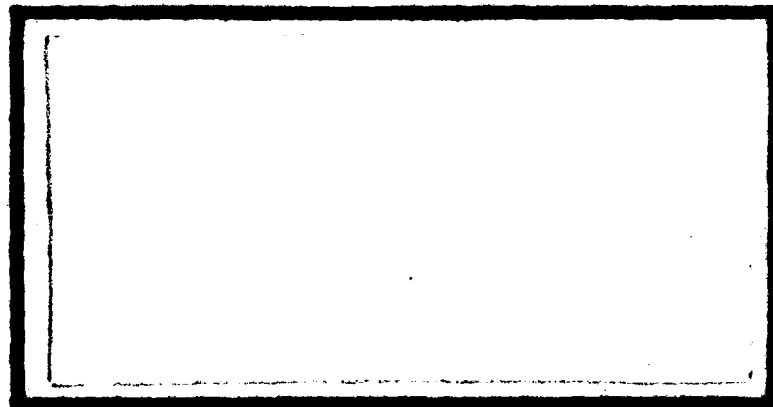


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THE ANALYSIS AND DESIGN OF  
CONSTRAINED LAYER DAMPING TREATMENTS

⑩ by  
Peter J. Torvik  
Professor of Mechanics  
⑭ AFIT-TR-80-4  
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## 1. INTRODUCTION

The extensive activity in the design, development, test, and application of constrained layer damping treatments was initiated by the 1959 paper (1) by E. M. Kerwin entitled "Damping of Flexural Waves by a Constrained Viscoelastic Layer". Dr. J. J. Baruch is identified in that paper as having suggested the shear damping mechanism. In the same year, a paper was presented at the Colloquium on Structural Damping sponsored by the Shock and Vibration Committee of the ASME. In that paper (2), Ross, Ungar and Kerwin gave a more complete analysis of the mechanism producing damping and identified several critical design parameters. These two papers may be regarded as the foundation of the body of literature which has developed since that time. Two other related works were published earlier but did not have the same degree of impact. H. J. Plass, Jr., considered in 1957 the damping of a sandwich plate with viscoelastic core. In his work (3), the face plates were treated as membranes. Thus the concept is fundamentally different than that of a beam with damping layer. In 1958, James Whittier investigated the use of a spaced and constrained viscoelastic layer as a means of damping a cantilever beam (4).

Since 1959, a number of reviews have been written of damping in general and of constrained layer damping in particular. Among these are the review (5) by Kerwin of early work (1964), a comprehensive review by Nakra (6), and one by Nelson (7) which emphasizes the applications to large structures.

In this paper, some of the fundamental principles of constrained layer damping methodology will be addressed. Simple analyses for significant classes of problems will be outlined, and a discussion of the principal contributions of the past two decades will be attempted. The use of constrained layer treatments in isolation systems will not be addressed, nor will the shock response of structures employing constrained layer treatments on sandwich beams with

viscoelastic cores, even though such analyses have been performed (8). Discussions of specific material properties, experimental methods, and applications will also not be addressed.

The analyses which have been reported in the literature have been of two types: the development and solution of the equations of motion for the composite system (4) or the deduction of an effective complex stiffness (1). Both approaches will be employed here.



## II. QUASI-STATIC ANALYSES

Even though it is well known that a significant frequency effect is present in constrained layer damping treatments, we will find it beneficial to begin by considering the quasi-static analysis of several simple but important configurations. In each of these first four examples, we assume that weightless and frictionless constraints are supplied so that no bending deformations occur. Following this, the coupled bending-axial deformation case will be explored.

### A. Axial Deformations

1. Uniform Surface Treatments. Materials having an inherently high damping capacity must be used effectively if optimal increases in damping are to be obtained. To demonstrate the truth of this assertion, let us consider several idealized examples. First, let us suppose that a layer of uniform thickness,  $t$ , of a material having a complex Young's modulus,  $E_1 + iE_2$ , is added over a width  $b$  of a tension member of thickness  $h$ , modulus  $E$ , and length  $L$ , which is subjected to a fluctuating load of amplitude  $P$ . For simplicity, we assume further that the frequency of loading (or the density) is such that inertial effects may be neglected.

If a tip deflection of amplitude  $\delta$  results from application of load  $P$  to the configuration shown in Figure 1a, we may assume the axial strain in the member and the layers is identical, i.e.

$$\epsilon = \delta/L \quad (1)$$

The amplitudes of force and displacement are related through

$$P = \{(Eh + E_1 t)^2 + (E_2 t)^2\}^{1/2} b \epsilon \quad (2)$$

The maximum strain energy stored during any one cycle is

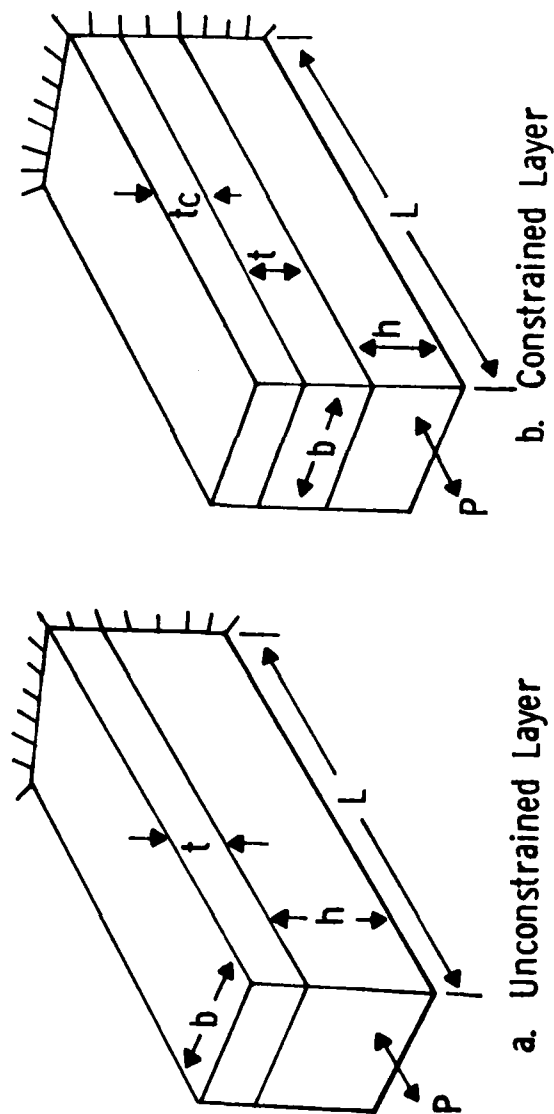


Figure 1 Damping Treatments for Bars

$$U_s = \frac{1}{2} (Eh + E_1 t) b \epsilon^2 L \quad (3)$$

and the energy dissipated is

$$D_s = \int_V \oint \sigma \dot{\epsilon} dt dv \quad (4)$$

$$= \pi t b L E_2 \epsilon^2 \quad (5)$$

Thus the loss factor for the entire structure is

$$\eta_s = \frac{D_s}{2\pi U_s} = \frac{t E_2}{Eh + E_1 t} \quad (6)$$

The form, in the nature of a rule of mixtures, arises only because the coupled bending deformation was not permitted to occur. As the storage modulus ( $E_1$ ) of most high damping materials is much less than of the structures to which they are to be applied,

$$\eta_s \approx \frac{E_2}{E} \frac{t}{h} \quad (7)$$

and high loss factors can be achieved only by adding large amounts of high loss materials.

2. Rigid Constraining Layer. It has been well known for more than twenty years that a more effective means of achieving high loss factors is to employ a mechanism for developing a high shear strain in the added material. Thus, we next consider the configuration depicted on Figure 1b. A cover sheet of thickness  $t_c$  and modulus  $E_c$  is placed over the added layer and restrained at one end. The fluctuating load,  $P$ , then induces a state of shear in the damping layer.

