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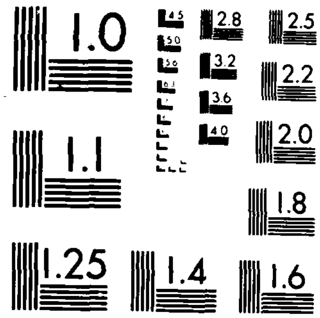

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HIGH FREQUENCY ANALYSIS OF CIRCULAR ARCHES

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September 1986

Final Report

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Air Force Systems Command
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This report presents the theory of the dynamic response of a linearly elastic, circular arch, including the effects of both shear deformation and rotary inertia. The arch central angle is arbitrary, and the end restraints are linearly elastic. The theory is extended to include linear viscoelastic behavior, in which both arch and end restraints are governed by the same single linear rate sensitivity parameter.

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SUMMARY

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1.0 INTRODUCTION

1.1 Purpose

This report presents the theory of the dynamic response of a linearly elastic, circular arch, including the effects of both shear deformation and rotary inertia. The arch central angle is arbitrary, and the end restraints are linearly elastic. The theory is extended to include linear viscoelastic behavior, in which both arch and end restraints are governed by the same single linear rate sensitivity parameter.

1.2 Motivation

The work described herein is an analytical extension of results presented in Reference 1. The approach was motivated by Dr. Timothy Ross' success in explaining direct shear failure of dynamically loaded reinforced concrete box roof slabs using a Timoshenko beam model and an appropriate reinforced concrete direct shear failure criterion (Ref. 2).

1.3 Application

When the theoretical results presented herein have been evaluated numerically for prescribed arch central angle, end restraint, and loading, they will apply to the KACHINA arches tested at the Air Force Weapons Laboratory (Refs. 3-5). Admittedly reinforced concrete is not always linearly elastic, so application of the theory to the KACHINA arches will require judgment. Nevertheless, the convenience of a closed form solution afforded by elastic theory is too attractive not to exploit for the perspective it gives on overall dynamic behavior and the influence of structural and loading parameters.

1.4 Point of Departure

The point of departure for the dynamic analysis is the set of equations for the static behavior of a circular planar member presented by Connor in Chapter 14 of Reference 6. Other related work is presented in References 7-11.

Details of the dynamic analysis are contained in Appendices A-L. The discussion in Section 2 below summarizes each appendix in order. Because of the detailed treatment and the many equations, a few letter symbols have been used differently in separate appendices. However, symbols are defined where first used in each appendix, so there should be no confusion. See Appendix M.

2.0 DISCUSSION

Figure 1 shows a portion of a circular arch, and defines the terms which control its dynamic response. The arch centerline radius is R, and the central angle defining a particular plane normal cross section, PQ, in the undeformed arch is θ. The width and depth of the constant rectangular arch cross section are b and d. Deformation of the arch causes the centroid of a plane cross section PQ to displace tangentially by an amount u_1 and radially inward by an amount u_2, and the cross section to rotate through an angle ψ; but the section is assumed to remain plane, so that P^*Q^* is a straight line. The distributed tangential, radial, and moment loads acting at point 0 in the undeformed arch are b_1, b_2, and m. Notice that the angle ψ is an independent displacement parameter, so that P^*Q^* is generally not normal to the deformed centroidal axis. The angular amount by which section P^*Q^* deviates from being normal to the deformed centroidal axis is the arch shear deformation at point 0.

Appendix A begins by formulating the equations of motion for a differential arch element, using the internal stress resultants (Equations A14-A16). Next, the form of the displacement field is defined by assuming that plane cross sections normal to the undeformed arch centroidal axis remain plane during deformation. This is equivalent to expanding the displacements in a Taylor series in two variables about the centroid (arc length and inward radial position) and retaining only the constant and linear terms. Having assumed the form of the displacement field (Equation A23), the extensional and shear strains are obtained as functions of the displacement field parameters and their derivatives with respect to arc length (Equations A31-A33). Three

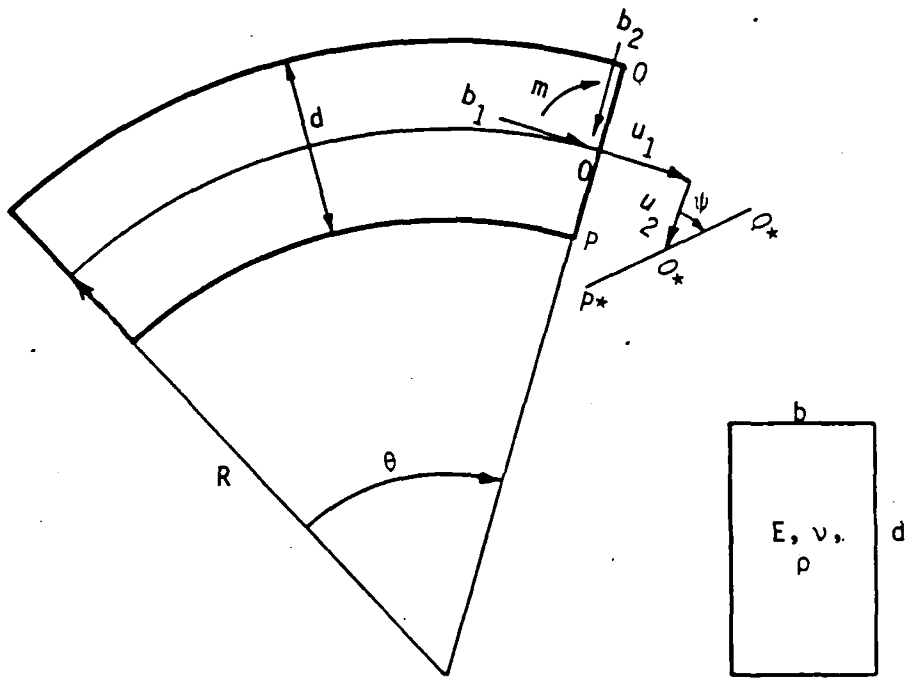

Figure 1. Definition of Terms Controlling the Dynamic Response of a Circular Arch.

strain parameters are defined at the centroid (Equations A34-A36), and then the strain components anywhere in the arch are expressed as functions of the strain parameters (Equations A37-A39). Normal and shear stresses are obtained as functions of the strain parameters, using elastic stress-strain equations (Equations A40 and A41), and then integrated over the cross section to obtain the internal stress resultants as functions of the strain parameters (Equations A42-A44). The stress resultant equations of motion, stress resultant-strain component equations, and strain component-displacement equations are summarized in matrix form in Equations A61, A63, and A64. Use of matrix notation greatly simplifies the entire analysis. It is compact, helps avoid algebraic errors, and even suggests analytical approaches not obvious when equations are written out in detail.

The equations of motion in terms of displacement components and their derivatives are obtained in matrix form in Appendix B, by substituting Equation A64 into Equation A63 and that result into Equation A61. The final result is Equation B17.

Appendix C considers the homogeneous (free vibration) form of the equations of motion (C2), and uses the classic separation of variables approach to define a vibration mode (Equation C3). Free vibration in a single mode is shown to be harmonic (Equation C15), and the mode shapes are shown to obey a coupled set of three, linear, ordinary, second order differential equations (C18). The three, coupled, mode shape differential equations are uncoupled by pre-multiplying them by the adjoint of the coefficient operator matrix (Ref. 12), but at the cost of having the resulting three separate identical linear ordinary differential equations be of order six instead of two (Equations C21-C23).

Equation C20 defines the sixth order, linear, ordinary differential operator, Δ, which appears in each of Equations C21-C23. These are the spatial differential equations which the mode shapes must satisfy. The operator, Δ, is defined as a determinant, which when expanded yields a sixth order polynomial in the frequency parameter, σ, (Equation D17) or a sixth order polynomial in the spatial differential operator, D, (Equation D18). Equation D18 is the more fundamental form, and shows the frequency dependence of its coefficients.

The last half of Appendix D discusses the way in which the coefficients in Equation D18 vary with frequency.

The operator, Δ, defined by Equation C20 is a sixth order, linear, ordinary differential operator, but the odd order derivatives are missing. Therefore, assuming the solution of the homogeneous equation $\Delta X = 0$ (Equation C21) to be of the form $X = Ke^{\lambda\theta}$ yields a cubic characteristic equation in λ^2 (E6). Whether the parameter λ is real, imaginary, or complex depends on the coefficients of the cubic equation. Appendix E discusses the algebraic solution of a cubic equation, as the basis for a detailed examination of the nature of the characteristic roots and their associated mode shape functions, which appears in Appendix F.

Appendix F examines the fifteen possible characteristic root combinations and their associated mode functions. The mode functions are a closed set of six linearly independent functions, in the sense that differentiation of any one yields a linear combination of the other five.

Appendix G establishes the derivative properties of symmetric and antisymmetric functions needed to define arch centerline boundary conditions for symmetric and antisymmetric modes.

Appendix H explains in detail how the natural frequencies and associated mode shapes are calculated for given arch boundary conditions. The feature which distinguishes this modal analysis from many others is the fact that the precise form of the frequency equation is frequency dependent.

Appendix I presents the arch modal orthogonality relations. The approach is that for a Sturm-Liouville problem, supplemented by Rayleigh's ingenious application of L'Hospital's Rule to find the integral of the inner product of a mode with itself over the arch length (Refs. 13 and 14).

Once the free vibration mode orthogonality relations have been established, the formal solution of a transient forced vibration problem becomes straightforward. Both the arch displacements and distributed loads are assumed to be expressible as modal series expansions, and the equations of motion are then scanned with a particular mode shape. The result is a single degree of freedom differential equation for the associated modal amplitude (J6), the solution of which is a Duhamel convolution integral (Equation J8).

Appendix K extends the above elastic arch analysis to the case of viscoelastic behavior. When both arch and boundary restraints are governed by the same single rate sensitivity parameter, the elastic modal analysis still applies.

Appendix L addresses the most difficult computational phase of arch modal analysis, finding the roots of the frequency equation (H7). The frequency equation is a complicated transcendental equation, usually involving both trigonometric and hyperbolic terms. The modal frequency parameter, σ, not only appears implicitly in several places, but also controls the precise form of the frequency equation. Considerable care is needed in calculating the higher modal frequencies, because some of them are apt to be closely spaced.

3.0 CONCLUSIONS

The distinguishing feature of the closed form, dynamic arch analysis presented in this report is its extensive use of matrix notation. The analysis includes the effects of both shear deformation and rotary inertia on the transient response of both an elastic and a viscoelastic circular arch, having an arbitrary central angle and corresponding elastic or viscoelastic end restraints. The resulting equations are presented at a level of detail sufficient for direct computer programming, and a computer program (a modification of ZEROIN) is presented for finding the modal frequencies.

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6.0 LIST OF SYMBOLS

Listed below are the symbols used in this report, and the page where each is introduced and defined. A semicolon indicates reuse of a symbol with a different definition.

Symbol	Page
A	23; 47; 57
A_1	58
A_2	58
A_3	58
A_j	59
A^*	102
B	48; 57
B_1	58; 75
B_2	58
B_3	59
B_j	59
B_k	61
$\underline{B}$	97
B_{ij}	100
C	48
C_1	35
C_2	35
C_3	36
$\underline{C}$	95
$\{C\}$	97
D	4, 36
E	3, 33; 48
$\underline{E}$	97
E^*	102
F	48
$\bar{F}$	23

Symbol	Page
S	64
$\underline{S}$	38
T	43
T_i	109
T_j	108
(U)	43
$\lceil \bar{U} \rfloor$	44
$(U)_i$	99
$(U)_j$	99
W_1	63
W_2	63
W_3	63
W_4	63
W_5	63
W_6	63
(W)	95
W_i	96
X	5, 43
X_i	101
X_j	101
Y	43
Y_i	101
Y_j	101
Z	43
Z_i	101
Z_j	101
a	51
b	2; 51
b_1	2, 23
b_2	2, 23
$\bar{b}$	23
(b)	38
c	41; 54

Symbol	Page
d	2
d^*	33
e	5
f	104
$f(x)$	92
$f_E(x)$	92
$f_O(x)$	92
i	57
k	31
k'	37
m	2, 23
$\bar{m}$	23
(m)	43
n	52
p	43
p_j	106
r	36; 62
$\bar{r}$	25
$\bar{r}_2$	25
$\bar{r}^*$	28
$\bar{r}_2^*$	28
s	23
$\bar{t}_1$	23
$\bar{t}_2$	23
$\bar{t}_3$	23
u_1	2, 23
u_2	2, 23
$\bar{u}$	23
$\bar{u}_2$	28
(u)	38
x	29; 34
y	56
y_1	60
y_2	25; 60
y_3	25; 60

Symbol	Page
y_j	59
z	34; 53
Γ_i	106
Δ	4, 45
Ω	100
α_2	27, 28
α^*	110
γ_{12}	30
δ_1	31
δ_2	31
(δ)	38
δ_{ij}	106
ϵ_1	29
ϵ_2	29
η	110
θ	2
λ	5, 56
λ_I	56
λ_{II}	56
λ_{III}	56
λ_{11}	62
λ_{21}	62
λ_{31}	62
λ_{12}	63
λ_{22}	63
λ_{32}	63
λ_1	65
λ_2	65
λ_3	65
λ_n	106
ν	3, 37
ξ	110
π	58
ρ	3, 23

APPENDIX A

GOVERNING EQUATIONS FOR THE DYNAMIC RESPONSE OF A CIRCULAR ARCH, INCLUDING BOTH SHEAR AND ROTARY INERTIA

Figure A1 shows a differential element of a circular arch. For the sign convention shown, with right-handed, orthogonal unit vectors $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_3$,

$$\bar{t}_3 = \bar{t}_1 \times \bar{t}_2 = \text{constant} \qquad [A1]$$

$$\frac{d\bar{t}_1}{ds} = \bar{t}_1' = \frac{1}{R}\,\bar{t}_2 \qquad [A2]$$

$$\frac{d\bar{t}_2}{ds} = \bar{t}_2' = -\frac{1}{R}\,\bar{t}_1 \qquad [A3]$$

If we set

$$\bar{F} = F_1\bar{t}_1 + F_2\bar{t}_2 \qquad \text{(internal force resultant)} \qquad [A4]$$

$$\bar{M} = M\bar{t}_3 \qquad \text{(internal moment resultant)} \qquad [A5]$$

$$\bar{b} = b_1\bar{t}_1 + b_2\bar{t}_2 \qquad \text{(external distributed load)} \qquad [A6]$$

$$\bar{m} = m\bar{t}_3 \qquad \text{(external distributed moment)} \qquad [A7]$$

$$\bar{u} = u_1\bar{t}_1 + u_2\bar{t}_2 \qquad \text{(centroidal displacement)} \qquad [A8]$$

$$\bar{\psi} = \psi\bar{t}_3 \qquad \text{(cross-section rotation)} \qquad [A9]$$

then the translational equation of motion for the element is

$$-\bar{F} + \left[\bar{F} + \frac{\partial \bar{F}}{\partial s}\,ds\right] + \bar{b}ds = \rho A ds\ddot{\bar{u}} \qquad [A10]$$

where ρ is mass density, and A is cross-sectional area.

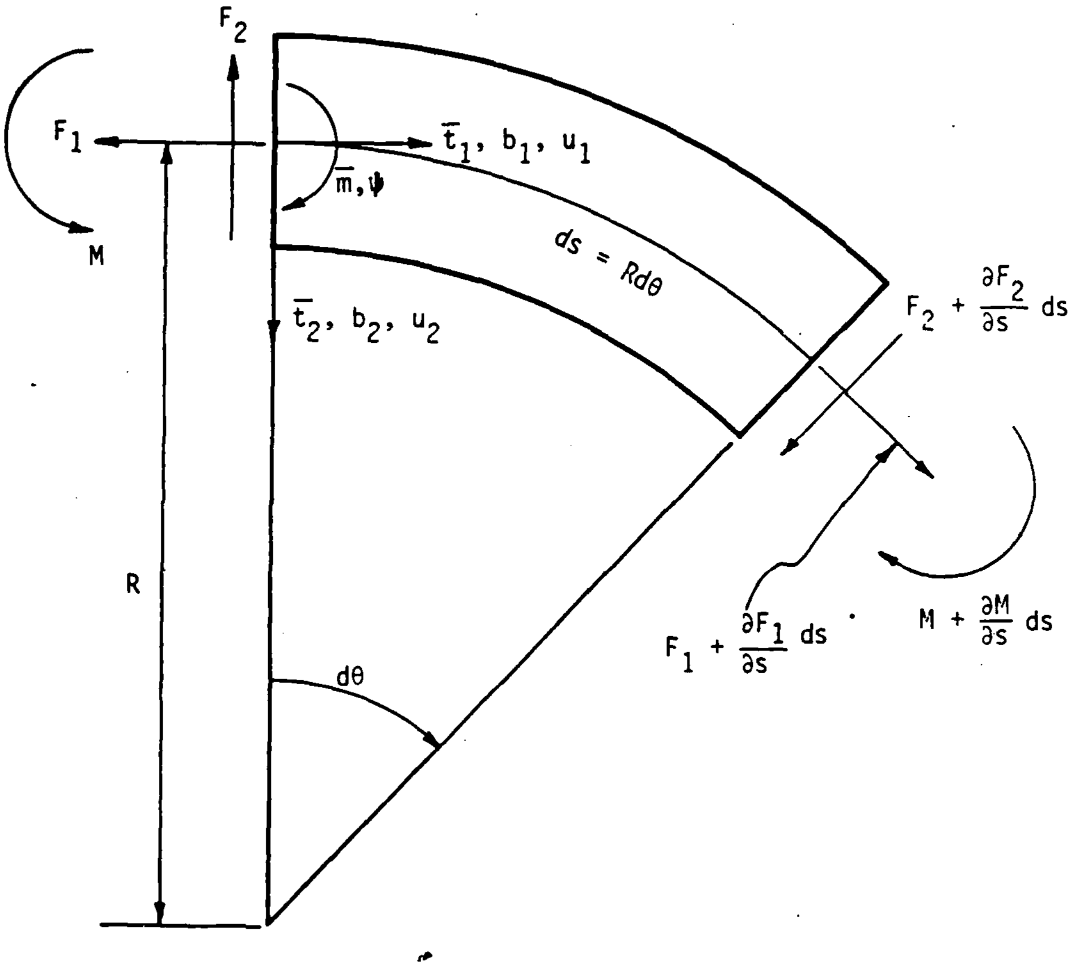

Figure A1. Differential Element of a Circular Arch.

or

$$\frac{\partial \bar{F}}{\partial s} + \bar{b} = \rho A \ddot{\bar{u}} \tag{A11}$$

and the rotational equation of motion for the element is

$$- \bar{M} + \left[\bar{M} + \frac{\partial \bar{M}}{\partial s} ds \right] + \overline{ds} \times \bar{F} + \bar{m} ds = \rho I ds \ddot{\bar{\psi}} \tag{A12}$$

where I is the moment of inertia. Then

$$\frac{\partial \bar{M}}{\partial s} + \bar{t}_1 \times \bar{F} + \bar{m} = \rho I \ddot{\bar{\psi}} \tag{A13}$$

Substitution of Equations A2 through A9 into Equations A11 and A13 yields

$$\frac{\partial F_1}{\partial s} - \frac{F_2}{R} + b_1 = \rho A \ddot{u}_1 \tag{A14}$$

$$\frac{\partial F_2}{\partial s} + \frac{F_1}{R} + b_2 = \rho A \ddot{u}_2 \tag{A15}$$

$$\frac{\partial M}{\partial s} + F_2 + m = \rho I \ddot{\psi} \tag{A16}$$

The next step in deriving the governing equations is to obtain expressions for the longitudinal extensional and shear strains at any point in an originally normal cross section. Consider a point Q in the undeformed arch shown in Figure A2, the position vector for which is denoted $\bar{r}_2$ in Figure A3. From Figures A2 and A3 we have

$$\bar{r}_2 = \bar{r} + y_2 \bar{t}_2 + y_3 \bar{t}_3 \tag{A17}$$

so that in Figure A3,

$$\overline{PP}_1 = \frac{\partial \bar{r}}{\partial s} ds = \bar{t}_1 \, ds \tag{A18}$$

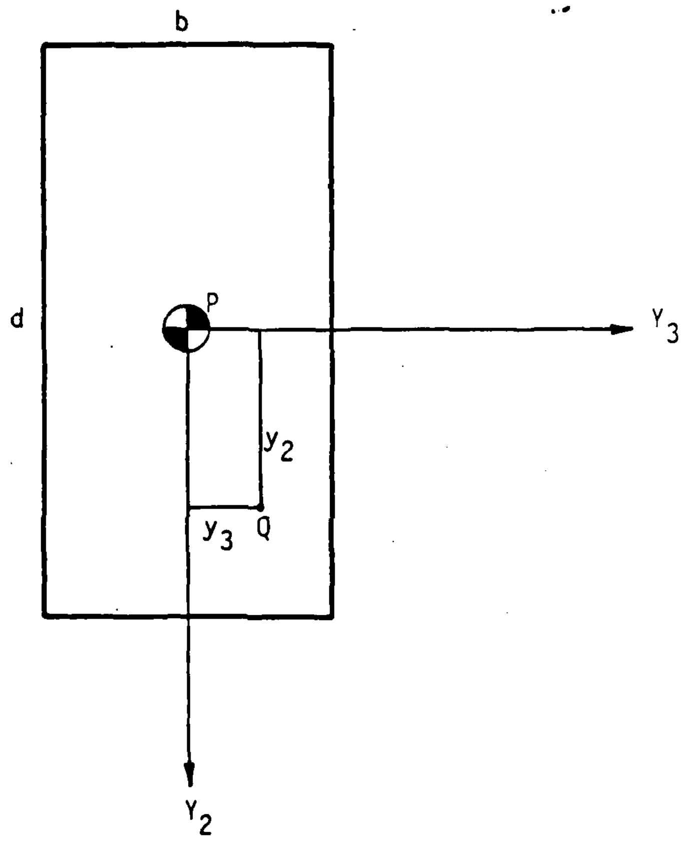

Figure A2. Normal Cross Section.

