_\theta)/V^2$	0·0145	K_3/V^2	$(m_\theta - M_\theta)/V^2$	0·25+2,192/V^2

D

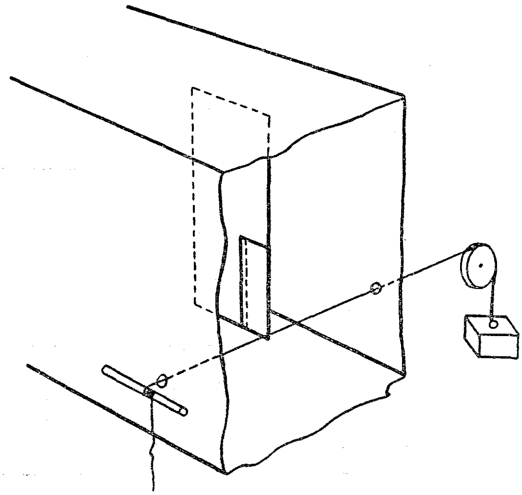

FIG. 20.—Diagram showing Aileron Adjustment.

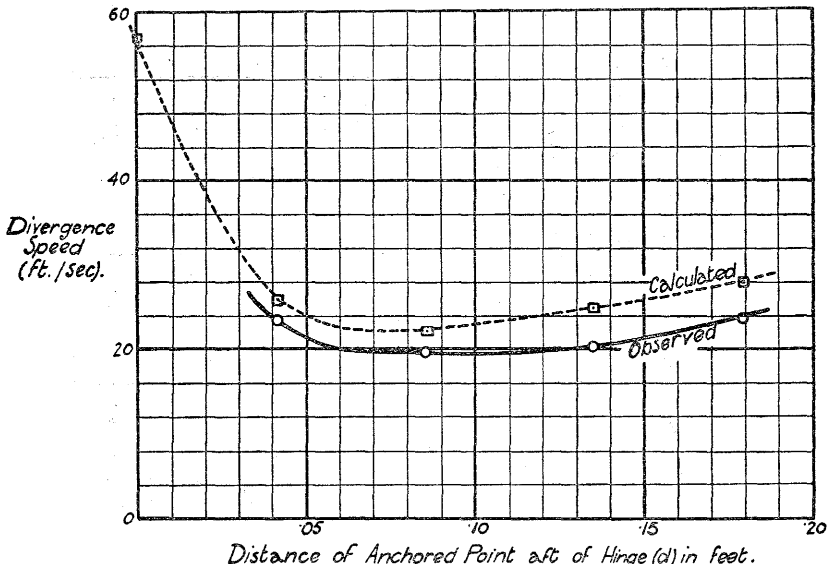

FIG. 21.—Observed and Calculated Divergence Speeds.

The observed phenomena were essentially the same for all the points of attachment selected. It was found that the earliest instability was a divergence of a rather peculiar character. At low wind speeds it was always possible to maintain the aileron at zero incidence by a suitable adjustment of the thread, but as the wind speed increased, a point was reached when this became impossible and the aileron always fell over to one side or the other. The divergence was not violent or destructive, but merely resulted in a loss of control of the aileron.

Table 12 gives a summary of the predicted and observed divergence speeds and critical flutter speeds. No critical flutter speeds could be observed since they were above the divergence speeds.*

TABLE 12.

Predicted and Observed Divergence and Flutter Speeds for Binary Motion.

Interaileron Strut at Wing Tip ($s' = s$ and $r' = r$).

d feet.	Divergence Speed. ft./sec.		Flutter Speed. ft./sec.	
	Calculated.†	Observed.	Calculated.	Observed.
0 ..	56·8	—‡	Above 100 if any	—
0·0417 ..	26·1	23·4	,, 100 ,,	—
0·0858 ..	22·2	19·6	88·0	—
0·1350 ..	24·9	20·3	75·6	—
0·1792 ..	28·2	23·7	69·5	—

The observed and calculated divergence speeds are plotted on a base of d in Fig. 21. The agreement is fair, for while all the calculated speeds are somewhat high, the type of variation with the position of the anchored point is quite correct. Possibly the lowness of the observed speeds may be partially due to the existence of finite disturbances in the tunnel.

For the sake of comparison with the tabulated figures, it may be mentioned that the ternary critical flutter speed for the wing was about 10 ft. per sec. and that the calculated ternary divergence speed was about 90 ft. per sec.

Additional calculations were made to find how the divergence speed varied as the interaileron strut was moved inwards from the wing tip. The procedure adopted in the calculations was as follows :— A position for the strut having been selected, the aileron angle ξ' was defined as the inclination of the aileron to the wing at this section. The equations of motion corresponding to the new

* Except in the case of $d = 0·0417$, flutter followed the divergence. The theory was then inapplicable on account of the large inclination of the aileron to the wing.

† On the basis of the *static* flexural and torsional stiffnesses. (*See* Appendix 4 to R. & M. 1155.)

‡ The observation was not attempted as the predicted speed was dangerously high.

aileron angle were found in the manner explained in Appendix 1, and the divergence speeds were then calculated by aid of the formulæ (137) as before. Values of the constants n, r' and s' were obtained with the help of photographs of free flexural and torsional oscillations of the wing in still air.

TABLE 13.

Variation of the Binary Divergence Speed with the Position of the Interaileron Strut in the Span.

$d = 0 \cdot 135$ ft. throughout.

Position of Interaileron Strut.	n	r'	s'	Calculated Divergence Speed ft./sec.
Wing Tip	0	0·663	2·25	24·9
Inboard end of aileron ..	0·493	0·336	1·049	36·8
Wing Root	1·0	0	0	63·7

The wing root was not a possible position for the interaileron strut since the aileron did not extend to the wing root, but the calculation has been added to show how the divergence speed rises when the strut is placed at the node. Experimental divergence speeds were not obtained as the model had been broken up before the calculations were made.

Experiments on a model biplane described in § 76 are in good general agreement with the theoretical results obtained. Hence the following conclusions regarding the binary motion can be stated :—

(1) When the interaileron strut is not near the interplane struts, the earliest instability will be a divergence.

(2) The binary divergence speed increases as the interaileron strut is moved inboard towards the interplane struts.

SECTION B.

EQUAL UNSTAGGERED BIPLANE WITHOUT RESTRICTION AS TO THE POSITION OF THE NODES.

§ 59. *Generalisation of the Basic Assumptions of the Theory.*— The experiments described in Part II show that it is not in general true that the vibratory motion of the outermost interplane struts in a flutter is negligibly small. Even when the bracing is stiff in relation to the spars, it is still possible for the amplitude of the oscillation of the struts to be large. Indeed, it is the *damping factor* of the motion involving a particular type of displacement and not *directly* the magnitude of the elastic resistance to the displacement which determines the importance of that motion.* Accordingly the theory must be generalised so as to take account of the movement of the interplane struts.

In conformity with the general ideas adopted in the discussion of wing flutter, it will still be assumed that the displacements of all points of the wings and ailerons can be specified by a finite number of variables, and the simplest system of co-ordinates compatible with the required freedom of the wing system will be adopted. It will be assumed that the outermost incidence truss can be treated as rigid,† and that bodily yawing oscillations of the wings in phase do not occur. Then the motions of the system will admit description in terms of the following eight co-ordinates :—

Φ Flexural or normal displacement of the outermost incidence truss.

Θ Torsional displacement of the same about an axis passing through its centre.

ϕ_1, ϕ_2 " Flexural " displacements of the upper and lower overhangs, respectively, measured relative to the positions which they assume in a pure Φ displacement (supposed produced by the application of a load at the incidence truss).

θ_1, θ_2 Torsional displacements of the upper and lower wings respectively, measured relative to the truss, at a certain chosen section.

ξ_1, ξ_2 Aileron angles for the upper and lower wings respectively.

The co-ordinates Φ and Θ thus determine the displacements of the wings as a whole, while the remainder are the relative co-ordinates of the separate wings and ailerons. On account of the interconnection of the ailerons, the number of degrees of freedom is seven.

Now suppose that Φ and Θ are initially zero, and let any equal but opposite displacements be given to the upper and lower wings. By symmetry, the displacements will be equal and opposite at all subsequent instants, i.e. both the interplane struts and the interaileron strut will remain at rest. Thus, the " binary motion " (as defined in § 55) is a component of the complete motion.

Next imagine the massless interaileron strut (*see* § 54) removed and let equal displacements be given to the upper and lower wings, i.e. initially let $\phi_1 = \phi_2$, $\xi_1 = \xi_2$, $\theta_1 = \theta_2$ while Φ and Θ may be finite. Clearly, on account of the symmetry of the system, the coupling between Φ and ϕ_1 will be the same as the coupling between Φ and ϕ_2 ; similarly the couplings between Θ and ϕ_1 and ϕ_2 will be

* Some experiments on the oscillations of a slender steel beam which strikingly illustrate this point are described in Appendix 3 (page 124).

† This assumption is reasonably correct according to Perring's experiments on an accurate model of a full scale biplane, even when the stagger is large. *See* R. & M. 1197, p. 8.

equal, and so on. On the other hand, it can be assumed that, on account of the manner in which the co-ordinates are defined, there is no *direct* coupling between ϕ_1 and ϕ_2, etc. Hence, during the subsequent motion $\phi_1 = \phi_2$, $\xi_1 = \xi_2$ and $\theta_1 = \theta_2$; consequently the massless strut can be replaced without alteration of the motion. The displacements of the two wings are thus always equal and the motion can be described by the five co-ordinates Φ, Θ, ϕ_1, ξ_1 and θ_1. This quinary motion is the second component of the complete motion. It may be regarded as an oscillation of a single wing of a generalised semi-rigid type, since the mass and elastic reaction of the bracing elements may be supposed equally divided between the upper and lower wings.

With regard to the binary component, the equations of motion already obtained in § 55 must hold good, together with the conclusions drawn therefrom. It remains to discuss the stability of the quinary motion in which the upper and lower wings move in phase. Now experiment shows that when the biplane is unstaggered the Θ motion is extremely small; accordingly, this motion will at first be neglected, but the case where it must be included will be considered briefly later.

§ 60. *Equations of Motion and Stability.*—The equations of small motions will be of the usual linear type and will be written in condensed notation as follows:—

Φ equation.—$L_0(\Phi) + A_0(\phi) + D_0(\xi) + G_0(\theta) = 0$, .. (138a)

ϕ equation.—$L_1(\Phi) + A_1(\phi) + D_1(\xi) + G_1(\theta) = 0$, .. (138b)

ξ equation.—$L_2(\Phi) + A_2(\phi) + D_2(\xi) + G_2(\theta) = 0$, .. (138c)

θ equation.—$L_3(\Phi) + A_3(\phi) + D_3(\xi) + G_3(\theta) = 0$, .. (138d)

where $L_0(\Phi) = L_0\ddot{\Phi} + M_0\dot{\Phi} + N_0\Phi$,

$A_0(\phi) = A_0\ddot{\phi} + B_0\dot{\phi} + C_0\phi$,

etc.

No attempt will be made to discuss the stability of the motion in general, and attention will be confined to the case where all the methods of flutter prevention as laid down in § 57 of R. & M. 1155, have been applied to the wings separately. It is there shown that ordinary ternary flutter (involving the co-ordinates ϕ, ξ, θ only) will be avoided provided that:—

(1) The flexural axis coincide with the axis of independence.

(2) The wing be mass balanced about the flexural axis.

(3) The flexural-aileron product of inertia be zero.

(4) The aileron compound damping coefficient B_2 ($= -H_{\dot{\phi}}$) be zero.

(5) The torsional-aileron motion be stable.

(6) The flexural motion be damped.

It will be convenient, as usual, to select the flexural axis as the transverse reference axis. Then conditions (1) and (2) ensure that $A_3\ (\phi) = 0$, while $A_2\ (\phi) = 0$ when conditions (3) and (4) are satisfied.* Thus the ternary motion resolves into a torsional-aileron motion and a stable flexural oscillation.

When conditions (1) and (2) are satisfied it will be reasonable to assume that $L_3\ (\Phi)$ vanishes. For, when the wing is mass balanced about the reference axis, the product of inertia L_3 of the $\Phi\,\theta$ motion must vanish. Likewise the aerodynamical derivative M_3 will be zero when the reference axis coincides with the axis of independence, and, lastly, N_3 will vanish since the reference axis coincides with the flexural axis.

It cannot in general be asserted that $L_2\ (\Phi)$ and $A_2\ (\phi)$ will vanish together. The coupling term $A_2\ (\phi)$ will be eliminated, as shown in § 57, when the line of attachment of the interplane struts approximately bisects the span of the aileron, irrespective of the position of the aileron hinge; but this condition obviously will not be sufficient to ensure the vanishing of $L_2\ (\Phi)$. However, if the aileron be mass balanced about its hinge and the hinge be located so that the derivative $M_2\ (= -\ H\dot{\phi})$ is zero, then $L_2\ (\Phi)$ will vanish.

Suppose now that the conditions are such that $A_3\ (\phi) = A_2\ (\phi) = L_3\ (\Phi) = L_2\ (\Phi) = 0$. Then the determinant of motion corresponding to the equations (138) becomes

$$\Delta\,(\lambda) \equiv \begin{vmatrix} L_0\,(\lambda), A_0\,(\lambda), D_0\,(\lambda), G_0\,(\lambda) \\ L_1\,(\lambda), A_1\,(\lambda), D_1\,(\lambda), G_1\,(\lambda) \\ 0\ ,\ \ 0\ , D_2\,(\lambda), G_2\,(\lambda) \\ 0\ ,\ \ 0\ , D_3\,(\lambda), G_3\,(\lambda) \end{vmatrix}$$

$$\equiv \begin{vmatrix} L_0\,(\lambda), A_0\,(\lambda) \\ L_1\,(\lambda), A_1\,(\lambda) \end{vmatrix} \times \begin{vmatrix} D_2\,(\lambda), G_2\,(\lambda) \\ D_3\,(\lambda), G_3\,(\lambda) \end{vmatrix}. \quad \cdot\cdot \quad (139)$$

The quaternary $\Phi\,\phi\,\xi\,\theta$ motion thus resolves into the ordinary torsional-aileron motion and a binary $\Phi\,\phi$ motion. Stability of the torsional-aileron motion is one of the conditions for the stability of the ordinary ternary $(\phi\,\xi\,\theta)$ motion. It remains to examine the stability of the $\Phi\,\phi$ motion.

§ 61. *Stability of the Binary Flexural Motion.*—The determinant of the binary flexural $(\Phi\,\phi)$ motion is, when written at length,

$$\begin{vmatrix} L_0\lambda^2+M_0\lambda+N_0, & A_0\lambda^2+B_0\lambda+C_0 \\ L_1\lambda^2+M_1\lambda+N_1, & A_1\lambda^2+B_1\lambda+C_1 \end{vmatrix}. \quad \cdot\cdot \quad (140)$$

* For C_2 vanishes when the angle of incidence is small (as is assumed throughout) and the aileron angle ξ is measured at the section in which the control operates.

On account of the manner in which the co-ordinates ϕ and Φ are defined, there can be no elastic cross-stiffness terms. For, if such a cross-stiffness existed, it would follow that a static displacement in Φ, produced by the application of a load at the interplane struts, would be associated with a displacement in ϕ. But no such associated displacement occurs, since by definition ϕ measures the displacement of the wing relative to the position assumed on the application of a load at the interplane struts. Provided, as usual, that the angle of incidence is small, there will be no aerodynamical contributions to the stiffness terms N and C. Thus, N_1 and C_0 will vanish, while the direct stiffnesses N_0 and C_1 will be of purely elastic origin.

Since the cross-stiffnesses are absent, the equation of energy takes the simple form (cp. R. & M. 1155, p. 42)

$$\frac{d}{dt}(\mathbf{T}+\mathbf{V})+2\mathbf{F}=0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (141)$$

where **T, V** and **F** are respectively the kinetic energy, potential and dissipation function, and are given by the equations

$$2\mathbf{T}=L_0\dot{\Phi}^2+2A_0\dot{\Phi}\dot{\phi}+A_1\dot{\phi}^2, \qquad \ldots \quad \ldots \quad \ldots \quad (142)$$

$$2\mathbf{V}=N_0\Phi^2+C_1\phi^2, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (143)$$

and

$$2\mathbf{F}=M_0\dot{\Phi}^2+(M_1+B_0)\dot{\Phi}\dot{\phi}+B_1\dot{\phi}^2. \qquad \ldots \quad \ldots \quad (144)$$

Since the potential **V** is of purely elastic origin, and is therefore necessarily positive, it follows that any motion of the system must decay provided that **F** is essentially positive. Thus a sufficient condition for the stability of the system is that the dissipation function shall be one-signed and positive.

The same conclusion regarding the stability can be deduced from a consideration of the test conic (*see* Chap. III of R. & M. 1155). Adopt a notation similar to that used in R. & M. 1155. Then the equation to the test conic can be written (*loc. cit.*, equation (56)) :—

$$q_1'(Xa_1+Yl_0+\alpha)(Xb_1+Ym_0)-q_0'(Xb_1+Ym_0)^2-XYq_1'^2=0, \quad (145)$$

where

$$q_0'=l_0a_1-a_0^2, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (146a)$$

$$q_1'=l_0b_1+a_1m_0-a_0(m_1+b_0), \qquad \ldots \quad \ldots \quad \ldots \quad (146b)$$

$$\alpha=m_0b_1-m_1b_0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (146c)$$

Since there are no aerodynamical stiffnesses, the "stiffness point" is at the origin. The conic (145) obviously passes through the origin, and its second intersections with the co-ordinate axes will be on the negative side of the origin provided that the conic is elliptic (which condition is the same as that the dissipation function shall be essentially positive). For, consider the second intersection with the axis OX. By equation (145) the abscissa is given by

$$q_1' (\mathrm{X}a_1 + \alpha) - q_0' \mathrm{X}\, b_1 = 0,$$

or

$$\mathrm{X}\ [a_1^2 m_0 - a_1 a_0 (m_1 + b_0) + a_0^2 b_1] + q_1' \alpha = 0. \quad .. \quad (147)$$

The coefficient of X in this equation will be positive provided that

$$4\, m_0 b_1 - (m_1 + b_0)^2 > 0, \quad .. \quad .. \quad .. \quad (148)$$

which is precisely the condition that the dissipation function shall be essentially positive. Under the same condition it readily follows that q_1' and α are necessarily positive.* Hence the second intersection of the conic with OX lies to the left of the origin and similarly the second intersection with OY lies below the origin. Thus the test conic is disposed as in Fig. 22, and, since the stiffness point is at the origin, there can be no finite critical speed.

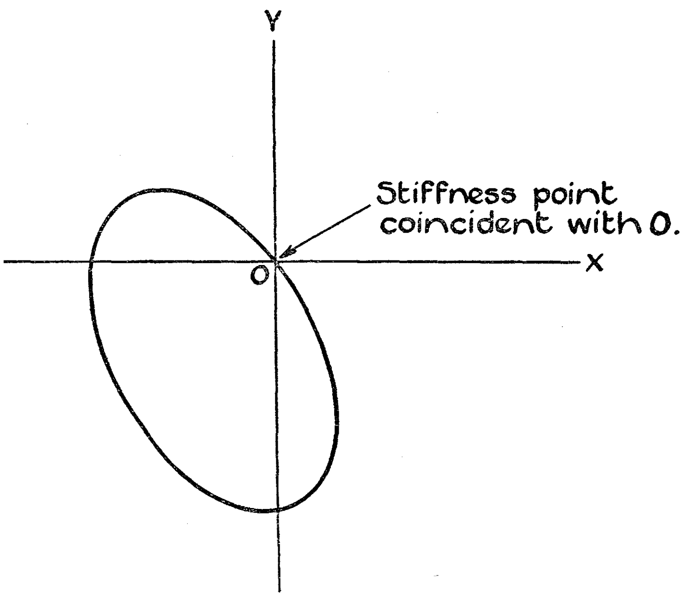

FIG. 22.—Test Conic for Binary Flexural Motion.

The only point which now requires consideration is whether the dissipation function is in fact essentially positive. The criteria are that

$$\mathrm{B}_1 > 0,$$

$$4\mathrm{M}_0\mathrm{B}_1 - (\mathrm{M}_1 + \mathrm{B}_0)^2 > 0.$$

Now at small angles of incidence B_1 and M_0 are certainly positive, and the values of all the derivatives can be estimated on the basis

* *See* footnote to p. 131 of R. & M. 1155.

of strip theory and substituted in the second inequality. Details of the calculations will be omitted, but the final result is that the inequality is satisfied for all ratios of the length of the overhang to the wing span. There is therefore a strong probability that the dissipation function is essentially positive,* and the argument will be continued on the assumption that this is true.

It can now be concluded that the binary $\Phi\ \phi$ motion is stable. Hence the quaternary $\Phi\ \phi\ \xi\ \theta$ motion will certainly be stable provided that

(1) The flexural axis of the wing coincide with the axis of independence.

(2) The wing be mass balanced about the flexural axis.

(3) The aileron be mass balanced about the hinge axis.

(4) The aileron hinge be so adjusted that the derivative $H_{\dot{\Phi}}$ vanishes.

(5) The torsional-aileron motion be stable.

It must be emphasised that the above conditions have merely been shown to be *sufficient* for the avoidance of instability. Many of them can in fact be violated without the introduction of instabilities at moderate speeds. Now condition (3) (modified as a *slight over-balance* of the aileron) is an essential condition for the satisfaction of (5)†. Hence theory strongly supports the advisability of the adoption of ailerons whose centres of gravity lie upon, or preferably slightly ahead of, the hinge axes (unless, of course, irreversible aileron controls are used). The model experiments described in Part II are in complete agreement with this view, and tend to show that the other conditions are comparatively unimportant.

§ 62. *Remarks on the General Torsional Motion.*—A displacement in Θ is resisted by the stiffnesses of both the interplane bracing wires and the drag bracing wires within the wings. Evidently the total stiffness in this type of motion must be very great.

The theory of the quinary $\Phi\,\Theta\phi\,\xi\theta$ motion can be developed on similar lines to that adopted in § 60 for the $\Phi\phi\ \xi\theta$ motion. Subject to conditions altogether analogous to those laid down in § 60, the quinary motion can be resolved into the stable $\Phi\phi$ motion and a ternary $\Theta\theta\ \xi$ motion. Even if the ordinary torsional-aileron ($\theta\ \xi$) motion be stable, it does not appear to be necessarily true that the $\Theta\ \theta\ \xi$ motion is stable. Thus apparently the great stiffness resisting a displacement in Θ is largely responsible for immunity from instability in this type of motion.

* A general proof that a purely flexural motion having any number of degrees of freedom has an essentially positive dissipation function can be easily obtained from the Lemma given in the Appendix to Chap. IV. It readily follows that any purely flexural motion is stable.

† Unless the line of struts bisects the aileron span and the staying is enormously stiff.

SECTION C.

UNSTAGGERED BIPLANE WITH UNEQUAL WINGS

§63. *Preliminary.*—Equations of motion of an unstaggered biplane with unequal wings can be obtained for the case where the outermost struts do not move by the method adopted in § 55. In general the motion does not resolve into binary and ternary components as in the case of equal upper and lower planes, and the equations are so complicated that useful conclusions cannot be deduced. However, there are some important special cases where definite deductions can be made, and these will now be discussed.

§64. *Case of Very Unequal Overhangs.*—Consider first the case where the upper overhang is long whereas the lower overhang is short. Experiments on a model biplane described in § 71 showed that when the upper overhang was twice as long as the lower overhang the only motion of the lower plane was a small flexural oscillation impressed upon it by the interplane struts. The same result was found by Perring* in the case of an accurate model of a full scale biplane where the upper overhang was only 45 per cent. greater than the lower overhang. Hence it may be concluded that for biplanes of the type considered the wing oscillations can be adequately described by the four co-ordinates Φ, ϕ_1, ξ_1, and θ_1.

Suppose first of all that the motion of the interplane struts is negligibly small. Then the lower plane will be stationary but the lower aileron will be forced to oscillate. The kinematical relation corresponding to equation (119) is now

$$\xi_2 = \xi_1 + \frac{s'}{d}\phi_1 + \frac{r'}{d}\theta_1. \quad \dots \quad \dots \quad \dots \quad (149)$$

Let additional suffices 1 and 2 be given to the dynamical constants appertaining to the upper and lower wings respectively, and let the dynamical constants of the combined system be written without additional suffices. The expression for the kinetic energy of the whole system is

$$\begin{aligned} 2T &= A_1\dot{\phi}_1^2 + D_2\dot{\xi}_1^2 + G_3\dot{\theta}_1^2 + 2G_2\dot{\xi}_1\dot{\theta}_1 + 2A_3\dot{\theta}_1\dot{\phi}_1 + 2D_1\dot{\phi}_1\dot{\xi}_1 \\ &= A_{11}\dot{\phi}_1^2 + D_{12}\dot{\xi}_1^2 + G_{13}\dot{\theta}_1^2 + 2G_{12}\dot{\xi}_1\dot{\theta}_1 + 2A_{13}\dot{\theta}_1\dot{\phi}_1 \\ &\quad + 2D_{11}\dot{\phi}_1\dot{\xi}_1 + D_{22}\left(\dot{\xi}_1 + \frac{s'}{d}\dot{\phi}_1 + \frac{r'}{d}\dot{\theta}_1\right)^2 \quad \dots \quad \dots \quad (150) \end{aligned}$$

Hence the expressions for the two aileron products of inertia for the combined system are :—

$$D_1 \equiv A_2 = D_{11} + \frac{s'}{d}D_{22}, \quad \dots \quad \dots \quad \dots \quad (151a)$$

$$G_2 \equiv D_3 = G_{12} + \frac{r'}{d}D_{22}. \quad \dots \quad \dots \quad \dots \quad (151b)$$

---

* Ref. 7.

Thus it will be seen that even if the ailerons be mass balanced about the hinge axes, the aileron products of inertia will be large unless r' and s' be zero. Large aileron products of inertia are, of course, most objectionable and it thus appears to be very desirable that the coefficients r' and s' should be made to vanish. This will actually be the case when the interaileron strut is opposite the interplane struts, and in the sequel it will be assumed that this arrangement has been adopted.