As a first approximation, we will regard the constraining layer to experience no axial deformation. The requirement that no bending moment may develop necessitates the rigid constraining layer be of infinite modulus and vanishing thickness. If we also neglect force transfer to the cover sheet, the shear strain distribution in the shear layer varies from zero at the fixed end to a maximum at the free end according to

$$\sigma = \frac{x}{t} \cdot \frac{\delta}{L} \quad (8)$$

Here  $\delta$  is again the tip displacement due to the axial load. If the shear layer has a complex modulus  $G^* = G_1 + iG_2$ , the energy dissipated per cycle in the layer is readily computed from Equation (4) to be

$$D_s = \pi G_2 \delta^2 L b / (3t) \quad (9)$$

The maximum strain energy stored during the cycle is

$$U_s = \left\{ \frac{Eh}{2} + \frac{G_1 L^2}{6t} \right\} \frac{b\delta^2}{L} \quad (10)$$

If  $G_1 \ll E$ , the energy stored in the damping material is negligible compared to that stored in the structure, and the approximate loss factor is found to be

$$\eta_s = \frac{2\pi}{3} \frac{G_2}{E} \frac{L^2}{th} \quad (11)$$

We observe that more effective use (higher loss factor) of the damping material is achieved through the use of less rather than more of the damping material. Comparison of Equation (11) with Equation (7) shows the effectiveness of the constrained layer mechanism. As  $t$  and  $h$  may be much smaller than  $L$ , the constrained shear layer has the potential for producing much higher levels of

damping than does the extensional surface treatment. The collection of system parameters appearing in Equation (11) will later be seen to play a dominant role in the effectiveness of more general treatments.

While this model and analysis serves well to demonstrate the effectiveness of the shear mechanism, Equation (11) is not adequate for design purposes. When the shear strains in the damping layer are made large, the influence of the shear layer on the deformations of the cover sheet and the member cannot be ignored. Their effect is to reduce the energy dissipated. Further, the shear strain in the layer cannot be made arbitrarily large (through making the thickness arbitrarily small) or the concomitant large shear stress will eventually produce failure of the cover sheet if not of the shear layer itself.

3. Complete Analysis: Axial Deformations. For these reasons, we require a complete deformation analysis of the three elements of this structural system: the member, the shear layer, and the cover sheet. From the free body diagrams of Figure 2, we may write the following

$$\frac{\partial T_c}{\partial x} + \tau b = 0 \quad (12a)$$

$$\frac{\partial T_m}{\partial x} - \tau b = 0 \quad (12b)$$

$$T_c = b E_c t_c \frac{\partial U_c}{\partial x} \quad (13a)$$

$$T_m = b E_m t_m \frac{\partial U_m}{\partial x} \quad (13b)$$

where  $T_c$ ,  $U_c$ ,  $T_m$ ,  $U_m$  are the tensile force and displacement in cover sheet and member, respectively. In the shear layer, of complex modulus  $G^*$ ,

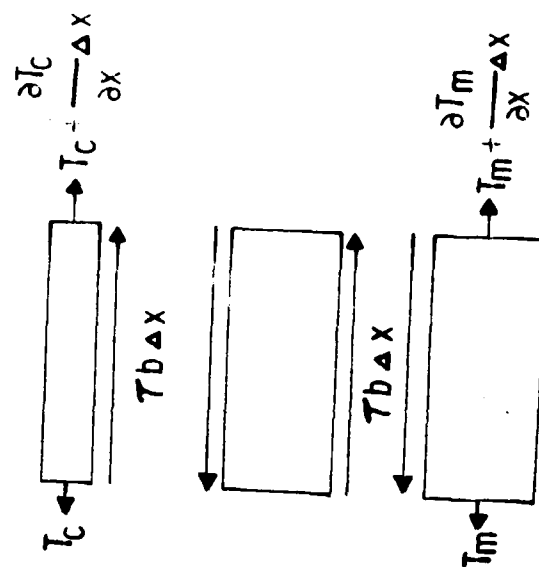


Figure 2 Free Body Diagram

$$\tau = (G_1 + iG_2) \frac{\Delta}{t} = G^* \psi \quad (14)$$

where

$$\Delta = U_m - U_c \quad (15)$$

The necessary boundary conditions are

$$U_m(0) = U_c(0) = T_c(L) = 0 \text{ and } T_m(L) = P \quad (16)$$

The temporal dependence  $\exp(i\omega t)$  has been suppressed in all variables, leaving only the  $x$  dependence. Elimination of  $T_c$ ,  $T_m$ , and  $\tau$  leads to

$$\frac{\partial^2 \Delta}{\partial x^2} - \frac{\beta^2 \Delta}{L^2} = 0 \quad (17)$$

$$\beta^2 = \frac{G^*}{t} \left\{ \frac{1}{hE} + \frac{1}{t_c E_c} \right\} L^2 \quad (18)$$

$$\Delta = -iA \sin(i\beta x/L) = A \sinh(\beta x/L) \quad (19)$$

$$A = \frac{PL}{Ebh\beta} \frac{1}{\cosh(\beta)} \quad (20)$$

$$T_c = \frac{PG^* L^2}{thE\beta^2} \left( 1 - \frac{\cosh(\beta x/L)}{\cosh\beta} \right) \quad (21)$$

$$T_m = P - T_c \quad (22)$$

Finally, the deflection at the tip is

$$\delta = U_m(L) = \frac{PL}{(t_c E_c + hE)b} + \frac{PG^* L^3 \tanh(\beta)}{\beta^3 b t (hE)^2} \quad (23)$$

Equation (21) may be used to determine the tensile force in the cover sheet, which must be kept below that value which produces failure. Equations (19) and (20) [together with (14)] may be used as a design constraint to keep the shear stress in the layers within allowable limits. The energy dissipated per cycle may be computed from

$$D_s = \int_v \pi G_2 |\gamma|^2 dv = \frac{\pi G_2 b}{t} \int_0^L |\Delta|^2 dx \quad (24)$$

$$D_s = \frac{\pi G_2 b L}{t} \left( \frac{PL}{bhE} \right)^2 F(\beta) \quad (25)$$

where

$$F(\beta) = -\frac{1}{2} \frac{I_m \{ \tan(i\beta) / (i\beta) \}}{R_e \{ \beta \} I_m \{ \beta \}} \quad (26)$$

The maximum energy stored during the cycle may be deduced from Equation (23).

Let

$$\delta = (C_1 + iC_2)P \quad (27)$$

Then the peak stored energy is

$$U_s = \frac{C_1 P^2}{2} \quad (28)$$

where

$$C_1 = \frac{L}{(t_c E_c + hE)b} + \frac{L^3}{bt(hE)^2} R_e \left\{ \frac{G^* \tan i\beta}{i\beta^3} \right\} \quad (29)$$



From the energy dissipated and energy stored, a structural loss factor may be computed from Equation (6) as

$$\eta_s = \frac{G_2}{E} \frac{L^2}{th} \left[ \frac{1}{1 + \frac{t_c E_c}{hE}} + \frac{L^2}{th} \operatorname{Re} \left\{ \frac{G^* \tan i\beta}{E} \right\} \right] \quad (30)$$

For design purposes, it is useful to note that only four parameters need be considered:

- (a) a characterization of the damping material

$$x_1 = G_1/G_2 \quad (31)$$

- (b) a comparison of the damping material and the structural material

$$x_2 = G_2/E \quad (32)$$

- (c) a comparison of the cover sheet stiffness to that of the structural member

$$x_3 = \frac{t_c E_c}{hE} \quad (33)$$

- (d) a geometric factor, incorporating the length and thickness of the damping layer

$$x_4 = \frac{L^2}{ht} \quad (34)$$

Of these,  $x_2$  and  $x_4$  appear only as a product, in the manner of Equation (11).

A satisfactory design must also take into account the maximum stress on the cover sheet and the maximum strain in the layer. The maximum stress occurs at

the fixed end. The tensile force is given by Equation (21) and may be expressed in terms of these same three parameters. The maximum shear strain occurs at the free tip and is

$$\gamma_{\max} = \frac{\Delta_{\max}}{t} = \frac{PL \tanh \beta}{tEb\beta} \quad (35)$$

Thus

$$\tau_{\max} = \frac{P}{Lb} \cdot \frac{L^2}{th} \left[ \frac{G^*}{E} \cdot \frac{\tanh i\beta}{i\beta} \right] \quad (36)$$

Here we note an explicit dependence on length not present in the previous expressions.