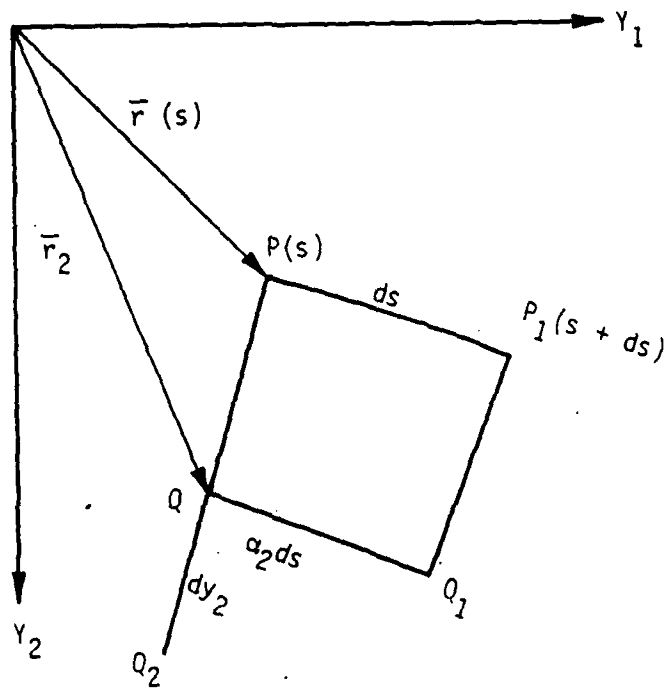

Figure A3. Differential Geometry in the Undeformed Arch.

$$\overline{QQ}_1 = \frac{\partial \bar{r}_2}{\partial s} ds = \left[1 - \frac{y_2}{R}\right] d s \bar{t}_1 = \alpha_2 d s \bar{t}_1 \qquad [A19]$$

$$\overline{QQ}_2 = \frac{\partial \bar{r}_2}{\partial y_2} dy_2 = dy_2 \bar{t}_2 \qquad [A20]$$

where

$$\alpha_2 = 1 - \frac{y_2}{R} \qquad [A21]$$

The centroidal displacement vector of point P in the deformed arch is

$$\bar{u} = u_1 \bar{t}_1 + u_2 \bar{t}_2 \qquad [A22]$$

and the assumption that plane cross sections normal to the centroidal axis before deformation remain plane, and normal to the 1-2 plane during deformation yields an expression for the displacement vector of point Q in the deformed arch

$$\bar{u}_2 = \bar{u} + \bar{\psi} \times y_2 \bar{t}_2 = (u_1 - \psi y_2)\bar{t}_1 + u_2 \bar{t}_2 \qquad [A23]$$

If P^*, Q^*, Q_1^*, and Q_2^* denote the displaced points P, Q, Q_1, and Q_2 in the deformed arch, then the position vectors of points P^* and Q^* are

$$\bar{r}^* = \bar{r} + \bar{u} \qquad [A24]$$

$$\bar{r}_2^* = \bar{r}_2 + \bar{u}_2 \qquad [A25]$$

so that in the deformed arch, Equations A25 and A19 yield

$$\overline{Q^* Q_1^*} = \frac{\partial \bar{r}_2^*}{\partial s} ds = \left[\alpha_2 \bar{t}_1 + \frac{\partial \bar{u}_2}{\partial s}\right] ds \qquad [A26]$$

and Equations A25 and A20 yield

$$\overline{Q^*Q_2^*} = \frac{\partial \bar{r}_2^*}{\partial y_2}\, dy_2 = \left[\bar{t}_2 + \frac{\partial \bar{u}_2}{\partial y_2}\right] dy_2 \qquad \text{[A27]}$$

The components of strain are, therefore, assuming small displacements and rotations*,

$$\epsilon_1 = \sqrt{\frac{\overline{Q^*Q_1^*} \cdot \overline{Q^*Q_1^*}}{\overline{QQ_1} \cdot \overline{QQ_1}}} - 1$$

$$= \sqrt{\frac{\alpha_2^{\,2} + 2\alpha_2\bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial s}}{\alpha_2^{\,2}}} - 1$$

$$\approx \frac{1}{\alpha_2}\bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{1}{2\alpha_2^{\,2}} \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial s} \qquad \text{[A28]}$$

$$\epsilon_2 = \sqrt{\frac{\overline{Q^*Q_2^*} \cdot \overline{Q^*Q_2^*}}{\overline{QQ_2} \cdot \overline{QQ_2}}} - 1$$

$$= \sqrt{\frac{1 + 2\bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{\partial \bar{u}_2}{\partial y_2} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{1}} - 1$$

$$\approx \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{2} \frac{\partial \bar{u}_2}{\partial y_2} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \qquad \text{[A29]}$$

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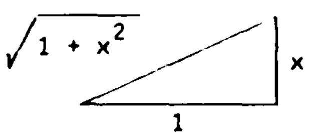
*The surveyor's slope correction states that when $x \ll 1$



$$\sqrt{1 + x^2} - 1 \approx \frac{x^2}{2}$$

$$\gamma_{12} = \frac{\overline{Q^{*}Q_1^{*}} \cdot \overline{Q^{*}Q_2^{*}}}{\left|\overline{Q^{*}Q_1^{*}}\right| \left|\overline{Q^{*}Q_2^{*}}\right|}$$

$$= \frac{\alpha_2 \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{\left|\alpha_2 \bar{t}_1 + \frac{\partial \bar{u}_2}{\partial s}\right| \left|\bar{t}_2 + \frac{\partial \bar{u}_2}{\partial y_2}\right|}$$

$$\approx \frac{\alpha_2 \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{\alpha_2 \cdot 1}$$

$$\approx \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{\alpha_2} \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{1}{\alpha_2} \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \qquad [A30]$$

Assuming the displacement derivatives are small with respect to unity permits neglect of the nonlinear terms in Equations A28, A29, and A30, which leaves, using Equations A21 and A23,

$$\epsilon_1 = \frac{1}{\alpha_2} \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s}$$

$$= \frac{1}{1 - \frac{y_2}{R}} \bar{t}_1 \cdot \left[\left[\frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s}\right]\bar{t}_1 + (u_1 - y_2\psi)\frac{\bar{t}_2}{R} + \frac{\partial u_2}{\partial s}\bar{t}_2 - u_2 \frac{\bar{t}_1}{R}\right]$$

$$= \frac{1}{1 - \frac{y_2}{R}} \left[\frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s} - \frac{u_2}{R}\right] \qquad [A31]$$

$$\epsilon_2 = \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} = \bar{t}_2 \cdot (-\psi \bar{t}_1) = 0 \qquad [A32]$$

$$\gamma_{12} = \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{\alpha_2}\bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} = \bar{t}_1 \cdot (-\psi \bar{t}_1) + \frac{1}{1 - \frac{y_2}{R}} \bar{t}_2$$

$$\cdot \left[\left[\frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s} \right] \bar{t}_1 + (u_1 - y_2\psi)\frac{\bar{t}_2}{R} + \frac{\partial u_2}{\partial s} \bar{t}_2 - u_2 \frac{\bar{t}_1}{R} \right]$$

$$= - \psi + \frac{1}{1 - \frac{y_2}{R}} \left[\frac{u_1 - y_2\psi}{R} + \frac{\partial u_2}{\partial s} \right]$$

$$= \frac{1}{1 - \frac{y_2}{R}} \left[- \psi + \frac{u_1}{R} + \frac{\partial u_2}{\partial s} \right] \qquad [A33]$$

Note that γ_{12} is positive when the dot product $\overline{Q^* Q^*_1} \cdot \overline{Q^* Q^*_2}$ is positive, which means that the angle $Q^*_1 Q^* Q^*_2$ is less than $\pi/2$. When this is the case, the direction of the shear stresses on a deformed arch element are as shown in Figure A4.

If we now set

$$\delta_1 = (\epsilon_1)_{y_2=0} = \frac{\partial u_1}{\partial s} - \frac{u_2}{R} \qquad [A34]$$

$$\delta_2 = (\gamma_{12})_{y_2=0} = - \psi + \frac{u_1}{R} + \frac{\partial u_2}{\partial s} \qquad [A35]$$

$$k = \frac{\partial \psi}{\partial s} \qquad [A36]$$

then Equations A31, A32, and A33 can be written in the form

$$\epsilon_1 = \frac{\delta_1 - y_2 k}{1 - \frac{y_2}{R}} \qquad [A37]$$

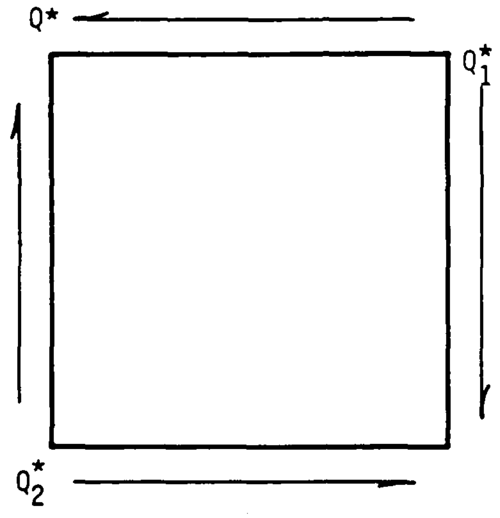


Figure A4. Shear Stress Directions for a Deformed Arch Element When the Angle $Q_1^* Q^* Q_2^*$ is Acute.

$$\epsilon_2 = 0 \qquad [A38]$$

$$\gamma_{12} = \frac{\delta_2}{1 - \frac{y_2}{R}} \qquad [A39]$$

Assuming zero initial strain, the longitudinal normal and transverse shear stresses are

$$\sigma_{11} = E\epsilon_1 = \frac{E}{1 - \frac{y_2}{R}} (\delta_1 - y_2 k) \qquad [A40]$$

$$\sigma_{12} = G\gamma_{12} = \frac{G}{1 - \frac{y_2}{R}} \delta_2 \qquad [A41]$$

where E is Young's modulus, and G is the shear modulus, so that the stress resultants, F_1, F_2, and M in Figure A1 are

$$F_1 = \iint \sigma_{11} dy_2 dy_3 = E\delta_1 \iint \frac{dy_2}{1 - \frac{y_2}{R}} dy_3 - Ek \iint \frac{y_2 dy_2}{1 - \frac{y_2}{R}} dy_3 \qquad [A42]$$

$$F_2 = \iint \sigma_{12} dy_2 dy_3 = G\delta_2 \iint \frac{dy_2}{1 - \frac{y_2}{R}} dy_3 \qquad [A43]$$

$$M = -\iint \sigma_{11} y_2 dy_2 dy_3 = -E\delta_1 \iint \frac{y_2 dy_2}{1 - \frac{y_2}{R}} dy_3 + Ek \iint \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} dy_3 \qquad [A44]$$

The three integrals appearing in Equations A42, A43, and A44 can be expressed in terms of a single integral. Referring to Figure A2, we write

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{dy_2}{1 - \frac{y_2}{R}} = - R\ln \left[1 - \frac{y_2}{R}\right] \Bigg|_{-\frac{d}{2}}^{\frac{d}{2}} = R\ln \left[\frac{1 + \frac{d}{2R}}{1 - \frac{d}{2R}}\right] = d* \qquad [A45]$$

Now

$$\frac{1}{1 - z} = \frac{1 - z + z}{1 - z} = 1 + z \left[\frac{1}{1 - z}\right] = 1 + z \left[1 + z \left[\frac{1}{1 - z}\right]\right]$$

so that

$$\frac{1}{1 - \frac{y_2}{R}} = 1 + \frac{\frac{y_2}{R}}{1 - \frac{y_2}{R}} = 1 + \frac{y_2}{R} + \frac{\left[\frac{y_2}{R}\right]^2}{1 - \frac{y_2}{R}} \qquad [A46]$$

Thus

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{y_2 dy_2}{1 - \frac{y_2}{R}} = R(d* - d) \qquad [A47]$$

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} = R^2(d* - d) \qquad [A48]$$

Now

$$\ln\left[\frac{1 + x}{1 - x}\right] = 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right] = 2 \sum_{j=1}^{\infty} \frac{x^{2j - 1}}{2j - 1} \qquad [A49]$$

so that if we set

$$\frac{d}{2R} = x \qquad [A50]$$

then Equation A45 yields

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{dy_2}{1 - \frac{y_2}{R}} = R\ln\left[\frac{1 + x}{1 - x}\right] = \left[\frac{R}{d}\ \ln\left[\frac{1 + x}{1 - x}\right]\right]d$$

$$= \frac{\ln\left[\frac{1 + x}{1 - x}\right]}{2x}\ d = \left[\frac{d^*}{d}\right]d = C_1 d \qquad [A51]$$

where

$$C_1 = \frac{d^*}{d} = \frac{\ln\left[\frac{1 + x}{1 - x}\right]}{2x} = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \ldots \qquad [A52]$$

and therefore

$$(C_1)_{d/R=0} = 1 \qquad [A53]$$

Equation A47 can be written in the form

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{y_2 dy_2}{1 - \frac{y_2}{R}} = \left[\frac{\frac{d^*}{d} - 1}{\frac{d}{R}}\right] d^2 = C_2 d^2 \qquad [A54]$$

where

$$C_2 = \frac{\frac{d^*}{d} - 1}{\frac{d}{R}} = \frac{C_1 - 1}{2x} = \frac{x}{6} + \frac{x^3}{10} + \frac{x^5}{14} + \ldots \qquad [A55]$$

and therefore,

$$(C_2)_{d/R=0} = 0 \qquad [A56]$$

Equation A48 can be written in the form

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} = 12 \left[\frac{\frac{d^*}{d} - 1}{\left[\frac{d}{R}\right]^2}\right] \frac{d^3}{12} = C_3 \, \frac{d^3}{12} \qquad [A57]$$

where

$$C_3 = \frac{12\left[\frac{d^*}{d} - 1\right]}{\left[\frac{d}{R}\right]^2} = \frac{3(C_1 - 1)}{x^2} = 1 + 3\left[\frac{x^2}{5} + \frac{x^4}{7} + \ldots\right] \qquad [A58]$$

and therefore,

$$(C_3)_{d/R=0} = 1 \qquad [A59]$$

Values of C_1, C_2, and C_3 are tabulated below to four significant figures as a function of d/R.

d/R	C_1	C_2	C_3
0	1.000	0	1.000
.10	1.001	.0083	1.002
.20	1.003	.0168	1.006
.30	1.008	.0253	1.014
.40	1.014	.0342	1.025
.50	1.022	.0433	1.039

If we define the angular differential operator, D, to be

$$\frac{\partial}{\partial\theta} = D \qquad [A60]$$

then Equations A14, A15, and A16 can be written in the form

$$\begin{bmatrix} \frac{D}{R} & -\frac{1}{R} & 0 \\ \frac{1}{R} & \frac{D}{R} & 0 \\ 0 & \frac{1}{r} & \frac{D}{R} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \frac{m}{r} \end{bmatrix} = \rho A \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ r\ddot{\psi} \end{bmatrix} \qquad [A61]$$

where

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\frac{bd^3}{12}}{bd}} = \frac{d}{2\sqrt{3}} \qquad [A62]$$

Equations A42, A43, and A44 can be written in the form

$$\begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} = EA \begin{bmatrix} C_1 & 0 & -C_2\frac{d}{2r} \\ 0 & \frac{C_1}{2(1+\nu)} & 0 \\ -C_2\frac{d}{2r} & 0 & C_3 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} \qquad [A63]$$

where ν is Poisson's ratio. Equations A34, A35, and A36 can be written in the form

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} = \begin{bmatrix} \frac{D}{R} & -\frac{1}{R} & 0 \\ \frac{1}{R} & \frac{D}{R} & -\frac{1}{r} \\ 0 & 0 & \frac{D}{R} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} \qquad [A64]$$

Equations A61, A63, and A64 are the governing equations for the dynamic response of a circular arch, including both shear and rotary inertia, with one exception. That exception is replacement of the constant C_1 in the second of Equations A63 by the transverse shear coefficient, k', to account for the fact that the arch cross section does not remain plane (Ref. 15).

Ref. 15. Oden, J. T. and E. A. Ripperger, Mechanics of Elastic Structures, 2nd Edition, McGraw-Hill, (1981).