When r' and s' are zero, the lower aileron behaves strictly as an appendage of the upper aileron even when the node does not lie at the interplane struts. Suppose then that both of the ailerons are mass balanced and that the other conditions for the avoidance of flutter detailed in § 60 are observed (in this instance the conditions affecting the flexural axis, etc., of course apply to the upper wing only). The equations of motion will be formally identical with equations (138) and again

$$A_3(\phi) = A_2(\phi) = L_3(\Phi) = L_2(\Phi) = 0.$$

Thus, as before, the motion will resolve into a stable $\Phi\phi$ motion and a torsional-aileron motion which can be rendered stable by the usual means.

The conclusions from the above argument can be summarised as follows :—

When the upper overhang is long but the lower overhang short, mass balance of the ailerons about their hinges may not prove effective in the suppression of flutter unless the interaileron strut lies close to the interplane struts. Provided that the latter condition is realised, flutter can be avoided by the same measures as for a biplane with equal overhangs.

A still greater simplification can be made in the theory when both the overhangs are short. Here, unless the stiffness of the bracing is enormously great in relation to the stiffness of the spars, the only important vibratory motions will be a general flexural (Φ) motion and an aileron (ξ) motion. This is in accord with the original simple theory of R. & M. 1042.* The important measure for the prevention of flutter is mass balance of the ailerons, and the position of the interaileron strut is here of little importance.

§ 65. *Some Remarks on a More General Case.*—Some little progress can be made in the discussion of the general case of the unequal biplane, subject to the conditions that the interaileron strut is opposite the interplane struts and that the latter do not move. In this case $\xi_1 = \xi_2$, and the equations of motion corresponding to (124) and (125) are (since the coefficients r' and s' are zero)

$$A_{11}(\phi_1) + D_{11}(\xi_1) + G_{11}(\theta_1) = 0\,, \qquad \ldots \quad (152a)$$

$$A_{12}(\phi_1) + D_{12}(\xi_1) + G_{12}(\theta_1) + dT = 0\,, \qquad \ldots \quad (152b)$$

$$A_{13}(\phi)_1 + D_{13}(\xi_1) + G_{13}(\theta_1) = 0\,, \qquad \ldots \quad (152c)$$

* Ref. 13.

and

$$A_{21}(\phi_2) + D_{21}(\xi_1) + G_{21}(\theta_2) = 0, \quad \ldots \quad (153a)$$

$$A_{22}(\phi_2) + D_{22}(\xi_1) + G_{22}(\theta_2) - d\,T = 0, \quad \ldots \quad (153b)$$

$$A_{23}(\phi_2) + D_{23}(\xi_1) + G_{23}(\theta_2) = 0. \quad \ldots \quad (153c)$$

Direct addition of (152b) and (153b) gives

$$A_{12}(\phi_1) + A_{22}(\phi_2) + (D_{12} + D_{22})(\xi_1) + G_{12}(\theta_1) + G_{22}(\theta_2) = 0. \quad (154)$$

The equations of motion are (152a), (152c), (153a), (153c) and (154).

When the usual measures of flutter prevention are adopted

$$A_{12}(\phi_1) = A_{13}(\phi_1) = A_{22}(\phi_2) = A_{23}(\phi_2) = 0,$$

and the determinant of motion can be written

$$\Delta(\lambda) = \begin{vmatrix} A_{11}(\lambda) & G_{11}(\lambda) & D_{11}(\lambda) & 0 & 0 \\ 0 & G_{13}(\lambda) & D_{13}(\lambda) & 0 & 0 \\ 0 & G_{12}(\lambda) & (D_{12}+D_{22})(\lambda) & G_{22}(\lambda) & 0 \\ 0 & 0 & D_{23}(\lambda) & G_{23}(\lambda) & 0 \\ 0 & 0 & D_{21}(\lambda) & G_{21}(\lambda) & A_{21}(\lambda) \end{vmatrix},$$

$$= A_{11}(\lambda)A_{21}(\lambda) \begin{vmatrix} G_{13}(\lambda) & D_{13}(\lambda) & 0 \\ G_{12}(\lambda) & (D_{12}+D_{22})(\lambda) & G_{22}(\lambda) \\ 0 & D_{23}(\lambda) & G_{23}(\lambda) \end{vmatrix}. \quad (155)$$

Thus the motion resolves into the separate flexural oscillations of the upper and lower wings and a ternary torsional-aileron $(\theta_1 \theta_2 \xi_1)$ motion. The determinant of the latter motion is, when expanded,

$$G_{13}(\lambda)\,[D_{22}(\lambda)\,G_{23}(\lambda) - D_{23}(\lambda)\,G_{22}(\lambda)]$$
$$+ G_{23}(\lambda)\,[D_{12}(\lambda)\,G_{13}(\lambda) - D_{13}(\lambda)\,G_{12}(\lambda)]. \quad \ldots \quad (156)$$

The expressions in the square brackets are the determinants of the torsional-aileron motions of the lower and upper wings respectively. Unfortunately it does not appear to be demonstrable that the motion represented by (156) is necessarily stable when the individual torsional-aileron motions are stable. All that can be said is that if the aileron couplings in the torsional-aileron motions be small, the terms $G_{12}(\lambda)$ and $G_{22}(\lambda)$ will be small and the predominant terms in (156) can be written

$$G_{13}(\lambda)\,G_{23}(\lambda)\,(D_{12} + D_{22})(\lambda).$$

The last expression represents a motion free from oscillatory instability.

SECTION D.

THE INFLUENCE OF STAGGER AND MISCELLANEOUS

§ 66. *Influence of Stagger.*—It will be assumed that the inter-aileron strut is pivoted on the centre lines of the aileron ribs and that these centre lines are parallel in the undisturbed position. Further, relative yawing motions of the wings will be neglected.

The investigation is of course limited to small oscillations. Consequently, neglecting second order quantities, the displacements of the pivots of the interaileron strut will be wholly normal to the wing chord. Thus the ends of the strut move in parallel straight lines, and the strut consequently does not rotate during the oscillation. Moreover, the normal displacements of the ends must be equal.

Since the strut does not rotate, it can be replaced by an ideal massless strut and a pair of massive particles at its ends, just as when stagger is absent. Also the kinematical equation is unaltered since the normal displacements of the ends of the strut are equal. Finally, if the normal component of the thrust in the ideal strut be substituted for the total thrust, the equations of motion (e.g. (124) and (125)) will be the same as when there is no stagger. Thus, subject to the validity of the assumptions made, stagger has no influence on the stability of biplanes, whether the planes are equal or not.

It is evident that when the stagger is large there will actually be a tendency to introduce a Θ motion, since the fore and aft separation of the planes must exaggerate any twisting moment due to the Φ motion. However, experiment shows that this effect is not large and confirms the theoretical conclusion that the influence of stagger on the critical flutter speed is in general trifling (*see* § 75).

§ 67. *Influence of Mass Loading within the Bay.*—Suppose that a mass m be attached to one of the wings in the bay at a place where the amplitude of the flexural oscillation is one *nth* part of the corresponding amplitude at the wing tip. Then, if the wing could be treated as semi-rigid, the mass m would be equivalent to a mass m/n^2 placed at the wing tip. For example, suppose 400 lb. of petrol to be placed in a wing tank where the amplitude is one-twentieth of that at the wing tip. According to the argument this would be merely equivalent to 1 lb. placed at the wing tip, and the influence of such a mass on the stability would probably be negligible.

There is no doubt, however, that the wing cannot be treated as semi-rigid where very massive concentrated loads are concerned. Such a mass offers great resistance to rapid vibration and tends to produce a node in its neighbourhood. Thus a secondary effect of a large mass in the bay may be a slight increase of the effective stiffness in an oscillation, with a concomitant small rise of the critical flutter speed. Experiments show that the influence of mass loading in the bay is in fact small, and is sometimes in the direction of greater stability (*see* § 74).

§ 68. *Summary of the Principal Theoretical Conclusions.*—In the case of a biplane with equal upper and lower planes, the general wing motion can be resolved into the following components :—

(1) A motion in which the upper and lower wings move equally and in phase.

(2) An effectively binary motion in which the upper and lower wings move equally but in opposition.

The first type of motion in general involves motion of the interplane struts, but the torsional component of the latter motion can usually be neglected. Flutter in this mode can be avoided by the same means as for a monoplane, and the most important of the preventive measures is mass balance of the ailerons (failing the adoption of irreversible aileron controls).

When the interaileron strut is not near the interplane struts, the earliest instability in the binary mode (2) will probably be a divergence. This divergence is not of a violent type, but causes loss of control of the ailerons. The divergence speed rises as the interaileron strut is moved inboard towards the interplane struts. Lastly, when the interaileron strut is opposite the interplane struts, the binary motion becomes of a purely flexural-torsional type and instability will only occur at very high speeds.

For a biplane having a long upper overhang and a short lower overhang, the usual measures of flutter prevention will not necessarily be effective unless the interaileron strut be opposite the interplane struts. In the contrary case the effective aileron products of inertia may be large, even when the C.G.'s of the ailerons lie upon, or slightly ahead of, the hinge axes. When both the overhangs are short, there are only two important types of displacement, namely, a general flexural displacement and rotation of the ailerons. Here again mass balance of the ailerons is the important flutter-preventive measure.

Lastly, the influence of stagger on the stability is small.

PART II.

WIND-TUNNEL EXPERIMENTS UPON THE WING FLUTTER OF BIPLANES.

§ 69. *Preliminary.*—The experiments described herein were undertaken in order to explore the general phenomena of the wing flutter of biplanes, and, more especially, in order to provide a check upon the theory and upon the efficacy of preventive measures. Accordingly, the model used was arranged to be modifiable so as to cover a wide variety of conditions, but in no condition was it an actual small scale representation of a particular machine. The correlation between the flutter characteristics of a biplane and of an accurate model of reduced elasticity had already been experimentally investigated by Perring,* and his results were in accord with dimensional theory. As this point had already been established, it was considered that the design of the model in the present instance should be governed by convenience of construction and of use in the wind-tunnel, and that exact adherence to any particular full-scale design was unnecessary.

§ 70. *Description of the Model.*—The model consisted of a pair of planes of thin section, each provided with an aileron, interconnected by struts and attached at the roots to a wooden block which was screwed to the floor of the wind tunnel. Fig. 23 is a photograph of the model in what may be considered its standard condition, i.e., with the planes equal and unstaggered. The two planes were made identical. The framework, consisting of a pair of slender solid wooden spars carrying wooden ribs, joined by thin wooden strips at the leading and trailing edges, was covered with silk fabric doped with a solution of vaseline in chloroform. At the roots the spars were gripped firmly between the base block A and the cover pieces B, B′. In order to provide for tests with upper and lower planes of unequal spans, the spars of the upper plane were continued inwards, and an additional section of wing was arranged to slide over these projections. In this manner the span of the upper wing could be increased and it will be noted that any slight discontinuity at the junction of the additional section occurred near

* Ref. 7.

[*To face page* 102.

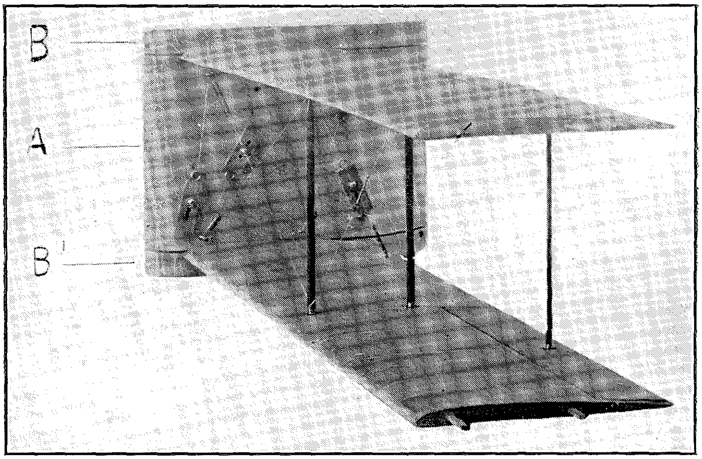

FIG. 23.—Photograph of Model Biplane.

the root where it was of no importance. Any desired degree of stagger could be given by merely loosening the cover pieces B, B′ and sliding the planes fore and aft.*

The interplane struts were made of rafwire, and were pivoted at their ends so as to permit adjustment for stagger, and diagonal bracing of the incidence truss was fitted as shown in the figure. Springs were inserted in the lift and antilift wires for two reasons :—

(1) Plain steel wires would have been disproportionately stiff in relation to the very flexible spars.

(2) It was desired to study the effect of varying the stiffness of the main bracing.

The interplane struts could be attached at a number of different positions in the span where extra thick ribs had been provided to receive them. Also an additional pair of interplane struts could be fitted if required. In many of the experiments diagonal bracing by the lift and antilift wires was abandoned in favour of the simpler direct staying to the tunnel wall through wires in line with the interplane struts and provided with springs. No drag bracing within the planes was provided.

The ailerons were of similar construction to the wings and were made as nearly as possible identical. Anti-friction hinges, consisting of hard steel points bearing in hard steel pivot plates, were used, and two hinge positions were available, situated at 0·1 and 0·2 of the aileron chord from its leading edge. The leading edge consisted of a rather thick solid wooden spar having a number of detachable wooden insets secured by screws. When it was desired to mass-balance the aileron about the hinge, the wooden insets were replaced by properly adjusted lead insets. A condition of slight over-balance with the hinge at 0·2 chord could also be obtained when required. The interaileron strut was of wood and adjustable in length ; it could be attached at any one of five positions in the span of the aileron. Spring control of the ailerons was not provided ; the influence of this control is now well understood as the result of theoretical and experimental work on monoplanes, and it was not considered necessary to increase the already lengthy programme of experiments by investigations on this matter.

Throughout the tests the angle of incidence at the wing root was maintained at 0°. The flutter of the wings could be controlled in amplitude or arrested by means of a wooden frame fitting over the small outer projections of the spars shown in Fig. 23. This frame was attached to a rod passing through the tunnel wall and its position could be varied as desired during the course of a test.

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* The length of the interplane struts was not varied so that the width of the block A had to be reduced by planing when the wings were staggered.

The principal dimensions and other particulars of the model are as follows :—

Span of upper wing (unextended)	27 in.
Span of upper wing (extended)	33·3 in.
Span of lower wing	27 in.
Wing chord	9 in.
Gap (with 0° stagger)	9 in.
Gap (with $6\frac{1}{2}$° stagger)	8·9 in.
Gap (with 30° stagger)	7·9 in.
Wing section	R.A.F. 15.
Aileron span	12·4 in.
Aileron chord	2·5 in.
Position of spar centres, measured from leading edge	1·25 in. and 5·25 in.
Depth of rectangular spar section	0·14 in.
Width of rectangular spar section	0·25 in.
Distance of forward interplane strut from L.E.	1·25 in.
Distance of rear interplane strut from L.E...	5·25 in.
Distance of interaileron strut from aileron L.E.	1·5 in.

Mass of one plane (27 in. span, with aileron having wooden insets, otherwise bare) 142 grams or $9 \cdot 73 \times 10^{-3}$ slug.

Mass of pair of interplane struts with sockets, screws, incidence wires and turnbuckles 50·5 grams or $3 \cdot 46 \times 10^{-3}$ slug.

Mass of bare aileron (with wooden insets) 23 grams or $1 \cdot 58 \times 10^{-3}$ slug.

Mass of lead balancing pieces less corresponding wooden insets (for one aileron) 18 grams or $1 \cdot 23 \times 10^{-3}$ slug.

Mass of lead overbalancing piece less corresponding wooden inset (for one aileron) 2·1 grams or $0 \cdot 14 \times 10^{-3}$ slug.

Mass of interaileron strut complete with pivot brackets and screws 4·1 grams or $0 \cdot 27 \times 10^{-3}$ slug.

Distance of C.G. of insets from leading edge of aileron 0·07 in.

With the ailerons hinged at 0·2 chord from the leading edge and the interaileron strut attached, the centres of gravity lay upon the hinge axes when the lead balancing masses were inserted in the leading edges. Under this condition the flexural-aileron product of inertia vanished when the interaileron strut was attached near mid-span of the aileron, but was slightly positive when the strut was attached near the tip and slightly negative when it was attached near the inboard end. When the small overbalancing masses were inserted, the centres of gravity of the ailerons lay 0·02 in. (i.e. 0·008 aileron chord) forward of the hinge axes.

Four sets of springs were used in order to vary the stiffness of the staying, each set consisting of four identical springs. The stiffnesses were as follows :—

Springs No. 1	..	17 lb. per ft.
,, No. 2	..	48 ,, ,,
,, No. 3	..	182 ,, ,,
,, No. 4	..	378 ,, ,,

In some experiments the strut points were attached directly to the tunnel wall through taut piano wires. This fixture was to all intents rigid.

All the tests were made in a 4 ft. wind tunnel at the National Physical Laboratory. The maximum wind speed available was 75 ft. per sec.

§ 71. *General Behaviour of the Model.*—In the earliest experiments the planes were equal (i.e. both 27 in. span) and unstaggered, and the interplane struts were placed opposite the inboard ends of the ailerons, thus giving a long overhang. For *all* stiffnesses of the staying, flutter was found to occur above a certain critical speed when the ailerons were not mass balanced. The general nature of the motion was as follows :—

(1) The upper and lower planes moved equally and in phase.

(2) The normal or flexural motion at the strut points was in general large and the only node was the wing root.

(3) No torsional motion at the strut points could be detected.

Conclusion (1) was suggested by visual observation, but a more rigorous demonstration resulted from experiments in which the upper and lower planes were connected together at the leading and trailing edges at the wing tip by taut linen threads. The planes were thus forced to move together and the critical speeds were found to be unaffected by the presence of the threads, as shown in Table 14.

TABLE 14.

Flutter Speeds With and Without Threads Connecting the Upper and Lower Wings of an Equal Biplane.

Planes unstaggered. Ailerons hinged at 0·1 chord from aileron leading edge. Interplane struts at inner end of ailerons. Interaileron strut at mid-span of ailerons.

Expt. No.	Bracing.	Critical Flutter Speed.	
		Without Threads.	With Threads.
1	Diagonal, Springs No. 2	14	13·9
2	Diagonal, Springs No. 3	18	18

In all conditions of the model the strut points exhibited considerable motion normal to the planes except when the staying was exceedingly stiff. For example, when the interplane struts were placed at midspan of the ailerons, a node close to the struts was only obtained when the staying was direct to the tunnel wall and with the stiffest springs (No. 4). In this condition the flexural stiffness of the biplane was 370 as against 12 for the wings merely connected by struts without staying. It may be remarked that a point which is not displaced in a static loading test is by no means necessarily near a vibrational node. In a vibration the deciding factor is the damping coefficient of a particular motion. A further discussion of the influence of bracing will be given in § 73.

The absence of torsional motion at the interplane struts was easily apparent when the planes were unstaggered. The struts were, of course, normal to the planes and it was quite clear that the motion was entirely in the direction of the axes of the struts. With a stagger of 30° there appeared to be a very slight torsional motion at the struts. The last result is in general agreement with the experiments of Perring.* The model tested by him had very unequal planes and a large stagger and it was found that a torsional node occurred in the top plane "close to where the struts interconnect the two wings." It may be concluded that Θ motion is absent when the stagger is small and is not important even with a large stagger.

When the planes were equal, the interplane struts at the inner end of the ailerons, and the ailerons mass balanced about their hinges, the earliest instability was found to be the binary divergence, as defined in § 58. When this divergence occurred, the ailerons, which had been lying in the planes of the wings, suddenly moved outwards together or inwards together, and took up new mean positions inclined to the planes at 20°–30°. Usually there was a slow and rather jerky oscillation about the new mean position. A further account of these experiments is given in § 76.

When the upper plane was extended, so that the overhang of the upper plane was long and that of the lower plane short, the motion of the lower plane in a flutter was not large, and appeared to be merely such as was impressed upon it by the interplane struts. The lower aileron was of course compelled to follow the motion of the upper aileron on account of the connection by the interaileron strut. To sum up, the lower wing and aileron appeared to move as more or less rigid bodies under the impulsion of the fluttering upper wing.

A striking feature of the experiments was the total absence of purely flexural-torsional flutter (i.e. flutter in which the ailerons played no essential part) even with the longest overhangs and weakest bracing. In some of the tests with the ailerons not mass balanced, a critical flutter speed as low as 11 ft. per sec. was found, but there was no flexural-torsional flutter within the range of wind speed

* Ref. 7.

TABLE 15.

Influence of Aileron Hinge Position and Mass Balance on the Critical Flutter Speed.

Expt. No.	General Condition of Model.					Critical Flutter Speed for Condition of Ailerons Stated.			Remarks.
	Planes.	Stagger (forward).	Distance struts from inner end upper aileron as fraction of aileron span.		Particulars of staying.	Hinge at 0·1 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. mass balanced.	
			Inter-plane.	Inter-aileron.					
1	Equal	None	0	0·5	Diagonal Springs No. 3	17·7	21·2	Binary diver-gence at 49·2	With statically overbalanced aileron, divergence occurred at same speed.
2	Equal	None	0·25	0·5	Straight Springs No. 3	39·1	42·3	70·8	No flutter when aileron slightly overbalanced.
3	Equal	None	0·25	0·25	Straight Springs No. 3	41·3	44·3	None	
4	Equal	None	0·25	0·25	Straight Springs No. 4	45·8	48·2	None	
5	Equal	None	0·25	0·25	Struts fixed by taut wires	50·6	56·0	None	

TABLE 15—(*continued*).

Expt. No.	General Condition of Model.					Critical Flutter Speed for Condition of Ailerons Stated.			Remarks
	Planes.	Stagger (forward).	Distance struts from inner end upper aileron as fraction of aileron span.		Particulars of staying.	Hinge at 0·1 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. mass balanced.	
			Inter-plane.	Inter-aileron.					
6	Equal	None	0·5	0·5	Diagonal Springs No. 3	26·2	29·9	67·2	
7	Equal	None	0·5	0·5	Diagonal Springs No. 4	36·8	38·8	None	
8	Equal	None	0·5	0·5	Straight Springs No. 4	72	None	None	
9	Equal	None	0·75	0·75	Straight Springs No. 2	43·6	49·6	None	
10	Equal	$6\frac{1}{2}°$	0·25	0·25	Straight Springs No. 3	41·4	45·8	None	
11	Equal	$6\frac{1}{2}°$	0·25	0·25	Straight Springs No. 4	45·5	50·6	None	

TABLE 15—(*continued*).

Expt. No.	General Condition of Model.					Critical Flutter Speed for Condition of Ailerons Stated.			Remarks.
	Planes.	Stagger (forward).	Distance struts from inner end upper aileron as fraction of aileron span.		Particulars of staying.	Hinge at 0·1 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. mass balanced.	
			Inter-plane.	Inter-aileron.					
12	Equal	30°	0·25	0·25	Straight Springs No. 3	39·7	44·2	None	
13	Equal	30°	0	0	Diagonal Springs No. 3	—	21	None	
14	Equal	30°	0	0·5	Diagonal Springs No. 3	—	19·4	Binary divergence at 44·1	With statically overbalanced aileron, divergence occurred at same speed. With inter-aileron strut slightly lengthened and aileron mass balanced—range of very mild flutter from 24·6 to 37·8, followed by divergence at 42·2.
15	Upper plane extended	None	0	0	Straight Springs No. 1	16·2	19·0	None	

TABLE 15—(*continued*).

Expt. No.	General Condition of Model.					Critical Flutter Speed for Condition of Ailerons Stated.			Remarks.
	Planes.	Stagger (forward).	Distance struts from inner end upper aileron as fraction of aileron span.		Particulars of staying.	Hinge at 0·1 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. under-balanced.	Hinge at 0·2 chord from L.E. mass balanced.	
			Inter-plane.	Inter-aileron.					
16	Upper plane extended	None	0	0	Straight Springs No. 2	23·4	29·1	71·6	No flutter when aileron slightly overbalanced.
17	Upper plane extended	None	0	0	Straight Springs No. 3	31·5	40·5	None	
18	Upper plane extended	None	0	0	Struts fixed by taut wires	34·7	44·5	None	
19	Upper plane extended	None	0	0	Straight Springs No. 2	24·3	30·8	None	Second set of interplane struts fitted at middle of bay. These connected straight to tunnel wall by springs No. 3.
20	Upper plane extended	$6\frac{1}{2}°$	0	0	Straight Springs No. 1	—	18·7	None	
21	Upper plane extended	30°	0	0	Straight Springs No. 1	—	18·9	None up to 60	Wind speed not taken above 60 on account of large lift on overhang.

available. This emphasises very strongly the desirability of eliminating the participation of the ailerons in the flutter, either by means of an irreversible control or by the adoption of mass balance (*see* § 72).

All of the general conclusions from the experiments are in satisfactory agreement with the theory developed in Part I.

§ 72. *Influence of Mass Balance of the Ailerons.*—Tests under a wide variety of conditions were carried out in order to investigate the effect of the mass distribution of the ailerons upon the critical flutter speed, and the results are summarised in Table **15**. It will be seen that when the centres of gravity of the ailerons lay upon their hinge axes, flutter was in most cases totally avoided, and that when it did occur, the critical speeds were very high (in the neighbourhood of 70 ft. per sec.). When the ailerons were very slightly overbalanced (i.e. with the C.G. at 0·008 of the aileron chord ahead of the hinge axis) flutter never appeared. On the other hand, partial mass balance merely raised the critical speed slightly. (Compare the columns in the Table for the underbalanced ailerons hinged at 0·1 and 0·2 chord respectively. The comparison is of course complicated by the alteration of the aileron aerodynamical stiffness and other derivatives, but the general conclusion is clear). The differences in the results of experiments 2 and 3 of the Table are probably attributable to the reduction of the aileron products of inertia consequent upon removal of the interaileron strut towards the wing root (*see* § 70).

It is held that the experiments afford convincing evidence of the effectiveness of mass balance (or, preferably, slight overbalance) of the ailerons of biplanes in the avoidance of wing flutter.

§ 73. *Influence of Bracing.*—Before entry upon a discussion of the experiments on the influence of bracing, the methods used in the measurement of the flexural stiffness of the model will be explained. The arrangement used when the planes were equal is shown in Fig. 24. Preliminary tests were made to determine the position of the flexural centre at the wing tip. A stout thread was attached to the two wings at the flexural centres and passed over an almost frictionless calibration pulley. When weights were placed in the scale pan, purely flexural displacements of the wings resulted. Let

s = wing span measured from root to tip, in feet.
W = load in scale pan in lb.
δ_A = mean of displacements of A and A′ in feet.
δ_B = ,, ,, ,, B and B′ ,,
l_ϕ = flexural stiffness in lb. ft. per radian.