4. Constrained Layer on Thick Substrate. For a layer attached to a very thick substrate (Figure 3), a modified analysis is appropriate. In this case, there is negligible load transfer to the lower member, and we may assume a uniform strain  $\epsilon_0$  over a length  $L$  of the substrate. It can be easily shown that the energy dissipated in the configuration is

$$D_s = \frac{\pi G_2 b L^3}{t} F(\beta_R) \epsilon_0^2 \quad (37)$$

where

$$\beta_R^2 = \frac{G^* L^2}{t t_c E_c} \quad (38)$$

The strain energy in cover sheet and shear layer are now negligible compared to that of the substrate. Hence

$$\eta_s = \frac{D_s}{2\pi U_s} = \frac{G_2}{E} \frac{L^2}{th} F(\beta_R) \quad (39)$$

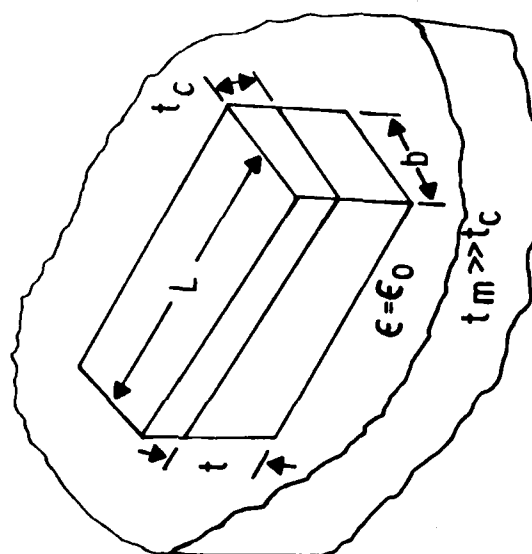


Figure 3 Layer on Substrate

where the strain energy is that of the length  $L$  of substrate only. Again, the factor  $G_2 L^2 / (E t h)$  is of major significance.

It should be noted that each of these analyses may be applied to a damping layer free on both ends by observing that either half of such a layer corresponds to the cases considered above.

## B. Flexural Deformations

1. Uniform Extensional Coatings (Oberst Beam). It is appropriate to begin our consideration of flexural deformations by considering the classic problem of the free layer treatment. The Oberst beam is the prototype of damping treatments and motivated the development of the constrained layer. Further, there has been significant attention devoted, as we shall see, to developing the concept of an equivalent homogeneous layer for multiple layer constrained treatments. Such an equivalent homogeneous layer is then used in the equations of the Oberst beam.

Jones has given (9) a particularly simple derivation for the vibrating Oberst beam. This approach may be easily adapted to provide a derivation appropriate to the quasi-static approach we are following in these sections. Let  $h$  and  $t$  be, respectively, the thickness of a beam and a viscoelastic covering on one side, as was depicted in Figure 1a. In bending, the neutral axis will be located a distance  $mh$  into the beam, measured from the interface. The net axial force acting on the section vanishes if

$$R_e \left\{ \int_{-(1-m)h}^{mh} \frac{E}{1-\nu^2} \frac{z}{R} dz + \int_{mh}^{(m+n)h} \frac{E^*}{1-\nu^2} \frac{z}{R} dz \right\} = 0 \quad (40)$$

Here  $n = t/h$ ,  $R$  is the radius of curvature, and  $z$  is measured from the neutral axis. This statement is appropriate for a beam, or for a unit width of a plate if Poisson's ratio for plate and viscoelastic layer are the same. Further, we let the ratio of moduli

$$e = R_e \left\{ \frac{E^*}{E} \right\} \quad (41)$$

Then,

$$m = \frac{1 - en^2}{2(1 + ne)} \quad (42)$$

The flexural rigidity of each component may be computed from:

For the elastic component

$$D = \frac{E}{1-\nu^2} \int_{-(1-m)h}^{mh} z^2 dz = \frac{Eh^3 A}{12(1-\nu^2)} \quad (43a)$$

For the damping layer,  $E^* = (1 + i\eta_D)R_e\{E^*\}$ , and

$$D_v = \frac{E^*}{1-\nu^2} \int_{mh}^{(m+n)h} z^2 dz = \frac{E^* h^3}{12(1-\nu^2)} B \quad (43b)$$

where

$$A = \{1 + 2ne + 4n^2 e^2 + 6n^3 e^2 + 3n^4 e^2\} / (1 + ne)^2 \quad (44a)$$

$$B = n\{3 + 6n + 4n^2 + 2n^3 e + n^4 e^2\} / (1 + ne)^2 \quad (44b)$$

Since both components of the two layer beam are undergoing the same curvature  $R$ , over some length  $L$ , the energy dissipated

$$D_s = \int_0^L \pi \operatorname{Im}\{D_v\} \frac{1}{R^2} dL \quad (45)$$

and energy stored

$$U_s = \frac{1}{2} \int_0^L D \frac{1}{R^2} dL + \frac{1}{2} \int_0^L R_e\{D_v\} \frac{1}{R^2} dL \quad (46)$$

may be computed and a loss factor determined. Hence,

$$\eta_s = \frac{D_s}{2\pi U_s} = \frac{\eta_D eB}{A + eB} \quad (47)$$

Substituting Equations (44a) and (44b), we find

$$\eta_s = \frac{n\eta_D}{(1 + en)} \left[ \frac{3 + 6n + 4n^2 + 2n^3e + n^4e^2}{1 + 2en(2 + 3n + 2n^2) + e^2n^4} \right] \quad (48)$$

This is the same version of the Oberst equation as given by Ross, Ungar, and Kerwin (2). There is in the literature some controversy over the interpretation of the term  $e$ , the ratio of stiffness. Ungar has given (10) Equation (48) in a form where the value of  $e$  which is to be used corresponds to  $|E^*/E|$ . Equation (40), as used here, leads to  $e = R_e \{E^*/E\}$ . The difference, of course, vanishes if  $\eta_D \ll 1$  as was assumed in the computed results given by Ross, Ungar, and Kerwin (2). Equation (40) as stated here is not entirely satisfying either. The question arises, can the imaginary part be nonzero? If  $|E^*| \ll E$ ,

$$\eta_s = \frac{E_2}{E} \left( 3 \frac{t}{h} + 6 \frac{t^2}{h^2} + 4 \frac{t^3}{h^3} \right) \quad (49)$$

which makes it once again apparent that a free layer treatment of a material with small loss modulus will not produce a large system loss factor.

2. The Constrained Shear Layer (Transverse Deformation). The next case of importance in the development of concepts is that of a beam of width  $b$ , or a strip, isolated from a plate undergoing bending about one axis only. The geometry of beam, shear layer, and cover sheet is shown in Figure 4. The thicknesses of beam, shear layer, and cover sheet are  $h$ ,  $t$ , and  $t_c$ , respectively, and each is assumed to remain constant throughout the deformation. The thicknesses are assumed to be small enough that the radius of curvature,  $R$ , of the cover sheet is essentially the same as that of the beam, and we will assume Poisson's ratio to be the same for both.

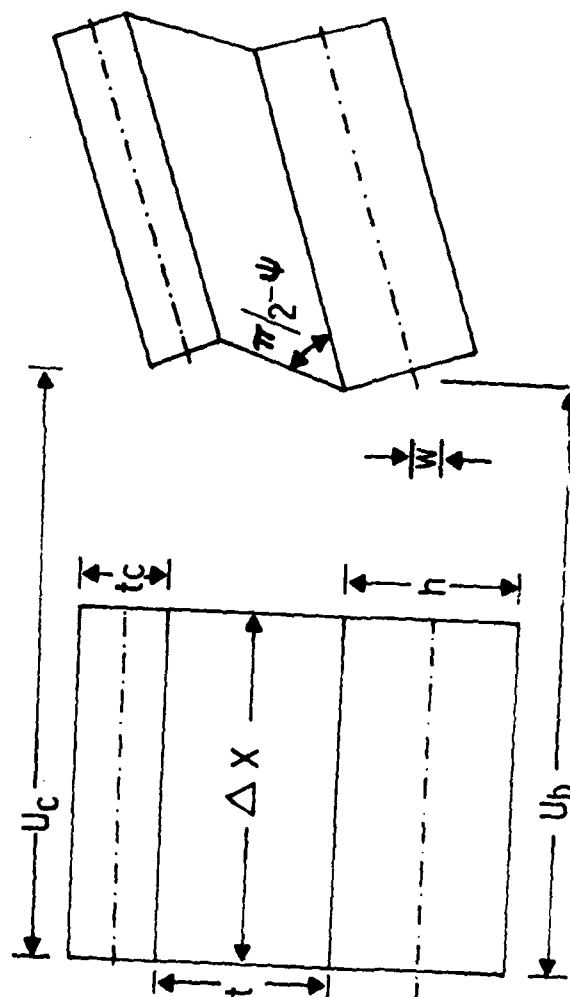


Figure 4 Constrained Layer in Bending



The assumed deformation field is as follows:

- (1) A transverse deflection,  $w$ , related to the curvature by

$$\frac{1}{R} = \frac{d^2 w}{dx^2} \quad (50)$$

which is a function of one coordinate only and the same throughout all layers,

- (2) an axial elongation of the beam centerline,  $u_b$ ,

- (3) an axial deformation of the coversheet centerline,  $u_c$ , and

- (4) a shear angle in the shear layer, assumed to be uniform over its thickness, varying only with axial coordinate,  $\psi(x)$ .