APPENDIX B

MATRIX FORM OF THE GOVERNING EQUATIONS

Equations A61, A63, and A64 can be written in matrix form by setting

$$\underline{G} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{R}{r} & 0 \end{bmatrix} \qquad [B1]$$

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -C_2\frac{d}{2r} \\ 0 & \frac{k'}{2(1+\nu)} & 0 \\ -C_2\frac{d}{2r} & 0 & C_3 \end{bmatrix} \qquad [B2]$$

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} \qquad [B3]$$

$$\{\delta\} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} \qquad [B4]$$

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} \qquad [B5]$$

$$\{b\} = \begin{bmatrix} b_1 \\ b_2 \\ \frac{m}{r} \end{bmatrix} \qquad [B6]$$

The equations of motion (A61) then take the form

$$\frac{1}{R}\,(D\underline{I} + \underline{G})\{F\} + \{b\} = \rho A\{\ddot{u}\} \qquad [B7]$$

where $\underline{I}$ is the identity matrix,

the stress resultant-strain equations (A63) take the form

$$\{F\} = EA\underline{S}\ \{\delta\} \qquad [B8]$$

and the strain-displacement equations (A64) take the form

$$\{\delta\} = \frac{1}{R}\,(D\underline{I} - \underline{G}^T)\{u\} \qquad [B9]$$

Substitution of Equations B8 and B9 into Equation B7 yields

$$\frac{EA}{R^2}\,(D\underline{I} + \underline{G})\underline{S}(D\underline{I} - \underline{G}^T)\{u\} + \{b\} = \rho A\ \{\ddot{u}\} \qquad [B10]$$

or

$$\frac{EA}{R^2}\,(D\underline{S} + \underline{GS})(D\underline{I} - \underline{G}^T)\{u\} + \{b\} = \rho A\ \{\ddot{u}\}$$

or

$$\frac{EA}{R^2}\left[D^2\underline{S} - D\underline{SG}^T + D\underline{GS} - \underline{GSG}^T\right]\{u\} + \{b\} = \rho A\ \{\ddot{u}\}$$

or

$$\frac{EA}{R^2}\left[D^2\underline{S} + D(\underline{GS} - \underline{SG}^T) - \underline{GSG}^T\right]\{u\} + \{b\} = \rho A\ \{\ddot{u}\} \qquad [B11]$$

The matrices in Equation B11 are computed below.

$$
\begin{array}{ccc|ccc}
 & & & C_1 & 0 & C_2 \frac{d}{r} \\
 & & \underline{S} & 0 & \frac{k'}{2(1+\nu)} & 0 \\
 & & & -C_2 \frac{d}{r} & 0 & C_3 \\
\hline
0 & -1 & 0 & 0 & -\frac{k'}{2(1+\nu)} & 0 \\
1 & 0 & 0 & C_1 & 0 & C_2 \frac{d}{r} \\
0 & \frac{R}{r} & 0 & 0 & \frac{k'R}{2r(1+\nu)} & 0 \\
 & \underline{G} & & & \underline{GS} &
\end{array}
\qquad [B12]
$$

$$
\underline{S}^T \underline{G}^T = \underline{SG}^T = \begin{bmatrix} 0 & C_1 & 0 \\ -\frac{k'}{2(1+\nu)} & 0 & \frac{k'R}{2r(1+\nu)} \\ 0 & -C_2 \frac{d}{r} & 0 \end{bmatrix} \qquad [B13]
$$

$$
\underline{H} = \underline{GS} - \underline{SG}^T = \begin{bmatrix} 0 & -\frac{k'}{2(1+\nu)} - C_1 & 0 \\ \frac{k'}{2(1+\nu)} + C_1 & 0 & -\frac{k'R}{2r(1+\nu)} - C_2 \frac{d}{r} \\ 0 & \frac{k'R}{2r(1+\nu)} + C_2 \frac{d}{r} & 0 \end{bmatrix} \qquad [B14]
$$

Note that the matrix $\underline{H}$ is antisymmetric.

$$\underline{SG}^T = \begin{bmatrix} 0 & C_1 & 0 \\ -\dfrac{k'}{2(1+\nu)} & 0 & \dfrac{k'R}{2r(1+\nu)} \\ 0 & -C_2\dfrac{d}{r} & 0 \end{bmatrix}$$

$$\underline{G} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \dfrac{R}{r} & 0 \end{bmatrix} \qquad \underline{GSG}^T = \underline{K} = \begin{bmatrix} \dfrac{k'}{2(1+\nu)} & 0 & -\dfrac{k'R}{2r(1+\nu)} \\ 0 & C_1 & 0 \\ -\dfrac{k'R}{2r(1+\nu)} & 0 & \dfrac{k'R^2}{2r^2(1+\nu)} \end{bmatrix} \qquad [B15]$$

If we set

$$c^2 = \frac{E}{\rho} \qquad [B16]$$

then Equation B11 can be written in the form

$$\{\ddot{u}\} - \left[\frac{c}{R}\right]^2 (D^2\underline{S} + D\underline{H} - \underline{K})\{u\} = \frac{1}{\rho A}\{b\} \qquad [B17]$$

Equation B17 is the governing equation for the dynamic response of a circular arch, including both shear and rotary inertia, written in matrix form. The matrices $\underline{S}$ and $\underline{K}$ are symmetric, and the matrix $\underline{H}$ is antisymmetric, i.e.,

$$\underline{S}^T = \underline{S} \qquad [B18]$$

$$\underline{K}^T = \underline{K} \qquad [B19]$$

$$\underline{H}^T = -\underline{H} \qquad [B20]$$

Note also that Equation A62 yields

$$\frac{d}{r} = 2\sqrt{3} \qquad [B21]$$

so that we can write

$$C_2 \frac{d}{r} = 2\sqrt{3}\ C_2 \qquad [B22]$$

APPENDIX C

UNCOUPLING THE EQUATIONS OF MOTION

For free vibration, we set

$$\{b\} = \{0\} \tag{C1}$$

in Equation B17, and obtain

$$\{\ddot{u}\} - \left[\frac{c}{R}\right]^2 (D^2\underline{S} + D\underline{H} - \underline{K})\{u\} = \{0\} \tag{C2}$$

Assuming a separable solution, we set

$$\{u\} = T\{U\} \tag{C3}$$

where T is a function of time only, and

$$\{U\} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{C4}$$

where X, Y, and Z are functions of θ only.

Substitution of Equation C3 into Equation C2 yields

$$\ddot{T}\{U\} - \left[\frac{c}{R}\right]^2 T(D^2\underline{S} + D\underline{H} - \underline{K})\{U\} = \{0\} \tag{C5}$$

or

$$\ddot{T}\{U\} - \left[\frac{c}{R}\right]^2 T\underline{L}\{U\} = \{0\} \tag{C6}$$

where $\underline{L}$ is the linear, spatial, matrix differential operator,

$$\underline{L} = D^2\underline{S} + D\underline{H} - \underline{K} \tag{C7}$$

It is convenient to define a diagonal matrix containing the elements X, Y, and Z,

$$\lceil \bar{U} \rfloor = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix} \qquad [C8]$$

(The notation $\lceil \ \rfloor$ denotes a diagonal matrix.)

as well as the identity vector

$$\{m\} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad [C9]$$

so that

$$\lceil \bar{U} \rfloor^{-1} \{U\} = \{m\} \qquad [C10]$$

and

$$\lceil \bar{U} \rfloor \{m\} = \{U\} \qquad [C11]$$

Premultiplication of Equation C6 by $\lceil \bar{U} \rfloor^{-1}$ yields

$$\ddot{T}\{m\} - \left[\frac{c}{R}\right]^2 T \lceil \bar{U} \rfloor^{-1} \underline{L}\{U\} = \{0\} \qquad [C12]$$

and division by T yields

$$\frac{\ddot{T}}{T}\{m\} = \left[\frac{c}{R}\right]^2 \lceil \bar{U} \rfloor^{-1} \underline{L}\{U\} \qquad [C13]$$

Since the LHS of Equation C13 is a function of time only, and the RHS is a function of θ only, for arbitrary values of T and θ, both sides must be constant, so that

$$\frac{T}{T}\{m\} = \left[\frac{c}{R}\right]^2 \lceil \bar{U} \rfloor^{-1} \underline{L}\{U\} = -p^2\{m\} \qquad [C14]$$

and therefore

$$\ddot{T} + p^2 T = 0 \qquad [C15]$$

The function T is therefore harmonic. Premultiplication of the last two parts of Equation C14 by $\lceil \bar{U} \rfloor$ now yields

$$\left[\frac{c}{R}\right]^2 \underline{L}\{U\} = -p^2\{U\}$$

or

$$\underline{L}\{U\} = -\frac{p^2}{\left[\frac{c}{R}\right]^2}\{U\} = -\sigma^2\{U\} \qquad [C16]$$

where

$$\sigma = \frac{p}{\left[\frac{c}{R}\right]} \qquad [C17]$$

Equation C16 can be written in the form

$$(\underline{L} + \sigma^2\underline{I})\{U\} = \{0\} \qquad [C18]$$

The linear differential operator coefficient matrix $(\underline{L} + \sigma^2\underline{I})$ in Equation C18 is coupled, but the system of ordinary differential equations can be uncoupled by premultiplying by the adjoint of the coefficient matrix.

$$(\underline{L} + \sigma^2\underline{I})^*(\underline{L} + \sigma^2\underline{I})\{U\} = \Delta\underline{I}\{U\} = \Delta\{U\} = \{0\} \qquad [C19]$$

where Δ is the determinant of the coefficient matrix in Equation C18.

$$\Delta = \left| \underline{L} + \sigma^2\underline{I} \right| \qquad [C20]$$

Equation C19 shows that X, Y, and Z each satisifies the same linear, ordinary differential equation:

$$\Delta X = 0 \tag{C21}$$

$$\Delta Y = 0 \tag{C22}$$

$$\Delta Z = 0 \tag{C23}$$

The system of equations is thus uncoupled.

APPENDIX D

EXPANSION OF THE CHARACTERISTIC DETERMINANT

In Equations C21, C22, and C23, the linear, ordinary, differential operator Δ represents the expansion of a determinant.

$$\Delta = \left| \underline{L} + \sigma^2 \underline{I} \right| \qquad [C20\ bis]$$

where

$$\underline{L} = D^2\underline{S} + D\underline{H} - \underline{K} \qquad [C7\ bis]$$

and

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -2\sqrt{3}C_2 \\ 0 & \frac{k'}{2(1+\nu)} & 0 \\ -2\sqrt{3}C_2 & 0 & C_3 \end{bmatrix} \qquad [D1]$$

$$\underline{H} = \begin{bmatrix} 0 & -\frac{k'}{2(1+\nu)} - C_1 & 0 \\ \frac{k'}{2(1+\nu)} + C_1 & 0 & -\frac{k'R}{2r(1+\nu)} - 2\sqrt{3}C_2 \\ 0 & \frac{k'R}{2(1+\nu)} + 2\sqrt{3}C_2 & 0 \end{bmatrix} \qquad [D2]$$

$$\underline{K} = \begin{bmatrix} \frac{k'}{2(1+\nu)} & 0 & -\frac{k'R}{2r(1+\nu)} \\ 0 & C_1 & 0 \\ -\frac{k'R}{2r(1+\nu)} & 0 & \frac{k'R^2}{2r^2(1+\nu)} \end{bmatrix} \qquad [B15\ bis]$$

To facilitate the expansion of Δ, let

$$C_1 = A \qquad [D3]$$

$$\frac{k'}{2(1+\nu)} = B \tag{D4}$$

$$C_3 = C \tag{D5}$$

$$2\sqrt{3}C_2 = E \tag{D6}$$

$$\frac{R}{r} = F \tag{D7}$$

Then Equations C7, D1, D2, and B15 yield

$$\underline{L} = D^2 \begin{bmatrix} A & 0 & -E \\ 0 & B & 0 \\ -E & 0 & C \end{bmatrix} + D \begin{bmatrix} 0 & -(A+B) & 0 \\ (A+B) & 0 & -(BF+E) \\ 0 & (BF+E) & 0 \end{bmatrix}$$

$$- \begin{bmatrix} B & 0 & -BF \\ 0 & A & 0 \\ -BF & 0 & BF^2 \end{bmatrix}$$

$$= \begin{bmatrix} AD^2-B & -(A+B)D & -ED^2+BF \\ (A+B)D & BD^2-A & -(BF+E)D \\ -ED^2+BF & (BF+E)D & CD^2-BF^2 \end{bmatrix} \tag{D8}$$

so that Equation C20 takes the form

$$\Delta = \begin{vmatrix} (AD^2-B)+\sigma^2 & -(A+B)D & -ED^2+BF \\ (A+B)D & (BD^2-A)+\sigma^2 & -(BF+E)D \\ -ED^2+BF & (BF+E)D & (CD^2-BF^2)+\sigma^2 \end{vmatrix} \tag{D9}$$

Equation D9 is very similar to the determinant which arises in calculating principal stresses, so that the expansion of Equation D9 is known to take the form

$$\Delta = \sigma^6 + I_1\sigma^4 + I_2\sigma^2 + I_3 \tag{D10}$$

where

$$I_1 = \mathrm{Tr}\,(\underline{L}) \qquad [D11]$$

$$I_2 = \begin{vmatrix} AD^2-B & -(A+B)D \\ (A+B)D & BD^2-A \end{vmatrix} + \begin{vmatrix} BD^2-A & -(BF+E)D \\ (BF+E)D & CD^2-BF^2 \end{vmatrix}$$

$$+ \begin{vmatrix} CD^2-BF^2 & -ED^2+BF \\ -ED^2+BF & AD^2-B \end{vmatrix} . \qquad [D12]$$

$$I_3 = |\,\underline{L}\,| = \begin{vmatrix} AD^2-B & -(A+B)D & -ED^2+BF \\ (A+B)D & BD^2-A & -(BF+E)D \\ -ED^2+BF & (BF+E)D & CD^2-BF^2 \end{vmatrix} \qquad [D13]$$

Equations D11, D12, and D13 yield

$$I_1 = (A + B + C^2)D - (A + B + BF^2) \qquad [D14]$$

$$I_2 = (AB + BC + AC - E^2)D^4$$

$$+ (2AB - AC + 4BEF + E^2 - BC - ABF^2)D^2 + (AB + ABF^2) \qquad [D15]$$

$$I_3 = (ABC - BE^2)(D^2 + 1)^2 D^2 \qquad [D16]$$

Substitution of Equations D14, D15, and D16 into Equation D10 yields

$$\Delta = \sigma^6 + \left[(A + B + C)D^2 - (A + B + BF^2)\right]\sigma^4$$

$$+ \left[(AB + BC + AC - E^2)D^4 + (2AB - AC + 4BEF + E^2 - BC - ABF^2)D^2\right.$$

$$\left. + (AB + ABF^2)\right]\sigma^2$$

$$+ \left[(ABC - BE^2)D^6 + 2(ABC - BE^2)D^4 + (ABC - BE^2)D^2\right] \qquad [D17]$$

Finally, Δ can be expressed as a polynomial in the operator D.

$$\Delta = (ABC - BE^2)D^6 + \left[(AB + BC + AC - E^2)\sigma^2 + 2(ABC - BE^2)\right]D^4$$
$$+ \left[(A + B + C)\sigma^4 + (2AB - AC + 4BEF + E^2 - BC - ABF^2)\sigma^2\right.$$
$$\left.+ (ABC - BE^2)\right]D^2 + \left[\sigma^6 - (A + B + BF^2)\sigma^4 + (AB + ABF^2)\sigma^2\right] \quad [D18]$$

All the coefficients in Equation D18 can be expressed in terms of the invariants of the matrices $\underline{S}$, $\underline{H}$, $\underline{K}$, and $\underline{S} - \underline{K}$, where $I_j\ (\underline{x})$ is the j^{th} invariant of the matrix, $\underline{x}$.

$$I_1\ (\underline{S}) = A + B + C \quad [D19]$$

$$I_2\ (\underline{S}) = AB + BC + AC - E^2 \quad [D20]$$

$$I_3\ (\underline{S}) = ABC - BE^2 \quad [D21]$$

$$I_1\ (\underline{H}) = 0 \quad [D22]$$

$$I_2\ (\underline{H}) = A^2 + 2AB + B^2 + B^2F^2 + 2BEF + E^2 \quad [D23]$$

$$I_3\ (\underline{H}) = 0 \quad [D24]$$

$$I_1\ (\underline{K}) = A + B + BF^2 \quad [D25]$$

$$I_2\ (\underline{K}) = AB + ABF^2 \quad [D26]$$

$$I_3\ (\underline{K}) = 0 \quad [D27]$$

$$\underline{S} - \underline{K} = \begin{bmatrix} A-B & 0 & BF-E \\ 0 & B-A & 0 \\ BF-E & 0 & C-BF^2 \end{bmatrix} \quad [D28]$$

$$I_2\ (\underline{S} - \underline{K}) = -A^2 + 2AB - B^2 - B^2F^2 + 2BEF - E^2 \quad [D29]$$

$$I_2\ (\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H})$$

$$= 2AB - AC + 4BEF + E^2 - BC - ABF^2 \quad [D30]$$

Thus, Equation D18 can be written in the form

$$\Delta = I_3\ (\underline{S})D^6 + \left[I_2(\underline{S})\sigma^2 + 2I_3(\underline{S})\right]D^4$$

$$+ \left[I_1(\underline{S})\sigma^4 + \left[\ I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H})\right]\sigma^2 + I_3(\underline{S})\right]D^2$$

$$+ \left[\sigma^6 - I_1(\underline{K})\sigma^4 + I_2(\underline{K})\sigma^2\right] \qquad [D31]$$

Substitution of Equations D3 - D7 into Equations D19, D20, D21, D25, D26, and D30 yields

$$I_1\ (\underline{S}) = \left[C_1 + C_3\right] + \frac{k'}{2(1 + \nu)} \qquad [D32]$$

$$I_2\ (\underline{S}) = \frac{k'}{2(1 + \nu)}\left[C_1 + C_3\right] + \left[C_1C_3 - 12C_2^2\right] \qquad [D33]$$

$$I_3\ (\underline{S}) = \frac{k'}{2(1 + \nu)}\left[C_1C_3 - 12C_2^2\right] \qquad [D34]$$

$$I_1\ (\underline{K}) = C_1 + \frac{k'}{2(1 + \nu)}\left[1 + \left[\frac{R}{r}\right]^2\right] \qquad [D35]$$

$$I_2\ (\underline{K}) = \frac{C_1k'}{2(1 + \nu)}\left[1 + \left[\frac{R}{r}\right]^2\right] \qquad [D36]$$

$$I_2\ (\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2\ (\underline{H})$$

$$= \frac{k'}{2(1 + \nu)}\left[2C_1 - C_3 + 8\sqrt{3}C_2\left[\frac{R}{r}\right] - C_1\left[\frac{R}{r}\right]^2\right]$$

$$- \left[C_1C_3 - 12C_2^2\right] \qquad [D37]$$

Therefore, if we set

$$a = \frac{I_2(\underline{S})}{I_3(\underline{S})}\ \sigma^2 + 2 \qquad [D38]$$

$$b = \frac{I_1(\underline{S})\sigma^4 + \left[I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H})\right]\sigma^2}{I_3(\underline{S})} + 1 \qquad [D39]$$

$$c = \frac{\sigma^6 - I_1(\underline{K})\sigma^4 + I_2(\underline{K})\sigma^2}{I_3(\underline{S})} \qquad [D40]$$

then Equation D31 can be written in the form

$$\Delta = I_3(\underline{S})\ (D^6 + aD^4 + bD^2 + c) \qquad [D41]$$

Inspection of Equations A52, A55, and A58 shows that $I_3(\underline{S})$, as defined by Equation D34, is always positive.