Then the angular flexural displacement at the wing tip is δ_A/s and the applied flexural moment is Ws. Hence the flexural stiffness is

$$l_\phi = \frac{Ws^2}{\delta_A}$$

The displacement ratio δ_B/δ_A serves to indicate the influence of the staying on the mode of deflection of the wings in static tests.

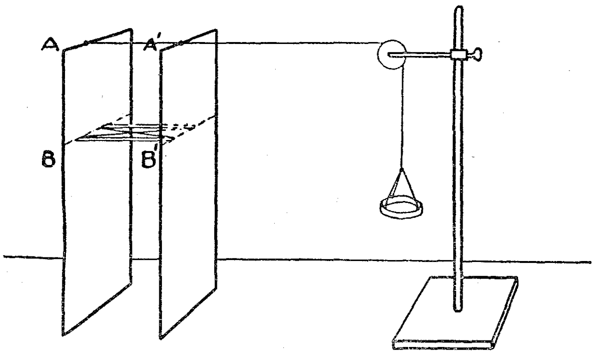

FIG. 24.—Method of Stiffness Measurement (upper and lower wings equal).

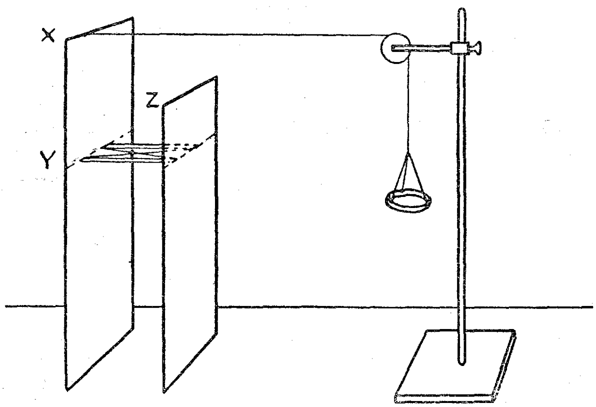

FIG. 25.—Method of Stiffness Measurement (upper wing extended).

When the upper wing was extended the load was applied entirely at the flexural centre of that wing at the tip. This flexural centre is defined by the condition that a normal load applied there produces no torsional displacement of the upper wing tip. In the actual tests such a load produced no perceptible torsion either at Y or Z (*see* Fig. 25). Here the flexural stiffness* is defined as

$$l'_\phi = \frac{W s_u^2}{\delta_x}$$

where s_u = span of upper wing measured from root
δ_x = displacement of X.

* The relation between the two measures of the flexural stiffness is given in Appendix 4 (*See* page 125).

TABLES 16, 17 and 18.

Influence of the Stiffness of Staying upon the Flutter Speed.

(Throughout, planes unstaggered and ailerons hinged at 0·1 aileron chord from aileron leading edge.)

Note.—The value of l_ϕ for the wings with interplane struts but no bracing is 12.

TABLE 16.

Interplane Struts at Inner End of Aileron.

Planes Equal. Interaileron Strut at mid-span of Aileron.

Expt. No.	Type of Staying.	Springs used.	l_ϕ	Deflection Ratio δ_B/δ_A	Critical Speed V_c	$\frac{V_c}{\sqrt{l_\phi}}$
1	diagonal ..	No. 1	25	0·34	11	2·2
2	diagonal ..	No. 2	42	0·27	14	2·16
3	diagonal ..	No. 3	64	0·18	18	2·25
4	straight ..	Taut Wires	85	0	30·5	3·31

TABLE 17.

Interplane Struts at Mid-span of Aileron.

Planes equal. Interaileron Strut at Mid-span of Aileron.

Expt. No.	Type of Staying.	Springs used.	l_ϕ	Deflection Ratio δ_B/δ_A	Critical Speed V_c	$\frac{V_c}{\sqrt{l_\phi}}$
1	diagonal ..	No. 3	160	0·50	26·2	2·07
2	diagonal ..	No. 4	243	0·31	36·8	2·36
3	straight ..	No. 4	370	0·13	72·0	3·75

TABLE 18.

Upper Plane Extended. All Struts at Inner End Upper Aileron.

Expt. No.	Type of Staying.	Springs used.	l'_ϕ	Deflection Ratios δ_Y/δ_X	Deflection Ratios δ_Z/δ_X	Critical Speed V_c	$\frac{V_c}{\sqrt{l'_\phi}}$
1	straight ..	No. 1	42	0·16	0·20	16·2	2·50
2	straight ..	No. 2	47	0·09	0·10	23·4	3·41
3	straight ..	No 3	53	0·03	0·03	31·5	4·33
4	straight ..	Taut Wires	58	0	0	34·7	4·56

The effect of bracing upon a wing structure may conveniently be considered under two heads :—

(1) Increase of the general rigidity of the structure.

(2) Modification of the mode of deflection.

The term " general rigidity " is rather vague, but will here be interpreted as the flexural stiffness as defined above. Changes in the mode of deflection are indicated by the variation of the deflection ratios δ_B/δ_A, δ_Y/δ_X and δ_Z/δ_X.

Tables 16, 17 and 18 give the results of some experiments on the influence of stiffness of staying upon the critical speeds. When all elastic stiffnesses increase proportionally and the mode of motion is unchanged, the ratio $V_c/\sqrt{l_\phi}$ is constant. A column giving the value of this ratio has been added to the tables and serves in a rough way to indicate the influence of change in the mode of motion on the critical flutter speed. The variation of the ratio is possibly in part due to changes in the ratio of the elastic stiffnesses in flexure and in torsion, but the exact analysis of the effect is impossible without complete knowledge of the dynamical constants of the system and will not be attempted here.

It will be seen that in all cases the ratio $V_c/\sqrt{l_\phi}$ did not vary widely until the relative stiffness of the staying became very great, as indicated by the smallness of the static deflection ratio δ_B/δ_A or δ_Y/δ_X; further, the value of the ratio $V_c/\sqrt{l_\phi}$ always increased as the stiffness of the bracing increased. Tables 16 and 18 show that (with a statically underbalanced aileron) when the interplane struts were situated near the inboard end of the upper aileron, flutter occurred at a moderate speed even when the bracing was completely rigid. On the other hand, when the line of strut centres bisected the aileron span (Table 17), the flutter speed was raised to the extreme limit of the tunnel range without the use of rigid bracing. In experiment No. 3 of Table 17 there was, in fact, a vibrational node at a short distance inboard from the struts, so that the conditions postulated in § 57 were closely approached. The experiments thus support the theoretical conclusion that flutter can be avoided by *sufficiently stiff* staying at mid-span of the aileron. However, it is improbable that the staying of a biplane can in practice be made sufficiently stiff to ensure that the node shall lie close to the struts. *Hence the adoption of interplane struts located at mid span of the ailerons cannot be considered as a safe alternative to mass balance of the ailerons.*

In the experiments hitherto described only a single pair of interplane struts were fitted, but the effects of additional staying within the bay are shown in Table 19. It was decided to proceed immediately to the case of rigid staying within the bay so that any effects might be exaggerated. Accordingly, in experiments 1 to 6 the selected ribs on the upper and lower planes were firmly fixed by three

TABLE 19.

Influence of Additional Staying within the Bay upon the Critical Flutter Speed.

Planes unstaggered and aileron hinge at 0·1 aileron chord from aileron leading edge.

Expt. No.	Planes.	Particulars of Original Staying.		Particulars of additional staying.	Critical Flutter Speed.		Remarks.
		Distance of struts from wing root. in.	Bracing.		Original Condition.	With additional staying.	
1	Equal	17·4	Springs No. 3 straight from interplane struts	Ribs at 6·1 in. from wing root fixed	41·3	40·0	Ribs fixed to tunnel wall by wooden struts
2	Equal	17·4	Springs No. 3 straight from interplane struts	Ribs at 8·1 in. ditto	41·3	41·5	Ribs fixed to tunnel wall by wooden struts
3	Equal	17·4	Springs No. 3 straight from interplane struts	Ribs at 10·1 in. ditto	41·3	39·2	Ribs fixed to tunnel wall by wooden struts
4	Equal	17·4	Springs No. 3 straight from interplane struts	Ribs at 12 in. ditto	41·3	39·3	Ribs fixed to tunnel wall by wooden struts
5	Equal	17·4	Springs No. 3 straight from interplane struts	Ribs at 14·1 in. ditto	41·3	36·0	Ribs fixed to tunnel wall by wooden struts
6	Equal	23·5	Springs No. 2 straight from struts	Ribs at 12 in. ditto	43·6	52·4	Ribs fixed to tunnel wall by wooden struts
7	Equal	23·5	Springs No. 2 straight from struts	Second set interplane struts at 12 in. from root stayed straight to wall through springs No. 4	43·6	48·8	
8	Upper extended	20·4	Springs No. 2 straight from struts	Second set struts at 10·2 in. Springs No. 3 straight	23·4	24·3	

pairs of wooden struts placed respectively between the wings themselves, and between each wing and the tunnel wall. The table shows that the influence of this additional staying upon the flutter speed was quite trifling and sometimes adverse.* The remaining tests were made with a second pair of interplane struts fitted at the middle of the bay. This strut panel was braced by incidence wires and by direct staying to the tunnel walls. Here again the influence on the critical speed was very slight. The conclusion drawn is that, while the stiffness of the wings at the outermost struts should be as great as possible, the precise manner in which this stiffness is attained is of secondary importance from the point of view of wing flutter.

§ 74. *Influence of Mass Loading within the Bay.*—Experiments were carried out to discover whether the variation of the mass of wing petrol tanks would be likely to cause large changes in the critical flutter speeds of biplanes. Table 20 gives a summary of the results and shows that the effect of even very great mass loading within the bay is slight. Similar results were obtained previously for a stayed monoplane wing.† Hence it may be concluded that variation of the mass of a wing petrol tank does not in general influence the flutter speed greatly.

TABLE 20.

Influence of Mass Loading within the Bay on the Flutter Speed.

Planes equal and unstaggered. Ailerons hinged at 0·1 chord from aileron leading edge. Additional mass loading of 537 grams attached to upper plane at 6 in. from root (middle of inner bay in experiment No. 3). (*Note.*—Mass of one wing and aileron only is 142 grams.)

Expt. No.	Particulars of Bracing.	Critical Flutter Speed.	
		Unloaded.	Loaded.
1	All struts at 17·4 in. from wing root. Springs No. 3 straight to tunnel wall.	41·3	40·4
2	As in expt. 1, but springs No. 4 ..	45·8	46·3
3	Outermost interplane struts and inter-aileron strut at 23·5 in. from root. This section stayed straight to tunnel wall through springs No. 2. Second set interplane struts at 12 in. from root stayed straight to wall through springs No. 4.	48·8	52·6

* In the experiments recorded the influence of the additional staying upon $l\varphi$ measured at the wing tip was not great. If the staying of the outermost struts had been so weak that the additional staying had made a large percentage alteration in $l\varphi$, the effect on the flutter speed would have been larger.

† *See* p. 178 of R. & M. 1155.

§ 75. *Influence of Stagger.*—The standard condition of the model was unstaggered, but a number of tests were made with a small forward stagger ($6\frac{1}{2}$ deg. or 1 in.) and with a large forward stagger (30 deg.). The critical speeds are given in Table 21, where they are compared with the corresponding figures for the unstaggered biplane. It will be seen that the influence of stagger is very slight, a result in accordance with the theory given in § 66. An anomalous effect of variation of the length of the interaileron strut was discovered in some of the experiments on the biplane with the upper wing extended and 30 deg. stagger. This effect will be discussed in § 77.

§ 76. *The Binary Divergence.*—Reference to the occurrence of binary divergence has already been made in § 71, and divergence speeds are recorded under Table 15, experiment 1, and Table 21, experiment 6. This phenomenon will now be considered in greater detail.

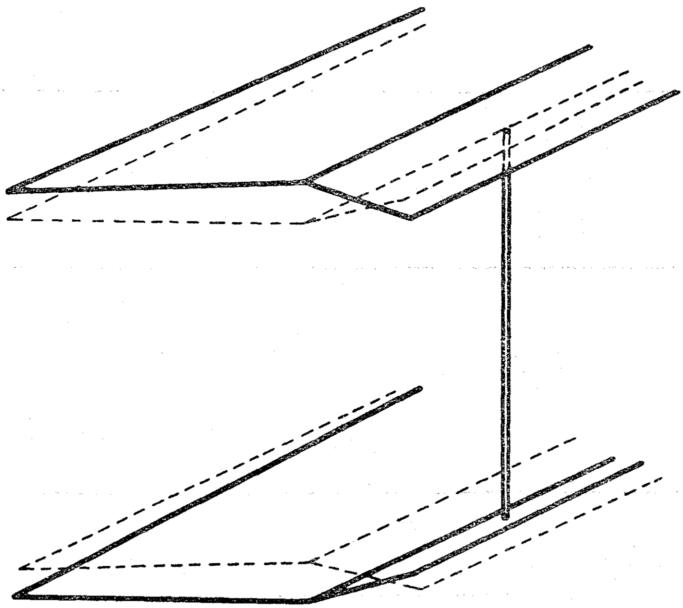

Fig. 26.—Diagrammatic Representation of Binary Divergence.

When the ailerons were mass balanced about their hinges and the interplane struts were situated at the inboard ends of the ailerons, it was found that the earliest instability was a divergence of the type illustrated diagrammatically in Fig. 26, where the full and dotted lines indicate the two alternative positions of equilibrium following the divergence from the initial condition with the ailerons in line with wings. When the length of the interaileron strut was properly adjusted, the divergence occurred suddenly at a speed which only

TABLE 21.

Influence of Stagger upon the Critical Speed.

Expt. No.	Planes.	Distance of struts from inner end of upper aileron as fraction of aileron span.		Particulars of Staying.	Distance of aileron hinge from aileron leading edge as fraction of aileron chord.	Critical speed with stagger as stated.			Remarks.
		Inter-plane.	Inter-aileron.			0°	$6\frac{1}{2}$°	30°	
1	Equal	0·25	0·25	Straight Springs No. 3	0·1	41·3	41·4	39·7	
2	Equal	0·25	0·25	Straight Springs No. 3	0·2	44·3	45·8	44·2	
3	Equal	0·25	0·25	Straight Springs No. 4	0·1	45·8	45·5	—	
4	Equal	0·25	0·25	Straight Springs No. 4	0·2	48·2	50·6	—	
5	Equal	0	0·5	Diagonal Springs No. 3	0·2	21·2	—	19·4	
6	Equal	0	0·5	Diagonal Springs No. 3	0·2	Binary divergence 49·2	—	Binary divergence 44·1	Ailerons mass balanced. *See* remarks expt. 14. Table 15.
7	Upper plane extended	0	0	Straight springs No. 1	0·2	19·0	18·7	18·9	

varied by a fraction of a foot per second in individual tests. On the other hand, when the length of the strut was slightly varied, the ailerons turned more gradually and the final positions were reached at rather lower speeds than before. A rather spasmodic and slow oscillation of the ailerons, always of the type in which the upper and lower ailerons moved in opposition, usually followed the divergence. Phenomena of the same type occurred even with 30 deg. stagger, but the divergence speed was rather lower (*see* Experiment 6 of Table 21).

The position of the interaileron strut in the span of the aileron has an important effect on the divergence speed, as shown in Table 22. It will be seen that the divergence speed rises as the

TABLE 22.

Influence of Position of Interaileron Strut upon the Binary Divergence Speed.

Planes equal and unstaggered. Interplane struts at inboard end of ailerons. Diagonal bracing with springs No. 3. Aileron hinges at 0·2 chord from aileron leading edge. Ailerons mass balanced.

Expt. No.	Distance interaileron strut from inner end of aileron as fraction of aileron span.	Binary divergence speed.	Remarks.
1	0	None	
2	0·5	49·2	Same divergence speed with statically overbalanced aileron.
3	1·0	38·6	Divergence at 39·3 with statically overbalanced aileron.

strut is moved inwards, and that there was no instability within the range of air speeds available when the interaileron strut was in line with the interplane struts. These results are in accord with the theory (*see* § 58). The influence of variation of the position of the interaileron strut in the chord of the aileron was not investigated on the biplane model, but the theoretical and experimental results quoted in § 58 show that the divergence speed is not sensitive to this variable within the practicable range.

Binary divergence certainly should not occur on a properly designed biplane, but there is at least one case on record where the ailerons of a biplane took up large angles to the wings whenever an attempt was made to operate them. Evidently here the divergence

TABLE 23.

Influence of Length of Interaileron Strut on Critical Speed.

Aileron hinged at 0·1 chord from leading edge throughout.

Expt. No.	Planes.	Stagger.	Distance struts from inner end upper aileron as fraction of aileron span.		Particulars of staying.	Critical Speed for Initial Aileron Angle.*				
			Inter-plane.	Inter-aileron.		See below.	—5°	0°	+5°	+10°
1	Equal	None	0·5	0·5	Straight Springs No. 2	35·4 at —8°	35·8	34·1	33·4	33·5
2	Equal	None	0·5	0·5	Straight Springs No. 3	57·4 at —7°	—	58·6	59·3	—
3	Upper plane extended	None	0	0	Straight Springs No. 3	32·5 at —7½°	31·5	30·7	31·5	33·5
4	Upper plane extended	30°	0	0	Straight Springs No. 3	32·4 at —10°	24·2	23·3	29·6	29·2

* Defined as downward angle of upper aileron when lower aileron at 0°. The angle was varied by alteration of the length of the interaileron strut.

speed was being approached. The divergence may be avoided by attention to the stiffness of the overhangs and by location of the interaileron strut in the same plane as the interplane struts.

§ 77. *Anomalous Effect of Variation of the Length of the Interaileron Strut.*—In certain tests of the model with the upper plane extended and a stagger of 30°, it was found that an adjustment of the length of the interaileron strut caused considerable variation of the critical flutter speed (*see* Expt. 4, Table 23). The explanation first suggested was that the initial setting of the ailerons slightly out of line with the wings had resulted in a variation of some of the aileron aerodynamical derivatives. However, this explanation had to be abandoned as the result of further experiments on the unstaggered model (*see* Expts. 1, 2 and 3 of Table 23). These showed that with zero stagger, variation of the length of the interaileron strut had a very slight effect upon the critical speed, so that large variations of the derivatives evidently did not occur.

The explanation of the effect which will now be suggested is that with a large stagger, the angular velocity ratio of the motions of the two connected ailerons depends upon the length of the interaileron strut. This alteration of the angular velocity ratio was exaggerated in the case of the model because the pivots of the interaileron strut were offset inwards from the centre lines of the ailerons to the extent of 0·2 in. A detailed examination of the kinematics of the arrangement is given in Appendix 5, and if it be admitted that the angular velocity ratio of the ailerons depends on the length of the inter-aileron strut, then a feasible explanation of the anomalies can be given. For, under the conditions of Expt. No. 4 of Table 23, the lower plane merely took up the motion impressed upon it by the interplane struts; likewise the lower aileron was virtually an appendage of the upper aileron. Clearly, therefore, what may be called " the equivalent dynamical coefficients " of the upper aileron depended on the angular velocity ratio of the ailerons. Consequently the critical speed depended upon the length of the interaileron strut.

An anomaly of a like nature is recorded under Expt. 14 of Table 15 and an explanation on similar lines to the above can be advanced.

APPENDICES TO CHAPTER III.

APPENDIX 1.

MODIFICATION OF THE EQUATIONS OF MOTION RESULTING FROM CHANGE OF THE SECTION AT WHICH THE AILERON ANGLE IS MEASURED.

Suppose that the co-ordinates φ, ξ and θ are originally all measured at one reference section and that the corresponding equations of motion are (in the condensed notation of §55 of the text) :—

$$A_1(\varphi) + D_1(\xi) + G_1(\theta) = 0 \,, \qquad \text{(1)}$$

$$A_2(\varphi) + D_2(\xi) + G_2(\theta) = 0 \,, \qquad \text{(2)}$$

$$A_3(\varphi) + D_3(\xi) + G_3(\theta) = 0 \,, \qquad \text{(3)}$$

Now let ξ' be the inclination of the aileron to the wing measured at some chosen section. On account of the twist of the wing between this section and the original reference section, ξ' will not be identical with ξ. The relation between these co-ordinates will be

$$\xi' = \xi + n\theta \,, \qquad \text{(4)}$$

where n is some proper fraction depending on the choice of the section. The equations of motion become by substitution :—

$$A_1(\varphi) + D_1(\xi') + (G_1 - n\,D_1)(\theta) = 0 \,, \qquad \text{(5)}$$

$$A_2(\varphi) + D_2(\xi') + (G_2 - n\,D_2)(\theta) = 0 \,, \qquad \text{(6)}$$

$$A_3(\varphi) + D_3(\xi') + (G_3 - n\,D_3)(\theta) = 0 \,, \qquad \text{(7)}$$

but they are not now in the symmetrical Lagrangian form. To obtain this form, (6) must be multiplied by n and subtracted from (7). The final equations of motion are

$$A_1(\varphi) + D_1(\xi') + (G_1 - n\,D_1)(\theta) = 0 \,, \qquad \text{(8)}$$

$$A_2(\varphi) + D_2(\xi') + (G_2 - n\,D_2)(\theta) = 0 \,, \qquad \text{(9)}$$

$$(A_3 - n\,A_2)(\varphi) + (D_3 - n\,D_2)(\xi') + (G_3 - n\,\overline{G_2 + D_3} + n^2 D_2)(\theta) = 0 \qquad \text{(10)}$$

The transformation is formally identical with that for change of the transverse reference axis, except that the rôles of the co-ordinates φ and ξ, and of the corresponding equations of motion, are interchanged. It can readily be shown (cp. R. & M. 1155, p. 39) that the determinant for ternary motion is absolutely invariant for transformations of the present type, and consequently that the coefficients in the determinantal equation and the test functions for stability are absolute invariants. The same is true for all the binary motions except flexural-torsional motion.

Suppose that originally the aileron control operated in the reference section and that it is desired to find the effect of operation of this control in some other section. Let ξ' be measured in this section and apply the transformation given above. The elastic control of the aileron at the reference section must be supposed removed, so that the term F_2 (which occurs in the operator $D_2(\;)$) is of purely aerodynamical origin. Finally, to allow for the operation of a control of stiffness $h\xi'$ at the new section, the term $\xi' h\xi'$ must be introduced in equation (9) only, for clearly, there are no aileron elastic cross-stiffnesses when the aileron angle is measured in the section at which the control operates. A change in the point of application of the aileron controls naturally does not leave the determinants of motion invariant (except in the single case of flexural-aileron motion).

APPENDIX 2.

STABILITY OF BINARY MOTION WITH SYMMETRICAL COUPLINGS.

Let the determinant of motion be

$$\Delta(\lambda) = \begin{vmatrix} \alpha_1\lambda^2+\beta_1\lambda+\gamma_1\ , & \alpha_2\lambda^2+\beta_2\lambda+\gamma_2 \\ \alpha_2\lambda^2+\beta_2\lambda+\gamma_2\ , & \delta_2\lambda^2+\varepsilon_2\lambda+\zeta_2 \end{vmatrix} = 0\,, \quad \ldots \quad (1)$$

or, when expanded,

$$q_0\lambda^4+q_1\lambda^3+q_2\lambda^2+q_3\lambda+q_4 = 0\,, \quad \ldots \quad (2)$$

where

$$q_0 = \alpha_1\delta_2 - \alpha_2^2\,, \quad \ldots \quad (3)$$
$$q_1 = \alpha_1\varepsilon_2 + \beta_1\delta_2 - 2\alpha_2\beta_2\,, \quad \ldots \quad (4)$$
$$q_2 = \alpha_1\zeta_2 + \gamma_1\delta_2 - 2\alpha_2\gamma_2 + \beta_1\varepsilon_2 - \beta_2^2\,, \quad \ldots \quad (5)$$
$$q_3 = \beta_1\zeta_2 + \gamma_1\varepsilon_2 - 2\beta_2\gamma_2\,, \quad \ldots \quad (6)$$
$$q_4 = \gamma_1\zeta_2 - \gamma_2^2. \quad \ldots \quad (7)$$

The necessary and sufficient criteria for stability, as laid down by Routh, are that all the coefficients q and T_3 shall be positive, where

$$T_3 = q_1q_2q_3 - q_0q_3^2 - q_4q_1^2\,. \quad \ldots \quad (8)$$

Consider first the coefficients q :—

$$\beta_1\delta_2q_1 = (\alpha_1\delta_2\beta_1\varepsilon_2 - \alpha_2^2\beta_2^2) + (\beta_1\delta_2 - \alpha_2\beta_2)^2\,, \quad \ldots \quad (9)$$
$$\gamma_1\delta_2q_2 = (\alpha_1\delta_2\gamma_1\zeta_2 - \alpha_2^2\gamma_2^2) + (\gamma_1\delta_2 - \alpha_2\gamma_2)^2 + \gamma_1\delta_2(\beta_1\varepsilon_2 - \beta_2^2)\,, \quad (10)$$
$$\gamma_1\varepsilon_2q_3 = (\beta_1\varepsilon_2\gamma_1\zeta_2 - \beta_2^2\gamma_2^2) + (\gamma_1\varepsilon_2 - \gamma_2\beta_2)^2 : \quad \ldots \quad (11)$$

Provided then that α_1, β_1, and γ_1 are positive and that

$$\alpha_1\delta_2 - \alpha_2^2 > 0\ , \quad \ldots \quad (12)$$
$$\beta_1\varepsilon_2 - \beta_2^2 > 0\ , \quad \ldots \quad (13)$$
$$\gamma_1\zeta_2 - \gamma_2^2 > 0\ , \quad \ldots \quad (14)$$

it follows that all the coefficients q must be positive. Now the expression $\alpha_1\delta_2 - \alpha_2^2$ is the discriminant of the kinetic energy and is necessarily positive for any actual physical system. Hence (13) and (14) are the effective conditions.