These four deformation measures are not, of course, independent. Rather,

$$\psi = \frac{dw}{dx} \left( 1 + \frac{t_c + h}{2t} \right) + \frac{u_c - u_b}{t} \quad (51)$$

Equilibrium of horizontal forces on the beam and cover sheet leads to the requirement that the axial tension on each varies according to:

$$\frac{\partial T_c}{\partial x} - b\tau = 0 \quad (52a)$$

$$\frac{\partial T_b}{\partial x} + b\tau = 0 \quad (52b)$$

Here  $\tau$  is the shear stress in the shear layer. As may be deduced from the free-body diagrams on Figure 5, equilibrium of vertical forces leads to

$$bq_c + bP_c + \frac{\partial V_c}{\partial x} = 0 \quad (53a)$$

$$bq_b - bP_b + \frac{\partial V_b}{\partial x} = 0 \quad (53b)$$

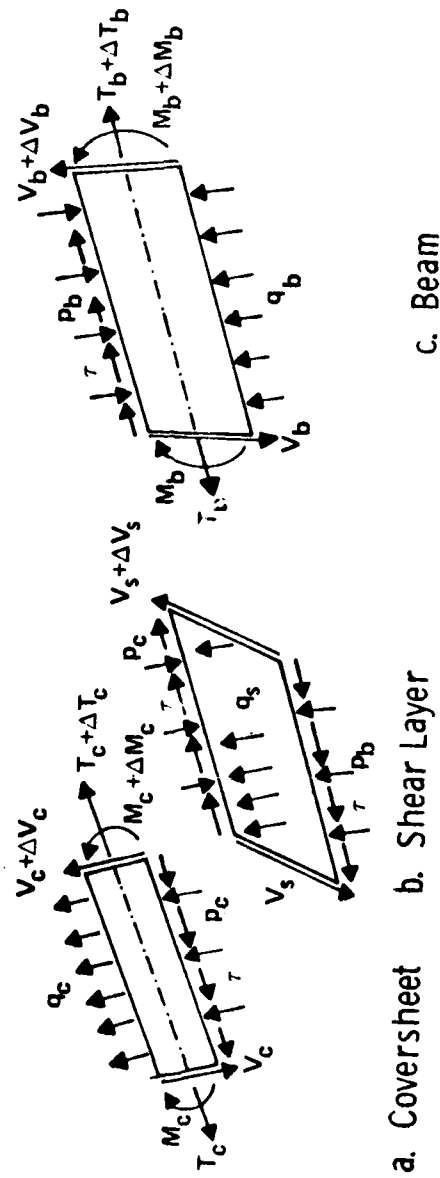


Figure 5 Free Body Diagrams

$$bq_s - bq_c + bP_b + \frac{\partial V_s}{\partial x} = 0 \quad (53c)$$

The pressures,  $P_c$  and  $P_b$ , act on cover sheet and beam respectively, and  $q_c$ ,  $q_b$ , and  $q_s$  are external loads applied to cover and beam and shear layer, respectively. These forces will be used to treat the inertial forces in a later section. Summation of moments on the three elements leads to

$$\frac{\partial M_c}{\partial x} + V_c - \frac{\tau b t_c}{2} = 0 \quad (54)$$

$$\frac{\partial M_b}{\partial x} + V_b - \frac{\tau b h}{2} = 0 \quad (55)$$

$$\tau b t = V_s \quad (56)$$

The measures of force and the measures of displacement are not independent, but are related through constitutive and geometric properties.

$$T_c = E_c b t_c \frac{\partial u_c}{\partial x} \quad (57a)$$

$$T_b = E_b b h \frac{\partial u_b}{\partial x} \quad (57b)$$

$$M_c = \frac{E_c t_c^3}{12} \frac{b}{(1 - \nu^2)} \frac{d^2 w}{dx^2} \quad (58a)$$

$$M_b = \frac{E_b h^3}{12} \frac{b}{(1 - \nu^2)} \frac{d^2 w}{dx^2} \quad (58b)$$

$$\tau = G*\psi = G* \left\{ \frac{(u_c - u_b)}{t} + \left( \frac{t_c + h}{2t} + 1 \right) \frac{dw}{dx} \right\} \quad (59)$$

Setting Poisson's ratio ( $\nu$ ) to zero gives the result for a beam. The complex shear modulus,  $G^*$ , is used, as simple harmonic motion is assumed. Substituting (59) into (54) and (55), and (58a) and (58b) into the results, leaves

$$\frac{E_c t_c^3}{12(1-\nu^2)} b \frac{d^3 w}{dx^3} + V_c - \frac{bt_c}{2} G^* \psi = 0 \quad (60)$$

$$\frac{E_b h^3}{12(1-\nu^2)} b \frac{d^3 w}{dx^3} + V_b - \frac{bh}{2} G^* \psi = 0 \quad (61)$$

Substitution of (57a), (57b) and (59) into (52a) and (52b) yields

$$E_c t_c \frac{\partial^2 u_c}{\partial x^2} - G^* \psi = 0 \quad (62)$$

$$E_b h \frac{\partial^2 u_b}{\partial x^2} + G^* \psi = 0 \quad (63)$$

Finally, the interlaminar pressures may be eliminated by combining Equations (53a), (53b), and (53c) and setting

$$q = q_c + q_b + q_s \quad (64)$$

The result is

$$\frac{\partial V_c}{\partial x} + bq + \frac{\partial V_b}{\partial x} + G^* tb \frac{\partial \psi}{\partial x} = 0 \quad (65)$$

These five equations [(60) - (63) and (65)], together with the definition of  $\psi$  [Equation (51)], are to be solved for the six unknowns  $w$ ,  $V_c$ ,  $\psi$ ,  $V_b$ ,  $u_c$ , and  $u_b$ .

Substitution of Equations (60) and (61) into (65) gives

$$q + G^* d \cdot \frac{d\psi}{dx} - D_t \frac{d^3 w}{dx^3} = 0 \quad (66)$$

where

$$d = t + \frac{t_c + h}{2} \quad (67)$$

$$D_t = \frac{E_c t_c^3 + E_b h^3}{12(1-\nu^2)} \quad (68)$$

and substitution of Equations (62) and (63) into (51) gives

$$\frac{d^2 \psi}{dx^2} - g \psi = \frac{d}{t} \frac{d^3 w}{dx^3} \quad (69)$$

where

$$g = G^*/t \left\{ \frac{1}{E_c t_c} + \frac{1}{E_b h} \right\} \quad (70)$$

The coupled equations of motion, Equations (66) and (69) define the motion of the system. The deformations of the shear layer may be identified by differentiating and substituting one into another to obtain:

$$q \frac{d}{t} + \left\{ \frac{G^* d^2}{t} + D_t \right\} \frac{d\psi}{dx} - D_t \frac{d^3 \psi}{dx^3} = 0 \quad (71)$$

An analogous process produces a single equation for the bending deformation, as will be considered in a later section.

For the beam without external loading, neglecting the inertial forces leads to the simple equation

$$\frac{d^3 \psi}{dx^3} - c^2 \frac{d\psi}{dx} = 0 \quad (72)$$

where

$$c^2 = \left\{ \frac{d^2}{t D_t} G^* + g \right\} \quad (73)$$

Thus  $\psi = A \sinh cx + B \cosh cx + D$ . Other dependent variables may be determined from the successive application of Equations (52a), (52b), (57a), (57b), and (51).