Referring to Equations D32-D37, we see that

$$\frac{I_2(\underline{S})}{I_3(\underline{S})} = \frac{C_1 + C_3}{C_1C_3 - 12C_2^2} + \frac{2(1+\nu)}{k'} \qquad [D42]$$

$$\frac{I_1(\underline{S})}{I_3(\underline{S})} = \frac{2(1+\nu)}{k'}\left[\frac{C_1 + C_3}{C_1C_3 - 12C_2^2}\right] + \frac{1}{C_1C_3 - 12C_2^2} \qquad [D43]$$

$$\frac{I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_3(\underline{H})}{I_3(\underline{S})}$$

$$= \frac{2C_1 - C_3 + 8\sqrt{3}C_2\left[\frac{R}{r}\right] - C_1\left[\frac{R}{r}\right]^2}{C_1C_3 - 12C_2^2} - \frac{2(1+\nu)}{k'} \qquad [D44]$$

$$\frac{I_1(\underline{K})}{I_3(\underline{S})} = \frac{2(1+\nu)}{k'}\left[\frac{C_1}{C_1C_3 - 12C_2^2}\right] + \frac{1 + \left[\frac{R}{r}\right]^2}{C_1C_3 - 12C_2^2} \qquad [D45]$$

$$\frac{I_2(\underline{K})}{I_3(\underline{S})} = \frac{C_1}{C_1C_3 - 12C_2^2}\left[1 + \left[\frac{R}{r}\right]^2\right] \qquad [D46]$$

and if we set

$$\frac{1}{C_1C_3 - 12C_2^2} = K_0 \qquad [D47]$$

$$\frac{C_1}{K_0} = K_1 \qquad [D48]$$

$$\frac{C_2}{K_0} = K_2 \qquad [D49]$$

$$\frac{C_3}{K_0} = K_3 \qquad [D50]$$

$$\frac{2(1 + \nu)}{k'} = n \qquad [D51]$$

$$\frac{R}{r} = z \qquad [D52]$$

then Equations D38-D40 can be written in the form

$$a = (K_1 + K_3 + n)\sigma^2 + 2 \qquad [D53]$$

$$b = \left[n(K_1 + K_3) + K_0\right]\sigma^4 + \left[K_1(2 - z^2) + 8\sqrt{3}K_2 z - K_3 - n\right]\sigma^2 + 1 \qquad [D54]$$

$$c = nK_0\sigma^6 - \left[nK_1 + K_0(1 + z^2)\right]\sigma^4 + K_1(1 + z^2)\sigma^2 \qquad [D55]$$

The coefficients K_0, K_1, K_2, and K_3 are functions of d/R.

d/R	C_1	C_2	C_3	K_0	K_1	K_2	K_3
0	1.000	0	1.000	1.000	1.000	0	1.000
.10	1.001	.0083	1.002	0.998	0.999	0.0083	1.000
.20	1.003	.0168	1.006	0.994	0.997	0.0167	1.000
.30	1.008	.0253	1.014	0.986	0.994	0.0249	1.000
.40	1.014	.0342	1.025	0.975	0.989	0.0334	1.000
.50	1.022	.0433	1.039	0.962	0.983	0.0417	1.000

Using the commonly accepted value

$$k' = \frac{5}{6} \approx \frac{\pi^2}{12} \qquad [D56]$$

Equation D51 yields

$$n = \frac{12(1 + \nu)}{5} \qquad [D57]$$

ν	n
0.10	2.64
0.15	2.76
0.20	2.88
0.25	3.00
0.30	3.12
0.35	3.24

Assuming, based on the values tabulated on the previous page, that

$$K_0 = K_1 = K_3 = 1$$

$$K_2 = 0$$

Equations D53, D54, and D55 reduce to

$$a = (2 + n)\sigma^2 + 2 \qquad [D58]$$

$$b = (2n + 1)\sigma^4 - (z^2 + n - 1)\sigma^2 + 1 \qquad [D59]$$

$$c = \left[n\sigma^4 - (z^2 + n + 1)\sigma^2 + (1 + z^2)\right]\sigma^2 \qquad [D60]$$

and if we assume that

$$n = 3$$

$$z^2 + n \pm 1 \approx z^2$$

Equations D58-D60 reduce to

$$a = 5\sigma^2 + 2 \qquad [D61]$$

$$b = 7\sigma^4 - z^2\sigma^2 + 1 \qquad [D62]$$

$$c = 3\sigma^4 - z^2\sigma^2 + z^2 \qquad [D63]$$

Equations D53-D55 and D58-D63 show the relation between structural parameters and frequency, and the coefficients of the characteristic equation, D41, under various assumed (approximate) conditions.

APPENDIX E

GENERAL SOLUTION OF THE CHARACTERISTIC DIFFERENTIAL EQUATION

The constant I_3 ($\underline{S}$) in Equation D41 is nonzero, and can therefore be ignored. We therefore seek the general solution of the homogeneous, linear, ordinary differential equation

$$LX = 0 \qquad [E1]$$

where

$$L = D^6 + aD^4 + bD^2 + c \qquad [E2]$$

$$D = \frac{d}{d\theta} \qquad [E3]$$

The solution of Equation E1 is assumed to be of the form

$$X = Ke^{\lambda\theta} \qquad [E4]$$

and substitution of Equation E4 into Equation E1 yields

$$(\lambda^6 + a\lambda^4 + b\lambda^2 + c)Ke^{\lambda\theta} = 0 \qquad [E5]$$

If a nontrivial solution exists, it must be that

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \qquad [E6]$$

Equation E6 is a cubic equation in λ^2, and therefore has three roots:

λ_I^2, λ_{II}^2, and λ_{III}^2

Equation E6 can therefore be written in the form

$$\left[\lambda^2 - \lambda_I^2\right]\left[\lambda^2 - \lambda_{II}^2\right]\left[\lambda^2 - \lambda_{III}^2\right] = \lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \qquad [E7]$$

Comparison of Equations E6 and E7 shows that

$$a = -\left[\lambda_I^2 + \lambda_{II}^2 + \lambda_{III}^2\right] \qquad [E8]$$

$$b = \left[\lambda_I^2\lambda_{II}^2 + \lambda_{II}^2\,\lambda_{III}^2 + \lambda_{III}^2\lambda_I^2\right] \qquad [E9]$$

$$c = -\,\lambda_I^2\,\lambda_{II}^2\lambda_{III}^2 \qquad [E10]$$

The classic algebraic solution of a cubic equation (E6) begins by setting

$$\lambda^2 = y - \frac{a}{3} \qquad [E11]$$

in order to obtain the reduced cubic equation

$$y^3 - Py - Q = 0 \qquad [E12]$$

where

$$P = \frac{a^2}{3} - b \qquad [E13]$$

$$Q = \frac{ab}{3} - \frac{2a^3}{27} - c \qquad [E14]$$

The solution of Equation E12 is obtained by assuming it to be of the form

$$y = A + B \qquad [E15]$$

which leads to the requirement that

$$AB = \frac{P}{3} \qquad [E16]$$

and

$$A^3 + B^3 = Q \qquad [E17]$$

Equations E16 and E17 generate a quadratic equation, the roots of which are

$$A^3 = \frac{Q}{2} + \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} \qquad [E18]$$

$$B^3 = \frac{Q}{2} - \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} \qquad [E19]$$

and if we set

$$R = \left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3 \qquad [E20]$$

then Equations E18 and E19 can be written in the form

$$A^3 = \frac{Q}{2} + \sqrt{R} \qquad [E21]$$

$$B^3 = \frac{Q}{2} - \sqrt{R} \qquad [E22]$$

The nature of the roots A^3 and B^3 is determined by the relative values of P and Q.

If

$$\left[\frac{P}{3}\right]^3 \leq \left[\frac{Q}{2}\right]^2 \qquad [E23]$$

then the quantity R defined by Equation E20 is nonnegative, so that $\sqrt{R}$, A^3, and B^3 are real. Then if we set

$$A = \left[\frac{Q}{2} + \sqrt{R}\right]^{1/3} \qquad [E24]$$

$$B = \left[\frac{Q}{2} - \sqrt{R}\right]^{1/3} \qquad [E25]$$

and then, using π as the constant, 3.14159..., set

$$A_1 = A \qquad [E26]$$

$$A_2 = Ae^{i\frac{2\pi}{3}} \qquad [E27]$$

$$A_3 = Ae^{-i\frac{2\pi}{3}} \qquad [E28]$$

$$B_1 = B \qquad [E29]$$

$$B_2 = Be^{i\frac{2\pi}{3}} \qquad [E30]$$

$$B_3 = Be^{-i\frac{2\pi}{3}} \qquad \text{[E31]}$$

then Equations E16 and E17 will be satisfied by setting

$$y_j = A_j + B_j \qquad (j = 1, 2, 3) \qquad \text{[E32]}$$

If, on the other hand,

$$\left[\frac{P}{3}\right]^3 > \left[\frac{Q}{2}\right]^2 \qquad \text{[E33]}$$

then the quantity R defined by Equation E20 is negative, so that $\sqrt{R}$ is imaginary and A^3 and B^3 are complex. Then if we set

$$A^3 = \frac{Q}{2} + i\sqrt{\left[\frac{P}{3}\right]^3 - \left[\frac{Q}{2}\right]^2} \qquad \text{[E34]}$$

$$B^3 = \frac{Q}{2} - i\sqrt{\left[\frac{P}{3}\right]^3 - \left[\frac{Q}{2}\right]^2} \qquad \text{[E35]}$$

then we can show the complex quantities A^3 and B^3 in an Argand diagram (Figure E). Note that Equation E33 guarantees that P is positive. Figure E yields

$$A^3 = \left[\frac{P}{3}\right]^{3/2} e^{i3\omega} \qquad \text{[E36]}$$

$$B^3 = \left[\frac{P}{3}\right]^{3/2} e^{-i3\omega} \qquad \text{[E37]}$$

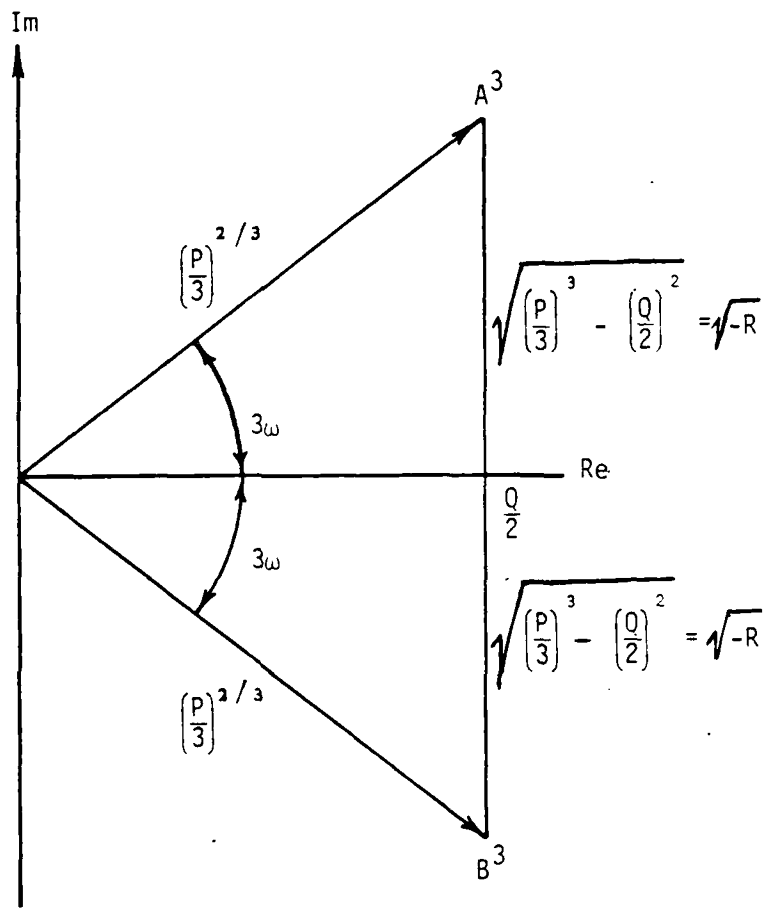


Figure E. Argand Diagram Showing A^3 and B^3 When $R < 0$.

where

$$\cos 3\omega = \frac{\frac{Q}{2}}{\left[\frac{P}{3}\right]^{3/2}} \qquad (0 \leq 3\omega \leq \pi) \qquad \text{[E38]}$$

If we set

$$\omega_1 = \omega \qquad \text{[E39]}$$

$$\omega_2 = \omega - \frac{2\pi}{3} \qquad \text{[E40]}$$

$$\omega_3 = \omega + \frac{2\pi}{3} \qquad \text{[E41]}$$

and then set

$$A_j = \left[\frac{P}{3}\right]^{1/2} e^{i\omega_j} \qquad (j = 1, 2, 3) \qquad \text{[E42]}$$

$$B_k = \left[\frac{P}{3}\right]^{1/2} e^{i\omega_k} \qquad (j = 1, 2, 3) \qquad \text{[E43]}$$

then Equations E16 and E17 will be satisfied by setting

$$y_j = A_j + B_j = 2\sqrt{\frac{P}{3}} \cos \omega_j \qquad (j = 1, 2, 3) \qquad \text{[E44]}$$

Finally, in keeping with Equation E11, we write

$$\lambda_I^2 = y_1 - \frac{a}{3} \qquad \text{[E45a]}$$

$$\lambda_{II}^2 = y_2 - \frac{a}{3} \qquad \text{[E45b]}$$

$$\lambda_{III}^2 = y_3 - \frac{a}{3} \qquad \text{[E45c]}$$

We now need to determine whether the roots λ_I^2, λ_{II}^2, and λ_{III}^2 are real or complex, and if real, whether they are positive or negative.

APPENDIX F

CHARACTERISTIC MODE FUNCTIONS

The exact nature of the three roots, $\lambda_I{}^2$, $\lambda_{II}{}^2$, and $\lambda_{III}{}^2$, defined by Equations E45, determines the forms of the arch mode shapes. The three roots are determined by the coefficients a, b, and c, which appear in Equation E2 and are defined by Equations D38-D40, D53-D55, D58-D60, or D61-D63.

First, note that Equation D53 guarantees the coefficient "a" will always be positive.

$$a > 0 \tag{F1}$$

Now consider Equation E12:

$$y^3 - Py - Q = 0 \qquad \text{[E12 bis]}$$

The various root combinations for Equation E12, and the arch mode shapes associated with each are discussed in detail below.

<u>Case 1: P = Q = 0</u>

In Case 1, Equation E12 reduces to

$$y^3 = 0 \tag{F2}$$

and the roots are

$$y_1 = y_2 = y_3 = 0 \tag{F3}$$

Therefore Equations E45 yield

$$\lambda_I^2 = \lambda_{II}^2 = \lambda_{III}^2 = -\frac{a}{3} \tag{F4}$$

so that

$$\lambda_{11} = \lambda_{21} = \lambda_{31} = i\sqrt{\frac{a}{3}} \tag{F5}$$

$$\lambda_{12} = \lambda_{22} = \lambda_{32} = -i\sqrt{\frac{a}{3}} \qquad [F6]$$

If we set

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda = \sqrt{\frac{a}{3}} \qquad [F7]$$

then because of the repeated nature of the roots, the mode functions are

$$W_1 = \cos \lambda\theta \qquad [F8]$$

$$W_2 = \sin \lambda\theta \qquad [F9]$$

$$W_3 = \theta \cos \lambda\theta \qquad [F10]$$

$$W_4 = \theta \sin \lambda\theta \qquad [F11]$$

$$W_5 = \theta^2 \cos \lambda\theta \qquad [F12]$$

$$W_6 = \theta^2 \sin \lambda\theta \qquad [F13]$$

<u>Case 2: P = 0; Q > 0</u>

In Case 2, Equation E12 reduces to

$$y^3 - |Q| = 0 \qquad [F14]$$

so that

$$y^3 = |Q| \qquad [F15]$$

Now let

$$r = |Q|^{1/3} > 0 \qquad [F16]$$

Then

$$y_1 = r \qquad [F17]$$

$$y_2 = re^{i\frac{2\pi}{3}} = \frac{r}{2}(-1 + i\sqrt{3}) \qquad [F18]$$

$$y_3 = re^{-i\frac{2\pi}{3}} = -\frac{r}{2}(1 + i\sqrt{3}) \qquad [F19]$$

and Equations E45 yield

$$\lambda_I^2 = r - \frac{a}{3} \qquad [F20]$$

$$\lambda_{II}^2 = -\left[\frac{r}{2} + \frac{a}{3}\right] + i\frac{r\sqrt{3}}{2} = Se^{i\phi} \qquad [F21]$$

$$\lambda_{III}^2 = -\left[\frac{r}{2} + \frac{a}{3}\right] - i\frac{r\sqrt{3}}{2} = Se^{-i\phi} \qquad [F22]$$

where

$$S^2 = \left[\frac{r}{2} + \frac{a}{3}\right]^2 + \frac{3r^2}{4} = r^2 + \frac{ra}{3} + \frac{a^2}{9} \qquad [F23]$$

$$\phi = \cos^{-1}\left[-\frac{\frac{r}{2} + \frac{a}{3}}{S}\right] \qquad [F24]$$

<u>Case 2a: $r - \frac{a}{3} \geq 0$</u>

For Case 2a, we set

$$\lambda_{11} = \sqrt{r - \frac{a}{3}} \qquad [F25]$$

$$\lambda_{12} = -\sqrt{r - \frac{a}{3}} \qquad [F26]$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\,\sin\frac{\phi}{2}\right] \qquad [F27]$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\phi}{2}} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\ \sin\frac{\phi}{2}\right] \tag{F28}$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\ \sin\frac{\phi}{2}\right] \tag{F29}$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\ \sin\frac{\phi}{2}\right] \tag{F30}$$

and then set

$$\lambda_1 = \sqrt{r - \frac{a}{3}} \tag{F31}$$

$$\lambda_2 = \sqrt{S}\ \cos\frac{\phi}{2} \tag{F32}$$

$$\lambda_3 = \sqrt{S}\ \sin\frac{\phi}{2} \tag{F33}$$

so that the mode functions are

$$W_1 = \cosh \lambda_1\theta \tag{F34}$$

$$W_2 = \sinh \lambda_1\theta \tag{F35}$$

$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \tag{F36}$$

$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \tag{F37}$$

$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \tag{F38}$$

$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \tag{F39}$$

Case 2b: $r - \frac{a}{3} < 0$

For Case 2b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - r} \qquad [F40]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - r} \qquad [F41]$$

$$\lambda_{21} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F27\ bis]$$

$$\lambda_{22} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F28\ bis]$$

$$\lambda_{31} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right] \qquad [F29\ bis]$$

$$\lambda_{32} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F30\ bis]$$

and then set

$$\lambda_{1} = \sqrt{\frac{a}{3} - r} \qquad [F42]$$

$$\lambda_{2} = \sqrt{S}\cos\frac{\phi}{2} \qquad [F32\ bis]$$

$$\lambda_{3} = \sqrt{S}\sin\frac{\phi}{2} \qquad [F33\ bis]$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \quad [F43]$$

$$W_2 = \sin \lambda_1\theta \quad [F44]$$

$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \quad [F36\ bis]$$

$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \quad [F37\ bis]$$

$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \quad [F38\ bis]$$

$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \quad [F39\ bis]$$

<u>Case 3: P = 0; Q < 0</u>

In Case 3, Equation E12 reduces to

$$y^3 + |Q| = 0 \quad [F45]$$

so that

$$y^3 = -|Q| \quad [F46]$$

Now let

$$r = |Q|^{1/3} > 0 \quad [F16\ bis]$$

Then

$$y_1 = -r \quad [F47]$$

$$y_2 = re^{i\frac{\pi}{3}} = \frac{r}{2}\left[1 + i\sqrt{3}\right] \quad [F48]$$

$$y_3 = re^{-i\frac{\pi}{3}} = \frac{r}{2}\left[1 - i\sqrt{3}\right] \quad [F49]$$

and Equations E45 yield

$$\lambda_I^2 = -\left[r + \frac{a}{3}\right] \quad [F50]$$

$$\lambda_{II}^2 = \left[\frac{r}{2} - \frac{a}{3}\right] + i\frac{r\sqrt{3}}{2} = Se^{i\phi} \qquad [F51]$$

$$\lambda_{III}^2 = \left[\frac{r}{2} - \frac{a}{3}\right] - i\frac{r\sqrt{3}}{2} = Se^{-i\phi} \qquad [F52]$$

where

$$S^2 = \left[\frac{r}{2} - \frac{a}{3}\right]^2 + \frac{3r^2}{4} = r^2 - \frac{ra}{3} + \frac{a^2}{9} \qquad [F53]$$

$$\phi = \cos^{-1}\left[\frac{\frac{r}{2} - \frac{a}{3}}{S}\right] \qquad [F54]$$