With regard to the test function T_3, consider first the case where the compound damping term β_2 is absent. A little manipulation leads to the result

$$T_3' = \beta_1\varepsilon_2\left\{(\alpha_1\varepsilon_2+\beta_1\delta_2)(\beta_1\zeta_2+\gamma_1\varepsilon_2) + (\alpha_1\zeta_2-\gamma_1\delta_2)^2\right\} + \left\{\alpha_2(\beta_1\zeta_2+\gamma_1\varepsilon_2) - \gamma_2(\alpha_1\varepsilon_2+\beta_1\delta_2)\right\}^2. \quad \ldots \quad (15)$$

The determinant (1) can be rewritten in the form

$$\begin{vmatrix} \alpha_1\lambda^2+\beta_1\lambda+\gamma_1 & ,\left(\alpha_2-\dfrac{\beta_2\alpha_1}{\beta_1}\right)\lambda^2+\left(\gamma_2-\dfrac{\beta_2\gamma_1}{\beta_1}\right) \\ \left(\alpha_2-\dfrac{\beta_2\alpha_1}{\beta_1}\right)\lambda^2+\left(\gamma_2-\dfrac{\beta_2\gamma_1}{\beta_1}\right), & \left(\delta_2-\dfrac{2\alpha_2\beta_2}{\beta_1}+\dfrac{\beta_2^2\alpha_1}{\beta_1^2}\right)\lambda^2+\left(\varepsilon_2-\dfrac{\beta_2^2}{\beta_1}\right)\lambda+\left(\zeta_2-\dfrac{2\gamma_2\beta_2}{\beta_1}+\dfrac{\beta_2^2}{\beta_1^2}\gamma_1\right) \end{vmatrix}.$$

The couplings are here symmetrical and the compound damping term is absent. Hence the modified coefficients can be substituted in the expression (15), and the final result after reduction is

$$\beta_1^2T_3 = (\beta_1\varepsilon_2 - \beta^2_2)\left[\beta_1^2q_1q_3 + \left\{\beta_1(\alpha_1\zeta_2-\gamma_1\delta_2) - 2\beta_2(\alpha_1\gamma_2-\alpha_2\gamma_1)\right\}^2\right] + \left[(\alpha_1\beta_2-\alpha_2\beta_1)q_3 + (\beta_1\gamma_2-\beta_2\gamma_1)q_1\right]^2. \quad \ldots \quad (16)$$

Evidently T_3 will be positive provided that the inequalities (13) and (14) are satisfied, and the final necessary and sufficient conditions for complete stability are that β_1 and γ_1 shall be positive and that (13) and (14) shall be valid.

Suppose that q_4 vanishes while the inequality (13) is still satisfied. From (9) and (11) it follows that q_1 and q_3 are positive and therefore, from (16), T_3 must be positive. Hence, so long as the damping coefficients satisfy the inequality (13), the earliest instability occurs when q_4 vanishes and is a divergence.

The above results may be of interest in relation to the theory of coupled electric circuits where the couplings are symmetrical unless thermionic valves are employed.

APPENDIX 3.

SOME EXPERIMENTS ON THE VIBRATION OF A SLENDER STEEL BEAM.

A steel beam of shallow rectangular section was clamped horizontally as a cantilever. A point on the beam towards the free end was connected to a rigid support through a vertical rubber band. On the application of a steady load at the tip of the beam there was found to be a large displacement of the point of attachment of the band. When, however, oscillation was permitted, it was found that when the point of attachment of the band lay within a certain range (in the neighbourhood of the natural node for the first overtone), this point became a flexural node after the initial two or three vibrations. On substitution of a steel spring of considerably greater stiffness for the rubber band there was no tendency for a node to be produced at the point of attachment. This result suggested that the property of the rubber which was responsible for the creation of the node was its hysteresis rather than its stiffness. To test this hypothesis, a circular disc of thin metal was suspended from the beam by a wire and arranged to be immersible in a dashpot at will. When the disc was not immersed there was of course no tendency to the production of a node, but a node almost instantly appeared on immersion of the disc, provided that the point of attachment of the wire lay in the range already mentioned. In other cases the dashpot soon damped out all motion.

APPENDIX 4.

RELATION BETWEEN TWO MEASURES OF THE FLEXURAL STIFFNESS.

In the case of a biplane with equal overhangs two methods of measuring the overall flexural stiffness might be used (*see* Figs. 1 and 2). In the first the load is equally divided between the upper and lower wing tips, while in the second it is wholly applied to the upper wing tip.

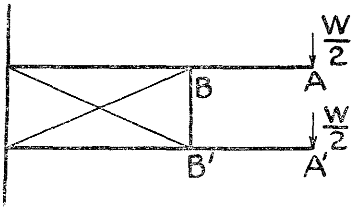

FIG. 1.

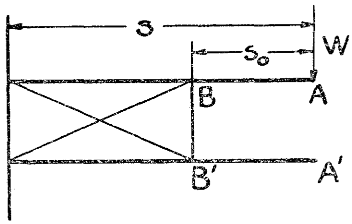

FIG. 2.

For the first case the flexural stiffness will be denoted as l_φ and the displacements of A, A′, B, and B′ will be denoted as δ_A, $\delta_{A'}$, δ_B, and $\delta_{B'}$ respectively. The same symbols with an accent will refer to the second case. Now in Fig. 2 the only load applied to the lower plane is at B′ and if the flexural rigidity be uniform (as for the model wing) it readily follows that

$$\frac{\delta'_{A'}}{\delta'_{B'}} = \frac{2\,s + s_0}{2(s - s_0)}\,, \qquad \text{(1)*}$$

or

$$\delta'_{A'} = \delta'_B \frac{2\,s + s_0}{2(s - s_0)}\,. \qquad \text{(1a)}$$

since

$$\delta'_{B'} = \delta'_B.$$

Consider now the conditions of Fig. 1. By the principle of superposition it follows that

$$\delta_A = \tfrac{1}{2}\,(\delta'_A + \delta'_{A'})\,, \qquad \text{(2)}$$

$$\delta_B = \delta'_B\,. \qquad \text{(3)}$$

Hence

$$\delta'_A = 2\,\delta_A - \delta'_{A}{}' = 2\,\delta_A - \delta_B\,\frac{2\,s + s_0}{2(s - s_0)}$$

$$= \delta_A\left[2 - \left(\frac{\delta_B}{\delta_A}\right)\frac{2\,s + s_0}{2(s - s_0)}\right],$$

and $$\frac{l_\varphi}{l'_\varphi} = \frac{\delta'_A}{\delta_A} = 2 - \left(\frac{\delta_B}{\delta_A}\right)\frac{2\,s + s_0}{2(s - s_0)}\,. \qquad \text{(4)}$$

As an example, take Expt. 2 of Table 16. The data are :—

$s = 2{\cdot}25 \qquad s_0 = 1{\cdot}033$

$l_\varphi = 42 \qquad \delta_B/\delta_A = 0{\cdot}27.$

Then equation (4) gives $l'_\varphi = 30{\cdot}2$.

* In the deduction of this formula it is assumed that the load at B′ is a pure normal force.

APPENDIX 5.

DEPENDENCE OF THE ANGULAR VELOCITY RATIO OF THE AILERONS UPON THE LENGTH OF THE INTERAILERON STRUT.

(*See* § 77).

If ABCD be a four bar mechanism, then, as is well known,

$$\frac{\omega_A}{\omega_D} = \frac{ED}{EA}$$

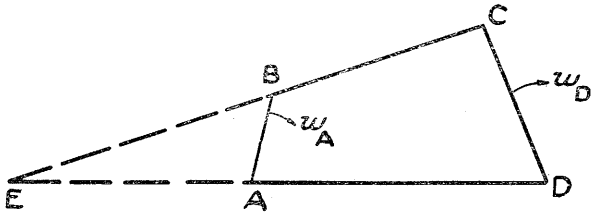

FIG. 1.

This relation has been applied to find ϱ, the ratio of the angular velocity of the lower aileron to that of the upper aileron. The pivots of the strut were offset inwards from the centre lines of the ailerons by 0·2 in., the distance between the aileron axes was 9 in., and the stagger 30°. The ratio is given in the following table for two positions of the aileron hinge and for the lower aileron at zero angle to the wing. Actually, in the wind, the latter angle will not be zero and may vary somewhat with the wind speed.

Setting of upper aileron.	Value of ϱ.	
	Hinge at 0·1 chord.	Hinge at 0·2 chord.
— 10°	0·94	0·93
0°	0·82	0·80
+ 10°	0·67	0·64

With zero stagger the value of ϱ does not differ appreciably from unity.

It will be seen that the variations of ϱ are considerable and they appear adequate to explain the observed effects, although a thorough analysis has not been attempted. The ratio tends towards unity for large negative angles and Table 23 of the text shows that under this condition the critical speed is normal.

CHAPTER IV.

THE FLUTTER OF AEROPLANE TAILS.*

By R. A. FRAZER, B.A., B.Sc., and W. J. DUNCAN, D.Sc., A.M.I.Mech.E.

PART I.

THEORETICAL INVESTIGATION.

§78. *Previous Investigations.*—In R. & M. 276† an analysis is given of the oscillations of the tail of an aeroplane in flight, the degrees of freedom assumed for the dynamical system being angular movement of the elevators about their hinges, and torsion of the fuselage. Amongst the measures suggested in that report for the elimination of flutter are :—(1) Connection of the two elevators by a tube stiff in torsion ; (2) Introduction of artificial elevator damping.

The oscillations of a tailplane in flexure and torsion are examined by an approximate theory in Report No. 285‡ of the National Advisory Committee for Aeronautics. Location of the centre of mass in, or forward of, the main supporting spar with the centre of pressure aft of this member is recommended—or, alternatively, an increase of structural rigidity.

A recent paper by Bolas§ deals with the static distortion of elastic tailplanes, and provides a basis for the calculation of the flexural-torsional divergence speed.

§79. *Range of Present Investigation.*—In the present chapter the problem of tail flutter will be treated by methods strictly analogous to those used for wing flutter in R. & M. 1155.‖ The underlying principle is the substitution of semi-rigid counterparts for such portions of the moving system as are likely to distort appreciably under the acting loads. For simplicity, only the tailplane, fin, and fuselage will be dealt with in this way, while elevators and rudder will be treated as rigid. A further limitation which will be imposed is that the only important motion of the fuselage is torsional.

These preliminary assumptions still leave the problem much too general for a detailed analysis. Nevertheless, before further simplifications are introduced, an explicit statement of the admissible

* Originally issued as R. & M. 1237 (Ref. 14).
† Ref. 2. ‡ Ref. 15. § Ref. 10. ‖ Ref. 1.

degrees of freedom will be useful. They may be classified as follows :—

(1) Fuselage twist Ω.
(2) Elevator* angle ξ_s (starboard), ξ_p (port).
(3) Flexure of tailplane ϕ_s (starboard), ϕ_p (port).
(4) Torsion of tailplane θ_s (starboard), θ_p (port).
(5) Rudder* angle ξ'.
(6) Flexure of fin ϕ'.
(7) Torsion of fin θ'.

The adoption of an irreversible type of control for the elevators and rudder would dispose of three important degrees of freedom, and the only possible flutter in that case would be one involving flexure and torsion of the tailplane and fin, and twist of the fuselage. Provided that the construction is reasonably stiff, the critical speed for that type of flutter would be expected to be very high. The advantages of irreversibility of control are emphasised in R. & M. 1155 in relation to the ailerons.

With the conventional construction and method of operation of the control surfaces two types of tail flutter are possible under the assumptions already made :—

(*a*) " *Symmetrical* " *Tail Flutter.*—In the first, the port and starboard elevators move up and down in step, the fuselage does not twist, and the rudder does not turn about its hinge. Tailplane flexibility is in general, essential for the production of this type of flutter,† which is the direct analogue of " ternary " wing flutter when the aileron is free from elastic constraint. Clearly, the recommendations of R. & M. 1155 regarding the prevention of flutter merely require reinterpretation to meet this new case.

(*b*) " *Antisymmetrical* " *Tail Flutter.*—The second type of tail flutter involves twist of the fuselage and, in general, oscillation of the rudder and of the two elevators. In this case the elevators move in opposition and are therefore subject to an elastic constraint. Flexibility of the tailplane itself is not essential for the occurrence of this type of flutter. However, if the flexibilities of the tailplane and of the fin are also included, the most general motion under the present heading embraces seven degrees of freedom. The disappearance of three of the original complement of ten coordinates arises from the circumstance that the port and the starboard displacements of tailplane and elevator (measured relative to the fuselage) are in exact opposition and equal in magnitude.

The investigation will be restricted mainly to an examination of conditions for the avoidance of tail flutter of the antisymmetrical type (*b*).

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* The standard symbols for elevator and rudder angle are η and ζ, respectively, but the use of these symbols here would tend to some confusion in the sequel.

† A preliminary condition, which will be assumed satisfied throughout the present report, is that the direct dampings involved are positive.

§80. *The Dynamical Equations.*—Let Y Y′ be the transverse reference axis of the starboard tail plane, Z Z′ the vertical reference axis in the fin, and X X′ the axis about which the fuselage twists (*see* Fig. 27). For simplicity it will be assumed that the axis of twist X X′ is at least approximately parallel to the chord of the tailplane. The flexural coordinate ϕ of the tailplane will be defined as the downward linear displacement of Y′ measured relative to axes fixed in the fuselage, divided by the tailplane semi-span Y Y′. Similarly, the flexural coordinate ϕ' of the fin will be the linear displacement of Z′ (to starboard) measured relative to the fuselage, divided by Z Z′.

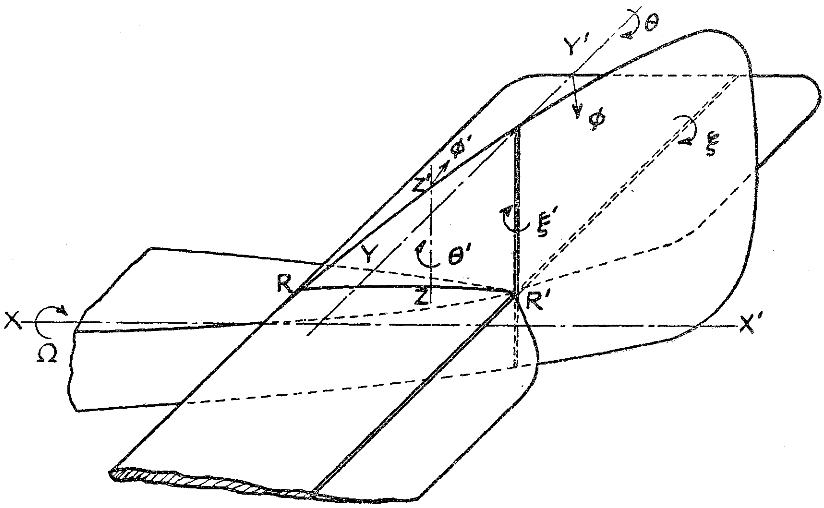

FIG. 27.—Diagram of Tail Unit.

As a preliminary to the statement of the equations of motion, it may be observed that tailplane or elevator movements cannot directly produce moments tending to change the fin or the rudder coordinates Reciprocally, no compound tailplane or elevator moments arise when fin or rudder movements occur. In other words, the two coordinate groups (ϕ, θ, ξ) and (ϕ' θ' ξ') can be treated as having only an indirect coupling due to twist of the fuselage. It will be supposed that the angles of incidence of the tailplane and fin are small. Then a pure twisting displacement Ω of the fuselage will produce no moments tending to alter any of the other coordinates. Similarly, no changes of aerodynamical moments will be produced by pure flexural displacements of the tailplane or fin. In view of these considerations the equations of motion may be exhibited schematically as in Table 24. The physical significance of the various coefficients will be obvious on inspection. Thus, typically the symbol A_1 represents the tailplane flexural moment per unit tailplane flexural acceleration (i.e., the tailplane flexural moment of inertia) ;

and, as a further example, E_0 denotes the twisting moment on the complete tail per unit angular velocity of the elevators (i.e., E_0 is the fuselage compound damping due to the elevators).

The determinant of motion $\Delta(\lambda)$ corresponding to the system of equations of Table 24 is most concisely presented as follows. Let expressions of the type $A_1\lambda^2 + B_1\lambda + C_1$ be written in the condensed form $A_1(\lambda)$; then

$$\Delta(\lambda) \equiv \begin{vmatrix} A_1(\lambda) & D_1(\lambda) & G_1(\lambda) & P_1(\lambda) & 0 & 0 & 0 \\ A_2(\lambda) & D_2(\lambda) & G_2(\lambda) & P_2(\lambda) & 0 & 0 & 0 \\ A_3(\lambda) & D_3(\lambda) & G_3(\lambda) & P_3(\lambda) & 0 & 0 & 0 \\ A_0(\lambda) & D_0(\lambda) & G_0(\lambda) & P_0(\lambda) & G_0'(\lambda) & D_0'(\lambda) & A_0'(\lambda) \\ 0 & 0 & 0 & P_3'(\lambda) & G_3'(\lambda) & D_3'(\lambda) & A_3'(\lambda) \\ 0 & 0 & 0 & P_2'(\lambda) & G_2'(\lambda) & D_2'(\lambda) & A_2'(\lambda) \\ 0 & 0 & 0 & P_1'(\lambda) & G_1'(\lambda) & D_1'(\lambda) & A_1'(\lambda) \end{vmatrix} \quad \ldots (158)$$

The coupling between the combination of tailplane and elevator, and the combination of fin and rudder, due to the fuselage twist, is represented by the terms in the fourth row and fourth column of (158).

In the sequel the dynamical coordinates will be supposed defined in a special manner which secures some simplification of the analysis. So far the two reference axes Y Y′ and Z Z′, used to define flexure of the tailplane and fin, respectively, have been left arbitrary. It will be convenient to adopt as their positions the flexural axes of the two surfaces concerned. Thus Y′ will be taken as that point of the reference section of the tailplane to which normal loads can be applied without the development of twist of that section; and a similar property will define the position of the point Z′. When the two flexural coordinates ϕ and ϕ' are defined in this way, the flexural-torsional elastic cross-stiffness of each stabilising surface vanishes, so that the coefficients C_3 and C_3' in Table 24 are zero. Again, the elevator angle ξ and the rudder angle ξ' will be assumed measured at the sections where the respective controls operate. In this case no elastic couplings will be introduced by the controls and the coefficients C_2, C_2' therefore vanish.

§81. *Stability of the Antisymmetrical Motion when Special Design Conditions are Observed.*—In the monograph R. & M. 1155 dealing with wing flutter, it is shown that when the design satisfies certain conditions, ternary wing motion can be resolved approximately into a damped flexural oscillation and a motion of the torsional-aileron type whose only instability is a divergence. It will now be assumed that the analogous conditions can be realised for the combinations consisting of tailplane and elevator, and of fin and rudder. Without a detailed recapitulation, it may be stated that one of the essential theoretical requirements is a suppression of all couplings due to the flexural motions. In the case of the tailplane and elevator the coefficients A_2, B_2, A_3, B_3, would strictly all have to be absent.*

* The coefficients C_2, C_3, have already been supposed eliminated (*see* end §80).

TABLE 24.

Schematic Representation of the Equations of Motion for Antisymmetrical Tail Flutter.

(A vacant space in the table indicates that the corresponding moment is absent.)

	Tailplane Flexure.	Elevator Angle.	Tailplane Torsion.	Fuselage Twist.	Fin Torsion.	Rudder Angle.	Fin Flexure.
	$\ddot{\phi}$ $\dot{\phi}$ ϕ	$\ddot{\xi}$ $\dot{\xi}$ ξ	$\ddot{\theta}$ $\dot{\theta}$ θ	$\ddot{\Omega}$ $\dot{\Omega}$ Ω	$\ddot{\theta}'$ $\dot{\theta}'$ θ'	$\ddot{\xi}'$ $\dot{\xi}'$ ξ'	$\ddot{\phi}'$ $\dot{\phi}'$ ϕ'
Flexural moments (L) on tailplane*	A_1 B_1 C_1	D_1 E_1 F_1	G_1 J_1 K_1	P_1 Q_1 —	— — —	— — —	— — —
Elevator hinge moments (H)*	A_2 B_2 C_2	D_2 E_2 F_2	G_2 J_2 K_2	P_2 Q_2 —	— — —	— — —	— — —
Torsional moments (M) on tailplane*	A_3 B_3 C_3	D_3 E_3 F_3	G_3 J_3 K_3	P_3 Q_3 —	— — —	— — —	— — —
Torsional moments on complete tail (T) ..	A_0 B_0 —	D_0 E_0 F_0	G_0 J_0 K_0	P_0 Q_0 R_0	G_0' J_0' K_0'	D_0' E_0' F_0'	A_0' B_0' —
Torsional moments (M′) on fin	— — —	— — —	— — —	P_3' Q_3' —	G_3' J_3' K_3'	D_3' E_3' F_3'	A_3' B_3' C_3'
Rudder hinge moments (H′)	— — —	— — —	— — —	P_2' Q_2' —	G_2' J_2' K_2'	D_2' E_2' F_2'	A_2' B_2' C_2'
Flexural moments on fin (L′)	— — —	— — —	— — —	P_1' Q_1' —	G_1' J_1' K_1'	D_1' E_1' F_1'	A_1' B_1' C_1'

* Actually *twice* the moments on starboard member: the duplication allows for the port members.

The typical equation of motion is to be written at length as follows :—

Flexural moments on tailplane. $A_1\ddot{\phi} + B_1\dot{\phi} + C_1\phi + D_1\ddot{\xi} + E_1\dot{\xi} + F_1\xi + G_1\ddot{\theta} + J_1\dot{\theta} + K_1\theta + P_1\ddot{\Omega} + Q_1\dot{\Omega} = 0$ .. .. (157)

The two inertial coefficients A_2, A_3, can be made to vanish by suitable distribution of the masses of the tailplane and of the elevator. A special location of the elevator hinge axis would be required for the elimination of B_2, and coincidence of the flexural axis of the tailplane with the axis of independence* would ensure the vanishing of B_3; however, the indications from wing flutter theory are that these two restrictions are not of the first importance. Since, with a normally constructed tail unit, twisting motions of the fuselage and flexural motions of the tailplane have a similar influence on the moments acting on the tailplane and elevator,† it is fair to assume that the coupling coefficients P_2, P_3, Q_2, Q_3, will vanish—or at least be negligibly small—when the corresponding coefficients A_2, A_3, B_2, B_3 are zero. Similar considerations apply to the fin and rudder, and it will accordingly be assumed that the design can be arranged so that neither torsional moments on the fin nor rudder hinge moments are produced by twist of the fuselage or flexure of the fin.

When the design accords with the foregoing conditions, the determinant of motion (158) simplifies on account of the vanishing of the elements $A_2(\lambda)$, $A_3(\lambda)$, $P_2(\lambda)$, $P_3(\lambda)$, $A_2'(\lambda)$, $A_3'(\lambda)$, $P_2'(\lambda)$ and $P_3'(\lambda)$. It is readily seen that the determinant now resolves into the following product :—

$$\begin{vmatrix} D_2(\lambda), & G_2(\lambda) \\ D_3(\lambda), & G_3(\lambda) \end{vmatrix} \begin{vmatrix} D_2'(\lambda), & G_2'(\lambda) \\ D_3'(\lambda), & G_3'(\lambda) \end{vmatrix} \begin{vmatrix} A_1(\lambda), & P_1(\lambda), & 0 \\ A_0(\lambda), & P_0(\lambda), & A_0'(\lambda) \\ 0, & P_1'(\lambda), & A_1'(\lambda) \end{vmatrix} \qquad (159)$$

The first factor of (159) represents a binary motion involving torsion of the tailplane and movement of the elevators ; whilst the second factor corresponds to a motion whose constituents are torsion of the fin and movement of the rudder. Both motions are analogous to the torsional-aileron motion discussed at length in §56 of R. & M. 1155. Hence it may be inferred that no flutter will arise due to the binary factors considered provided that the following conditions are satisfied in addition to those already imposed :—

(*a*) All direct dampings large.
(*b*) Elevators and rudder definitely underbalanced aerodynamically.
(*c*) Moments of inertia of elevators and rudder about their hinges small.

It remains to deal with the ternary factor of (159), which when written in full is

$$\begin{vmatrix} A_1\lambda^2+B_1\lambda+C_1, & P_1\lambda^2+Q_1\lambda, & 0 \\ A_0\lambda^2+B_0\lambda, & P_0\lambda^2+Q_0\lambda+R_0, & A_0'\lambda^2+B_0'\lambda \\ 0, & P_1'\lambda^2+Q_1'\lambda, & A_1'\lambda^2+B_1'\lambda+C_1' \end{vmatrix} \qquad (160)$$

* For definition, *see* R. & M. 1155, p. 15.

† Even when the axis of twist of the fuselage is situated at an appreciable distance below the tailplane, the lateral motion due to the twist has a negligible influence on the moments considered.

The constituents of this motion are flexure of the tailplane, torsion of the fuselage, and flexure of the fin. Since stiffness couplings are entirely absent, the equation of energy takes the simple form (*see* §21 of R. & M. 1155)

$$\frac{d}{dt}(\mathbf{T}+\mathbf{V})+2\mathbf{F}=0, \quad \ldots \quad \ldots \quad (161)$$

where $\mathbf{T}$ is the kinetic energy, $\mathbf{V}$ is the necessarily positive elastic potential, and $\mathbf{F}$ is the dissipation function. In the Appendix to the present chapter it is shown that the dissipation function is necessarily positive in the present case (at least for small angles of incidence), and it may immediately be deduced that the ternary motion in question is necessarily stable.

The general conclusion is that the adoption of the design recommendations of R. & M. 1155 (judiciously interpreted in relation to the components of the tail unit) would suffice to prevent flutter of the antisymmetrical type, irrespective of the value of the torsional stiffness of the fuselage.

§82. *Divergent Instability*.—The discussion of divergence for the most general type of motion presents no difficulty. At a divergence speed the term independent of λ in the determinantal equation vanishes (*see* §17 of R. & M. 1155). Hence, on reference to equation (158), the condition for the case of antisymmetrical motion is

$$C_1C_1'R_0 \begin{vmatrix} F_2 & K_2 \\ F_3 & K_3 \end{vmatrix} \begin{vmatrix} F_2' & K_2' \\ F_3' & K_3' \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (162)$$

Since the coefficients C_1, C_1', R_0 are purely of elastic origin and therefore necessarily positive, divergence occurs when either

$$\begin{vmatrix} F_2 & K_2 \\ F_3 & K_3 \end{vmatrix} = 0, \quad \ldots \quad \ldots \quad \ldots \quad (163a)$$

or

$$\begin{vmatrix} F_2' & K_2' \\ F_3' & K_3' \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (163b)$$

It is readily seen that the condition for divergence in the symmetrical motion is

$$C_1 \begin{vmatrix} F_2 & K_2 \\ F_3 & K_3 \end{vmatrix} = 0,$$

which is effectively identical with (163a).