For the particular case where both beam and cover sheet are clamped at  $x = 0$  and free of external shear force at  $x = L$ , the results are

$$\psi = A \sinh cx \quad (74)$$

$$T_c = -T_b = bG^* \frac{A}{c} (\cosh cx - \cosh cL) \quad (75)$$

$$u_c = \frac{G^*}{E_c t_c} \frac{A}{c^2} (\sinh cx - cx \cosh cL) \quad (76)$$

$$w = \frac{At}{cd} \left[ (\cosh cx - 1) - g \left\{ \frac{\cosh cx - 1}{c^2} - \frac{x^2}{2} \cosh cL \right\} \right] \quad (77)$$

Thus, for a tip moment  $M(L) = M_o$ , properly distributed over the beam and cover sheet, we may evaluate the second derivatives of (77) and use (58a) and (58b) to find

$$A = \frac{M_o d}{b_t D_t} \cdot \frac{1}{c \cosh cL} \quad (78)$$

As the shear strain distribution [Equation (74)] is now known, the energy dissipation may be evaluated as in the previous section. Since we have computed the deformation of the composite under a tip moment, we may evaluate the rotation at the point of loading and deduce a compliance,  $C^*$ ,

$$\left. \frac{dw}{dx} \right|_{x=L} = (C_1 + iC_2) M_o \quad (79)$$

From this compliance, we may deduce the peak value of stored energy to be

$$u_s = \frac{C_1}{2} M_o^2 \quad (80)$$

where

$$C_1 = \frac{M_o}{bD_t} K_e \left\{ \frac{\tanh cL}{c} - \frac{g}{c^3} (\tanh cL - cL) \right\} \quad (81)$$

The energy dissipated is

$$D_s = \frac{\pi G_2}{bt} L \left\{ \frac{M_o L d^2}{D_t} \right\}^2 F(\beta) \quad (82)$$

where  $\beta = cL$ , and  $F(\beta)$  was defined by Equation (26).. The loss factor is then found from

$$\eta_s = \frac{D_s}{2\pi U_s} \quad (83)$$

If the beam rigidity in both bending and tension is much larger than that of the cover sheet, and if the shear layer is thin compared to the beam thickness,

$$A = \frac{6(1-\nu^2)}{E_b t h^2 b c} \frac{M_o}{\cosh cL} \quad (84)$$

$$c^2 = \frac{G^*}{E_c t_c t} \quad (85)$$

$$w = \frac{6M_o (1-\nu^2) x^2}{E_b b h^3} \quad (86)$$

$$M = M_0$$

$$\psi = \frac{6M_0(1-\nu^2) \sinh cx}{E_b t h^2 b c \cosh cL} \quad (87)$$

The energy dissipated and peak stored energy follow, and the loss factor is found to be

$$\eta_s = 3(1-\nu^2) \left( \frac{G_2 L^2}{E_b t h} \right) F(cL) \quad (88)$$

However, the quantity  $(L^2/th)$  cannot be made arbitrarily large or the tensile force in the cover sheets [Equation (57a)] will become large enough to produce excessive deformation. Further, several investigators have observed the existence of an optimum length for maximum damping (11).

Plunkett and Lee (12) considered the single constrained layer treatment under the simplified loading criteria described in Section A.4. By assuming a uniform surface strain on a substrate, they produced results applicable to tension members or beams, subject to reasonable assumptions concerning the relative dimensions of various components. They found that a constraining layer, free on both ends, would have the maximum loss coefficient if its length,  $L$ , satisfied a relationship

$$L_1 = 3.28 \sqrt{\frac{t_t E_c}{G_1}} \quad (89)$$

We note once again the occurrence of the combination of terms we have identified as dominating constrained layer analysis. For stiff bands, or for very soft shear layers, this maximum length may not be obtainable. Conversely, segmenting may improve the performance of some configurations when the total length of the



application exceeds  $L_1$ . It should be noted that this length,  $L_1$ , is for a layer free on both ends. This should be compared with twice the length of the layers in the preceding analyses, as they were fixed on one end and free on the other.

What Nokes and Nelson described (11) as a boundary layer effect is made evident by the preceding analysis. The distribution of shear strain in the layer

$$\psi = \frac{M_o dL}{D_t b t c L} \frac{\sinh cx}{\cosh cL} \quad (90)$$

is strongly dependent on the value of  $cL$ . For large  $cL$ ,  $\psi$  is very nearly zero, except as  $x \rightarrow L$ . Thus, only that portion of the shear layer near to the end contributes significantly to the energy dissipation. For small  $cL$ , the maximum shear remains at the end, but a greater relative contribution is made by the remainder of the layer. It is also important to note that we have found this effect through the quasi-static analysis. Consequently, we can hardly view it as a frequency effect. Rather, it is fundamental to the geometry of the constrained layer configuration.

### C. Equivalent Homogeneous Material

We have seen in earlier sections that the loss factor is not generally increased through the use of more of the damping material in a single constrained layer, either by increasing thickness or length. Thus, attempts to produce greater added damping have turned to the use of more than one constrained layer. Because of the complexity of such analyses when a complete solution is attempted in the manner of the preceding sections, simpler approaches have been sought. This led to the concept of an equivalent homogeneous material.

The concept may be effectively demonstrated by another quasi-static analysis, this time of a large number of layers of damping material which are separated, as shown in Figure 6a, by a large number of segmented constraining layers. As a first approximation, we will treat these constraining layers as being rigid and of negligible thickness. Let each constraining segment be of length  $L$ , separated from its neighbor by a gap  $s$ . Let a thickness  $t$  of material with complex modulus  $G^*$  occupy the space between the constraining layers, which are presumed to be staggered, as shown. Isolating the segment ABCD by bisecting the constraining bands, and assuming a uniform shear stress  $\tau = G^*\gamma$  in each of the four parallelograms, we deduce from the free body diagrams of Figure 6b that

$$T/2 = \frac{L}{2} \tau b \quad (91)$$

for a width  $b$ . The segment AB, of length  $L + s$ , undergoes an elongation

$$\Delta = 2\gamma t \quad (92)$$

where the shear strain  $\gamma$  is assumed uniform. The total axial force on the cell is

$$T = bLG^*\gamma \quad (93)$$

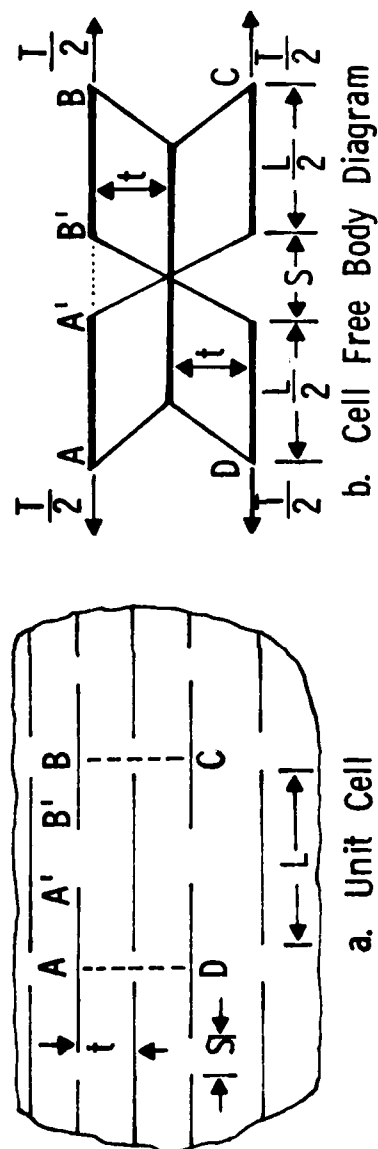


Figure 6 Unit Cell in Segmented Multiple Layer Treatment

Forming the average (homogeneous) strain by dividing the cell elongation by its length and deducing an average (homogeneous) stress by dividing the force,  $T$ , by the cell area  $2bt$  enables the computation of an effective axial modulus for the cell or

$$E_{\text{eff}}^* = \frac{\sigma_{\text{ave}}}{\epsilon_{\text{ave}}} = \frac{bLG^*\gamma}{2bt} \cdot \frac{L+s}{2\gamma t} = \frac{G^*}{4} \frac{(L+s)L}{t^2} \quad (94)$$

If the constraining layer thickness,  $t_c$ , is taken into account, and the separation distance,  $s$ , is neglected, the corresponding result is

$$E_{\text{eff}}^* = \frac{G^*L^2}{4t(t+t_c)} \quad (95)$$

For a surface treatment which is many such cells in length and depth, an effective modulus such as this may be used, together with the total thickness, in the Oberst equations. The assumption of rigid constraining layers is not, of course, appropriate but was made to develop a simple theory which demonstrates well the concept.