We now set

$$\lambda_{11} = i\sqrt{r + \frac{a}{3}} \qquad [F55]$$

$$\lambda_{12} = -i\sqrt{r + \frac{a}{3}} \qquad [F56]$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F27\ bis]$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\phi}{2}} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F28\ bis]$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right] \qquad [F29\ bis]$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\Phi}{2}} = \sqrt{S}\left[-\cos\frac{\Phi}{2} + i\ \sin\frac{\Phi}{2}\right] \qquad \text{[F30 bis]}$$

and then set

$$\lambda_1 = \sqrt{r + \frac{a}{3}} \qquad \text{[F57]}$$

$$\lambda_2 = \sqrt{S}\ \cos\frac{\Phi}{2} \qquad \text{[F32 bis]}$$

$$\lambda_3 = \sqrt{S}\ \sin\frac{\Phi}{2} \qquad \text{[F33 bis]}$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad \text{[F43 bis]}$$
$$W_2 = \sin \lambda_1\theta \qquad \text{[F44 bis]}$$
$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F36 bis]}$$
$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F37 bis]}$$
$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F38 bis]}$$
$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F39 bis]}$$

<u>Case 4: P > 0; Q = 0</u>

In Case 4, Equation E12 reduces to

$$y^3 - |P|y = y[y^2 - |P|] = 0 \qquad \text{[F58]}$$

Now let

$$r = \sqrt{|P|} \qquad \text{[F59]}$$

Then

$$y_1 = 0 \qquad \text{[F60]}$$
$$y_2 = r \qquad \text{[F61]}$$
$$y_3 = -r \qquad \text{[F62]}$$

and Equations E45 yield

$$\lambda_I^2 = -\frac{a}{3}$$ [F63]

$$\lambda_{II}^2 = r - \frac{a}{3}$$ [F64]

$$\lambda_{III}^2 = -\left[r + \frac{a}{3}\right]$$ [F65]

Case 4a: $r - \frac{a}{3} \geqslant 0$

For Case 4a, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3}}$$ [F66]

$$\lambda_{12} = -i\sqrt{\frac{a}{3}}$$ [F67]

$$\lambda_{21} = \sqrt{r - \frac{a}{3}}$$ [F68]

$$\lambda_{22} = -\sqrt{r - \frac{a}{3}}$$ [F69]

$$\lambda_{31} = i\sqrt{r + \frac{a}{3}}$$ [F70]

$$\lambda_{32} = -i\sqrt{r + \frac{a}{3}}$$ [F71]

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \qquad [F72]$$

$$\lambda_2 = \sqrt{r - \frac{a}{3}} \qquad [F73]$$

$$\lambda_3 = \sqrt{r + \frac{a}{3}} \qquad [F74]$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad [F43\ bis]$$

$$W_2 = \sin \lambda_1\theta \qquad [F44\ bis]$$

$$W_3 = \cosh \lambda_2\theta \qquad [F75]$$

$$W_4 = \sinh \lambda_2\theta \qquad [F76]$$

$$W_5 = \cos \lambda_3\theta \qquad [F77]$$

$$W_6 = \sin \lambda_3\theta \qquad [F78]$$

<u>Case 4b: $r - \frac{a}{3} < 0$</u>

For Case 4b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3}} \qquad [F66\ bis]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3}} \qquad [F67\ bis]$$

$$\lambda_{21} = i\sqrt{\frac{a}{3} - r} \qquad [F79]$$

$$\lambda_{22} = -i\sqrt{\frac{a}{3} - r} \qquad \text{[F80]}$$

$$\lambda_{31} = i\sqrt{r + \frac{a}{3}} \qquad \text{[F70 bis]}$$

$$\lambda_{32} = -i\sqrt{r + \frac{a}{3}} \qquad \text{[F71 bis]}$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \qquad \text{[F72 bis]}$$

$$\lambda_2 = \sqrt{\frac{a}{3} - r} \qquad \text{[F81]}$$

$$\lambda_3 = \sqrt{r + \frac{a}{3}} \qquad \text{[F74 bis]}$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad \text{[F43 bis]}$$
$$W_2 = \sin \lambda_1\theta \qquad \text{[F44 bis]}$$
$$W_3 = \cos \lambda_2\theta \qquad \text{[F82]}$$
$$W_4 = \sin \lambda_2\theta \qquad \text{[F83]}$$
$$W_5 = \cos \lambda_3\theta \qquad \text{[F77 bis]}$$
$$W_6 = \sin \lambda_3\theta \qquad \text{[F78 bis]}$$

<u>Case 5: $P < 0$; $Q = 0$</u>

In Case 5, Equation E12 reduces to

$$y^3 + |P|y = y[y^2 + |P|] = 0 \qquad \text{[F84]}$$

Now let

$$r = \sqrt{|P|} \qquad \text{[F59 bis]}$$

Then

$$y_1 = 0 \qquad \text{[F60 bis]}$$

$$y_2 = ir \qquad \text{[F85]}$$

$$y_3 = -ir \qquad \text{[F86]}$$

and Equations E45 yield

$$\lambda_I^2 = -\frac{a}{3} \qquad \text{[F63 bis]}$$

$$\lambda_{II}^2 = -\frac{a}{3} + ir = Se^{i\phi} \qquad \text{[F87]}$$

$$\lambda_{III}^2 = -\frac{a}{3} - ir = Se^{-i\phi} \qquad \text{[F88]}$$

where

$$S^2 = \frac{a^2}{9} + r^2 \qquad \text{[F89]}$$

$$\phi = \cos^{-1}\left[-\frac{a}{3S}\right] \qquad \text{[F90]}$$

We now set

$$\lambda_{11} = i\sqrt{\frac{a}{3}} \qquad \text{[F66 bis]}$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3}} \qquad \text{[F67 bis]}$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\Phi}{2}} = \sqrt{S}\left[\cos\frac{\Phi}{2} + i\sin\frac{\Phi}{2}\right] \qquad \text{[F27 bis]}$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\Phi}{2}} = -\sqrt{S}\left[\cos\frac{\Phi}{2} + i\ \sin\frac{\Phi}{2}\right] \qquad \text{[F28 bis]}$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\Phi}{2}} = \sqrt{S}\left[\cos\frac{\Phi}{2} - i\ \sin\frac{\Phi}{2}\right] \qquad \text{[F29 bis]}$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\Phi}{2}} = \sqrt{S}\left[-\cos\frac{\Phi}{2} + i\ \sin\frac{\Phi}{2}\right] \qquad \text{[F30 bis]}$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \qquad \text{[F72 bis]}$$

$$\lambda_2 = \sqrt{S}\ \cos\frac{\Phi}{2} \qquad \text{[F32 bis]}$$

$$\lambda_3 = \sqrt{S}\ \sin\frac{\Phi}{2} \qquad \text{[F33 bis]}$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad \text{[F43 bis]}$$
$$W_2 = \sin \lambda_1\theta \qquad \text{[F44 bis]}$$
$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F36 bis]}$$
$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F37 bis]}$$
$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F38 bis]}$$
$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F39 bis]}$$

<u>Case 6: $P < 0;\ Q \neq 0$</u>

In Case 6,

$$\left[\frac{P}{3}\right]^3 < 0 \qquad \text{[F91]}$$

so that Equation E20 yields

$$\sqrt{R} = \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} > \left|\frac{Q}{2}\right| \quad \text{[F92]}$$

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} > 0 \quad \text{[F93]}$$

$$B^3 = \frac{Q}{2} - \sqrt{R} < 0 \quad \text{[F94]}$$

Now let

$$A = \left[\frac{Q}{2} + \sqrt{R}\right]^{1/3} \quad \text{[F95]}$$

$$B = \left|\frac{Q}{2} - \sqrt{R}\right|^{1/3} \quad \text{[F96]}$$

and then set

$$A_1 = A \quad \text{[F97]}$$

$$A_2 = Ae^{i\frac{2\pi}{3}} = \frac{A}{2}(-1 + i\sqrt{3}) \quad \text{[F98]}$$

$$A_3 = Ae^{-i\frac{2\pi}{3}} = -\frac{A}{2}(1 + i\sqrt{3}) \quad \text{[F99]}$$

$$B_1 = -B \quad \text{[F100]}$$

$$B_2 = Be^{i\frac{\pi}{3}} = \frac{B}{2}(1 + i\sqrt{3}) \quad \text{[F101]}$$

$$B_3 = Be^{-i\frac{\pi}{3}} = \frac{B}{2}(1 - i\sqrt{3}) \quad \text{[F102]}$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = A - B \tag{F103}$$

$$y_2 = A_2 + B_2 = \frac{1}{2}\left[(-A + B) + i\sqrt{3}(A+B)\right] \tag{F104}$$

$$y_3 = A_3 + B_3 = \frac{1}{2}\left[(-A + B) - i\sqrt{3}(A + B)\right] \tag{F105}$$

so that Equations E45 yield

$$\lambda_I^2 = (A - B) - \frac{a}{3} \tag{F106}$$

$$\lambda_{II}^2 = \left[\frac{-A + B}{2} - \frac{a}{3}\right] + i\frac{\sqrt{3}}{2}(A + B) = Se^{i\phi} \tag{F107}$$

$$\lambda_{III}^2 = \left[\frac{-A + B}{2} - \frac{a}{3}\right] - i\frac{\sqrt{3}}{2}(A + B) = Se^{-i\phi} \tag{F108}$$

where

$$S^2 = \frac{A^2 - 2AB + B^2}{4} - \frac{(-A + B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A^2 + 2AB + B^2)$$

$$= (A^2 + AB + B^2) + \frac{(A - B)a}{3} + \frac{a^2}{9} \tag{F109}$$

$$\phi = \cos^{-1}\left[-\frac{\frac{A - B}{2} + \frac{a}{3}}{S}\right] \tag{F110}$$

<u>Case 6a: $(A - B) - \frac{a}{3} \geq 0$</u>

For Case 6a, we set

$$\lambda_{11} = \sqrt{(A - B) - \frac{a}{3}} \tag{F111}$$

$$\lambda_{12} = -\sqrt{(A - B) - \frac{a}{3}} \tag{F112}$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\,\sin\frac{\phi}{2}\right] \tag{F27 bis}$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\phi}{2}} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\,\sin\frac{\phi}{2}\right] \tag{F28 bis}$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\,\sin\frac{\phi}{2}\right] \tag{F29 bis}$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\,\sin\frac{\phi}{2}\right] \tag{F30 bis}$$

and then set

$$\lambda_1 = \sqrt{(A - B) - \frac{a}{3}} \tag{F113}$$

$$\lambda_2 = \sqrt{S}\cos\frac{\phi}{2} \tag{F32 bis}$$

$$\lambda_3 = \sqrt{S}\sin\frac{\phi}{2} \tag{F33 bis}$$

so that the mode functions are

$$W_1 = \cosh\lambda_1\theta \tag{F34 bis}$$
$$W_2 = \sinh\lambda_1\theta \tag{F35 bis}$$
$$W_3 = \cosh\lambda_2\theta\cos\lambda_3\theta \tag{F36 bis}$$
$$W_4 = \cosh\lambda_2\theta\sin\lambda_3\theta \tag{F37 bis}$$
$$W_5 = \sinh\lambda_2\theta\cos\lambda_3\theta \tag{F38 bis}$$
$$W_6 = \sinh\lambda_2\theta\sin\lambda_3\theta \tag{F39 bis}$$

Case 6b: $(A - B) - \frac{a}{3} < 0$

For Case 6b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - (A - B)} \qquad [F114]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - (A - B)} \qquad [F115]$$

$$\lambda_{21} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F27\ bis]$$

$$\lambda_{22} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F28\ bis]$$

$$\lambda_{31} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right] \qquad [F29\ bis]$$

$$\lambda_{32} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad [F30\ bis]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - (A - B)} \qquad [F116]$$

$$\lambda_2 = \sqrt{S}\cos\frac{\phi}{2} \qquad [F32\ bis]$$

$$\lambda_3 = \sqrt{S}\sin\frac{\phi}{2} \qquad [F33\ bis]$$

so that the mode functions are

$$W_1 = \cos\lambda_1\theta \qquad [F43\ bis]$$
$$W_2 = \sin\lambda_1\theta \qquad [F44\ bis]$$
$$W_3 = \cosh\lambda_2\theta\cos\lambda_3\theta \qquad [F36\ bis]$$
$$W_4 = \cosh\lambda_2\theta\sin\lambda_3\theta \qquad [F37\ bis]$$

$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F38 bis]}$$

$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F39 bis]}$$

<u>Case 7: $0 < P \leqslant 3\left[\frac{Q}{2}\right]^{2/3}$; $Q > 0$</u>

In Case 7,

$$0 < \left[\frac{P}{3}\right]^3 \leqslant \left[\frac{Q}{2}\right]^2 \qquad \text{[F117]}$$

so that Equation E20 yields

$$0 \leqslant \left[\sqrt{R} = \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3}\ \right] < \left|\frac{Q}{2}\right| \qquad \text{[F118]}$$

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} > 0 \qquad \text{[F119]}$$

$$B^3 = \frac{Q}{2} - \sqrt{R} > 0 \qquad \text{[F120]}$$

Now let

$$A = \left[\frac{Q}{2} + \sqrt{R}\right]^{1/3} \qquad \text{[F121]}$$

$$B = \left[\frac{Q}{2} - \sqrt{R}\right]^{1/3} \qquad \text{[F122]}$$

and then set

$$A_1 = A \qquad \text{[F97 bis]}$$

$$A_2 = Ae^{i\frac{2\pi}{3}} = \frac{A}{2}(-1 + i\sqrt{3}) \qquad \text{[F98 bis]}$$

$$A_3 = Ae^{-i\frac{2\pi}{3}} = -\frac{A}{2}(1 + i\sqrt{3}) \qquad \text{[F99 bis]}$$

$$B_1 = B \qquad \text{[F123]}$$

$$B_2 = Be^{i\frac{2\pi}{3}} = \frac{B}{2}(-1 + i\sqrt{3}) \qquad \text{[F124]}$$

$$B_3 = Be^{-i\frac{2\pi}{3}} = -\frac{B}{2}(1 + i\sqrt{3}) \qquad \text{[F125]}$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = A + B \qquad \text{[F126]}$$

$$y_2 = A_2 + B_2 = \frac{A + B}{2}(-1 + i\sqrt{3}) \qquad \text{[F127]}$$

$$y_3 = A_3 + B_3 = -\frac{A + B}{2}(1 + i\sqrt{3}) \qquad \text{[F128]}$$

so that Equations E45 yield

$$\lambda_I^2 = (A + B) - \frac{a}{3} \qquad \text{[F129]}$$

$$\lambda_{II}^2 = -\left[\frac{A + B}{2} + \frac{a}{3}\right] + i\frac{\sqrt{3}}{2}(A + B) = Se^{i\phi} \qquad \text{[F130]}$$

$$\lambda_{III}^2 = -\left[\frac{A + B}{2} + \frac{a}{3}\right] - i\frac{\sqrt{3}}{2}(A + B) = Se^{-i\phi} \qquad \text{[F131]}$$

where

$$S^2 = \frac{(A + B)^2}{4} + \frac{(A + B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A + B)^2$$

$$= (A + B)^2 + \frac{(A + B)a}{3} + \frac{a^2}{9} \qquad \text{[F132]}$$

$$\phi = \cos^{-1}\left[-\frac{\frac{A+B}{2} + \frac{a}{3}}{S}\right] \qquad \text{[F133]}$$

<u>Case 7a: $(A + B) - \frac{a}{3} \geqslant 0$</u>

For Case 7a, we set

$$\lambda_{11} = \sqrt{(A + B) - \frac{a}{3}} \qquad \text{[F134]}$$

$$\lambda_{12} = -\sqrt{(A + B) - \frac{a}{3}} \qquad \text{[F135]}$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad \text{[F27 bis]}$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\phi}{2}} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad \text{[F28 bis]}$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right] \qquad \text{[F29 bis]}$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \qquad \text{[F30 bis]}$$

and then set

$$\lambda_{1} = \sqrt{(A + B) - \frac{a}{3}} \qquad \text{[F136]}$$

$$\lambda_{2} = \sqrt{S}\cos\frac{\phi}{2} \qquad \text{[F32 bis]}$$

$$\lambda_{3} = \sqrt{S}\sin\frac{\phi}{2} \qquad \text{[F33 bis]}$$

so that the mode functions are

$$W_1 = \cosh \lambda_1\theta \qquad \text{[F34 bis]}$$

$$W_2 = \sinh \lambda_1\theta \qquad \text{[F35 bis]}$$

$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F36 bis]}$$

$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F37 bis]}$$

$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F38 bis]}$$

$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F39 bis]}$$

<u>Case 7b: $(A + B) - \frac{a}{3} < 0$</u>

For Case 7b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - (A + B)} \qquad \text{[F137]}$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - (A + B)} \qquad \text{[F138]}$$

$$\lambda_{21} = \sqrt{S}\left[\cos \frac{\phi}{2} + i \sin\frac{\phi}{2}\right] \qquad \text{[F27 bis]}$$

$$\lambda_{22} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i \sin\frac{\phi}{2}\right] \qquad \text{[F28 bis]}$$

$$\lambda_{31} = \sqrt{S}\left[\cos\frac{\phi}{2} - i \sin\frac{\phi}{2}\right] \qquad \text{[F29 bis]}$$

$$\lambda_{32} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i \sin\frac{\phi}{2}\right] \qquad \text{[F30 bis]}$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - (A + B)} \qquad \text{[F139]}$$

$$\lambda_2 = \sqrt{S} \cos\frac{\phi}{2} \qquad \text{[F32 bis]}$$

$$\lambda_3 = \sqrt{S}\,\sin\frac{\phi}{2} \qquad \text{[F33 bis]}$$

so that the mode functions are

$$W_1 = \cos\lambda_1\theta \qquad \text{[F43 bis]}$$

$$W_2 = \sin\lambda_1\theta \qquad \text{[F44 bis]}$$

$$W_3 = \cosh\lambda_2\theta\,\cos\lambda_3\theta \qquad \text{[F36 bis]}$$

$$W_4 = \cosh\lambda_2\theta\,\sin\lambda_3\theta \qquad \text{[F37 bis]}$$

$$W_5 = \sinh\lambda_2\theta\,\cos\lambda_3\theta \qquad \text{[F38 bis]}$$

$$W_6 = \sinh\lambda_2\theta\,\sin\lambda_3\theta \qquad \text{[F39 bis]}$$

Case 8: $0 < P \leq 3\left[\frac{Q}{2}\right]^{2/3}$; $Q < 0$

In Case 8,

$$0 < \left[\frac{P}{3}\right]^3 \leq \left[\frac{Q}{2}\right]^2 \qquad \text{[F117 bis]}$$

so that Equation E20 yields

$$0 \leq \left[\sqrt{R} = \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3}\;\right] < \left|\frac{Q}{2}\right| \qquad \text{[F118 bis]}$$