The foregoing discussion shows that, as far as concerns divergences, the tailplane and elevator, and the fin and rudder, can be treated as independent combinations. It will suffice, therefore, to consider in detail merely one of these pairs—the tailplane and elevator, for instance. The divergence speed in this case is given by the equation (163a), which may be written in the form

$$\begin{vmatrix} h_\xi - H_\xi, & -H_\theta \\ -M_\xi, & m_\theta - M_\theta \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (164)$$

Here h_ξ denotes the elastic stiffness of the elevators, and m_θ the torsional elastic stiffness of the tailplane *measured relative to the flexural axis.* The remaining symbols express, in the usual notation, the relevant aerodynamical derivatives : thus, $-H_\xi$ is the rate of change of elevator hinge moment with elevator angle, $-M_\theta$ is the aerodynamical torsional stiffness of the tailplane, and the remainder are compound aerodynamical stiffnesses.

A discussion of equations of the type (164) is given in §53(c) of R. & M. 1155, where it is shown that an increase of h_ξ always raises the divergence speed. Thus, the divergence speed will be lowest when $h_\xi = 0$ (corresponding to the condition of symmetrical tail flutter, as defined in §79 of the present chapter) and will then be given by the equation

$$V_d = \sqrt{\frac{m_\theta\,(-H_\xi/V^2)}{\left[(-H_\theta/V^2)\,(-M_\xi/V^2) - (-M_\theta/V^2)\,(-H_\xi/V^2)\right]}} \quad \ldots (165)$$

Since the elevators may be assumed underbalanced, the derivative $-H_\xi/V^2$ is positive; hence, divergence will occur unless the denominator of (165) is negative. From numerical illustrations given in R. & M. 1155 it appears likely that the conditions for the avoidance of divergence would only be realised if the flexural axis of the tailplane lay very close to the leading edge of the tailplane. This conclusion is supported by the numerical example which follows on p. 135 (*see* also Fig. 28).

Divergent instability of tailplanes has been discussed by H. Bolas,* but his treatment is very different from that adopted in the present report. His fundamental assumptions can be summarised as follows :—

(1) The tailplane is supposed supported by uniform elastic spars, and the flexural and torsional stiffnesses are entirely attributed to the *flexural* rigidities of these two spars.†

(2) The elevator is supposed to remain at a fixed angle to the fuselage during the motion of the tailplane.

(3) It is tacitly assumed that each fore-and-aft strip of the elevator transmits its aerodynamical load to the corresponding strip of the tailplane.

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* Ref. 10.

† A discussion of the flutter of a wing without aileron based on the same assumption has been given by S. B. Gates (Ref. 9) ; *see also* Chapter II.

It has been considered instructive to apply the formula (165) to the tailplane whose divergence speed is calculated by Bolas. The particulars of this tailplane are :—

Overhang, fuselage side to tip . .	5 ft.
Total chord	4 ft.
Distance from nose to front spar	0·875 ft.
Distance between spars . .	1·39 ft.
Flexural rigidity of each spar . .	$8{\cdot}72 \times 10^6$ (units—length in inches and force in pounds).

The flexural axis will here be supposed to lie midway between the equal spars (i.e., at 0·392 chord from the leading edge), and the torsional stiffness m_θ (assumed entirely due to the flexural stiffnesses of the spars) works out as 1,400. There is considerable uncertainty as to the values of the aerodynamical derivatives, but these have been estimated from the results obtained for a cantilever model wing of 27 in. span (*see* R. & M. 1155) ; allowance has been made for the different proportions of the parts.

TABLE 25.

Aerodynamical Derivatives for Tailplane and Elevator.

(*Coefficients referred to leading edge.*)

Derivative Coefficient.	Value.
$-M_\theta/V^2$	19×10^{-3}
$-L_\theta/V^2$	95×10^{-3}
$-M_\xi/V^2$	41×10^{-3}
$-L_\xi/V^2$	84×10^{-3}
$-H_\xi/V^2$	6×10^{-3}
$-H_\theta/V^2$	2×10^{-3}

The divergence speed has been calculated for the actual position of the flexural axis, and for other positions both fore and aft of this, the torsional stiffness m_θ being taken as constant. Values of the aerodynamical derivatives appropriate to the several axes have been deduced from the values tabulated by means of the transformation formulæ given in Table 6 of R. & M. 1155. Finally, in order to allow for the effect of interference due to the fuselage, etc., all the derivatives have been reduced by 35 per cent., which is the figure adopted by Bolas. The results of the calculations are exhibited in Fig. 28, where the divergence speed is plotted against the position of the flexural axis. For the actual position of this axis the divergence speed is 367 ft. per sec., which agrees closely with the figure of 356 calculated by Bolas. In view of the many uncertainties in the data, and of the differences in the fundamental assumptions, this close agreement should be viewed as accidental.

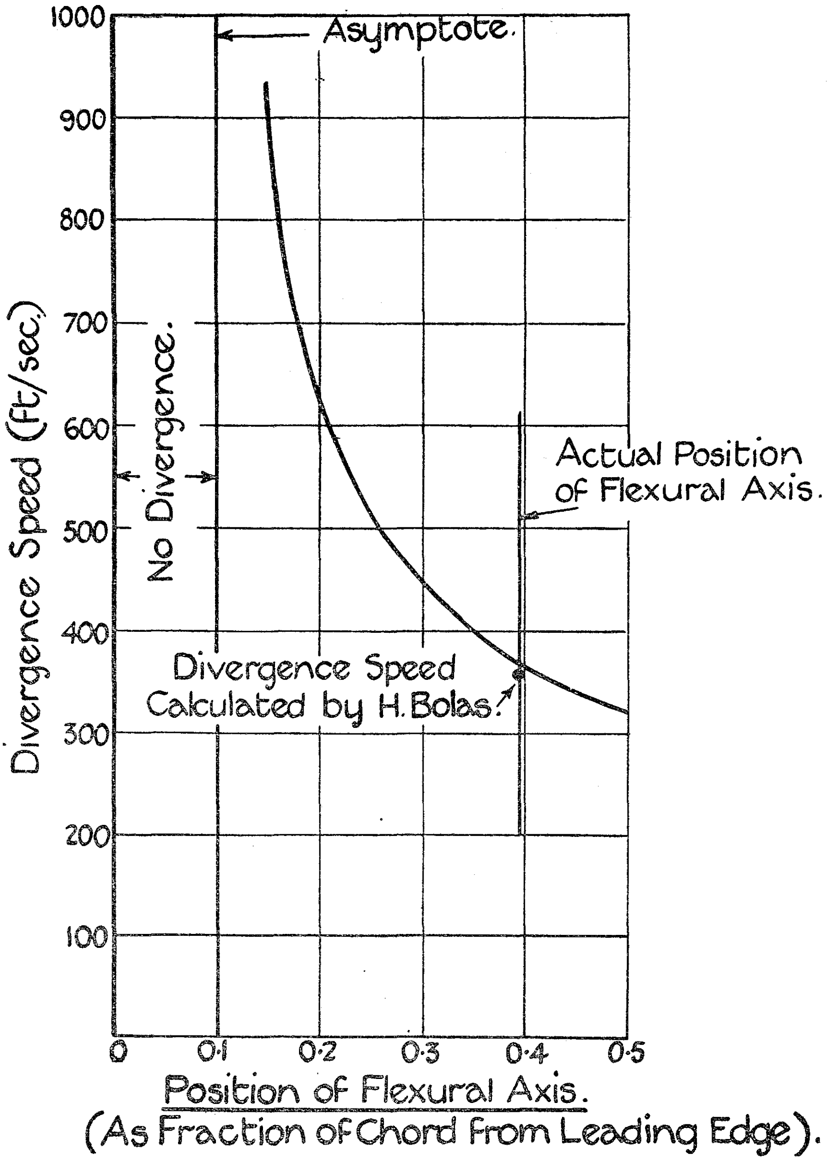

Fig. 28.—Divergence Speed for a Particular Tailplane.

It will be seen from Fig. 28 that no divergence would occur if the flexural axis were placed at 0·1 chord from the leading edge. This position is probably not practicable, but the diagram shows clearly the advantage from the standpoint of divergence of a flexural axis placed as far forward as possible.

§83. *Antisymmetrical Flutter when Tailplane and Fin can be Treated as Rigid.*—When the tailplane and fin are both extremely stiff the important motion is tenary and involves only elevator and rudder movements and fuselage twist. The determinantal equation corresponding to (158) now reduces to

$$\begin{vmatrix} D_2(\lambda), & P_2(\lambda), & 0 \\ D_0(\lambda), & P_0(\lambda), & D_0'(\lambda) \\ 0, & P_2'(\lambda), & D_2'(\lambda) \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (166)$$

Inspections shows immediately that this determinant would resolve into a simple and a binary factor if either of the two coupling terms $P_2'(\lambda)$ or $P_2(\lambda)$ were to vanish.* When $P_2'(\lambda)$ vanishes, the factors correspond to an independent rudder motion and a binary elevator-fuselage motion; whereas when $P_2(\lambda)$ vanishes, the constituents are an independent elevator motion and a binary rudder-fuselage motion. The two cases will now be discussed under separate headings.

§84a. *Tailplane and Fin Rigid and Rudder Motion Independent.*—Independence of the rudder motion would strictly be ensured if the rudder were symmetrically bisected by the axis of fuselage twist XX′, as indicated in Fig. 29. Clearly, in this case no rudder hinge moments

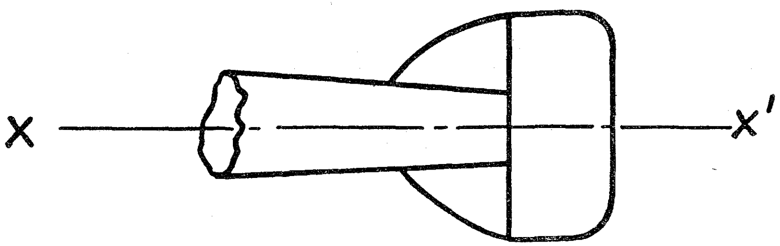

FIG. 29.—Diagram of Symmetrical Rudder.

whatever could be produced by twisting movements of the fuselage,† so that both coefficients P_2' and Q_2' would vanish. In actual practice however, it is rarely convenient to adopt a symmetrically disposed rudder. Nevertheless, even for the usual design of rudder, the product of inertia P_2' can be made to vanish by suitable mass loading. The aerodynamical coefficient Q_2' measures the rudder hinge moment due to unit twisting velocity of the fuselage, and could always be made to vanish by choice of a special hinge position.

* The resolution can, of course, be effected in other ways, but less conveniently.

† It is here assumed that the masses of the rudder are also symmetrically disposed about XX′.

This hinge position may be near to the position for complete aerodynamical balance of the rudder, and therefore prove objectionable from other points of view* (cp. §92). However, some tolerance on the hinge position is certainly allowable, since an exact elimination of both P_2' and Q_2' is not actually essential for stability.

When the design is such that both P_2' and Q_2' vanish, the determinant (166) resolves into the factors $D_2'(\lambda) = 0$ (representing the independent motion of the rudder), and the binary factor :—

$$\begin{vmatrix} D_2(\lambda) & P_2(\lambda) \\ D_0(\lambda) & P_0(\lambda) \end{vmatrix} = 0. \quad .. \quad .. \quad (167)$$

This corresponds to the coupled motion of the elevators and fuselage—a type already considered in R. & M. 276.† A re-examination of this case may be useful, to include the influence of the elevator product of inertia $D_0 = P_2$, which term was neglected in the analysis of that report.

The determinant (167) is exactly analogous to the determinant for the "flexural aileron" motion investigated in R. & M. 1155, and the graphical representation of the stability criteria there developed may be adopted. For the application of this method, it will be convenient to introduce a supplementary set of symbols to denote the non-dimensional equivalents of the dynamical coefficients D_2, E_2, etc. The full list is given in Table 26, where ρ denotes the air density, S the tailplane area, s the tailplane semi-span, t_Ω the torsional stiffness of the fuselage, and h_ξ the elastic stiffness of the elevators in antisymmetrical motion.

TABLE 26.

Coefficients for the Elevator-Fuselage Motion.

Fuselage Twisting Moments T.			Elevator Hinge Moments H.		
Coeff.	Significance.	Non-dimensional Form.	Coeff.	Significance.	Non-dimensional Form.
P_0	Inertia	$\rho\, Ss^3 p_0$	P_2	Inertia	$\rho\, Ss^3 P$
Q_0	$-T_{\dot\Omega}$	$\rho\, V\, Ss^2 q_0$	Q_2	$-H_{\dot\Omega}$	$\rho V\, Ss^2 q_2$
R_0	t_Ω	$\rho V^2 Ss\left(\dfrac{t_\Omega}{\rho V^2 Ss}\right)$	R_2	0	0
D_0	Inertia	$\rho\, S s^3 P$	D_2	Inertia	$\rho S s^3 d_2$
E_0	$-T_{\dot\xi}$	$\rho\, V\, Ss^2 e_0$	E_2	$-H_{\dot\xi}$	$\rho\, V\, Ss^2 e_2$
F_0	$-T_\xi$	$\rho\, V^2 Ssf_0$	F_2	$h_\xi - H_\xi$	$\rho V^2 Ss\left(\dfrac{h_\xi}{\rho V^2 Ss}+f_2\right)$

* An advantage of the "symmetrical" rudder is that Q_2' vanishes for any hinge position. Any projection of the rudder below the fuselage will assist elimination of the couplings. Another method for the elimination of Q_2' is the provision of some balancing area at the top of the rudder. It is probable that the area required is considerably less than that required for complete aerodynamical balance.

† Ref. 2.

As usual, the two total stiffness coefficients will be treated as current coordinates, X, Y, in a plane ; thus

$$X \equiv \frac{t_\Omega}{\rho V^2 Ss}, \quad \ldots \quad (168a)$$

$$Y = \frac{h_\xi}{\rho V^2 Ss} + f_2. \quad \ldots \quad (168b)$$

The "stiffness point" Z has, accordingly, the coordinates $(0, f_2)$, and the slope of the "stiffness line" is h_ξ / t_Ω.

The types of test diagram obtained will depend upon the relative magnitudes of the various coefficients,* and some representative cases are sketched in Figs. 30(*a*) and 30(*b*). Alternative possible positions for the stiffness point are shown as Z_1 and Z_2.

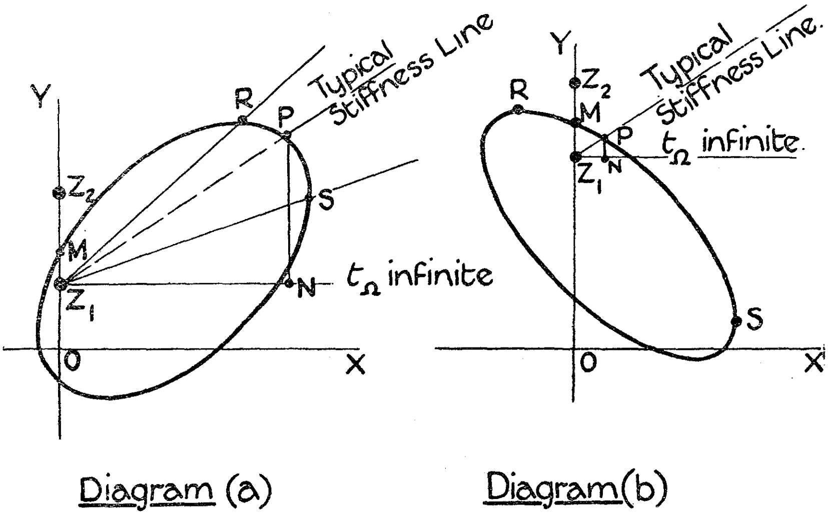

FIG. 30. Test Conics for Elevator-Fuselage Motion.

Diagram (*a*).—In diagram (*a*) the slope of the tangent at M to the test conic is positive, and the point R whose ordinate is a maximum accordingly lies to the right of OY. Moreover the point S corresponding to the maximum abscissa is taken to lie above Z_1N.

Suppose, firstly, that the stiffness point lies within the test conic, e.g. at Z_1, and let a typical stiffness line intersect the test conic in P, as shown. From (168) it follows that the critical flutter speed V_c

* The values of the derivatives required for a thorough discussion are not available. However, values appropriate to the case of tailplane and elevator have been roughly estimated from data supplied in R. & M. 1155, and these show that the condition for an elliptic test conic is likely to be satisfied.

corresponding to this stiffness line will be given by either of the equations

$$\rho\, Ss\, V_c^2 = t_\Omega/Z_1N = h_\xi/NP. \quad \ldots \quad (169)$$

Thus, when h_ξ is regarded as fixed, and t_Ω as variable, the critical speed will increase or diminish according as NP diminishes or increases. Now an increase of torsional stiffness results in a reduction of the slope of the stiffness line. If, therefore, the slope of the line is initially *less* than that of Z_1R, then *any* increase of torsional stiffness will increase V_c. On the other hand, if the slope is initially greater than that of Z_1R, the first effect of an increase in the torsional stiffness is a reduction of critical speed, and a great increase may be required to effect an increase of the critical speed.

Now, suppose that t_Ω is regarded as fixed, and h_ξ as variable: then V_c will only increase if Z_1N diminishes. Since an increase of h_ξ results in a steepening of the stiffness line, it follows that V_c will increase continuously with h_ξ only if, initially, the stiffness line is steeper than Z_1S.

The foregoing argument is equally applicable when the stiffness point lies above M (e.g. at Z_2). It suggests that, when the test diagram is of the type considered, an increase of only one elastic stiffness is not usually advisable as a remedial measure, unless the increase can be made really drastic. Theoretically, the safest condition appears to be when h_ξ is large, and t_Ω *small*; this provides a nearly vertical stiffness line. With the stiffness point at Z_1, the ordinate NP is now small, h_ξ is large, and the critical speed is, therefore, high; whereas, with the stiffness point at Z_2, flutter is prevented entirely. In the reverse conditions, with h_ξ small and t_Ω large, flutter occurs even when the stiffness point is at Z_2. The detailed analysis given in §55 (*b*) of R. & M. 1155 for the corresponding type of wing flutter suggests as a rough guide that the ratio h_ξ/t_Ω should preferably be so large that the natural frequency of the elevators in still air is well in excess of that of the fuselage in torsion.* This measure would entirely eliminate flutter of the type discussed, provided that the stiffness point could definitely be brought above M.

The conditions that Z shall lie above M are that the product of inertia of the elevators shall be small, and that

$$T_\Omega H_\xi > H_\Omega T_\xi. \quad \ldots \quad (170)$$

---

* This criterion is merely qualitative, owing to the extreme indefiniteness of the term "natural frequency of the fuselage." In the measurement of t_Ω the rational procedure would be to assume the fuselage fixed in torsion at approximately the C.G. of the aeroplane. The value of h_ξ is appropriate to the condition where the control stick is locked, and any direct connection between the two elevator planes (*e.g.*, a torsion tube) is rigidly clamped midway between these planes.

If the elevators are definitely underbalanced aerodynamically, (170) will almost certainly be satisfied, particularly in view of the relatively large direct damping of the fuselage. On the other hand, the inequality might well fail in the case where the hinge is close to the position for complete aerodynamical balance.

Diagram (*b*).—This represents the optimum condition, in which the slope of the tangent at M to the test conic is negative, and neither of the points R, S falls within the positive quadrant YZ_1 N.

With the stiffness point at Z_1 flutter will sooner or later occur, but any increase of either elastic stiffness will now ensure an increase of the critical speed. In the alternative case, with the stiffness point at Z_2, flutter is avoided completely.

The theoretical conditions which ensure a diagram of the desirable type (*b*) and a stiffness point above M, are deduced in §55 of R. & M. 1155. They may be interpreted in relation to the present problem as follows :—

(i) Elevators definitely underbalanced aerodynamically.

(ii) Product of inertia (P_2) of each elevator zero.

(iii) Moment of inertia (D_2) of each elevator small.

(iv) Elevators heavily damped.

§84b. *Tailplane and Fin Rigid and Elevator Motion Independent.*—In the alternative method of resolution of the determinant (166), the coupling term $P_2(\lambda)$ is made to vanish. The product of inertia P_2 can always be eliminated by suitable mass distribution, and Q_2 could be made zero by special location of the hinge. However, as for the rudder, this hinge position may have certain disadvantages.* A much more satisfactory expedient is interconnection of the elevators by a tube so stiff in torsion that no significant relative twist of the elevators can occur. This measure alone is sufficient to prevent participation of the elevators in the antisymmetrical motion. It must, however, be remembered that if the tailplane has appreciable flexibility, mass balance of the elevators (and possibly a special location of the hinge) may still be advisable as a safeguard against flutter of the symmetrical type (*see* §79).

When the elevator motion has been eliminated in either of the ways suggested, the residual factor of (166) is

$$\begin{vmatrix} P_0(\lambda), & D_0'(\lambda) \\ P_2'(\lambda), & D_2'(\lambda) \end{vmatrix} = 0, \quad .. \quad (171)$$

* The elimination of Q_2 could probably be effected by use of some balancing area at the tip (*cp.*, footnote to p. 138).

which represents the coupled rudder-fuselage motion. The discussion is analogous to that detailed in §84a, and leads to the following sufficient conditions for stability :—

(i) Rudder definitely underbalanced aerodynamically.

(ii) Product of inertia P_2' of the rudder zero.

(iii) Moment of inertia D_2' of the rudder small.

(iv) Rudder heavily damped.

§85. *Summary of Conclusions, and Suggestions Regarding Design.* —The preceding theoretical review of the problem of tail flutter is based on the assumptions that the tailplane and fin may be treated as semi-rigid, and that the elevators and the rudder are rigid. The degrees of freedom permitted are (1) angular movements of the elevators and rudder (2) flexure and torsion of tailplane and fin (3) twist of the fuselage.

Two types of tail flutter, described as "symmetrical" and "antisymmetrical," are considered. In the first, the elevators move freely and in phase, the rudder does not turn, and the fuselage does not twist; in the second, the elevators are elastically constrained and have a "scissors" motion, and all seven degrees of freedom may be involved. The principal conclusions drawn from the theory are :—

(*a*) Symmetrical flutter will not occur except at very high speeds if the tailplane is of a stiff construction. If the tailplane is appreciably flexible, flutter could be avoided by an observance of measures analogous to those recommended in R. & M.1155 for wings and ailerons.

(*b*) Antisymmetrical flutter will normally involve both the elevators and the rudder, but the rudder flutter would be eliminated completely if the rigid rudder were to extend symmetrically above and below the axis of fuselage twist.

(*c*) If the rudder is very unsymmetrically disposed about the axis of twist, the rudder flutter can be disposed of by suitable mass distribution and hinge location.

(*d*) Interconnection of the two elevator planes by a tube very stiff in torsion disposes of the antisymmetrical elevator flutter.*

(*e*) Merely moderate increase of the torsional stiffness of the fuselage appears to offer no advantage, and may actually tend to promote flutter.

(*f*) The divergences of the tailplane elevator combination, and of the fin-rudder combination, can be treated as independent. Complete immunity from divergence at all speeds (however high) could only be obtained at the risk of flutter, but stiff construction will ensure high divergence speeds.

* This assumes that the elevators themselves are very stiff in torsion. In the contrary case the measure will clearly be ineffective.

The following list of suggestions regarding the design of the tail unit is based purely on a survey of the conditions for the avoidance of flutter, and no weight has been attached to the merits or demerits of the proposals from other standpoints.

Features of Design favourable to the Avoidance of Tail Flutter.

Group I.—*General.*

(*a*) Tailplane and fin very stiff both in flexure and torsion.

(*b*) Rigidity of elevator planes and rudder.

(*c*) Irreversibility of elevator and rudder controls.

Failing (*c*)—

Group II.—*Elevators.*

(*d*) Interconnection of elevators by tube very stiff in torsion.*

(*e*) Product of inertia of each elevator zero (*see* Notes, §85a).

(*f*) Moment of inertia of elevator about hinge axis small.

(*g*) Elevators definitely underbalanced aerodynamically.

(*h*) Elevators heavily damped.

Group III.—*Rudder.*

(*i*) Projection of part of rudder below fuselage. Optimum condition is rudder symmetrically bisected by centre line of fuselage.

(*j*) Product of inertia of rudder zero (*see* Notes, §85a).

(*k*) Moment of inertia of rudder about hinge axis small.

(*l*) Rudder definitely underbalanced aerodynamically.

(*m*) Rudder heavily damped.

Failing (*a*)—

Group IV.—*Tailplane.*

(*n*) Balance of masses of each half of tailplane (including elevator) about its flexural axis.

(*o*) Flexural axis close to axis of independence.

§85a. *Some Notes on the Measures for the Avoidance of Tail Flutter.*

Group I.—When items (*a*), (*b*) and (*c*) are all observed, the remaining Groups can be ignored.

When the requirements (*a*) and (*b*) are met, but (*c*) is not, then a judicious observance of Groups II and III is necessary.

* This recommendation is due to Bairstow and Fage (Ref. 2).

When only condition (*b*) is satisfied, Group IV will also require attention.

The elevator control will be irreversible in the sense of recommendation (*c*) if, when the elevator is set at any desired angle, it remains locked to the tailplane in that position, until again purposely moved by the pilot. Similar remarks apply to irreversibility of the rudder control.

Group II.—If the tailplane is extremely stiff, and if (*d*) is observed, then items (*e*) and (*f*) may be ignored.

The product of inertia referred to under heading (*e*) is the constant P_2 of Table 24, and is given by the formula

$$P_2 = \Sigma\, m x_e y,$$

where m is an element of mass of the elevator, x_e is the distance of the element measured downstream from the elevator hinge axis, and y is the perpendicular distance from the plane of symmetry of the machine. No allowance is made in the formula for the "virtual" product of inertia due to aerodynamical action (acceleration derivative). To compensate for this the product of inertia as calculated should preferably be slightly negative (*cp.* R. & M. 1155, p. 172).