Plunkett and Lee took into account the deformations of the constraining layer and arrived at a result (12)

$$E_{\text{eff}}^* = E_c \left( \frac{t_c}{t_c + t} \right) \left( 1 + \frac{\cosh \alpha^*}{\alpha^*} \right)^{-1} \quad (96)$$

where

$$\alpha^* = \frac{L}{2} \sqrt{\frac{G^*}{t_c t E_c}} \quad (97)$$

and  $E_c$  is the extensional modulus of the constraining layer. For  $E_c \rightarrow \infty$ ,

$\alpha^* \rightarrow 0$ , and

$$E_{\text{eff}}^* \rightarrow \frac{E_c t_c}{t_c + t} \alpha^{*2} = \frac{G^* L^2}{4t(t_c + t)} \quad (98)$$

which is in agreement with the preceding result.

Nashif and Nicholas (13) took a somewhat different approach to the concept of an equivalent homogeneous material. They hypothesized an equivalent homogeneous but anisotropic material, the complex moduli of which can be developed from a special rule of mixtures.

Effective shear compliance = shear compliance of shear layer x volume  
 fraction of shear layer

Effective loss factor in shear = loss factor of shear layer x volume  
 fraction of shear layer

Effective longitudinal stiffness = sum of products of individual stiffness  
 x volume fractions

Effective longitudinal loss factor = longitudinal loss factor of shear  
 layer x longitudinal stiffness  
 fraction of shear layer

This approach does not appear to provide for two important features of constrained layers. First, that the properties of a multiple layer configuration will depend on the wavelength, and secondly, that the thickness of individual layers are important.

#### D. Other Modifications

In addition to the use of multiple layers to increase damping, two other avenues have been explored. These are the spaced layer and the anchored treatment.

Ross, Ungar, and Kerwin (2) observed that the introduction of a shear-stiff material between the beam and the damping layer would serve to amplify the deformation in the damping layer. Whittier (4) analyzed and tested such a configuration. It was demonstrated by Ungar (14) that the inclusion of an ideal spacer in the design of a constrained layer treatment did not lead to significant complications in the analysis. Nonetheless, it is useful to consider a rigid band approximation in order to better understand the effectiveness of the mechanism. We will consider the idealized design shown in Figure 7.

Two bands are employed, AB and CD, assumed to have negligible thickness and bending rigidity. Since the relative displacement of two rigid bands is constant, a uniform shear strain is developed, and the tensile force in each band varies linearly from zero at A and D respectively, to a maximum value at B and C of

$$T = \tau bL \quad (99)$$

The relative displacement when the beam is deformed to a radius of curvature  $R$  is

$$\Delta = \frac{L}{R} \left( s + \frac{h}{2} + \frac{t}{2} - \delta \right) \quad (100)$$

where  $\delta$  is the location of the neutral axis above the centerline of the beam. The beam thickness  $h$  and spacer thickness  $s$  are assumed constant. If the shear layer is of thickness  $t$ ,

$$\tau = G^* \frac{\Delta}{t} \quad (101)$$

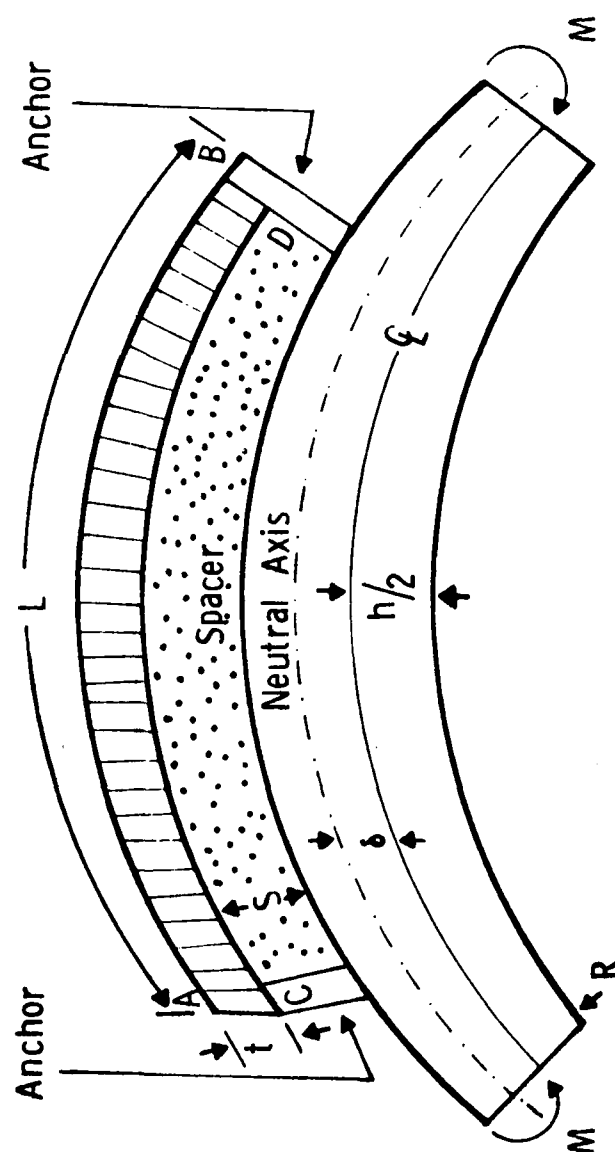


Figure 7 Ideal Spaced and Anchored Constrained Layer Treatment

The total moment is

$$M = b \int_{-\delta-h/2}^{-\delta+h/2} z E_b \frac{z}{R} dz + T \cdot \left( s + \frac{h}{2} + \frac{t}{2} - \delta \right) \quad (102)$$

The location of the neutral axis follows from the vanishing of horizontal force.

Solving for  $\delta$ , combining with the previous gives

$$\delta = \frac{2TR}{E_b h b} \quad (103)$$

$$T = \frac{b L^2 G^*}{R t} \left( s + \frac{h}{2} + \frac{t}{2} \right) \left( 1 + \frac{2 G^* L^2}{E_b h t} \right)^{-1} \quad (104)$$

and

$$M = \frac{E_b b h^3}{12 R} + \left( \frac{b L^2 G^*}{R t} \right) \left( s + \frac{h}{2} + \frac{t}{2} \right)^2 \frac{\left( 1 + \frac{4 G^* L^2}{E_b h t} \right)}{\left[ 1 + \frac{2 G^* L^2}{E_b h t} \right]^2} \quad (105)$$

But  $G^* L^2 / E_b h t \ll 1$  if the rigid band theory is to be appropriate, thus we compute the complex stiffness to be

$$(EI)^* = \frac{E_b h^3 \cdot b}{12} + \left( s + \frac{h}{2} + \frac{t}{2} \right)^2 \frac{b L^2 G^*}{t} \quad (106)$$

and deduce a loss factor

$$\eta_s = \frac{I_m (EI)^*}{R_e (EI)^*} \quad (107)$$



$$\eta_s = \frac{\eta_v}{1 + \frac{E_b h^3 t}{12L^2 G_1 (s + \frac{t}{2} + \frac{h}{2})^2}} \quad (108)$$

The available loss factor is seen to be driven by  $(s/h)^2$ . The effectiveness of the spacer may be evaluated by setting  $s = 0$ . Then

$$\frac{\eta(\text{Spaced Configuration})}{\eta(\text{No Spacer})} = \frac{(s + \frac{h}{2} + \frac{t}{2})^2}{(\frac{h}{2} + \frac{t}{2})^2} \quad (109)$$

Such increases are not fully achievable due to the difficulty in constructing the ideal spacer—one which contributes nothing to the bending rigidity (and nothing to the undesired shift in the neutral axis), yet maintains the full separation distance. In addition to the previously mentioned spaced configuration, other attempts were made by Lazan and co-workers. An anchored constraining layer with spacer, as shown in Figure 8, was designed and tested (15). Substantial amounts of damping were achieved through the geometric amplification of strain obtained with this configurational addition.

It has also been observed that a design wherein ends of the constraining layers are appropriately fixed can be considerably more effective than a design for which they are not. Consider, for example, the two layer configurations shown in Figure 9a. If the bands are very stiff, there will be no shear in the upper layer and therefore no energy dissipation. A very thin lower constraining layer acts as if it were not present at all, which has the effect of doubling the thickness of the damping layer. This, of course, reduces the energy dissipation.