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} < 0 \qquad \text{[F140]}$$

$$B^3 = \frac{Q}{2} - \sqrt{R} < 0 \qquad \text{[F141]}$$

Now let

$$A = \left|\frac{Q}{2} + \sqrt{R}\right|^{1/3} \qquad \text{[F142]}$$

$$B = \left|\frac{Q}{2} - \sqrt{R}\right|^{1/3} \qquad \text{[F143]}$$

and then set

$$A_1 = -A \qquad [F144]$$

$$A_2 = Ae^{i\frac{\pi}{3}} = \frac{A}{2}(1 + i\sqrt{3}) \qquad [F145]$$

$$A_3 = Ae^{-i\frac{\pi}{3}} = \frac{A}{2}(1 - i\sqrt{3}) \qquad [F146]$$

$$B_1 = -B \qquad [F147]$$

$$B_2 = Be^{i\frac{\pi}{3}} = \frac{B}{2}(1 + i\sqrt{3}) \qquad [F148]$$

$$B_3 = Be^{-i\frac{\pi}{3}} = \frac{B}{2}(1 - i\sqrt{3}) \qquad [F149]$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = -(A + B) \qquad [F150]$$

$$y_2 = A_2 + B_2 = \frac{A + B}{2}(1 + i\sqrt{3}) \qquad [F151]$$

$$y_3 = A_3 + B_3 = \frac{A + B}{2}(1 - i\sqrt{3}) \qquad [F152]$$

so that Equations E45 yield

$$\lambda_I^2 = -\left[(A + B) + \frac{a}{3}\right] \qquad [F153]$$

$$\lambda_{II}^2 = \left[\frac{A + B}{2} - \frac{a}{3}\right] + i\frac{\sqrt{3}}{2}(A + B) = Se^{i\phi} \qquad [F154]$$

$$\lambda_{III}^2 = \left[\frac{A + B}{2} - \frac{a}{3}\right] - i\frac{\sqrt{3}}{2}(A + B) = Se^{-i\phi} \qquad [F155]$$

where

$$S^2 = \frac{(A + B)^2}{4} - \frac{(A + B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A + B)^2$$

$$= (A + B)^2 - \frac{(A + B)a}{3} + \frac{a^2}{9} \quad \text{[F156]}$$

$$\phi = \cos^{-1}\left[\frac{\frac{A + B}{2} - \frac{a}{3}}{S}\right] \quad \text{[F157]}$$

We then set

$$\lambda_{11} = i\sqrt{(A + B) + \frac{a}{3}} \quad \text{[F158]}$$

$$\lambda_{12} = -i\sqrt{(A + B) + \frac{a}{3}} \quad \text{[F159]}$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \quad \text{[F27 bis]}$$

$$\lambda_{22} = -\sqrt{S}e^{i\frac{\phi}{2}} = -\sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \quad \text{[F28 bis]}$$

$$\lambda_{31} = \sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right] \quad \text{[F29 bis]}$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right], \quad \text{[F30 bis]}$$

and then set

$$\lambda_1 = \sqrt{(A + B) + \frac{a}{3}} \quad \text{[F160]}$$

$$\lambda_2 = \sqrt{S}\cos\frac{\phi}{2} \quad \text{[F32 bis]}$$

$$\lambda_3 = \sqrt{S}\,\sin\frac{\Phi}{2} \qquad \text{[F33 bis]}$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad \text{[F43 bis]}$$

$$W_2 = \sin \lambda_1\theta \qquad \text{[F44 bis]}$$

$$W_3 = \cosh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F36 bis]}$$

$$W_4 = \cosh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F37 bis]}$$

$$W_5 = \sinh \lambda_2\theta \cos \lambda_3\theta \qquad \text{[F38 bis]}$$

$$W_6 = \sinh \lambda_2\theta \sin \lambda_3\theta \qquad \text{[F39 bis]}$$

<u>Case 9: $P > 3\left[\frac{Q}{2}\right]^{2/3}$; $Q \neq 0$</u>

In Case 9,

$$\left[\frac{P}{3}\right]^3 > \left[\frac{Q}{2}\right]^2 \qquad \text{[F161]}$$

so that Equation E20 yields

$$R = \left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3 < 0 \qquad \text{[F162]}$$

We therefore define

$$\omega = \frac{1}{3}\cos^{-1}\left[\frac{\left[\frac{Q}{2}\right]}{\left[\frac{P}{3}\right]^{3/2}}\right] \quad (0 \leqslant \omega \leqslant \frac{\pi}{3}) \qquad \text{[F163]}$$

and then set

$$\omega_1 = \omega \qquad \text{[E39 bis]}$$

$$\omega_2 = \omega - \frac{2\pi}{3} \qquad \text{[E40 bis]}$$

$$\omega_3 = \omega + \frac{2\pi}{3} \qquad \text{[E41 bis]}$$

and, following Equation E44, set

$$y_1 = 2\sqrt{\frac{P}{3}}\cos\omega_1 \qquad \text{[F164]}$$

$$y_2 = 2\sqrt{\frac{P}{3}}\cos\omega_2 \qquad \text{[F165]}$$

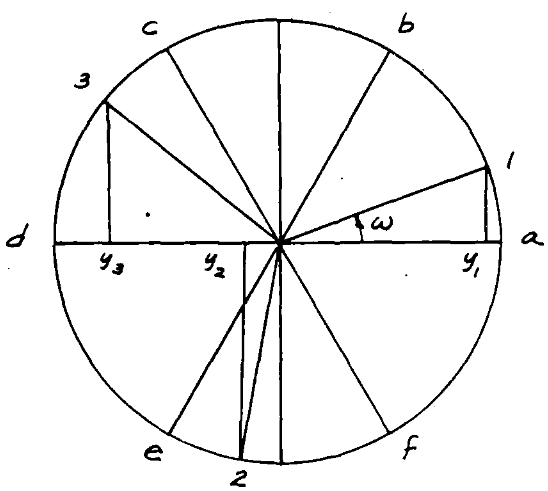
$$y_3 = 2\sqrt{\frac{P}{3}}\cos\omega_3 \qquad \text{[F166]}$$

The following diagram shows that, since

$$0 \leqslant \omega \leqslant \frac{\pi}{3} \qquad \text{[F167]}$$

then

$$y_1 \geqslant y_2 \geqslant y_3 \qquad \text{[F168]}$$



We then set

$$\lambda_I^2 = y_1 - \frac{a}{3} \quad [F169]$$

$$\lambda_{II}^2 = y_2 - \frac{a}{3} \quad [F170]$$

$$\lambda_{III}^2 = y_3 - \frac{a}{3} \quad [F171]$$

These cases are possible, as shown below.

		$\lambda_I^2 \geq 0$	$\lambda_I^2 < 0$
λ_{II}^2	≥ 0	9a	
	< 0	9b	9c

Case 9a: $\lambda_I^2 \geq 0$; $\lambda_{II}^2 \geq 0$

In Case 9a, we set

$$\lambda_{11} = \sqrt{y_1 - \frac{a}{3}} \quad [F172]$$

$$\lambda_{12} = -\sqrt{y_1 - \frac{a}{3}} \quad [F173]$$

$$\lambda_{21} = \sqrt{y_2 - \frac{a}{3}} \quad [F174]$$

$$\lambda_{22} = -\sqrt{y_2 - \frac{a}{3}} \quad [F175]$$

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3} \quad [F176]$$

$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3} \quad [F177]$$

and then set

$$\lambda_1 = \sqrt{y_1 - \frac{a}{3}} \qquad \text{[F178]}$$

$$\lambda_2 = \sqrt{y_2 - \frac{a}{3}} \qquad \text{[F179]}$$

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3} \qquad \text{[F180]}$$

so that the mode functions are

$$W_1 = \cosh \lambda_1\theta \qquad \text{[F34 bis]}$$
$$W_2 = \sinh \lambda_1\theta \qquad \text{[F35 bis]}$$
$$W_3 = \cosh \lambda_2\theta \qquad \text{[F75 bis]}$$
$$W_4 = \sinh \lambda_2\theta \qquad \text{[F76 bis]}$$
$$W_5 = \cos \lambda_3\theta \qquad \text{[F77 bis]}$$
$$W_6 = \sin \lambda_3\theta \qquad \text{[F78 bis]}$$

Case 9b: $\lambda_I^2 \geq 0;\ \lambda_{II}^2 < 0$

In Case 9b, we set

$$\lambda_{11} = \sqrt{y_1 - \frac{a}{3}} \qquad \text{[F172 bis]}$$

$$\lambda_{12} = -\sqrt{y_1 - \frac{a}{3}} \qquad \text{[F173 bis]}$$

$$\lambda_{21} = i\sqrt{\frac{a}{3} - y_2} \qquad \text{[F181]}$$

$$\lambda_{22} = -i\sqrt{\frac{a}{3} - y_2} \qquad \text{[F182]}$$

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3} \qquad \text{[F176 bis]}$$

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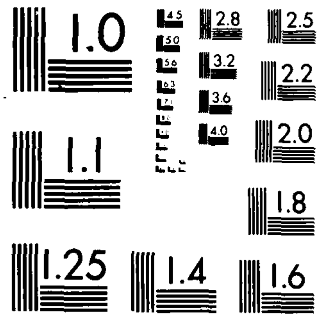

MICROCOPY RESOLUTION TEST CHART
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$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3} \qquad \text{[F177 bis]}$$

and then set

$$\lambda_1 = \sqrt{y_1 - \frac{a}{3}} \qquad \text{[F178 bis]}$$

$$\lambda_2 = \sqrt{\frac{a}{3} - y_2} \qquad \text{[F183]}$$

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3} \qquad \text{[F180 bis]}$$

so that the mode functions are

$$W_1 = \cosh \lambda_1\theta \qquad \text{[F34 bis]}$$
$$W_2 = \sinh \lambda_1\theta \qquad \text{[F35 bis]}$$
$$W_3 = \cos \lambda_2\theta \qquad \text{[F82 bis]}$$
$$W_4 = \sin \lambda_2\theta \qquad \text{[F83 bis]}$$
$$W_5 = \cos \lambda_3\theta \qquad \text{[F77 bis]}$$
$$W_6 = \sin \lambda_3\theta \qquad \text{[F78 bis]}$$

Case 9c: $\lambda_I^2 < 0$

In Case 9c, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - y_1} \qquad \text{[F184]}$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - y_1} \qquad \text{[F185]}$$

$$\lambda_{21} = i\sqrt{\frac{a}{3} - y_2} \qquad \text{[F181 bis]}$$

$$\lambda_{22} = -i\sqrt{\frac{a}{3} - y_2} \qquad \text{[F182 bis]}$$

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3} \qquad \text{[F176 bis]}$$

$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3} \qquad \text{[F177 bis]}$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - y_1} \qquad \text{[F186]}$$

$$\lambda_2 = \sqrt{\frac{a}{3} - y_2} \qquad \text{[F183 bis]}$$

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3} \qquad \text{[F180 bis]}$$

so that the mode functions are

$$W_1 = \cos \lambda_1\theta \qquad \text{[F43 bis]}$$
$$W_2 = \sin \lambda_1\theta \qquad \text{[F44 bis]}$$
$$W_3 = \cos \lambda_2\theta \qquad \text{[F82 bis]}$$
$$W_4 = \sin \lambda_2\theta \qquad \text{[F83 bis]}$$
$$W_5 = \cos \lambda_3\theta \qquad \text{[F77 bis]}$$
$$W_6 = \sin \lambda_3\theta \qquad \text{[F78 bis]}$$

Cases 1-9 (Summary)

For convenience, the above cases are summarized below.

Case	Criteria	
1	$P = Q = 0$	
2	$P = 0;\ Q > 0$	
2a		$r - a/3 \geq 0$
2b		$r - a/3 < 0$
3	$P = 0;\ Q < 0$	

Case	Criteria	
4	$P > 0;\ Q = 0$	
4a		$r - a/3 \geqslant 0$
4b		$r - a/3 < 0$
5	$P < 0;\ Q = 0$	
6	$P < 0;\ Q \neq 0$	
6a		$(A - B) - a/3 \geqslant 0$
6b		$(A - B) - a/3 < 0$
7	$0 < P \leqslant 3\left[\frac{Q}{2}\right]^{2/3};\ Q > 0$	
7a		$(A + B) - a/3 \geqslant 0$
7b		$(A + B) - a/3 < 0$
8	$0 < P \leqslant 3\left[\frac{Q}{2}\right]^{2/3};\ Q < 0$	
9	$P > 3\left[\frac{Q}{2}\right]^{2/3};\ Q \neq 0$	
9a	$\lambda_I^2 \geqslant 0;\ \lambda_{II}^2 \geqslant 0$	
9b	$\lambda_I^2 \geqslant 0;\ \lambda_{II}^2 < 0$	
9c	$\lambda_I^2 < 0$	

APPENDIX G

SYMMETRY

Any function, $f(x)$, can be expressed as the sum of a symmetric (even) function, $f_E(x)$, and an antisymmetric (odd) function, $f_o(x)$,

$$f(x) = f_E(x) + f_o(x) \qquad [G1]$$

where

$$f_E(x) = \frac{f(x) + f(-x)}{2} \qquad [G2]$$

$$f_o(x) = \frac{f(x) - f(-x)}{2} \qquad [G3]$$

From Equations G2 and G3 come the definitions of an even and an odd function.

$$f_E(-x) = f_E(x) \qquad [G4]$$

$$f_o(-x) = -f_o(x) \qquad [G5]$$

From the definition of a derivative, and using the notation

$$L = \lim_{\Delta x \to o} \qquad [G6]$$

we obtain

$$f_E'(-x) = L\left[\frac{f_E(-x) - f_E(-x-\Delta x)}{\Delta x}\right]$$

$$= L\left[\frac{f_E(x) - f_E(x+\Delta x)}{\Delta x}\right] = -f_E'(x) \qquad [G7]$$

$$f_o'(-x) = L\left[\frac{f_o(-x) - f_o(-x-\Delta x)}{\Delta x}\right]$$

$$= L\left[\frac{-f_o(x) + f_o(x+\Delta x)}{\Delta x}\right] = f_o'(x) \qquad [G8]$$

It follows that the even and odd derivatives of even and odd functions are even or odd according to the table below.

		DERIVATIVE	
		EVEN	ODD
FUNCTION	EVEN	EVEN	ODD
	ODD	ODD	EVEN

APPENDIX H

MODE SHAPE DETERMINATION

Equations B5, C3, and C4 express the centroidal translation and rotation of a plane cross section of a circular arch vibrating in a single mode in separable form,

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} = T\ \{U\} = T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad [H1]$$

where, referring to Figure A1

- u_1 = centroidal tangential displacement, in the direction of increasing θ
- u_2 = centroidal inward radial displacement
- ψ = clockwise rotation
- r = arch cross-section radius of gyration, defined in Equation A62
- T = modal amplitude, a function of time only
- X, Y, Z = mode shapes, functions of the arch angle, θ, only

Equation H1 is the general definition of a structural vibratory mode, i.e., a condition in which the relative displacements of all points on the structure remain constant, and the absolute displacements are proportional to a single scalar function of time. The configuration defined is that of the arch cross-section centroidal axis, not its neutral axis, because the centroidal axis location is independent of cross-section rotation.

When the arch undergoes free vibration, Equation C15 shows that the modal amplitude, T, is harmonic with angular frequency p.

$$\ddot{T} + p^2 T = 0 \qquad [C15\ bis]$$

Appendix F shows that the three mode shapes, X (for centroidal tangential displacement), Y (for centroidal radial displacement), and Z (for plane cross-section rotation) can each be expressed as a linear combination of six mode functions,

$$\{U\} = \underline{C}\ \{W\} \qquad [H2]$$

where

$\{U\}$ = 3 x 1 column matrix of mode shapes

$\{W\}$ = 6 x 1 column matrix of mode functions

$\underline{C}$ = 3 x 6 rectangular matrix of mode shape coefficients

Each of the six mode functions, W_i (i=1,6), satisfies Equation E2, which is a homogeneous, linear, ordinary differential equation with constant coefficients, of order six. Equation E2 is derived from Equations C18, which are three independent, homogeneous, linear, ordinary differential equations with constant coefficients, of order two. Equations C18 are the fundamental equations describing arch free vibration, and Equation E2 was derived from them for the sake of convenience, to uncouple the system (cf. Equations C21-C23). However, the price of uncoupling the system is that Equation H2 is too general. Unless restrictions are placed on the mode shape coefficients, Equations C18 will not be satisfied. To determine what restrictions must be placed on the mode shape coefficients, we substitute Equation H2 into Equation C18 and obtain

$$(\underline{L} + \sigma^2 \underline{I})\ \{U\} = \underline{C}(\underline{L} + \sigma^2 \underline{I})\{W\} = \{0\} \qquad [H3]$$

where

$$\sigma = \frac{p}{\left[\frac{c}{R}\right]} \qquad [C17\ bis]$$

$$c^2 = \frac{E}{\rho} \qquad [B16\ bis]$$

and

E = Young's elastic modulus

ρ = mass density

R = arch centroidal radius

The six mode functions, W_i, are a closed set, in the sense that differentiation of any one mode function yields a linear combination of the other five. Equations H3 are thus a set of three homogeneous linear equations in the six mode functions. But the six mode functions are linearly independent, so the coefficient matrix in the resulting equations must be the null matrix (all elements zero), i.e.,

$$\underline{C}(\underline{L} + \sigma^2 \underline{I})\{W\} = \underline{C}\ \underline{E}\ \{W\} = \{0\} \qquad [H4]$$

where

$$\underline{C}\ \underline{E} = \underline{0} \qquad [H5]$$

However, since Equation E2 already represents all three of Equations C18 taken together, only two rows of Equation H4 are linearly independent. Therefore only 12 of the 18 scalar equations represented by Equation H5 are independent. Equation H5 therefore yields expressions for 12 of the 18 mode shape coefficients as linear combinations of the remaining 6.

The relative values of the six remaining mode shape coefficients are determined from the six arch boundary conditions, which take the form

$$\underline{B}\ \{C\} = \{0\} \qquad [H6]$$

where

$\{C\}$ = 6 x 1 column matrix of remaining mode shape coefficients
$\underline{B}$ = 6 x 6 square matrix

The elements of the $\underline{B}$ matrix are linear combinations of the six mode functions, defined in Appendix F, evaluated at an arch boundary, the precise forms of which depend on the frequency parameter, σ. For a nontrivial solution to Equation [H6] to exist, the determinant of the $\underline{B}$ matrix must vanish.

$$|\underline{B}| = 0 \qquad [H7]$$

When this happens, the frequency parameter, σ, corresponds to a natural or modal frequency.