Group III.—In item (*i*) the optimum condition is strictly when the rudder is symmetrically bisected by the axis of fuselage twist. When the symmetrical type of construction can be adopted, then (*j*) will necessarily be satisfied provided that the mass distribution is also symmetrical.

The product of inertia referred to under heading (*j*) is the constant P_2' of Table 24, and is to be calculated from the formula

$$P_2' = -\Sigma\, m x_r z,$$

where m is an element of mass of the rudder, x_r is the distance of the element measured downstream from the rudder hinge axis, and z is the perpendicular distance above the centre line (axis of twist) of the fuselage. For reasons already stated in connection with the elevators the constant determined by the formula should preferably be slightly positive.

Group IV.—Attention to the measures in this group will probably be unnecessary with normally stiff construction of the tail unit. For detailed information on the "flexural axis" and "axis of independence" reference should be made to R. & M. 1155.

PART II.

EXPERIMENTAL INVESTIGATION.

§86. *Preliminary.*—The experiments here described were undertaken in order to test some of the theoretical conclusions reached in Part I. As already explained in §79, " symmetrical " tail flutter is directly analogous to " ternary " wing flutter. Since the latter has already been studied experimentally in some detail (*see* R. & M. 1155) the present investigation has been confined almost entirely to " antisymmetrical " tail flutter.

§87. *Description of the Apparatus.*—In antisymmetrical flutter the important motions are (1) " rolling " of the tail, (2) flapping of the elevators and (3) oscillation of the rudder about its hinge. All these motions can be exhibited by a model whose separate components are rigid ; accordingly these components were constructed of solid wood, and elastic constraint was provided where necessary by means of steel springs.

The arrangement of the apparatus will be clear from Figs. 31 and 32. Reproduction of a complete fuselage was not attempted, and the bare tail unit used was mounted on a horizontal wooden rod provided with pivots at its ends, so as to admit rolling or torsional motion. Torsional stiffness of the fuselage was represented by helical springs, and could be varied by movement of the point of attachment of the springs along the span of the tailplane or by change of the springs themselves. The elevators were freely hinged, but were elastically connected by a strip of watch spring, equivalent to a not very stiff direct connection of the elevators for full scale. This elastic constraint was only operative for antisymmetrical motion of the elevators. Two rudders of widely different design were used :—the first (Fig. 31) was disposed entirely above the axis of torsional motion, whereas the second (Fig. 32) was symmetrically bisected by this line. In some of the experiments the rudder was completely free, but in others it was securely locked to the fin to simulate the conditions of an irreversible rudder control. The effect of elastic constraint of the rudder was not examined.

The principal dimensions of the parts were as follows :—

Tailplane span (overall), $15\frac{3}{4}$ in.
Tailplane chord (fixed part only), 3 in.
Elevator span, $5\frac{3}{8}$ in.
Elevator chord, $2\frac{5}{8}$ in.
Height of top of fin above torsional axis, 4 in.
Unsymmetrical rudder—total height of rudder surface, 6 in.
Unsymmetrical rudder—maximum chord, $4\frac{1}{4}$ in.
Symmetrical rudder—height, 8 in.
Symmetrical rudder—chord $2\frac{3}{4}$ in.
Sections of surfaces—all thin and symmetrical.

§88. *Influence of the Elastic Stiffnesses of the Fuselage and of the Connection between the Elevators.*—The effect of variations of the elastic stiffnesses of the system are exhibited in Tables 27, 28, 29 and 30.

TABLE 27.

Critical Flutter Speeds with Locked Symmetrical Rudder.

Fuselage Stiffness t_Ω (lb. ft. per radian).	Elevator Stiffness h_ξ (lb. ft. per radian).	Critical Speed (ft. per sec.)
0·79	0·039	None
1·94	0·039	None
3·91	0·039	27·5
6·78	0·039	25·6

TABLE 28.

Critical Flutter Speeds with Free Symmetrical Rudder.

Fuselage Stiffness t_Ω (lb. ft. per radian.)	Elevator Stiffness h_ξ (lb. ft. per radian).	Critical Speed (ft. per sec.).
0·79	0·093	None
1·94	0·093	None
3·91	0·093	None
6·78	0·093	None

TABLE 29.

Critical Flutter Speeds with Free Symmetrical Rudder (Elevators Partially Mass Balanced.)

Fuselage Stiffness t_Ω (lb. ft. per radian).	Elevator Stiffness h_ξ (lb. ft. per radian).	Critical Speed (ft. per sec.).
0·79	0·039	None
1·94	0·039	None
3·91	0·039	None
6·78	0·039	21·5

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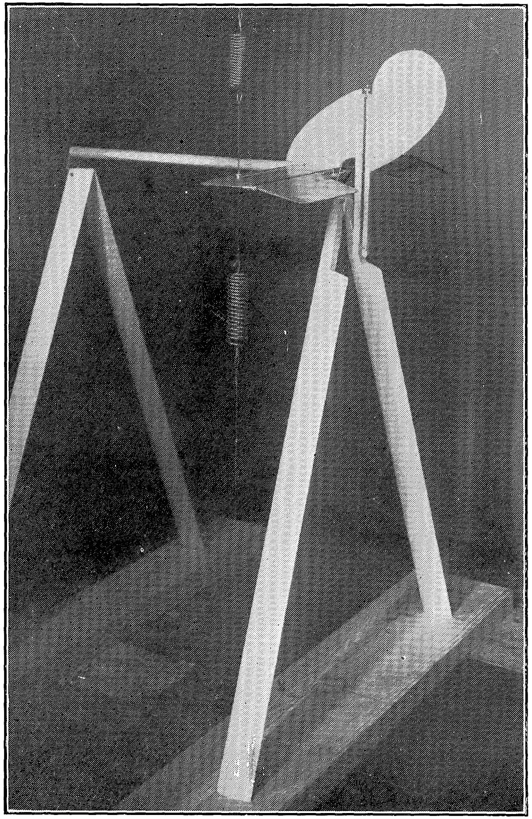


FIG. 31.—Model Tail with Unsymmetrical Rudder.

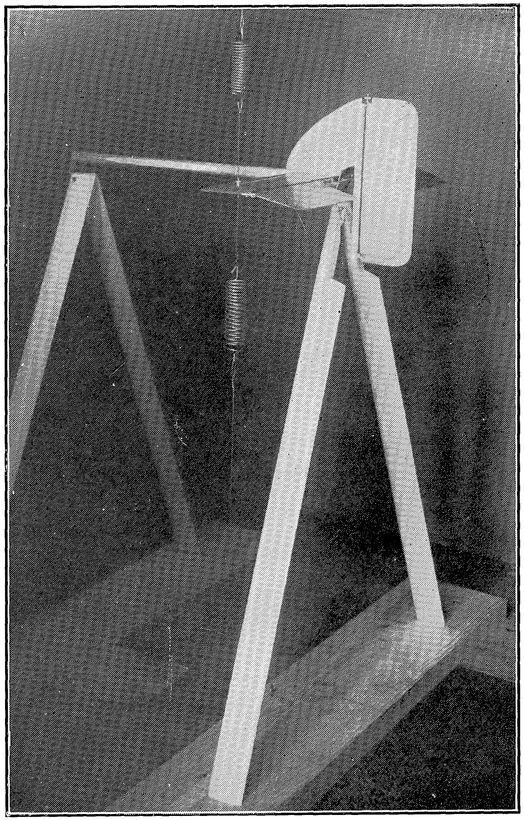


FIG. 32.—Model Tail with Symmetrical Rudder.

[To face page 146.

TABLE 30.

Critical Speeds with Free Unsymmetrical Rudder.
(*Elevators as for Tables* 27 *and* 28.)

Fuselage Stiffness t_Ω (lb. ft. per radian).	Elevator Stiffness h_ξ (lb. ft. per radian).	Critical Speed (ft. per sec.).
0·61	0·039	31·6
1·87	0·039	43·0
3·76	0·039	31·2
6·60	0·039	26·5

The results given in Tables 27, 28 and 29 may be considered first. In all three cases the motion was essentially binary, since in accordance with theory the symmetrical rudder, when free, behaved as if locked to the fin. Table 27 corresponds to a rather flexible elevator connection, and flutter only occurred for the *higher* values of the fuselage stiffness. A similar effect is exhibited in Table 29 for the case where the elevators were partially mass balanced. Table 28 shows that with a stiffer connection between the elevators there was no flutter within the range of wind speeds available (maximum 75 ft. per sec.) even with the stiffest fuselage. These facts are in general accord with the theoretical deductions of §84a.

In the experiments with the free unsymmetrical rudder the motion was obviously ternary. Here there was a definite optimum stiffness ratio t_Ω/h_ξ (*see* Table 30) and increase of t_Ω beyond the value corresponding to this ratio led to a pronounced fall of the critical speed.

No evidence of upper critical speeds was obtained in the experiments already described. This indicates that with the special model used the test conics for the fuselage-elevator motion were either hyperbolas or elongated ellipses. In further experiments the damping of the fuselage was increased by the addition of a large false fin. Upper critical speeds were then obtained, above which the system became stable.

§89. *Influence of Mass Loading of the Elevators.*—Experiments were carried out in which the elevator product of inertia was reduced by the attachment of weights at the tips forward of the hinges. The free symmetrical rudder was fitted throughout. When the elevator product of inertia was zero, or even considerably negative, no flutter occurred for any value of the fuselage stiffness. This was still true when the elevators were entirely free from elastic constraint.

§90. *Influence of the Type of Rudder.*—As already remarked, the symmetrical rudder when free behaved in flutter as if locked to the fin. Accordingly, the critical speeds were found to be uninfluenced by freedom or fixture of the symmetrical rudder. When the unsymmetrical rudder was fitted the behaviours in the two conditions were sometimes widely different.

TABLE 31.

Critical Speeds with Symmetrical Rudder Fixed and Free.
(h_ξ = 0·039 *in all the tests.*)

Fuselage Stiffness t_Ω (lb. ft. per radian.)	Critical Speed (ft. per sec.).	
	Rudder Fixed.	Rudder Free.
0·79	None	None
1·94	None	None
3·91	27·5	27·2
6·78	25·6	25·8

TABLE 32.

Critical Speeds with Unsymmetrical Rudder Fixed and Free.
(h_ξ = 0·039 *in all the tests.*)

Fuselage Stiffness t_Ω (lb. ft. per radian).	Critical Speed (ft. per sec.)	
	Rudder Fixed.	Rudder Free.
0·61	None	31·6
1·87	None	43·0
3·76	30·8	31·2
6·60	27·3	26·5

In the first two tests recorded in Table 32, the flutters which occurred with the free rudder were of almost pure binary fuselage-rudder type. For the two higher values of the fuselage stiffness, however, the motions were predominantly of the fuselage-elevator type, and here freedom of the rudder had little influence on the critical speeds.

§91. *Influence of Mass Loading on the Behaviour of the Unsymmetrical Rudder.*—The product of inertia of the rudder was reduced approximately to zero by attachment of a weight *forward* of the hinge axis and *above* the axis of torsional motion. Results of the flutter tests are given in Table 33.

TABLE 33.

Critical Speeds with Unsymmetrical Rudder.
(*Product of inertia reduced to zero.*)

Fuselage Stiffness $t\Omega$ (lb. ft. per radian).	Elevator Stiffness h_ξ (lb. ft. per radian).	Critical Speed (ft. per sec.).
0·43	0·039	None
1·76	0·039	None
3·73	0·039	None
6·34	0·039	36

It will be seen that flutter was avoided except for the greatest value of the fuselage stiffness, and the motion was then predominantly of the fuselage-elevator type. Hence mass balance appears to be effective in the elimination of rudder flutter. A large increase of the rudder moment of inertia caused a reappearance of rudder flutter, even when the product of inertia was maintained zero.

§92. *Influence of Aerodynamical Balance of the Elevators.*—Some experiments were carried out with horn balanced elevators. On account of the horizontal attitude of the tailplane it was necessary in this instance to load the elevators until their centres of gravity lay upon the hinge axes. In this condition the product of inertia was *negative.* The unsymmetrical rudder was fitted and kept locked to the fin throughout the tests. When the elevators were slightly underbalanced aerodynamically no spontaneous flutter occurred, but the damping of the elevators was very small, and unstable *symmetrical* oscillations followed a large disturbance. When very slightly overbalanced, the elevators diverged at a low speed ; this divergence was followed by a symmetrical oscillation of very large amplitude. These phenomena are no doubt closely connected with the " snatching " reported as occurring with certain early types of horn-balanced elevators.

§93. *General Conclusions.*—The results of the tests are in general accord with the theoretical conclusions of Part I. Antisymmetrical flutters of the following types were demonstrated :—

(1) Binary elevator-fuselage flutter.
(2) Binary rudder-fuselage flutter.
(3) Ternary elevator-rudder-fuselage flutter.

The experiments indicate that, at least for the case where the tailplane and fin are very stiff, rudder flutter can be avoided by suitable mass loading or by adoption of a rudder symmetrically disposed about the axis of torsion of the fuselage : further, that flutter of the elevators can be eliminated by the provision of a very stiff direct connection between the elevators, or by mass loading. A high torsional stiffness of the fuselage does not appear to be particularly advantageous. Lastly, very close approach to aerodynamical balance of the control surfaces may lead to instability.

APPENDIX TO CHAPTER IV.

ON THE DISSIPATION FUNCTION FOR A SPECIAL TYPE OF TERNARY MOTION.

In order to complete the argument of §81 it is necessary to show that the dissipation function corresponding to the particular motion represented by the determinant (160) is one-signed and positive. A proof can be supplied, based on the following auxiliary proposition.

Lemma.—If the motion of every possible simple system obtained by the imposition of any $(n-1)$ constraints upon a system having n *degrees of freedom is damped, then the dissipation function for the* n*-ary system is one-signed and positive—and conversely.*

Let the generalised coordinates of the n-ary system be $x_1, x_2, \cdots x_n$. Also, for conciseness write

$$A_{11}(x_1) \equiv A_{11}\ddot{x}_1 + B_{11}\dot{x}_1 + C_{11}x_1, \quad \text{etc.} \qquad (1)$$

The equations of motion of the unconstrained n-ary system can then be written

$$\left.\begin{array}{l} A_{11}(x_1) + A_{12}(x_2) + \cdots\cdots + A_{1n}(x_n) = 0, \\ A_{21}(x_1) + A_{22}(x_2) + \cdots\cdots + A_{2n}(x_n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ A_{n1}(x_1) + A_{n2}(x_2) + \cdots\cdots + A_{nn}(x_n) = 0. \end{array}\right\} \qquad (2)$$

Suppose now that $(n-1)$ constraints are introduced in accordance with the independent equations :—

$$\left.\begin{array}{l} \mu_{11}x_1 + \mu_{12}x_2 + \cdots\cdots + \mu_{1n}x_n = 0, \\ \mu_{21}x_1 + \mu_{22}x_2 + \cdots\cdots + \mu_{2n}x_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_{n-1,1}x_1 + \mu_{n-1,2}x_2 + \cdots\cdots + \mu_{n-1,n}x_n = 0. \end{array}\right\} \qquad (3)$$

No constants appear on the right of these equations since the coordinates are supposed measured from an equilibrium position of the constrained system. At least one of the coordinates (say x_1) is still variable after the imposition of the constraints. Hence the equations (3) can be solved for $x_2, x_3, \cdots x_n$ in terms of x_1 and replaced by

$$\left.\begin{array}{l} x_2 = \lambda_2 x_1/\lambda_1, \\ x_3 = \lambda_3 x_1/\lambda_1, \\ \cdots\cdots \\ x_n = \lambda_n x_1/\lambda_1, \end{array}\right\} \qquad (4)$$

where λ_1, λ_2, etc., are determinants involving the elements μ. Corresponding to the $(n-1)$ constraints there are $(n-1)$ constraining forces $T_1, T_2, \cdots T_{n-1}$. Accordingly equations (2) must be replaced by the set

$$\begin{array}{l} A_{11}(x_1) + A_{12}(x_2) + \cdots + A_{1n}(x_n) \\ \qquad = a_{11}T_1 + a_{12}T_2 + \cdots + a_{1,n-1}T_{n-1}, \\ \qquad\qquad \text{etc.,} \end{array} \qquad (5)$$

where the coefficients a are constants depending on the points of application of the constraints. To obtain the equation of motion of the constrained system the quantities T must be eliminated from the equations (5). An application of the principle of energy shows at once how the elimination can be performed. The equation of energy is obtained by multiplication of the equations (5) by $\dot{x}_1$, $\dot{x}_2$, etc., respectively and addition. Since the forces of constraint do no work they must disappear from the result. Hence these forces can be eliminated from the set (5) by multiplication of the equations by λ_1, λ_2, etc. (which, by (4), are proportional to $\dot{x}_1$, $\dot{x}_2$, etc.) and addition. The dynamical equation obtained in this way reduces to the form

$$\ddot{x}_1\begin{bmatrix}A_{11}\lambda_1^2 + A_{22}\lambda_2^2 + \cdot \cdot \cdot + A_{nn}\lambda_n^2 \\ +2A_{12}\lambda_1\lambda_2 + 2A_{13}\lambda_1\lambda_3 + 2A_{23}\lambda_2\lambda_3 + \text{etc.}\end{bmatrix}$$
$$+\dot{x}_1\begin{bmatrix}B_{11}\lambda_1^2 + B_{22}\lambda_2^2 + \cdot \cdot \cdot + B_{nn}\lambda_n^2 \\ +(B_{12}+B_{21})\lambda_1\lambda_2 + (B_{13}+B_{31})\lambda_1\lambda_3 + (B_{23}+B_{32})\lambda_2\lambda_3 + \text{etc.}\end{bmatrix}$$
$$+x_1\begin{bmatrix}C_{11}\lambda_1^2 + C_{22}\lambda_2^2 + \cdot \cdot \cdot + C_{nn}\lambda_n^2 \\ +(C_{12}+C_{21})\lambda_1\lambda_2 + (C_{13}+C_{31})\lambda_1\lambda_2 + (C_{23}+C_{32})\lambda_2\lambda_3 + \text{etc.}\end{bmatrix}$$
$$= 0 \ldots\ldots \qquad (6)$$

The coefficient of $\ddot{x}_1$ in (6) is the kinetic energy of the n-ary system when $\dot{x}_1 = \lambda_1$, $\dot{x}_2 = \lambda_2$, etc., and *is therefore necessarily positive*. It can thus be taken as the equivalent inertia of the constrained system. The damping coefficient of (6) is simply the dissipation function of the n-ary system with λ_1 substituted for $\dot{x}_1$, etc., as before. Hence, finally, the condition that every constrained system should be damped is exactly equivalent to the condition that the dissipation function of the n-ary system should be necessarily positive.

Return now to the particular motion under examination, of which the three constituents are flexure of the tailplane and of the fin, and twist of the fuselage. In whatever manner these constituents may be compounded, each fore and aft strip of the tailplane (or fin) moves bodily about an axis at least approximately parallel to the centre line of the machine. Let any two constraints of the type considered in the Lemma be imposed, so that in effect the three constituent motions are now geared together, and the displacements of all points depend upon a single generalised coordinate (say x_1). Then the component displacement of any typical strip of the tailplane (or fin) normal to the latter can be expressed as $c\,x_1$, where c is some function of position of the strip. Corresponding to the generalised velocity $\dot{x}_1$ there will be an incidence change at the strip, and therefore also a normal aerodynamical force, proportional to $c\,\dot{x}_1$. By the principle of virtual work it readily follows that the contribution of the strip to the damping coefficient is proportional to c^2. Moreover, for normal angles of incidence the factor of proportionality will be positive. The total damping coefficient in the constrained motion is therefore positive. It follows from the Lemma that the dissipation function for the unconstrained ternary motion is one-signed and positive.

CHAPTER V.

TAIL FLUTTER OF A PARTICULAR AEROPLANE *

By W. J. Duncan, D.Sc., A.M.I.Mech.E., and A. R. Collar, B.A., B.Sc.

§ 94. *Reasons for Enquiry.*—In the course of a test flight of the aeroplane violent rudder oscillations occurred, leading to fracture of the sternpost, and ultimately to destruction of the machine through crashing. The accident was considered both by the Accidents Investigation Sub-Committee and by the Flutter Sub-Committee, and attributed to tail flutter. An investigation, based upon the theory of tail flutter developed in Chapter IV. was authorised by the Aeronautical Research Committee, and an account of the work will now be given.

§ 95. *Outline of the Investigation.*—It was considered probable that the flutter which caused the accident was predominantly of the rudder-fuselage type, in which the important motions are oscillation of the rudder and " rolling " of the tail unit due to torsion of the fuselage. In order to calculate a critical flutter speed for comparison with the observed speed, two groups of numerical coefficients were required. The first consisted of the appropriate inertial and stiffness constants of the actual machine; these were determined at the Royal Aircraft Establishment by direct experiment where possible, and in other cases by calculation from the drawings. The second group comprised the relevant aerodynamical derivative coefficients. As it was not considered possible to make reliable theoretical estimates of these derivatives, their values were deduced from the results of experiments upon a one-eighth scale model of the aeroplane. The measurements were conducted in a 4-foot wind tunnel at the National Physical Laboratory; a description of the technique used and of the model itself are given in the sequel.

§ 96. *Description of the Model and Apparatus.*—The general features of the design of the aeroplane will be evident from the plan and side elevation reproduced in Fig. 33. It will be noted that the rudder lay entirely above the axis of the fuselage and that a horn balance was provided. The rudder was in fact only very slightly underbalanced aerodynamically (c.p. § 99a) Some of the principal dimensions of the tail unit were as follows :—

Total span of tailplane..	12 ft. 8 in.
Total chord of tailplane (including elevators)	4 ft. 3·95 in.
Height of highest point of rudder above fuselage axis	5 ft. 11·25 in.
Total height of rudder surface	5 ft. 1·5 in.
Distance from lower edge of horn balance to top of rudder	2 ft. 8·65 in.

* Originally issued as R. & M. 1247 (Ref. 16).

Radius from rudder axis to tip of horn balance . . . 1 ft. 3·52 in.
Radius from rudder axis to trailing edge at same section . . . . 2 ft. 1·92 in.
Maximum radius from rudder axis to trailing edge . . . . 2 ft. 3·08 in.

The model was of one-eighth scale, and constructed of wood. It was mounted horizontally in the wind tunnel as shown in Fig. 34. The main planes were supported from the tunnel walls at the tips (slightly truncated), and were stiffened by steel wires. In the case of the actual machine the mobility of the tailplane in torsion or

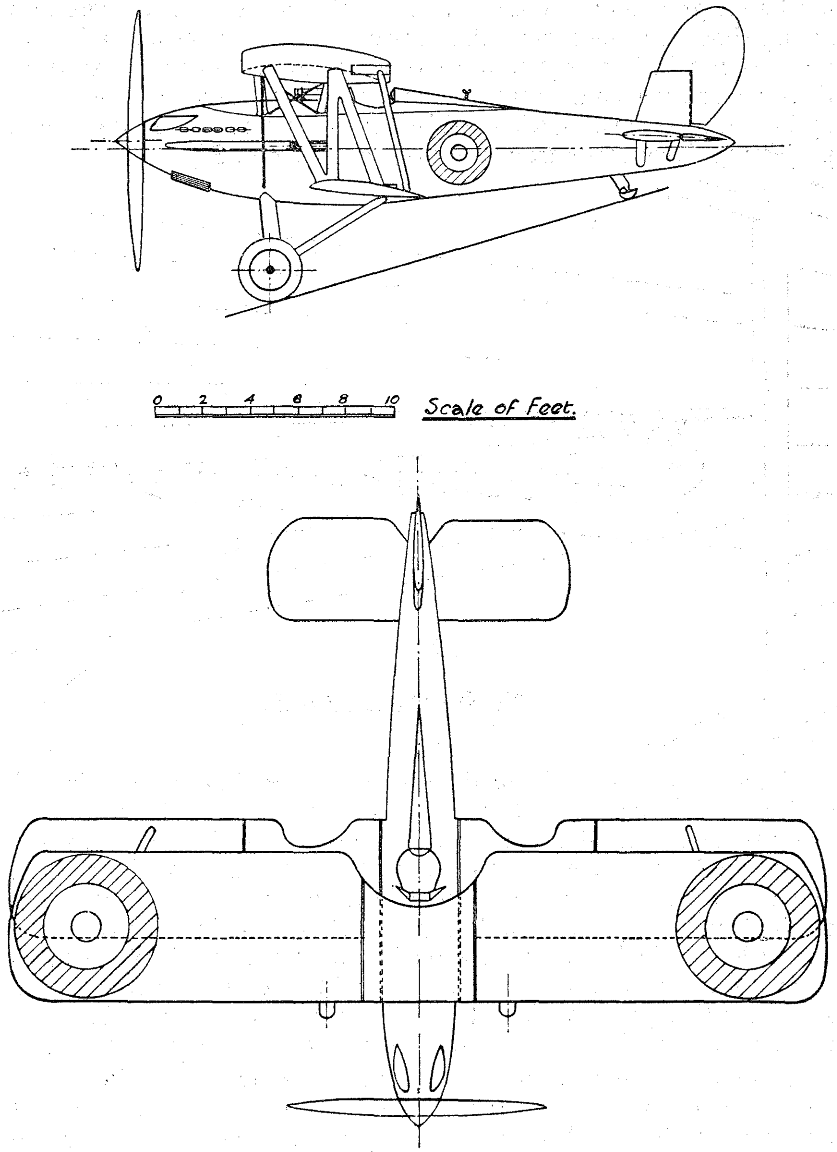
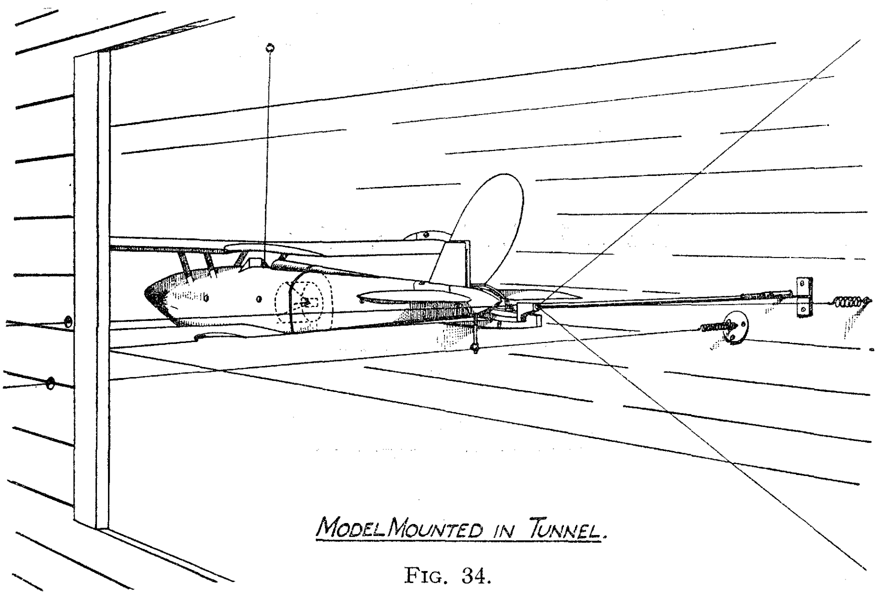


FIG. 33.—General Arrangement of the Aeroplane.