By way of contrast, consider the two-layer configuration of Figure 9b. If the layers are sufficiently stiff, the damping is more than doubled by the added

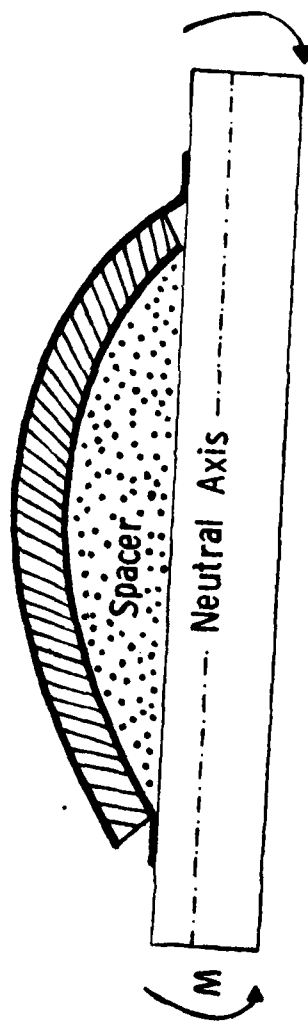


Figure 8 Possible Spaced and Anchored Configuration

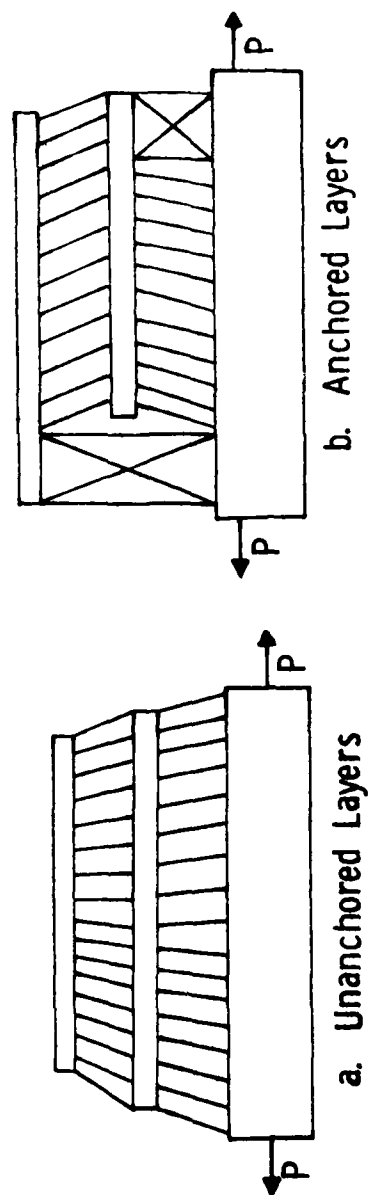


Figure 9 Multiple Layers in Tension

layer. Adding further layers continues to increase the damping, as the same maximum strain is induced in each layer. Each added layer is, in fact, even more effective than the first layer, for the shear strain in the first layer is not uniform. Such anchored configurations were considered by Lazan, Metherell, and Sokol (16) who performed analysis and experiments on two and four layer configurations of the class shown in Figure 9b undergoing bending deformations. It should also be noted that the segmented and staggered constraining layers discussed in the previous section constitute an anchored configuration. Adjacent layers in the segmented treatment experience shear strains of alternating sign, as in the anchored configuration of Lazan, et al.

### III. VIBRATING SYSTEMS

In the previous sections, we considered a number of quasi-static analyses as convenient means of gaining familiarity with basic concepts. With this background, we may proceed with examples of vibrating systems. The first of these will be the beam, or plate, covered with a free layer.

#### A. The Oberst Beam

Let a beam of finite or infinite length, or a strip isolated from a plate, be again covered on one side with a viscoelastic layer. The expressions for the bending stiffness previously given [Equations (43a) and (43b)] remain valid. Thus, the differential equation of motion is

$$D_e \nabla^4 W + (\rho h)_e \ddot{W} = 0 \quad (110)$$

where  $D_e = D + D_v$ , and we define an effective density,

$$(\rho h)_e = \rho_v t + \rho_b h, \quad (111)$$

and the transverse displacement is assumed to have modal amplitudes satisfying

$$\ddot{T}_n + \omega_n^2 (1 + i\eta_n) T_n = 0 \quad (112)$$

For any real mode shape,

$$D_e \nabla^4 W_n = (\rho h)_e \omega_n^2 (1 + i\eta_n) W_n \quad (113)$$

Taking the real and imaginary part, we find from the ratio that

$$\eta_n = I_m \{D_e\} / R_e \{D_e\} \quad (114)$$

which is the previously obtained result. Equation (114) and Equation (6) give the same result only if the stored energy is computed by using the real part of

the modulus, rather than the absolute value, as was noted by Ungar and Kerwin (17). In writing (110), the shearing deformations of both components have been neglected, as have the influences of rotatory inertias. The particular significances of Equation (114) are as follows: (1) it is independent of the particular mode; thus the loss factors of all modes are the same, and there is no dependence on the resonant frequency; (2) the loss factor of beam and plate are the same; and (3) the result is independent of the boundary conditions.

In the quasi-static analysis of a finite beam covered with a free layer, deforming axially or in bending, the length of the configuration did not appear explicitly in the loss factor. From this, we might have expected the loss factor in vibration to be independent of the frequency. In a vibrating system, the wavelength serves to determine the effective length. Since the loss factor was found independent of the length, it must also be independent of wavelength and natural frequency. Similar reasoning may be pursued to deduce that results derived for a beam may be applied directly to the panel. The loss factor for a low wave number mode of bending about one axis is the same as a high wave number mode bending about that, or a transverse, axis. Thus, the loss factor for all modes of the infinite plate is the same as of the beam. Further, the details of planform and boundary condition are irrelevant to the determination of the loss factor, for any mode of the finite plate with classical boundary conditions may be put as a superposition of the modes of the infinite plate.

Unfortunately, this happy state of affairs does not survive the addition of a constraining layer. It was found that the loss factor of constrained layer systems depends upon the length of the constraining layer and that an optimal length exists. For vibrating systems, then, the loss factor will depend on the wavelength and on the natural frequency. Further, since continuous systems may have many natural frequencies and modes within a range of interest, a design

which is optional for one mode will not be optimal for another, even for frequency-independent materials. A second consequence of the dependence on wavelength is, for finite configurations, a dependence on boundary condition. Early investigators avoided this complexity by considering a beam of infinite extent. Let us also begin in this manner.

### B. Infinite Beam; Single Constrained Layer

Actually, we have derived the governing equations in an earlier section. Equations (66) and (69) for the bending displacement and shear displacement may be adapted to the present problem by putting

$$q = -(\rho_b h + \rho_v t + \rho_c t_c) \ddot{w} \quad (115)$$

The mode shapes are:

$$W_n = A_n \sin k_n x \sin \Omega t \quad (116a)$$

$$\Psi_n = B_n \cos k_n x \sin \Omega t \quad (116b)$$

where

$$\Omega^2 = \omega_n^2 (1 + i\eta_n) \quad (117)$$

$\omega_n$ ,  $k_n$ , and  $\eta_n$  are the frequency, wave-number, and loss factor, respectively, for the  $n$ th mode. Let

$$g = g_1 (1 + i\eta_v) \quad (118)$$

$$\frac{1}{Y} = \frac{D_t}{d^2} \left( \frac{1}{E_c t_c} + \frac{1}{E_b t_b} \right) \quad (119)$$

$$X_n = g_1 / k_n^2 \quad (120)$$

Equations (66) and (69) then become

$$\{m\omega_n^2 (1 + i\eta_n) - D_t k_n^4\} A_n - k_n G dB_n = 0 \quad (121a)$$

$$t_s \{k_n^2 + g\} B_n - dk_n^3 A_n = 0 \quad (121b)$$



Combining these to eliminate amplitudes and then requiring that real and imaginary parts vanish separately leads to two real equations for the natural frequencies and modal damping coefficients. Solving these for loss factors, we find:

$$\eta_n = \frac{\eta_v Y x_n}{1 + (2 + Y) X_n + (1 + Y)(1 + \eta_v^2) X_n^2} \quad (122)$$

This is the accepted expression for the loss factor of a beam with single constraining layer.

This problem was first considered by Kerwin (1) and by Ross, Ungar, and Kerwin (2) with this same result given in 1959. The equation was presented in this form by Ungar (14), and a derivation substantially along the lines of that given here was obtained by Mead and Ditaranto (18) using earlier work of Mead and Markus (19) and Ditaranto (20). Ruzicka has studied (21) the physical significance of the various terms and has produced results for a wide variety of engineering applications.