What makes the arch vibration problem more complicated than some other structural vibration problems is the fact that the precise form of the transcendental frequency equation generated by Equation H7 depends on the frequency parameter, σ. Thus in a trial and error solution to find the natural frequencies and mode shapes of a circular arch, the form of the frequency equation varies with the trial frequency parameter, σ.

APPENDIX I

ORTHOGONALITY RELATIONS

The basic equations for the mode shapes of a circular arch are

$$(\underline{L} + \sigma^2 \underline{I})\{U\} = \{0\} \qquad \text{[C18 bis]}$$

where

$$\underline{L} = \begin{bmatrix} AD^2-B & -(A+B)D & -ED^2+BF \\ (A+B)D & BD^2-A & -(BF+E)D \\ -ED^2+BF & (BF+E)D & CD^2-BF^2 \end{bmatrix} \qquad \text{[D8 bis]}$$

and

$$D = \frac{d}{d\theta} \qquad \text{[A60 bis]}$$

$$A = C_1 \qquad \text{[D3 bis]}$$

$$B = \frac{k'}{2(1+\nu)} \qquad \text{[D4 bis]}$$

$$C = C_3 \qquad \text{[D5 bis]}$$

$$E = 2\sqrt{3}C_2 \qquad \text{[D6 bis]}$$

$$F = \frac{R}{r} \qquad \text{[D7 bis]}$$

The constants C_1, C_2, and C_3 are functions of the ratio of arch depth to radius, as shown in the table following Equation A58.

To investigate the modal orthogonality relations we evaluate Equation C18 for modes i and j.

$$(\underline{L} + \sigma_i^2\underline{I})\ \{U\}_i = \{0\} \qquad \text{[I1]}$$

$$(\underline{L} + \sigma_j^2\underline{I})\ \{U\}_j = \{0\} \qquad \text{[I2]}$$

Premultiplying Equation I1 by $\{U\}_j^T$, and premultiplying Equation I2 by $\{U\}_i^T$ yields

$$\{U\}_j^T \underline{L} \{U\}_i + \sigma_i^2 \{U\}_j^T \{U\}_i = 0 \qquad [I3]$$

$$\{U\}_i^T \underline{L} \{U\}_j + \sigma_j^2 \{U\}_i^T \{U\}_j = 0 \qquad [I4]$$

Subtracting Equation I4 from Equation I3 yields

$$\{U\}_j^T \underline{L} \{U\}_i - \{U\}_i^T \underline{L} \{U\}_j + \left[\sigma_i^2 - \sigma_j^2\right] \{U\}_i^T \{U\}_j = 0$$

or

$$\left[\sigma_i^2 - \sigma_j^2\right] \{U\}_i^T \{U\}_j = \{U\}_i^T \underline{L} \{U\}_j - \{U\}_j^T \underline{L} \{U\}_i \qquad [I5]$$

It remains to be shown that the RHS of Equation I5 is the derivative of a function which has equal values at $\theta = 0$ and $\theta = \Omega$, so that

$$\{U\}_i^T \underline{L} \{U\}_j - \{U\}_j^T \underline{L} \{U\}_i = B'_{ij} \qquad [I6]$$

and

$$\left[B_{ij}(\theta)\right]_0^\Omega = 0 \qquad [I7]$$

If Equations I6 and I7 are satisfied, then integration of Equation I5 on θ between 0 and Ω yields

$$\left[\sigma_i^2 - \sigma_j^2\right] \int_0^\Omega \{U\}_i^T \{U\}_j \, d\theta = \int_0^\Omega B'_{ij} \, d\theta = \left[B_{ij}(\theta)\right]_0^\Omega = 0 \qquad [I8]$$

When $i \neq j$, so that $\sigma_i^2 \neq \sigma_j^2$, Equation I8 yields

$$\int_0^\Omega \{U\}_i^T \{U\}_j \, d\theta = 0 \qquad (i \neq j) \qquad [I9]$$

Thus the i^{th} and j^{th} modes are orthogonal over the closed interval $0 \leq \theta \leq \Omega$, provided Equations I6 and I7 are satisfied.

To prove Equation I6 we expand its LHS using Equation D8, and obtain

$$
\begin{aligned}
B'_{ij} &= A\left[X_iX_j'' - X_jX_i''\right] - (A+B)\left[X_iY_j' - X_jY_i'\right] - E\left[X_iZ_j'' - X_jZ_i''\right] \\
&\quad + (A+B)\left[Y_iX_j' - Y_jX_i'\right] + B\left[Y_iY_j'' - Y_jY_i''\right] - (BF+E)\left[Y_iZ_j' - Y_jZ_i'\right] \\
&\quad - E\left[Z_iX_j'' - Z_jX_i''\right] + (BF+E)\left[Z_iY_j' - Z_jY_i'\right] + C\left[Z_iZ_j'' - Z_jZ_i''\right] \\
&= A\left[X_iX_j'' - X_jX_i''\right] + B\left[Y_iY_j'' - Y_jY_i''\right] + C\left[Z_iZ_j'' - Z_jZ_i''\right] \\
&\quad - (A+B)\left[\left[X_i'Y_j + X_iY_j'\right] - \left[X_j'Y_i + X_jY_i'\right]\right] \\
&\quad - (BF+E)\left[\left[Y_i'Z_j + Y_iZ_j'\right] - \left[Y_j'Z_i + Y_jZ_i'\right]\right] \\
&\quad + E\left[\left[Z_i''X_j - Z_iX_j''\right] - \left[Z_j''X_i - Z_jX_i''\right]\right]
\end{aligned}
\qquad [I10]
$$

Now

$$
\left[X_iX_j'\right]' - \left[X_i'X_j\right]' = X_i'X_j' + X_iX_j'' - X_i''X_j - X_i'X_j' = X_iX_j'' - X_jX_i'' \qquad [I11]
$$

and

$$
\left[Z_i'X_j\right]' - \left[Z_iX_j'\right]' = Z_i''X_j + Z_i'X_j' - Z_i'X_j' - Z_iX_j'' = Z_i''X_j - Z_iX_j'' \qquad [I12]
$$

Therefore, Equation I10 can be written in the form

$$
\begin{aligned}
B'_{ij} &= A\left[\left[X_iX_j'\right]' - \left[X_i'X_j\right]'\right] + B\left[\left[Y_iY_j'\right]' - \left[Y_i'Y_j\right]'\right] \\
&\quad + C\left[\left[Z_iZ_j'\right]' - \left[Z_i'Z_j\right]'\right] - (A+B)\left[\left[X_iY_j\right]' - \left[X_jY_i\right]'\right]
\end{aligned}
$$

$$- (BF + E) \left[\left[Y_i Z_j \right]' - \left[Y_j Z_i \right]' \right]$$

$$+ E \left[\left[Z_i' X_j \right]' - \left[Z_i X_j' \right]' - \left[Z_j' X_i \right]' + \left[Z_j X_i' \right]' \right] \qquad [I13]$$

Each term on the RHS of Equation I13 is a derivative, so Equation I6 is proven.

To prove Equation I7 we expand Equations B8 and B9.

$$\{F\} = E^{*} A^{*} \underline{S} \; \{\delta\} \qquad [B8 \text{ bis}]$$

$$\{\delta\} = \frac{1}{R} (D\underline{I} - \underline{G}^{T}) \; \{u\} \qquad [B9 \text{ bis}]$$

where, to avoid confusion with symbols already used in this appendix

E^{*} = Young's elastic modulus

A^{*} = arch cross-sectional area

Substituting Equations B9 and C3 into Equation B8 yields

$$\{F\} = \frac{E^{*}A^{*}}{R} \; \underline{S} \; (D\underline{I} - \underline{G}^{T}) \; \{u\} = \frac{E^{*}A^{*}}{R} \; (D\underline{S} - \underline{SG}^{T}) \; \{u\}$$

$$= \frac{E^{*}A^{*}}{R} \; T \; (D\underline{S} - \underline{SG}^{T}) \; \{U\} \qquad [I14]$$

Now

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -2\sqrt{3}C_2 \\ 0 & \frac{k'}{2(1+\nu)} & 0 \\ -2\sqrt{3}C_2 & 0 & C_3 \end{bmatrix} \qquad [B2, B22 \text{ bis}]$$

$$\underline{SG}^T = \begin{bmatrix} 0 & C_1 & 0 \\ -\dfrac{k'}{2(1+\nu)} & 0 & \dfrac{k'R}{2r(1+\nu)} \\ 0 & -2\sqrt{3}C_2 & 0 \end{bmatrix} \qquad [B13\ bis]$$

so that

$$\underline{DS} - \underline{SG}^T = \begin{bmatrix} C_1 D & -C_1 & -2\sqrt{3}C_2 D \\ \dfrac{k'}{2(1+\nu)} & \dfrac{k'}{2(1+\nu)}D & -\dfrac{k'R}{2r(1+\nu)} \\ -2\sqrt{3}C_2 D & 2\sqrt{3}C_2 & C_3 D \end{bmatrix}$$

$$= \begin{bmatrix} AD & -A & -ED \\ B & BD & -BF \\ -ED & E & CD \end{bmatrix} \qquad [I15]$$

Now if

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ \underline{M} \\ r \end{bmatrix} = \{N\}\ T = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} T \qquad [I16]$$

then substituting Equations I15, I16, and C3 into Equation I14 yields

$$N_1 = \frac{E^* A^*}{R} \left[A(X' - Y) - EZ'\right] \qquad [I17]$$

$$N_2 = \frac{E^* A^*}{R} \left[B(X + Y' - FZ)\right] \qquad [I18]$$

$$N_3 = \frac{E^* A^*}{R} \left[E(Y - X') + CZ'\right] \qquad [I19]$$

By rearranging terms, Equation I13 can be written in the form

$$B'_{ij} = \left[X_i\left[A\left[X'_j - Y_j\right] - EZ'_j\right]\right]' - \left[X_j\left[A\left[X'_i - Y_i\right] - EZ'_i\right]\right]'$$

$$+ \left[Y_i B\left[X_j + Y_j' - FZ_j\right]\right]' - \left[Y_j B\left[X_i + Y_i' - FZ_i\right]\right]'$$

$$+ \left[Z_i\left[E\left[Y_j - X_j'\right] + CZ_j'\right]\right]' - \left[Z_j\left[E\left[Y_i - X_i'\right] + CZ_i'\right]\right]' \qquad [I20]$$

Substituting Equations I17, I18, and I19 into Equation I20 yields

$$B_{ij}' = \frac{R}{E^*A^*}\left[\{U\}_i^T\{N\}_j - \{U\}_j^T\{N\}_i\right]' \qquad [I21]$$

and substituting Equation I21 into Equation I8 yields

$$\left[\sigma_i^{\,2} - \sigma_j^{\,2}\right]\int_0^{\Omega}\{U\}_i^T\{U\}_j\,d\theta = \frac{R}{E^*A^*}\left[\{U\}_i^T\{N\}_j - \{U\}_j^T\{N\}_i\right]_0^{\Omega} = 0 \qquad [I22]$$

The conditions under which Equation I22 is valid for $i \neq j$ must still be determined.

Consider the elastic restraint conditions

$$\{N\} = \lceil K \rfloor \{U\} \qquad (\theta = 0) \qquad [I23a]$$

$$\{N\} = -\lceil K \rfloor \{U\} \qquad (\theta = \Omega) \qquad [I23b]$$

Substituting Equations I23 into Equation I22 yields

$$\left[\sigma_i^{\,2} - \sigma_j^{\,2}\right]\int_0^{\Omega}\{U\}_i^T\{U\}_j\,d\theta = \frac{R}{E^*A^*}\left[\{U\}_i^T\lceil K \rfloor\{U\}_j - \{U\}_j^T\lceil K \rfloor\{U\}_i\right]_{\Omega}^{0} = 0 \qquad [I24]$$

Thus Equations I23 are the conditions under which Equation I9 is valid.

One last task remains to complete the orthogonality analysis. That is to find the integral of the inner product of a mode shape with itself over the closed interval $0 \leq \theta \leq \Omega$. When $i = j$, Equation I22 reduces to the indeterminate form

$$0 \cdot \int_0^\Omega \{U\}_i^T \{U\}_i \, d\theta = 0 \qquad [I25]$$

As suggested by Rayleigh (Refs. 13 and 14), L'Hospital's Rule can be used to evaluate the integral in question by considering a set of functions identical in form to $\{U\}$ and $\{N\}$, except with σ the independent variable instead of θ. Thus if we set

$$\sigma_j = \sigma \qquad [I26]$$

$$\sigma_i = \sigma + d\sigma \qquad [I27]$$

$$\frac{d(\;)}{d\sigma} = (\bar{\;}) \qquad [I28]$$

$$\{U\}_j = \{U\} \qquad [I29]$$

$$\{U\}_i = \{U\} + \{\bar{U}\} \, d\sigma \qquad [I30]$$

$$\{N\}_j = \{N\} \qquad [I31]$$

$$\{N\}_i = \{N\} + \{\bar{N}\} \, d\sigma \qquad [I32]$$

then applying L'Hospital's Rule to Equation I22 yields

$$2\sigma \, d\sigma \int_0^\Omega \{U\}^T \{U\} \, d\theta = \frac{R d\sigma}{E^* A^*} \left[\{\bar{U}\}^T \{N\} - \{U\}^T \{\bar{N}\} \right]_0^\Omega$$

or

$$\int_0^\Omega \{U\}^T \{U\} \, d\theta = \frac{R}{2E^* A^* \sigma} \left[\{\bar{U}\}^T \{N\} - \{U\}^T \{\bar{N}\} \right]_0^\Omega \qquad [I33]$$

The derivatives with respect to σ on the RHS of Equation I33 can be evaluated as follows. All elements of $\{U\}$ and $\{N\}$ have the general form

$$f = f\,(\lambda_1\theta, \lambda_2\theta, \lambda_3\theta) \qquad [I34]$$

where

$$\lambda_n = \lambda_n(\sigma) \qquad (n = 1,2,3) \tag{I35}$$

therefore

$$\frac{df}{d\sigma} = \frac{\partial f}{\partial \lambda_n} \bar{\lambda}_n \qquad \text{(sum on n)} \tag{I36}$$

The derivatives with respect to λ are easily evaluated by the chain rule.

$$\frac{\partial f}{\partial \lambda_n} = \theta \frac{\partial f}{\partial(\lambda_n \theta)} \tag{I37}$$

The derivative of λ_n with respect to σ can be evaluated by differentiating Equation E6, which yields

$$(6\lambda^5 + 4a\lambda^3 + 2b\lambda)\,\bar{\lambda} + (\bar{a}\lambda^4 + \bar{b}\lambda^2 + \bar{c}) = 0 \tag{I38}$$

so that

$$\bar{\lambda}_n = - \frac{\bar{a}\lambda_n^{\,4} + \bar{b}\lambda_n^{\,2} + \bar{c}}{2\lambda_n\left[3\lambda_n^{\,4} + 2a\lambda_n^{\,2} + b\right]} \tag{I39}$$

Because of the factor θ in Equation I37, the RHS of Equation I33 vanishes at the lower limit, and therefore Equation I33 reduces to

$$\int_0^{\Omega} \{U\}_i^T \{U\}_i \, d\theta = \Gamma_i = \frac{R}{2E^* A^* \sigma_i} \left[\{\bar{U}\}_i^T \{N\}_i - \{U\}_i^T \{\bar{N}\}_i \right]_{\theta = \Omega} \tag{I40}$$

It is convenient to normalize the mode shapes so that

$$\Gamma_i = 1 \tag{I41}$$

and in that case

$$\int_0^{\Omega} \{U\}_i^T \{U\}_j \, d\theta = \delta_{ij} \tag{I42}$$

The orthogonality condition, I42, is used in Appendix J to obtain transient vibration solutions by expressing both the arch displacements and the forcing functions in modal series form. See Equations J2 and J3.

APPENDIX J

TRANSIENT ANALYSIS

Equations B17 and C7 describe the transient response of a circular arch. The basic equation is

$$\{\ddot{u}\} - \left[\frac{c}{R}\right]^2 \underline{L}\ \{u\} = \frac{1}{\rho A}\{b\} \qquad [J1]$$

Both the displacements and the externally applied loads are assumed to be expressible in a modal series expansion of the form

$$\{u\} = \sum_{j=1}^{\infty} T_j \{U\}_j \qquad [J2]$$

$$\{b\} = \sum_{j=1}^{\infty} H_j \{U\}_j \qquad [J3]$$

where the mode shapes satisfy the relations

$$\underline{L}\{U\}_j = -\sigma_j^2 \{U\}_j = -\frac{p_j^2}{\left[\frac{c}{R}\right]^2} \{U\}_j \qquad \text{(no sum)} \qquad [C16]$$

and

$$\int_0^{\Omega} \{U\}_i^T \{U\}_j d\theta = \delta_{ij} \qquad [I42]$$

Premultiplying Equation J3 by $\{U\}_i^T$ and integrating on θ from 0 to Ω yields

$$H_i = \int_0^{\Omega} \{U\}_i^T \{b\} d\theta \qquad [J4]$$

Substituting Equations J2, J3, and C16 into Equation J1 yields

$$\sum_{j=1}^{\infty} \ddot{T}_j (U)_j + \sum_{j=1}^{\infty} p_j^2 T_j (U)_j = \frac{1}{\rho A} \sum_{j=1}^{\infty} H_j (U)_j \qquad [J5]$$

Premultiplying Equation J5 by $(U)_i^T$ and integrating on θ from 0 to Ω yields

$$\ddot{T}_i + p_i^2 T_i = \frac{1}{\rho A} H_i \qquad [J6]$$

so that, assuming

$$T_i(0) = \dot{T}_i(0) = 0 \qquad [J7]$$

the solution is

$$T_i(t) = \frac{1}{\rho A} \int_0^t H_i(\tau) \sin p_i(t - \tau) d\tau \qquad [J8]$$

which completes the formal analysis.

APPENDIX K

VISCOELASTIC ANALYSIS

The dynamic analysis of a viscoelastic circular arch turns out to be a fairly simple extension of the results for an undamped elastic arch, when the rate-sensitive behavior is governed by a single parameter.