" roll " was of course due to the flexibility of the fuselage. However, the construction of a torsionally flexible fuselage for the model would have been troublesome, and was viewed as unnecessary. The construction actually adopted was as follows. The fuselage was made of solid wood and was divided in two by a transverse cut just aft of the main planes (*see* Fig. 34). A short three-wire suspension concealed within the fuselage itself was fitted to the forward end of the rear portion of the fuselage, and the suspension at the tail end was as illustrated in Fig. 35. The stout rafwire shown on the right of the figure had a hardened knife-edge bearing in a hard steel groove on the model. In conjunction with the wire and spring shown in the figure, this rafwire entirely prevented a lateral oscillation which was found to occur with the ordinary type of three-wire suspension. Stiffness in the torsional motion was provided by helical steel springs attached by fine steel wires to a brass lever projecting downwards from the fuselage (*see* Fig. 34).



Model Mounted in Tunnel.

Fig. 34.

A scale drawing of the model rudder is given in Fig. 36. Special anti-friction bearings, as shown in the figure, were adopted. The lower pivot consisted of a steel needle bearing in a glass cup, while the upper bearing consisted of a steel needle fixed in the rudder, and passing through an easy hole in a thin metal plate attached to the top of the fin. The lower end of the rudder was provided with a lever which carried a vertically adjustable needle for attachment to the thread connecting the rudder to its control springs and recording gear (*see* Fig. 35). The needle was adjusted so that this thread passed exactly through the axis of motion of the fuselage.

In many of the measurements it was necessary to obtain precise records of the motions of the fuselage and rudder which occurred under controlled conditions. The apparatus used for this purpose is illustrated diagrammatically in Fig. 37, and was a modification of the recording gear used in previous work on wing flutter. This gear is described in some detail in R. & M. 1155, so that a very brief account will here suffice. The rudder and fuselage were connected by wires or threads to the recording levers. Upon the spindles of these levers were mounted small stainless steel mirrors which reflected beams of light from a " Pointolite " lamp on to the surface of the photographic paper carried by a drum revolving inside the camera box. A time scale was provided by means of an electrically maintained tuning fork, and an automatic camera shutter was used which ensured exposure for exactly one complete revolution of the drum.

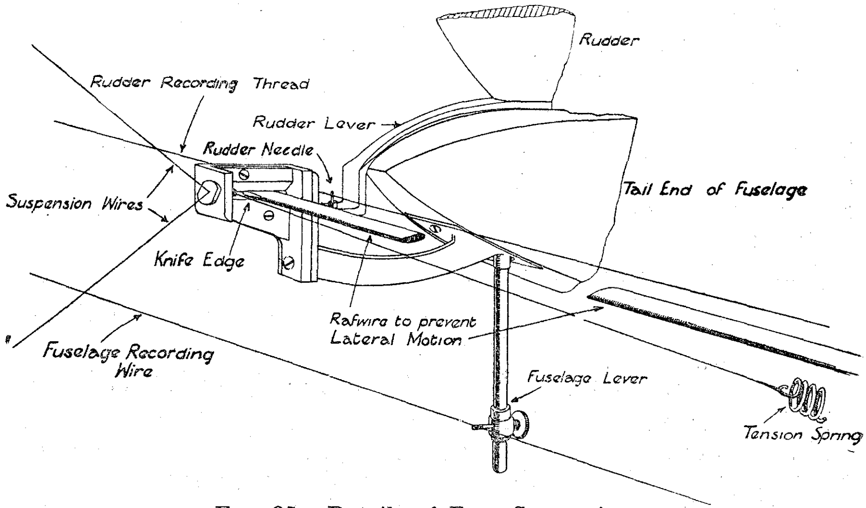


FIG. 35.—Details of Rear Suspension.

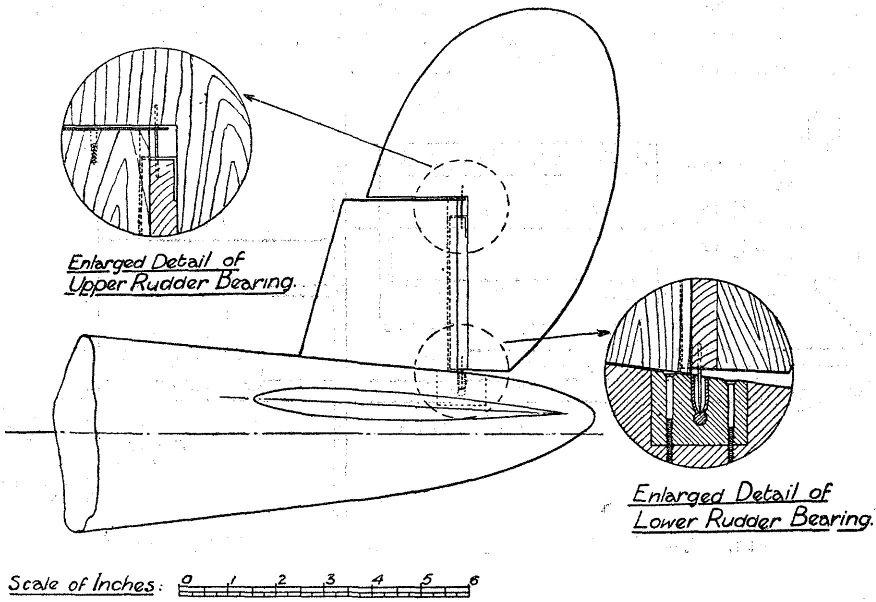


FIG. 36.—Details of Rudder.

§ 97. *Methods of Measurement.*—The present section will be devoted to a brief account of the theoretical basis of the measurements. The latter are described in greater detail under separate headings below. In general the notation is that adopted in Chapter IV for rudder-fuselage flutter, but for simplicity accents will be omitted from certain of the symbols.

Let

Ω = angular displacement of the tail unit in torsion or roll (positive when starboard tip of tailplane moves downwards).

ξ = angular displacement of rudder about its hinge (positive when trailing edge moves to port).

T = torsional or rolling moment on tail unit.

H = rudder hinge moment.

The moments T and H are positive when they tend to increase the corresponding angles Ω and ξ. Then the equations of motion of the rudder-fuselage system (including the recording gear, etc.) will be written :—

Equation of Torsional Moments.

$$P_0\ddot{\Omega} + Q_0\dot{\Omega} + R_0\Omega + D_0\ddot{\xi} + E_0\dot{\xi} + F_0\xi = 0, \quad \text{.. (172a)}$$

Equation of Rudder Hinge Moments.

$$P_2\ddot{\Omega} + Q_2\dot{\Omega} + R_2\Omega + D_2\ddot{\xi} + E_2\dot{\xi} + F_2\xi = 0. \quad \text{.. (172b)}$$

The significance of the various coefficients in the dynamical equations will be evident on inspection of Table 34, in which the notation for the aerodynamical derivatives conforms with that adopted in R. & M. 1155. All the stiffnesses are composed of two terms ; the first term—written with a small letter—represents the influence of elasticity and gravity, while the second—written with a capital letter—is the aerodynamical contribution.

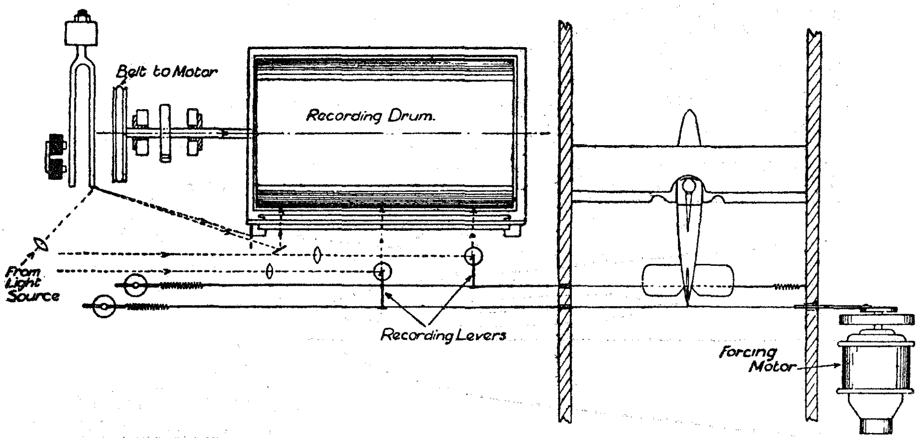


FIG. 37.—Diagrammatic Plan of Apparatus. (*Not to scale.*)

TABLE 34.

Dynamical Coefficients for Rudder-Fuselage Motion.

Equation of Torsional Moments.			Equation of Rudder Hinge Moments.		
Coefft.	Equivalent.	Significance.	Coefft.	Equivalent.	Significance.
P_0	—	Moment of inertia	P_2	$\equiv D_0$	Product of inertia
Q_0	$-T_{\dot{\Omega}}$	Direct damping	Q_2	$-H_{\dot{\Omega}}$	Compound damping
R_0	$t_\Omega - T_\Omega$	Direct stiffness	R_2	$h_\Omega - H_\Omega$	Cross stiffness
D_0	$\equiv P_2$	Product of inertia	D_2	—	Moment of inertia
E_0	$-T_{\dot{\xi}}$	Compound damping	E_2	$-H_{\dot{\xi}}$	Direct damping
F_0	$t_\xi - T_\xi$	Cross stiffness	F_2	$h_\xi - H_\xi$	Direct stiffness

The quantities actually required for the purpose of the full-scale calculations were the aerodynamical derivatives. In outline the methods used for their determination were as follows. All the stiffness derivatives (T_Ω, T_ξ, H_Ω, and H_ξ) were deduced from direct moment measurements taken over a range of wind speeds. The two direct damping derivatives ($T_{\dot{\Omega}}$ and $H_{\dot{\xi}}$) were obtained from records of the damped motions in each of the two degrees of freedom separately. Lastly, the two compound damping derivatives ($T_{\dot{\xi}}$ and $H_{\dot{\Omega}}$) were found by an analysis of the records of motions which occurred when a simple harmonic motion in one or other of the two degrees of freedom was imposed by aid of a forcing motor. As remarked in § 96, the rudder recording thread passed exactly through the axis of rotation of the fuselage, so that a force applied by this thread, as in a forcing experiment, did not produce (directly) any moment upon the fuselage. Moreover, the displacement of the thread was directly proportional to ξ and independent of Ω, and this greatly facilitated analysis of the records.

As preliminaries to the measurements of the derivatives the values of various stiffnesses and inertial coefficients of the model were required, and the determination of these is briefly described in the next section.

§ 98. *Inertial Coefficients and Elastic and Gravitational Stiffnesses.*— Elastic stiffnesses were determined by measurement of the displacements produced by suitably applied static loads. The displacements were measured by micrometer, and as a rule the loads were applied through horizontal threads. It was found that the recording threads or wires extended appreciably under load, so that it was always necessary to apply the load and to measure the displacement directly at the fuselage or rudder. This extensibility of the threads also affected the magnification ratios of the photographic records, but the correct ratios could be found from static calibrations, since the inertial loads due to the recording gear itself were quite trifling at the oscillation frequencies actually employed.

The gravitational cross-stiffness ($t_\xi \equiv h_\Omega$) was found by measurement of the angular displacement of the rudder produced by a known rotation of the fuselage in still air. For the conditions of the experiment equation (172b) becomes (*see* Table 34)

$$\Omega h_\Omega + \xi h_\xi = 0, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (173)$$

from which h_Ω can be found when h_ξ is known. In an actual experiment the data were :—$h_\xi = 0 \cdot 0340$, $\xi/\Omega = -0 \cdot 496$. Accordingly, the value of h_Ω for the rudder with its recording gear is $0 \cdot 0169$.

Both of the moments of inertia P_0 and D_2 (*see* Table 34) were deduced at once from the frequencies of the corresponding free oscillations in still air. In order to measure the product of inertia of the rudder with its recording gear ($P_2 \equiv D_0$), one end of the fuselage recording wire was connected to a crosshead driven by the forcing motor. The speed of the motor was adjusted until the oscillation of the rudder which occurred in still air was of suitable amplitude, and a photographic record was taken when the motion had become steady. Since no extraneous force is applied to the rudder or its recording gear equation (172b) is applicable, with the simplification that Q_2 and E_2 can be taken to vanish for still air. In the absence of these damping coefficients the phase difference of the oscillations in Ω and ξ is zero or 180 degrees. Accordingly let

$$\Omega = \alpha \sin pt, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (174a)$$

$$\xi = \beta \sin pt. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (174b)$$

Then equation (172b) becomes

$$\alpha\,(h_\Omega - p^2 P_2) + \beta\,(h_\xi - p^2 D_2) = 0,$$

or

$$P_2 = \frac{h_\Omega}{p^2} + \frac{\beta}{\alpha}\left(\frac{h_\xi}{p^2} - D_2\right). \qquad \ldots \qquad \ldots \qquad (175)$$

In a particular experiment the data were :—
$h_\Omega = 0 \cdot 0169$, $h_\xi = 0 \cdot 034$, $D_2 = 1 \cdot 33 \times 10^{-4}$, $\beta/\alpha = -0 \cdot 884$, period $0 \cdot 8875$ sec., giving $p = 7 \cdot 085$.

Hence from equation (175)

$$P_2 = \frac{0 \cdot 0169}{50 \cdot 2} - 0 \cdot 884\left(\frac{0 \cdot 034}{50 \cdot 2} - 1 \cdot 33 \times 10^{-4}\right)$$
$$= 3 \cdot 367 \times 10^{-4} - 4 \cdot 813 \times 10^{-4}$$
$$= -1 \cdot 446 \times 10^{-4} \text{ slug ft.}^2.$$

A similar experiment with period 0·6288 gave the result $-1{\cdot}404 \times 10^{-4}$. The value of P_2 accepted is the mean, say $-1{\cdot}42 \times 10^{-4}$.

§ 99. *Stiffness Derivatives.*—Since the tailplane, fin, and rudder were always at small angles of incidence, it was assumed that no changes in the aerodynamical moments were caused by a rotational displacement of the fuselage, i.e., it was assumed that the derivatives T_Ω and H_Ω were zero. The truth of this assumption as regards T_Ω is supported by the constancy of the period of the free torsional oscillations at various wind speeds (*see* § 100b.). Thus the only derivatives in this group to be determined were H_ξ and T_ξ.

§ 99a. *Rudder Direct Aerodynamical Stiffness* H_ξ.—On account of the large horn balance provided, the rudder was only very slightly underbalanced aerodynamically. The derivative H_ξ was accordingly very small and correspondingly difficult to measure with accuracy.

The fuselage was fixed and one end of the rudder recording thread anchored through a very extensible helical spring. On the other side of the rudder the thread was passed over an almost frictionless pulley and connected to a light scale pan. The procedure consisted in measuring the rudder displacements for a series of loads for still air and for various wind speeds. For each wind speed the load was plotted against the displacement, and the slope of the graph at the point of zero displacement gave the total stiffness, from which the aerodynamical stiffness was obtained by difference. The final results are given in Table 35.

TABLE 35.

Derivative H_ξ.

Wind Speed V ft./sec.	$\frac{H_\xi}{V^2}$
20	$-6{\cdot}55 \times 10^{-6}$
25	$-7{\cdot}31$
25	$-7{\cdot}66$
30	$-7{\cdot}52$
30	$-6{\cdot}00$
30	$-6{\cdot}29$
35	$-7{\cdot}90$
40	$-6{\cdot}78$

The mean of the tabulated values is $-7{\cdot}00 \times 10^{-6}$.

It was noted that the rudder thread was blown downstream perceptibly, and it was thought that the drag on the thread might be contributing appreciably to the measured value of the derivative. Some repeat measurements were accordingly made with streamlined tin guards fitted over the thread, and the results are given in Table 36.

TABLE 36.

Derivative H_ξ.

(Guards fitted over thread.)

Wind Speed V ft./sec.	$\frac{H_\xi}{V^2}$
20	$-7 \cdot 45 \times 10^{-6}$
25	$-6 \cdot 82$
30	$-6 \cdot 30$
35	$-7 \cdot 31$

The mean of these results is $-6 \cdot 97 \times 10^{-6}$ and agrees with the previous mean very closely. The accepted value is -7×10^{-6}.

§ 99b. *Fuselage Aerodynamical Cross-Stiffness* T_ξ.—The rudder could be fixed at various angles by means of a small divided quadrant fitted to the fuselage. The rudder thread was disconnected, and one end of the fuselage thread was anchored through a helical spring. On the other side of the fuselage this thread was passed over a pulley and connected to a scale pan and dashpot. Weights were placed in the scale pan until the angular displacement of the fuselage vanished, as indicated by a beam of light reflected on to a fixed scale from a mirror attached to the fuselage. This was done in still air and for a number of wind speeds; in each case the aerodynamical moment was obtained by difference. A series of such tests provided a set of corresponding rudder angles and aerodynamical moments for each of the selected wind speeds. It was found that the moments plotted linearly against rudder angle within the range $\pm 5°$, and from the slopes of these graphs the values of the derivative were calculated. The results are given in Table 37.

TABLE 37.

Derivative T_ξ.

Wind Speed V ft./sec.	$\frac{T_\xi}{V^2}$
20	$2 \cdot 08 \times 10^{-4}$
25	$1 \cdot 96$
30	$1 \cdot 96$
35	$1 \cdot 92$
40	$1 \cdot 96$

The value accepted is the mean, $1 \cdot 98 \times 10^{-4}$.

§ 100. *Direct Damping Derivatives.*—The derivatives $H_{\dot{\xi}}$ and $T_{\dot{\Omega}}$ were obtained by measurement of the logarithmic decrement of the free oscillation in the corresponding degree of freedom. Motion in the remaining degree of freedom was prevented, and photographic records of the damped oscillations for a number of different wind speeds were obtained.

§ 100a. *Rudder Direct Damping Derivative* $H_{\dot{\xi}}$.—The equation governing the free oscillation of the rudder is (*see* equation (172b))

$$D_2\ddot{\xi} + E_2\dot{\xi} + F_2\xi = 0. \quad \ldots \quad \ldots \quad (176)$$

Let T be the periodic time and ξ_n, ξ_{n+1} consecutive maxima (or minima) as given by the record. Then

$$E_2 = -H_{\dot{\xi}} = \frac{2D_2}{T}\log_e\left(\xi_n/\xi_{n+1}\right). \quad \ldots \quad \ldots \quad (177)$$

In Fig. 38 the values of $H_{\dot{\xi}}$, corrected for the hysteresis damping of the springs, etc., is plotted against wind speed. For wind speeds above 10 ft./sec. the value of the derivative is given by

$$\frac{H_{\dot{\xi}}}{V} = -8{\cdot}3 \times 10^{-6}$$

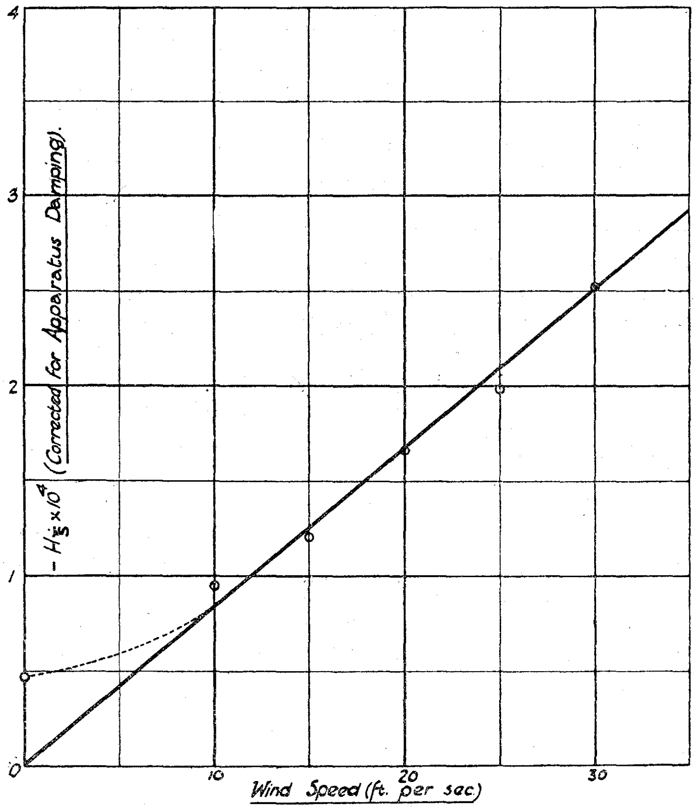


FIG. 38.—Rudder Direct Damping Derivative $H_{\dot{\xi}}$.

In various experiments described in §§ 101a and 102, allowance has been made for the contribution to $H_{\dot{\xi}}$ due to the elastic hysteresis of the rudder springs employed.

§ 100b. *Fuselage Direct Damping Derivative* $T_{\dot{\Omega}}$.—The derivative was calculated from a formula analogous to (177), namely

$$Q_0 = -T_{\dot{\Omega}} = \frac{2P_0}{T} \log_e (\Omega_n/\Omega_{n+1}) \quad \ldots \quad \ldots \quad (178)$$

In the present instance the hysteresis damping due to the springs, etc., was negligibly small in comparison with the aerodynamical damping. The results of the measurements are given in Table 38 and Fig. 39.

TABLE 38.

Derivative $T_{\dot{\Omega}}$.

Wind Speed V ft./sec.	Periodic Time Sec.	$T_{\dot{\Omega}}$.
0	0·4318	$-0·162 \times 10^{-2}$
10	0·4306	$-0·443$
15	0·4305	$-0·635$
25	0·4283	$-1·131$
30	0·4300	$-1·311$
35	0·4338	$-1·464$

Since the same springs were used throughout, the constancy of the periodic time serves to confirm that the stiffness derivative T_{Ω} is zero (cp. § 99). The value of the damping derivative is given by

$$\frac{T_{\dot{\Omega}}}{V} = -4·32 \times 10^{-4}.$$

§ 101. *Compound Damping Derivatives.*—As is usually the case, these derivatives were more troublesome to measure than the remainder. The general method employed was to impress a definite simple harmonic motion in one degree of freedom by means of the forcing motor, and to record photographically the motions of the system which occurred at a known wind speed. Analysis of a record leads to a pair of equations, one of which gives a value of the derivative required, while the other serves as a check upon the quantities already measured.

§ 101a. *Rudder Compound Damping Derivative* $H_{\dot{\Omega}}$.—In this instance the forcing motor was connected to the fuselage as for the measurement of the product of inertia (*see* § 98). The dynamical equation (172b) is still applicable, but on account of the presence of

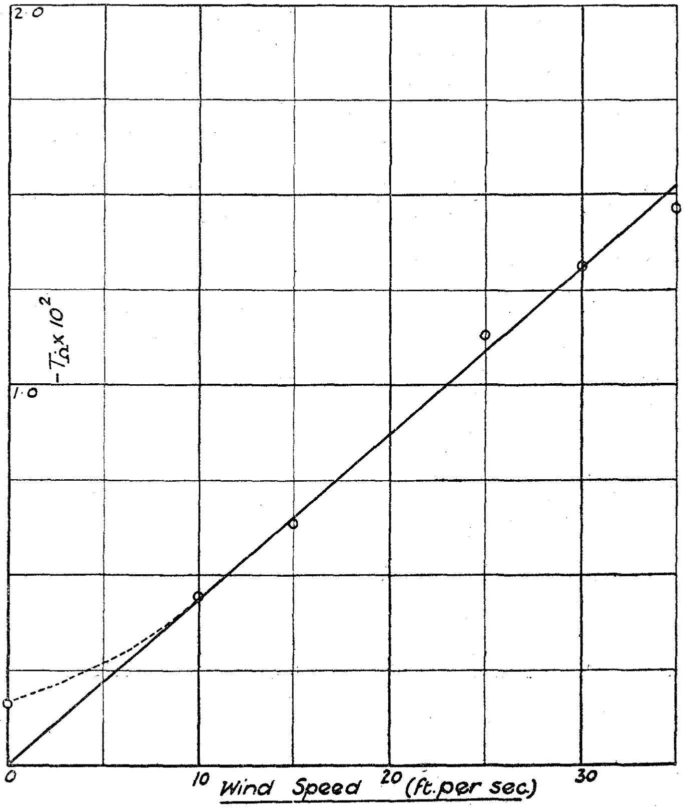


FIG. 39.—Fuselage Direct Damping Derivative $T_{\dot{\Omega}}$.

damping coefficients the oscillations in the two degrees of freedom are no longer in phase. Suppose that analysis of the record yields the results

$$\Omega = \alpha \sin pt, \quad \text{.. .. .. ..} \quad (179a)$$

$$\xi = \beta \sin pt + \gamma \cos pt. \quad \text{.. ..} \quad (179b)$$

Substitute these values in (172b) and equate separately to zero the coefficients of sin pt and of cos pt:—

$$R_2 - p^2P_2 + \frac{\beta}{\alpha}(F_2 - p^2D_2) - \frac{\gamma}{\alpha}pE_2 = 0, \quad \text{..} \quad (180a)$$

$$pQ_2 + \frac{\beta}{\alpha}pE_2 + \frac{\gamma}{\alpha}(F_2 - p^2D_2) = 0. \quad \text{.. ..} \quad (180b)$$

Equation (180a) is the check equation, while (180b) gives the required derivative $H_{\dot{\Omega}}$ $(= - Q_2)$.