When the arch is viscoelastic, Equations A40 and A41 have the forms

$$\sigma_{11} = E\epsilon_1 + \alpha^{*}\mathring{\epsilon}_1 = E\left[\epsilon_1 + \frac{\alpha^{*}}{E}\mathring{\epsilon}_1\right] \qquad [K1]$$

$$\sigma_{12} = G\gamma_{12} + \eta\mathring{\gamma}_{12} = G\left[\gamma_{12} + \frac{\eta}{G}\mathring{\gamma}_{12}\right] \qquad [K2]$$

If we assume that

$$\frac{\alpha^{*}}{E} = \frac{\eta}{G} = \xi \qquad [K3]$$

then Equations K1 and K2 can be written in the form

$$\sigma_{11} = E\,(\epsilon_1 + \xi\mathring{\epsilon}_1) \qquad [K4]$$

$$\sigma_{12} = G\,(\gamma_{12} + \xi\mathring{\gamma}_{12}) \qquad [K5]$$

and therefore Equation B8 has the form

$$(F) = EA\underline{S}\,(\delta) + \xi(\mathring{\delta}) \qquad [K6]$$

and Equations B17 and C7 have the form

$$(\ddot{u}) - \left[\frac{c}{R}\right]^2 \underline{L}\left\{(u) + \xi(\mathring{u})\right\} = \frac{1}{\rho A}\,(b) \qquad [K7]$$

Equation C6, which describes free vibration in a normal mode, has the form

$$\ddot{T}(U) - \left[\frac{c}{R}\right]^2 (T + \xi\dot{T})\underline{L}(U) = (0) \qquad [K8]$$

and Equations C12 and C14 have the forms

$$\ddot{T}(m) - \left[\frac{c}{R}\right]^2 (T + \xi\dot{T}) \lceil \bar{U} \rfloor^{-1} \underline{L}\ (U) = (0) \qquad [K9]$$

$$\frac{\ddot{T}}{T + \xi\dot{T}}(m) = \left[\frac{c}{R}\right]^2 \lceil \bar{U} \rfloor^{-1} \underline{L}\ (U) = -p^2(m) \qquad [K10]$$

Equation C15 has the form

$$\ddot{T} + \xi p^2 \dot{T} + p^2 T = 0 \qquad [K11]$$

but Equation C16 remains the same.

$$\underline{L}\ (U) = -\frac{p^2}{\left[\frac{c}{R}\right]^2}(U) = -\sigma^2(U) \qquad [C16\ bis]$$

Equations I14 and I16 have the forms

$$(F) = \frac{E^*A^*}{R}(T + \xi\dot{T})(D\underline{S} - \underline{SG}^T)(U) \qquad [K12]$$

$$(F) = (T + \xi\dot{T})(N) \qquad [K13]$$

Equations I23 remain the same, and substituting Equations I23 into Equation K13 yields

$$(F) = (T + \xi\dot{T})\lceil K \rfloor (U) = \lceil K \rfloor\ (u) + \xi(\dot{u}) \qquad (\theta = 0) \qquad [K14a]$$

$$(F) = -(T + \xi\dot{T})\lceil K \rfloor (U) = -\lceil K \rfloor\ (u) + \xi(\dot{u}) \qquad (\theta = \Omega) \qquad [K14b]$$

Thus, in order for the modal orthogonality conditions to remain valid, the support restraints must also be viscoelastic, with ξ as the single rate sensitivity parameter. This will guarantee that the form of Equation I24 will be preserved.

The transient analysis problem is approached by substituting Equations J2 and J3 into Equation K7, which yields

$$\sum_{j=1}^{\infty} \ddot{T}_j \{U\}_j + \sum_{j=1}^{\infty} p_j^2 \left[T_j + \xi \dot{T}_j \right] \{U\}_j = \frac{1}{\rho A} \sum_{j=1}^{\infty} H_j \{U\}_j \qquad [K15]$$

Premultiplying Equation [K15] by $\{U\}_i^T$ and integrating on θ from 0 to Ω yields

$$\ddot{T}_i + \xi p_i^2 \dot{T}_i + p_i^2 T_i = \frac{1}{\rho A} H_i \qquad [K16]$$

The form of the convolution integral which is the solution to Equation K16 depends on the values of ξ and p_i. When $\xi p_i < 2.0$ the modal damping is less than critical; when $\xi p_i = 2.0$ the modal damping is critical; and when $\xi p_i > 2.0$ the modal damping is greater than critical.

APPENDIX L

NUMERICAL ANALYSIS

The most difficult computational phase of the arch modal analysis is finding the roots of the frequency equation (H7). A sample problem for a Timoshenko beam was solved numerically on the AFWL CRAY. The solution involved eleven program segments, which were stored in six files. These program segments are described below. The function used was Equation 5-14 of Reference 2.

<u>Program TEST</u>

This is the main program in the solution. It performs all of the I/O, determines the interval and tolerance for root finding, and specifies the number of roots sought.

<u>Function PRPCOM</u>

This function's purpose is to prepare the common region PARAM, which contains the function parameters C12, C22, R, L, and THETA. The meanings of these variables are given below, along with the other variables of PRPCOM.

C12	c_1^2	square of longitudinal wavespeed
C22	c_2^2	square of shear wavespeed
R	r	radius of gyration
L	L	length of beam
E	E	beam modulus of elasticity
G	G	beam elastic shear modulus
RHO	ρ	beam density
I	I	beam moment of inertia
KPRIME	K'	shear deformation coefficient
RBIG	R	rotational beam-end restraint
THETA	θ	EI/RBIG

Subroutine ZEROIN

This is a modified version of the ZEROIN routine found in Reference 16. The modifications were necessary to allow the routine to search for more than one root. The roots are stored in the array ROOT as they are found, and the routine continues searching until it has found the desired number of roots, or until it cannot find any more. For a discussion of the methods used to find the roots, see Reference 16.

Function SGN

This simple function returns 1.0 if its argument is positive, 0.0 if the argument is zero, and -1.0 if the argument is negative. This is used by ZEROIN.

Function G

This routine uses deflation to allow ZEROIN to find more than one root of the function. Deflation is a technique by which previously found roots are eliminated, but other roots remain. The basic idea is to find the roots of $g(x) = f(x)/\Pi(x - r_i)$, where r_i indicates a previously found root, and Π denotes a product. In thisway, the roots are eliminated. Note that extreme care must be taken when evaluating the function at points close to a previous root.

Function FACT

This simple function returns the factorial of its integer argument. It is used by Function G.

Function F

This is the function whose roots are of interest. The function parameters are contained in the common region PARAM. For the trial function they were used to find XI (ξ) and GAMMA (γ), which are used to find K1 and K2. The function is then evaluated piece by piece, since it is quite complicated.

Function F1

This function is used to find XI and GAMMA.

Function F2

This function is used to find K1 and K2.

Ref. 16. Forsythe, G. E., M. A. Malcolm, and C. B. oler, Computer Methods for Mathematical Computations, Prentice-Hall (1977).

Function DERIV

As mentioned above, care must be taken when evaluating the function $g(x) = f(x)/\pi(x - r_i)$ near $x = r_i$, a previous root. At such a point, L'Hospital's Rule must be applied. This means that the derivative of $f(x)$ at $x = r_i$ must be found. This function calculates the n^{th} derivative of $f(x)$ at some point.

A Closer Look at G and DERIV

As mentioned above,

$$g(x) \equiv f(x)/\pi(x - r_i) \qquad [L1]$$

where $f(x)$ is the function of interest and r_i is a previously found root. This definition allows us to ignore all previous roots without losing any others. A simple example will help illustrate. If

$$f(x) \equiv (x-1)(x-2)(x-3) \qquad [L2]$$

then the roots are $x = 1, 2, 3$. Suppose that in our first search we find that $x = 3$ is a root. Then

$$g(x) \equiv [(x-1)(x-2)(x-3)]/(x-3) = (x-1)(x-2) \qquad [L3]$$

We have eliminated the root $x = 3$, but have retained $x = 1$ and $x = 2$, so that one of them will be discovered. Note that once all the roots have been discovered, $g(x) \equiv 1$, so no more roots will be mistakenly discovered.

This method works equally well for functions other than polynomials, but it is more difficult to demonstrate. The problem with the method is that when ZEROIN tries to evaluate the function at a previous root, it obtains

$$g(x) \equiv f(x)/(x-r_i) = 0/0 \qquad [L4]$$

which the computer cannot evaluate. However, through the application of L'Hospital's Rule, we obtain

$$g(x) \equiv f(x)/\pi(x - r_i) \equiv \left[f'(x)/\left[\frac{d}{dx}[\pi(x-r_i)]\right]\right]_{x=r} \qquad [L5]$$

The method used to compute derivatives is quite simple. It is based on the slope interpretation of the derivative:

$$\left.\frac{df}{dx}\right|_{x=x_0} \doteq \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \qquad [L6]$$

or

$$\doteq \frac{f(x_1) - f(x_1 - h)}{h} \qquad [L7]$$

where $x_1 = x_0 + \Delta x$, $h = 2\Delta x$.

Higher order derivatives are found simply from lower order derivatives

$$\left.\frac{df^{(n)}}{d^{(n)}x}\right|_{x = x_0} \doteq \frac{\left[\left.\frac{df^{(n-1)}}{d^{(n-1)}x}\right|_{x = x_1} - \left.\frac{df^{(n-1)}}{d^{(n-1)}x}\right|_{x = x_1 - h}\right]}{h} \qquad [L8]$$

This procedure is implemented easily using the array DFNDNX. In the first step of the routine, enough points (n + 1) are chosen to evaluate the n^{th} derivative. The points are evenly spaced (h is the interval width for adjacent points), and the point at which the n^{th} derivative is desired is at the midpoint of the interval. As the points are chosen, the function is evaluated at each point; and these values are stored in DFNDNX(1,i), i = 1, n + 1. In the third step, each adjacent pair of values is used to compute a single value for the derivative. This step is repeated until there is only one value. This is the n^{th} derivative.

For example, if f(x) $x^3 + x^2 + x$, and the 4^{th} derivative of f is desired at x = 1.0, then DFNDNX will contain (if h = 0.01)

DFNDNX

	1	2	3	4	5	
	f(.98)	f'(0.985)	f''(0.99)	f'''(0.995)	f^{iv}(1.00)	. . . .
1	2.8816	5.8807	7.9400	6.0000	0.0000	
	f(.99)	f'(0.995)	f''(1.00)	f'''(1.005)	. . . .	
2	2.9404	5.9601	8.0000	6.0000		
	f(1.00)	f'(1.005)	f''(1.01)	. . . .		
3	3.0000	6.0401	8.0600			
	f(1.01)	f'(1.015)	. . . .			
4	3.0604	6.1207				
	f(1.02)	. . . .				
5	3.1216					
	.					
	.					
	.					
	.					

These values are correct, since

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

$$f'''(x) = 6$$

$$f^{iv}(x) = 0$$

Values for less well-behaved functions will have greater error.

Solving for Roots of Other Functions

When seeking the roots of other functions, the user should use the function f to compute the function. Any necessary parameters can be calculated or assigned in PRPCOM, and passed through the common region PARAM. The function f must be a function of one variable. The user must be aware of the meanings of the arguments of ZEROIN.

a	the lower bound of the interval
b	the upper bound of the interval
f	the function of interest
tol	the tolerance (maximum uncertainty) of the independent variable (root)

nroot	the number of roots desired
root	the roots
iroot	the number of roots found

Thus, in order to solve another function, both f and PRPCOM must be rewritten, as must lines 9-14 of TEST. All other code is completely compatible.

```fortran
      real function deriv(f,n,x,h)
      implicit none
      real f,x,h
      integer n
c
      common /machin/ eps
      real eps
c
      external fndeps
c
c n cannot be greater than 10 unless array dimensions are increased
c
      real x1(11),dfndnx(11,11)
      real tol,xtemp,dfn,dnx
      integer i,j,maxj
c
      if(eps.eq.0.0)call fndeps
      tol=2.0*eps*x
      x1(1)=x-(float(n)/2.0)*h
      xtemp=x1(1)
      dfndnx(1,1)=f(xtemp)
c
      do 10 i=2,n+1
        x1(i)=x1(1)+float(i-1)*h
        xtemp=x1(i)
        dfndnx(1,i)=f(xtemp)
   10 continue
c
      do 15 i=1,n+1
   15 continue
c
      do 30 i=2,n+1
        maxj=n+2-i
        do 20 j=1,maxj
          dfndnx(i,j)=dfndnx(i-1,j+1)/h-dfndnx(i-1,j)/h
   20   continue
   30 continue
c
      do 50 i=1,n+1
   50 continue
c
      deriv=dfndnx(n+1,1)
      return
      end

      subroutine fndeps
c
      common /machin/ eps
      real eps
c
      real tol
c
      eps=1.0
   10 eps=eps/2.0
      tol=1.0+eps
      if(tol.gt.1.0)goto 10
c
      return
      end
```

```
************************************************************

      real function f(p)
      implicit none
      real p
c
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real f1,f2
      external f1,f2
c
      real xi,gamma,k1,k2
      real fsign,temp
      real xil,gl,co1,co2,co3,co4,co5,co6,t1,t2,t3,t4
c
      f=0.0
      if (p.eq.0.0)goto 99
      fsign=-1.0
      xi=f1(fsign,p)
      fsign=+1.0
      gamma=f1(fsign,p)
c
      fsign=-1.0
      temp=xi
      k1=f2(fsign,temp,p)
      fsign=+1.0
      temp=gamma
      k2=f2(fsign,temp,p)
c
      xil=xi*l
      if (xil.gt.5000.0)xil=5000.0
      gl=gamma*l
c
      co1=2.0
      co2=2.0*theta*(gamma+xi*k2/k1)
      co3=2.0*theta*(xi+gamma*k1/k2)
      co4=2.0*theta*theta*gamma*xi
      co5=(k2/k1)*(1.0+((theta*xi)**2))
      co6=(k1/k2)*(((theta*gamma)**2)-1.0)
c
      t1=co1*(1-cosh(xil)*cos(gl))
      t2=co2*cosh(xil)*sin(gl)
      t3=co3*sinh(xil)*cos(gl)
      t4=(co4+co5+co6)*sinh(xil)*sin(gl)
c
      f=t1+t2-t3+t4
99    continue
c
      return
      end

      real function f1(s,p)
```

```
      real s,p
c
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real t1,t2,t3,t4,t5
c
      t1=s*(1.0/c12+1.0/c22)*p*p/2.0
      t2=(1.0/c22-1.0/c12)*(p**4)/4.0
      t3=((p/r)**2)/c12
      t4=t2+t3
      t5=t1+sqrt(t4)
c
      f1=sqrt(t5)
c
      return
      end

      real function f2(s,t,p)
      real s,t,p
c
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real t1,t2,t3,t4
c
      t1=c22/(r*r)
      t2=t1-rho*rho
      t3=s*c12*t*t
      t4=t1*t
c
      f2=(t2+t3)/t4
c
      return
      end
```

```
*********************************************************
```

```
      real function g(f,x,root,nroot)
      implicit none
      real f,x
      real root(*)
      integer nroot
      external f
c
      external fndeps
      common /machin/ eps
      real eps
c
      real deriv
      integer fact
      external deriv,fact
c
```

```
      real tol,h
      integer i,n,nfact
      logical atroot
c
      if (eps.eq.0.0) call fndeps
      tol=abs(2.0*eps*x)
      if (tol.lt.eps) tol=eps
c
      atroot=.false.
      n=0
      do 10 i=1,nroot
        if(abs(x-root(i)).gt.tol) goto 10
        n=n+1
        atroot=.true.
   10 continue
c
      nfact=fact(n)
      h=0.001
      write(7,100)nfact
  100 format(' evaluating limit',i5)
      if (atroot) g=deriv(f,n,x,h)/float(nfact)
      if (.not.(atroot)) g=f(x)
c
      do 40 i=1,nroot
        if (abs(x-root(i)).lt.tol) goto 40
      write(7,200)
  200 format(' deflating function')
        g=g/(x-root(i))
   40 continue
c
      return
      end

      integer function fact(n)
      integer n
c
      integer i
c
      fact=1
      do 10 i=2,n
        fact=fact*i
   10 continue
c
      return
      end
```

```
*******************************************************
```

```
      real function prpcom(dummy)
      implicit none
      real dummy
c
```

```
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real e,g,rho,b,i,kprime,rbig
c
      e=4.08e06
      g=1.70e06
      rho=2.247e-04
      r=1.386
      b=1.0
      l=48.0
      i=9.216
      kprime=0.822467
      rbig=1.0e06
c
      theta=e*i/r
      c12=e/rho
      c22=kprime*g/rho
c
      prpcom=sqrt(c22/(r*r))
c
c
      return
      end
```

```
***************************************************************
```

```
      subroutine zeroin(ax,bx,f,tol,nroot,root,iroot)
      implicit none
      real ax,bx,f,tol,root(*)
      integer nroot,iroot
c
      external fndeps
      common /machin/ eps
      real eps
c
      real g,sgn
      external g,sgn
c
      real a,b,c,d,e
      real tol1,xm
      real fa,fb,fc
      real p,q,r,s
c
c compute eps, the relative machine precision
c
      if (eps.eq.0.0)call fndeps
c
c initialization
c
      iroot=0
      a=ax
```

```
      b=bx
c
15    fa=g(f,a,root,iroot)
      fb=g(f,b,root,iroot)
c
c begin step
c
20    c=a
      fc=fa
      d=b-a
      e=d
30    if (abs(fc).ge.abs(fb)) goto 40
      a=b
      b=c
      c=a
      fa=fb
      fb=fc
      fc=fa
c
c convergence test
c
40    tol1=2.0*eps*abs(b)+0.5*tol
      xm=.5*(c-b)
      if (abs(xm).le.tol1) goto 90
      if (fb.eq.0.0) goto 90
c
c is bisection necessary?
c
      if (abs(e).lt.tol1) goto 70
      if (abs(fa).le.abs(fb)) goto 70
c
c is quadratic interpolation possible?
c
      if (a.ne.c) goto 50
c
c linear interpolation
c
      s=fb/fa
      p=2.0*xm*s
      q=1.0-s
      goto 60
c
c inverse quadratic interpolation
c
50    q=fa/fc
      r=fb/fc
      s=fb/fa
      p=s*(2.0*xm*q*(q-r)-(b-a)*(r-1.0))
      q=(q-1.0)*(r-1.0)*(s-1.0)
c
c adjust signs
c
60    if (p.gt.0.0)q=-q
      p=abs(p)
c
c is interpolation acceptable?
c
      if ((2.0*p).ge.(3.0*xm*q-abs(tol1*q))) goto 70
      if (p.ge.abs(0.5*e*q)) goto 70
      e=d
```

```
      d=p/q
      goto 80
c
c bisection
c
70    d=xm
      e=d
c
c complete step
c
80    a=b
      fa=fb
      if (abs(d).gt.tol1)b=b+d
      if (abs(d).le.tol1)b=b+sign(tol1,xm)
      fb=g(f,b,root,iroot)
      if ((fb*(fc/abs(fc))).gt.0.0) goto 20
      goto 30
c
c done
c
90    if (sgn(fa).ne.sgn(fb))goto 95
      if (sgn(fc).ne.sgn(fb))goto 95
      if (fb.eq.0.0)goto 95
      return
c
95    iroot=iroot+1
      root(iroot)=b
      if (iroot.eq.nroot)return
c
      a=ax
      b=bx
      goto 15
      end

      real function sgn(a)
      implicit none
      real a
      if (a.eq.0.0) goto 10
        sgn=a/abs(a)
        return
10    sgn=0.0
      return
      end
```

```
 ***********************************************************************
```

```
      program test
      implicit none
      real prpcom
      external f,zeroin,prpcom
      real a,wprime,tol,root(100),dummy
      integer i,nroot,iroot
```

```
c
      open(unit=7,file='output')
      a=0.0
      wprime=prpcom(dummy)
      tol=1.0e-6
c
      nroot=20
      call zeroin(a,wprime,f,tol,nroot,root,iroot)
      write(7,90)iroot
      do 10 i=1,iroot
        write(7,100)root(i)
   10 continue
c
   90 format(' zeroin found ', i5, ' roots')
  100 format(e12.6)
      stop
      end
```

```
 ***************************************************************
```

```
 zeroin found     8 roots
0.000000e+00
0.000000e+00
0.142336e+05
0.369554e+04
0.147244e+04
0.750657e+03
0.202099e+03
0.288620e+02
```

END

12-86

DTIC