The derivative was found to be very small and difficult to measure. When the forcing frequency was not near resonance, the phase difference was very small and it was found to be impossible to measure it to a sufficient percentage accuracy. Accordingly the forcing frequency was adjusted to be very close to the resonance value. Under this condition the phase difference was large, but a very small forcing amplitude had to be used in order to prevent the amplitude of the rudder oscillations from becoming excessive.

Experiment 1. Data :– $P_2 = -1 \cdot 42 \times 10^{-4}$, $R_2 = 0 \cdot 0169$, $D_2 = 1 \cdot 33 \times 10^{-4}$, $E_2 = 2 \cdot 38 \times 10^{-4}$, $F_2 = 0 \cdot 0384$, $V = 25$, $p = 18 \cdot 74$, $\frac{\beta}{\alpha} = 6 \cdot 59$, $\frac{\gamma}{\alpha} = 4 \cdot 234$.

In the first place the data will be substituted in the check equation (180a). The best way to present the result is to give separately the positive and negative items and to compare the aggregates, which should strictly be equal.

Positive Terms.		*Negative Terms.*	
$-p^2P_2$	0·0499	$-\frac{\beta}{\alpha}p^2D_2$	0·3079
R_2	0·0169	$-\frac{\gamma}{\alpha}pE_2$	0·0188
$\frac{\beta}{\alpha}F_2$	0·2529		
	0·3197		0·3267

The agreement of the two aggregates is reasonably close. Next, equation (180b) gives

$$\begin{aligned} Q_2 &= -\frac{\beta}{\alpha}E_2 - \frac{1}{p}\frac{\gamma}{\alpha}(F_2 - p^2D_2) \\ &= -15 \cdot 68 \times 10^{-4} + 18 \cdot 75 \times 10^{-4} \\ &= 3 \cdot 07 \times 10^{-4}. \end{aligned}$$

Hence $$\frac{H_{\dot{Q}}}{V} = -1 \cdot 23 \times 10^{-5}.$$

Experiment 2. Data :– $P_2 = -1 \cdot 42 \times 10^{-4}$, $R_2 = 0 \cdot 0169$, $D_2 = 1 \cdot 33 \times 10^{-4}$, $E_2 = 2 \cdot 80 \times 10^{-4}$, $F_2 = 0 \cdot 0403$, $V = 30$, $p = 19 \cdot 10$, $\frac{\beta}{\alpha} = 5 \cdot 70$, $\frac{\gamma}{\alpha} = 4 \cdot 326$.

Check Equation.

Positive Terms.		*Negative Terms.*	
$-p^2P_2$	0·0518	$-\frac{\beta}{\alpha}p^2D_2$	0·2765
R_2	0·0169	$-\frac{\gamma}{\alpha}pE_2$	0·0231
$\frac{\beta}{\alpha}F_2$	0·2298		
	0·2985		0·2996

The agreement of the positive and negative totals is satisfactory.

By equation (180b)

$$Q_2 = -\ 15{\cdot}96 \times 10^{-4} + 18{\cdot}57 \times 10^{-4}$$
$$= 2{\cdot}61 \times 10^{-4},$$

and $$\frac{H_{\dot{\Omega}}}{V} = -\ 0{\cdot}87 \times 10^{-5}.$$

The mean of the two values of Q_2/V is $1{\cdot}05 \times 10^{-5}$ and the round figure of $1{\cdot}0 \times 10^{-5}$ will be accepted. It is shown in § 104a that complete neglect of $H_{\dot{\Omega}}$ only introduces a small error in the calculated full scale critical flutter speed.

For a normal rudder without horn balance the derivative $H_{\dot{\Omega}}$ will clearly be positive. Thus, although the present rudder is slightly underbalanced aerodynamically, the balance area provided is sufficient to change the sign of $H_{\dot{\Omega}}$. Hence the interesting conclusion emerges that the objectionable "coupling" derivative $H_{\dot{\Omega}}$ can be eliminated by the use of a horn balance without very close approach to aerodynamical balance. This confirms the footnote to p. 138.

§ 101b. *Fuselage Compound Damping Derivative* $T_{\dot{\xi}}$.—Measurement of this derivative presented no particular difficulty. The general procedure was similar to that described for the case of $H_{\dot{\Omega}}$, but here the forcing motor was connected to the rudder, and it was not found necessary to work in the neighbourhood of resonance.

The dynamical equation for the fuselage motions is (172a). Let the result of the analysis of the record of the motion be expressed in the form

$$\xi = \alpha \sin pt, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (181a)$$

$$\Omega = \beta \sin pt + \gamma \cos pt \quad \ldots \quad \ldots \quad \ldots \quad (181b)$$

Then substitution in (172a) gives the following equations, which correspond to (180)

$$F_0 - p^2 D_0 + \frac{\beta}{\alpha}(R_0 - p^2 P_0) - \frac{\gamma}{\alpha} p\, Q_0 = 0, \quad \ldots \quad (182a)$$

$$pE_0 + \frac{\beta}{\alpha} pQ_0 + \frac{\gamma}{\alpha}(R_0 - p^2 P_0) = 0 \quad \ldots \quad \ldots \quad (182b)$$

The first of these is the check equation, and the second determines the required derivative $T_{\dot{\xi}}$ $(= -\ E_0)$.

Experiment 1. Data :– $D_0 = -\ 1{\cdot}42 \times 10^{-4}$, $F_0 = -\ 6{\cdot}23 \times 10^{-2}$, $P_0 = 7{\cdot}93 \times 10^{-3}$, $Q_0 = 8{\cdot}7 \times 10^{-3}$, $R_0 = 1{\cdot}686$, $V = 20$, $p = 16{\cdot}12$, $\frac{\beta}{\alpha} = -\ 0{\cdot}0684$, $\frac{\gamma}{\alpha} = -\ 0{\cdot}0625$.

Check Equation.	*Positive Terms.*		*Negative terms.*	
	$-p^2D_0$	0·0369	F_0	0·0623
	$-\frac{\beta}{\alpha}p^2P_0$	0·1410	$\frac{\beta}{\alpha}R_0$	0·1153
	$-\frac{\gamma}{\alpha}pQ_0$	0·0088		0·1776
		0·1867		

By equation (182b) :—

$$E_0 = -\frac{\beta}{\alpha}Q_0 - \frac{\gamma}{\alpha}\frac{1}{p}(R_0 - p^2P_0)$$
$$= 5 \cdot 95 \times 10^{-4} - 14 \cdot 5 \times 10^{-4}$$
$$= -8 \cdot 55 \times 10^{-4},$$

and

$$\frac{T\xi}{V} = 4 \cdot 27 \times 10^{-5}.$$

Experiment 2. Data :- $D_0 = -1{\cdot}42 \times 10^{-4}$, $F_0 = -22{\cdot}57 \times 10^{-2}$, $P_0 = 7{\cdot}93 \times 10^{-3}$, $Q_0 = 1{\cdot}522 \times 10^{-2}$, $R_0 = 1{\cdot}686$, $V = 35$, $p = 17{\cdot}98$, $\frac{\beta}{\alpha} = -0{\cdot}1882$, $\frac{\gamma}{\alpha} = -0{\cdot}0934$.

Check Equation	*Positive Terms.*		*Negative Terms.*	
	$-p^2D_0$	0·0459	F_0	0·2257
	$-\frac{\beta}{\alpha}p^2P_0$	0·4824	$\frac{\beta}{\alpha}R_0$	0·3172
	$-\frac{\gamma}{\alpha}pQ_0$	0·0255		0·5429
		0·5538		

$$E_0 = 28 \cdot 66 \times 10^{-4} - 45 \cdot 56 \times 10^{-4}$$
$$= -16 \cdot 9 \times 10^{-4},$$

and

$$\frac{T\xi}{V} = 4 \cdot 83 \times 10^{-5}.$$

The value of $T\xi/V$ accepted is the mean, namely $4 \cdot 55 \times 10^{-5}$.

§ 102. *Flutter Tests of the Model.* —To provide a check upon the derivative measurements, and as a preliminary to the calculations for full scale, critical flutter speeds for the model under two different conditions were calculated and compared with the experimental values. In the first test the rudder was connected to the recording gear and weak control springs were used, while in the second the rudder was completely free to swing on its hinges.

The theoretical critical flutter speed for very small disturbances is that speed at which the Routh test function corresponding to the dynamical equations (172) changes sign from positive to negative. The test function is (*see* R. & M. 1155, § 18)

$$T_3 = q_1 q_2 q_3 - q_4 q_1^2 - q_0 q_3^2, \quad \ldots \quad \ldots \quad (183)$$

where the coefficients q are given by the equations :—

$$q_0 = P_0 D_2 - P_2^2, \quad \ldots \quad (184a)$$

$$q_1 = P_0 E_2 + Q_0 D_2 - P_2(E_0 + Q_2), \quad \ldots \quad (184b)$$

$$q_2 = P_0 F_2 + R_0 D_2 - P_2(F_0 + R_2) + Q_0 E_2 - Q_2 E_0, \quad (184c)$$

$$q_3 = Q_0 F_2 - Q_2 F_0 + R_0 E_2 - R_2 E_0, \quad \ldots \quad (184d)$$

$$q_4 = R_0 F_2 - R_2 F_0. \quad \ldots \quad (184e)$$

The frequency f of the flutter which occurs just at the critical speed can be calculated from the formula (*see* R. & M. 1155, equation (29)).

$$f = \frac{1}{2\pi} \sqrt{\frac{q_3}{q_1}}. \quad \ldots \quad \ldots \quad (185)$$

Table 39 summarises the data for flutter test No. 1, and Table 40 gives a comparison between the observed and calculated flutter speeds and frequencies. The agreement is considered satisfactory.

TABLE 39.

Data for Flutter Test No. 1.

(Rudder connected to recording gear. Control springs weak).

Coefft.	Value × 10³.	Coefft.	Value × 10³.
P_0	7·93	P_2	— 0·142
Q_0	0·432 V	Q_2	0·01 V
R_0	1700	R_2	16·9
D_0	— 0·142	D_2	0·133
E_0	— 0·0455 V	E_2	0·03 + 0·0083 V
F_0	16·9 — 0·198 V²	F_2	34 + 0·007 V²

TABLE 40.
Results of Flutter Test No. 1.

Critical Flutter Speed. ft./sec.			Flutter Frequency at Critical Speed. cycles per sec.	
Observed.		Calculated.		
Wind Speed Rising.	Wind Speed Falling.		Observed.	Calculated.
20·9 21·2 (Repeat)	19·5 19·6	19·7	2·59	2·57

Tables 41 and 42 give, respectively, the data and results of the second flutter test, in which the rudder was disconnected from the recording gear. The calculated critical speed is in good agreement with the observed speed at which a steady oscillation of moderate amplitude appeared.

TABLE 41.
Data for Flutter Test No. 2.
(Rudder free.)

Coefft.	Value × 10^3.	Coefft.	Value × 10^3.
P_0	7·93	P_2	— 0·142
Q_0	0·432 V	Q_2	0·01 V
R_0	1700	R_2	16·9
D_0	— 0·142	D_2	0·0935
E_0	— 0·0455 V	E_2	0·0083 V
F_0	16·9 — 0·198 V^2	F_2	0·007 V^2

TABLE 42.
Results of Flutter Test No. 2.

Critical Speed. ft./sec.		Frequency. cycles per sec.		Remarks.
Observed.	Calculated.	Observed.	Calculated.	
27·7 27·4 (Repeat)	26·0	2·05	2·13	Small oscillations steadily maintained.
29·5 29·3 (Repeat)	—	—	—	Very large oscillations generated.
22·3 17·3 (Repeat)	—	—	—	Existing large oscillations died out.

§ 103. *Flutter Speed for the Full Scale Machine.*—The values of the inertial constants and elastic stiffnesses of the aeroplane were supplied by the Mechanical Test Department of the Royal Aircraft Establishment and by the Airworthiness Department. All the constants for the rudder were obtained experimentally, while the elastic stiffness of the fuselage was calculated from the drawings. The estimation of the fuselage moment of inertia P_0 calls for some comment. Since the tailplane, elevators, fin, rudder and tail end of the fuselage partake of the whole torsional motion Ω, they contribute the whole of their moments of inertia about the fuselage axis to the coefficient P_0. On the other hand, the angular displacement of the middle portion of the fuselage is graded from zero to the full amount. An exact calculation of the effective moment of inertia of this part would be very troublesome, but it was decided to adopt one third of the total moment of inertia of the central part of the fuselage, extending from the leading edge of the tailplane to the trailing edge of the wings. Since this contribution only amounts to one-eighth of P_0, the estimate is probably sufficiently accurate.

The gravitational cross-stiffness h_Ω ($\equiv t_\xi$) was calculated from the relation

$$h_\Omega = W\bar{X}, \quad \ldots \quad \ldots \quad \ldots \quad (186)$$

where W = weight of rudder in lb.,

and $\bar{X}$ = distance of C.G. measured downstream from hinge

The values of the aerodynamical derivatives were deduced from those measured on the model, scale effect being assumed negligible. It is shown in R. & M. 1155, § 23 and Appendix VI, that stiffness and damping derivatives are proportional respectively to the third and fourth powers of the typical linear dimension. Hence in the present instance the stiffness derivatives are obtained by multiplication of the corresponding values for the model by the factor 8^3, while the factor for the damping derivatives is 8^4.

A rather heavy tail lamp bracket was attached to the rudder recovered from the wreckage of the machine. In the present section the critical speed is calculated for the case of tail lamp bracket and standard tail lamp fitted to the rudder, but the effect of removal of these appendages will be considered in § 104. Hysteresis damping is neglected, since its magnitude is quite unknown. However, some calculations on the effect of hysteresis are included in § 104.

No elastic control of the rudder is allowed for, i.e., the critical speed corresponds to the case where the pilot has removed his feet from the rudder bar. It is shown in § 104 that a weak elastic control reduces the critical speed (cp. the behaviour of the model, § 102).

TABLE 43.

Data for Full Scale Flutter Calculation.

Rudder free. Tail lamp and bracket fitted.

Coefft	Value.	Coefft.	Value.
P_0	44·7	P_2	— 1·15
Q_0	1·77 V	Q_2	0·041 V
R_0	33700	R_2	10·4
D_0	— 1·15	D_2	0·745
E_0	— 0·186 V	E_2	0·034 V
F_0	10·4 — 0·101 V^2	F_2	0·00358 V^2

Complete data for the calculation of the critical flutter speed are given in Table 43, and the final results are :—

Critical Flutter speed 238·6 ft. per sec., or 141·3 knots.

Flutter frequency at critical speed 4·07 cycles per sec.

The actual behaviour of the machine can be judged from the following extracts from the report on the accident :—

"The pilot next proceeded to test the rudder for 'hunting' by taking his feet off the rudder bar and gradually increasing speed with the engine while the machine descended at a gliding angle. Rudder oscillations started immediately and developed steadily as the aeroplane gathered speed, until, at about 120 knots, the oscillations suddenly became so violent that the pilot shut off the engine and pulled out of the dive, at the same time putting his feet on the rudder bar and checking its movement.

After recovering normal flight . . . . the pilot. . . . put the aeroplane into a dive as before, but this time he held the rudder bar stationary. Up to a speed of 140–150 knots the machine behaved normally, but almost as soon as the pilot opened up the engine a little more, in order to gain further speed, the rudder started to flutter very violently."

It is evident from the report that the critical flutter speed lay somewhere in the range 120–150 knots. The calculated speed is thus in substantial agreement with fact, and there can be no reasonable doubt that the accident was due to the development of tail flutter, predominantly of the rudder-fuselage type.

§ 104. *Influence of Individual Factors on the Full Scale Critical Speed.*—The present section will be devoted to the examination of the influence of certain hypothetical changes in the dynamical constants of the system upon the critical flutter speed.

§ 104a. *Rejection of the Derivative* $H_{\dot{\Omega}}$.—As stated in § 101a, the derivative $H_{\dot{\Omega}}$ was very small and difficult to measure with accuracy. It has accordingly been considered advisable to repeat the calculation for the critical speed with this derivative assumed zero. The data are as in Table 43, except that Q_2 is zero, and the calculated critical speed is 147·1 knots, as against 141·3 knots with Q_2 retained. Hence it appears that the experimental errors in Q_2 cannot affect the critical speed materially. It must not be supposed, however, that the neglect of Q_2 would, in general, be legitimate.

§ 104b. *The Influence of Hysteresis.*—The actual magnitude of the hysteresis damping of the torsional motion of the fuselage was unknown, but a representative calculation of the critical speed has been made on the supposition that the hysteresis is equivalent to the aerodynamical damping at 50 ft. per sec. The data are as for Table 43, except that Q_0 is $(1 \cdot 77\ V + 88 \cdot 5)$. It is found that the critical speed is raised from 141·3 knots to 152·5 knots, so that the influence of a moderate amount of hysteresis is not great.

§ 104c. *Neglect of the Gravitational Cross-Stiffness.*—When t_ξ $(= h_\Omega)$ is rejected, the term 10·4 disappears from F_0 and R_2 in Table 43. The critical speed is now 141·9 knots, (239·8 ft./sec.), so that the influence of the gravitational cross-stiffness is negligible.

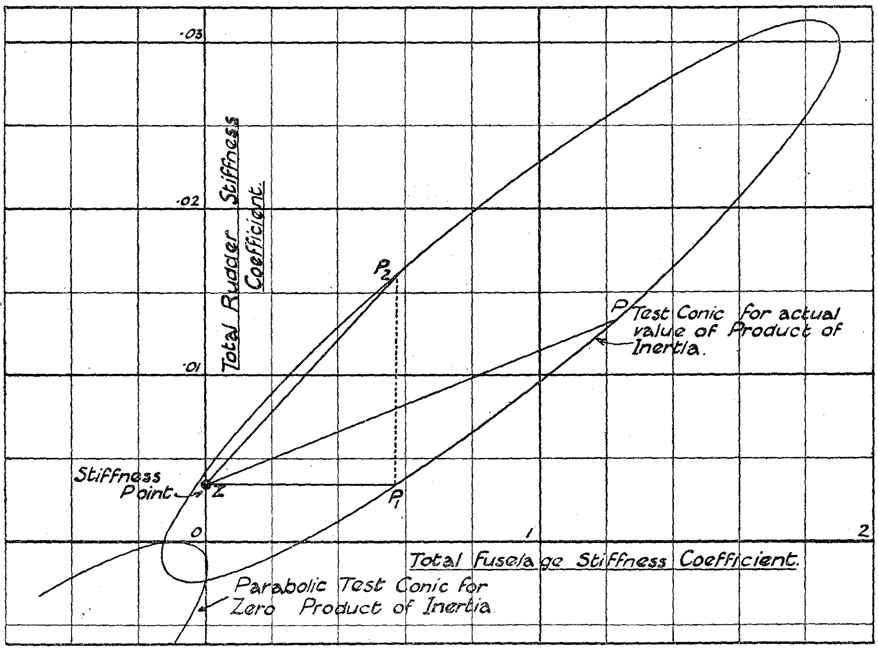


Fig. 40.—Test Conics for Rudder-Fuselage Motion.

This term was omitted in the theory developed in Chapter IV, and was also omitted, for simplicity, from all the calculations which are described later in the present report.

§ 104d. *Elimination of the Product of Inertia: Test Conics.*—When hysteresis and the gravitational cross-stiffness are absent, the critical flutter speed for any values of the direct elastic stiffnesses can be found from the " test conic " (*see* R. & M. 1155, Chap. III, and present text § 84a). Test conics are shown plotted in Fig. 40 for the original data as per Table 43 (except that the gravitational term 10·4 is omitted from F_0 and R_2), and for the case where the product of inertia D_0 ($= P_2$) is reduced to zero. It will be seen that no stiffness line* can be drawn through the stiffness point Z so as to intersect the parabolic test conic in a point having a positive abscissa. Thus, when the product of inertia is zero and the other data remain unchanged, there can be no flutter. When, however, the product of inertia has its measured value, the stiffness point lies inside the elliptic test conic, so that flutter sooner or later occurs whatever may be the elastic stiffnesses of the fuselage and of the rudder control. Consider the stiffness line ZP and let X be the abscissa of P. Then the critical speed V_c corresponding to the particular stiffnesses in use is given by

$$V_c = \sqrt{\frac{t_\Omega}{X}} \quad \ldots \quad \ldots \quad \ldots \quad (187)$$

As an example, let t_Ω have the actual value of 33,700 and let the rudder stiffness be zero, so that the stiffness line is horizontal. Then,

$$V_c = \sqrt{\frac{t_\Omega}{Z\,P_1}} = \sqrt{\frac{33700}{0 \cdot 578}} = 241 \cdot 5 \text{ ft. per sec.}$$

This agrees closely with the calculated figure of 239·8 (*see* § 104c). The effect of an elastic control of the rudder will only be to raise the critical speed above that corresponding to the free rudder, if the abscissa of P is less than the abscissa of P_1. Hence, to secure an elevation of the critical speed the stiffness ratio must be greater than the slope of Z P_2. From the figure this slope is 0·0218, and if t_Ω is 33,700, then the value of the rudder stiffness h_ξ is 735. This stiffness is almost certainly higher than any which could be produced by the application of the pilot's feet to the rudder bar. In a comparison with observed effects (*see* § 103) it must, however, be remembered that the forces exerted by the pilot are not simply of the nature of elastic restoring forces. It may be added that an irreversible rudder control would give rise to a vertical stiffness line in the diagram, and therefore to an infinite critical speed (*see* equation (187)).

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* The slope of the stiffness line is necessarily positive.

It has already been shown that no flutter can occur when the rudder is dynamically balanced and all other factors remain unaltered. A supplementary investigation of the stability of the system has been made for the case where the rudder is mass balanced, but has no horn balance area. The values of some of the derivatives can only be roughly estimated, but calculation shows that with these values, flutter would be avoided with a considerable margin of safety. This result suggests that mass balance of the rudder is usually sufficient as a preventative for rudder-fuselage flutter, and that special measures for the suppression of $H_{\dot{\Omega}}$ will not generally be necessary.

§ 104e. *Change of the Mass of the Tail Lamp and Bracket.*—Since the tail lamp and bracket contributed largely to the inertial constants of the rudder, it was considered instructive to calculate the effect of both increase and decrease of the mass of these appendages. Let a unit of mass be equal to the mass of the tail lamp with bracket (actually 2·69 lb.). Then the modified inertial coefficients corresponding to the case where N such units are concentrated at the original position are

$$P_0 = 43\cdot7 + 0\cdot98\ N,$$
$$D_2 = 0\cdot238 + 0\cdot507\ N,$$
$$P_2 = D_0 = -\ 0\cdot441 - 0\cdot708\ N.$$

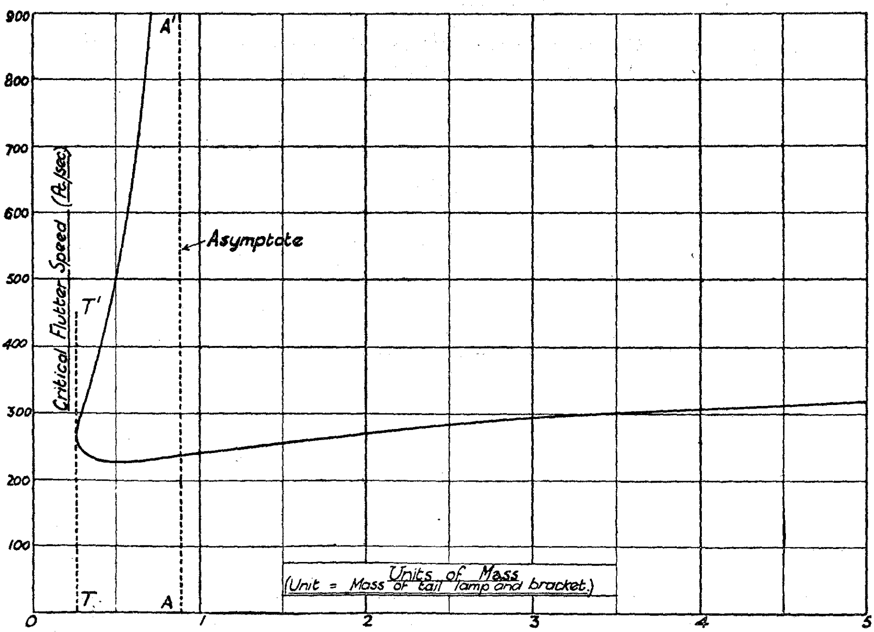

Fig. 41. Influence of Mass of Tail Lamp on Flutter Speed.

Calculations of the critical flutter speeds have been made for values of N ranging from 0 to 4, and the results are plotted in Fig. 41. For values of N less than that corresponding to the vertical tangent TT′ there can be no flutter; in particular, when the tail lamp and bracket are removed there can be no flutter. The lamp bracket alone corresponds roughly to $N = 0 \cdot 35$. The corresponding point is near the bend of the curve in Fig. 41, and there are two critical speeds, but the lower critical speed is almost identical with that for $N = 1$. For values of N greater than $0 \cdot 882$ (the abscissa of the asymptote AA′) there is only a single critical speed.

§ 105. *General Conclusions.*—The principal conclusions of the investigation may be summarised as follows :—

(*a*) The accident was due to tail flutter, essentially of the rudder-fuselage type.

(*b*) Flutter would not have occurred if the rudder had been dynamically balanced (product of inertia zero).

(*c*) Calculation indicates that flutter would even have been avoided in the present instance if the tail lamp and bracket had not been fitted to the rudder.

(*d*) It is possible to eliminate the rudder compound damping $H_{\dot{\Omega}}$ (which if large might be objectionable) by use of a horn balance without very close approach to aerodynamical balance of the rudder.*

(*e*) The gravitational cross-stiffness of the rudder has a negligible influence on the critical flutter speed.

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* As pointed out in Chapter IV, very close approach to aerodynamical balance is in certain circumstances dangerous.

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15	N.A.C.A. No. 285.	" A Study of Wing Flutter ". A. F. Zahm and R. M. Bear. 1928 (Revise of earlier Report, dated 1926).
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Ref. Number.	Official Reference.	Title.
17	R. & M. 1257	" Comparison of Calculated and Measured Elasticity of the Wings of an Aircraft, etc." K. T. Spencer and D. Seed. April, 1929.
18	—	" Über das Leitwerkflattern und die Mittel zu seiner Verhütung ". F. N. Scheubel, Berichte und Abhandlungen der wissenschaftlichen Gesellschaft für Luftfahrt. December, 1926.

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