The factor Y has been observed to be strictly a geometric parameter, depending only on the moduli and thicknesses of the beam and the constraining layer. Ungar has interpreted Y (14) as

$$Y = \frac{(EI)_\infty}{(EI)_0} - 1 \quad (123)$$

i.e., the fractional increase in bending stiffness which would result if the beam and constraining layer were completely coupled (bending as a single beam about the composite neutral axis) compared to the bending stiffness which occurs when the two beams are completely uncoupled (each bending about its own neutral axis). To reiterate, Y is independent of the material properties of the shear layer and

the wave length of the vibration. The shear parameter  $x$  takes these into account. Ungar has also shown that the complex stiffness of the composite beam is

$$(EI)^* = (EI)_0 \left\{ 1 + \frac{X^*}{1 + X^*} Y \right\} \quad (124)$$

where

$$X^* = X(1 + i\beta) \quad (125)$$

The loss factor is then readily extracted from Equation (114) and produces the same result as Equation (123).

In addition to Kerwin (1), Ruzicka performed early experiments with this configuration (22). Experimental confirmation of the predictions of Equation (123) was given (23) by Yin, Kelly, and Barry and by many others since that time.

#### IV. ADVANCES IN ANALYSIS AND DESIGN

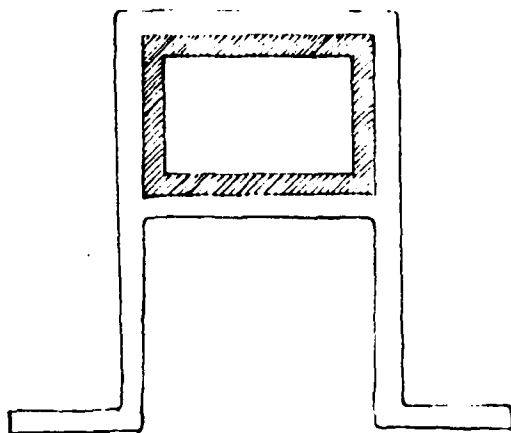
##### A. Beams with Constrained Layers

The shear damping mechanism was quickly applied to other geometries by Ruzicka (24), Kerwin (5), Ungar (14), and others (25). For configurations which are prismatic beams, the basic analysis was found to be readily adaptable to provide the necessary predictive ability. A number of damped configurations have been suggested by various authors; a few are depicted in Fig. 10 as a stimulus to thinking. More recently, laminated hollow circular beams (26), composite elastic-viscoelastic torsion members (27), and curved bars and helical springs (28) have been considered. An innovative damping addition for beams was recently described by Patel, et al (29). Several of these investigators have developed analyses of their designs, using the now well-established principles, to provide a method for design optimization.

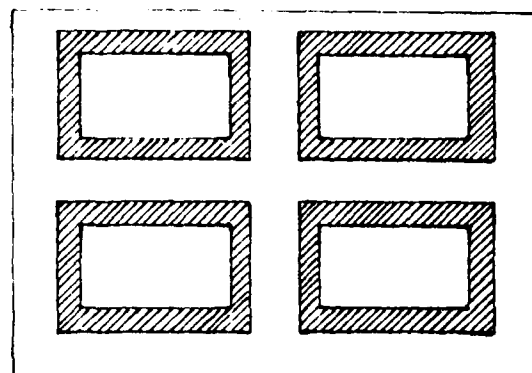
The constrained layer result, Equation (122), was derived by assuming a beam of infinite extent. The resulting sinusoidal mode shape is also appropriate to a finite beam with pinned ends but to no other case. It is now well recognized that the original equation cannot be expected to give good results for any mode which differs too significantly from a sinusoid. Unfortunately, the first mode of a cantilever is such a mode and is frequently encountered. If we return to Equations (66) and (69) and use again Equation (115), we may solve for the general displacements of the coupled equations. These substitutions produce

$$\frac{\partial^4 w}{\partial x^4} - g(1 + \gamma) \frac{\partial^4 w}{\partial x^4} + \frac{(\rho h) e}{D_t} \left( \frac{\partial^2 \ddot{w}}{\partial x^2} - \ddot{w} \right) = 0 \quad (126)$$

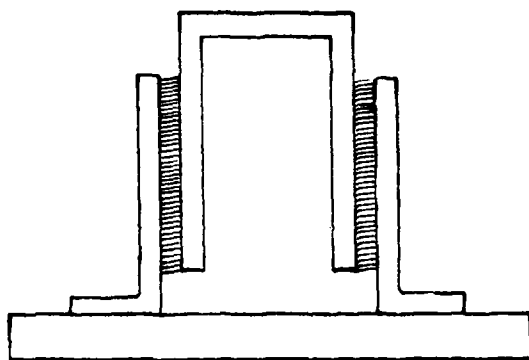
A coupled (sixth order) theory in a different dependent variable was first given by Ditaranto (20), but a coupled theory is implied in the work of Whittier (4). Mead and Markus (19) derived this particular sixth order equation, provided



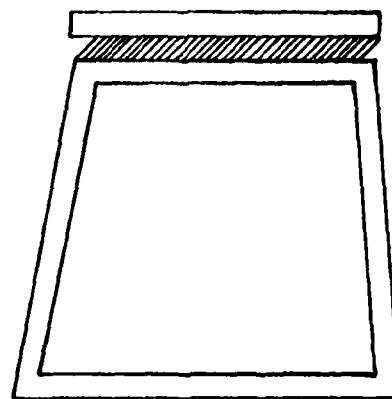
a. Ruzicka, 1961



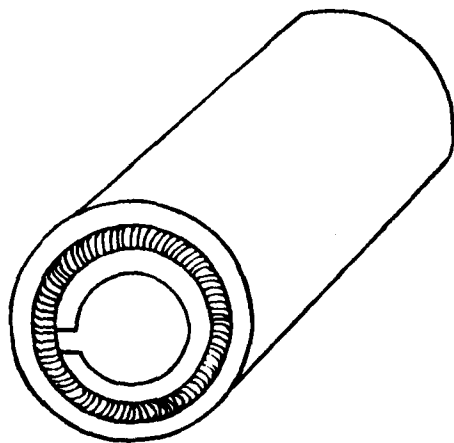
b. Ruzicka, 1961



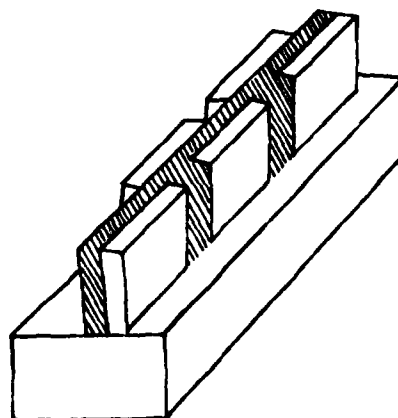
c. Kerwin, 1964



d. Kerwin, 1964



e. Yin, Kelly, Barry, 1967



f. Patel, Warnaka, Mead, 1978

Figure 10 Suggested Damped Beam Configurations

solutions for a number of boundary conditions, and demonstrated the orthogonality of the modes. Finite element methods have recently been applied (30,31) to the analysis of damped beams and enable the satisfaction of general boundary conditions.

As was noted in an earlier section, multiple constrained layers have been applied to beams. The work of Plunkett and Lee (12) with the equivalent homogeneous layer has already been mentioned. Derby and Ruzicka (25) gave general results for a N layer laminate, while Nakra and Grootenhuys (32) gave specific results for a four layer sandwich using two viscoelastic layers of differing properties in an attempt to improve the frequency effectiveness. Nashif and Nicholas (33) and Jones, Nashif, and Parin (34) provided experimental data on beams with up to 14 layers. Finally, theoretical work with large numbers of layers on beams has been undertaken by Asnani and Nakra (35). Miles recently presented (36) a theory along the lines of earlier work (25) and showed that predictions are in satisfactory agreement with available experimental data for multi-layer configurations. Ditaranto and Blasingame have contributed computations for three and five layer beams obtained from the Ross, Ungar, and Kerwin theory (37) and from the sixth order theory (38) and have provided useful approximations and design rules. Particularly notable in Ditaranto's contributions (26) is the finding that the loss factor vs frequency curve is independent of the choice of boundary condition, among all nondissipative boundary conditions. The boundary condition does, of course, influence the natural frequency and therefore the values of the modal damping coefficients. Grootenhuys performed computations for five layer beams and found (39) that symmetric five layer beams could be reduced to an equivalent three layer configuration. Useful design guides were given, including that when two differing viscoelastic materials are

used in an attempt to improve the frequency response there is no need to separate them with a third elastic layer.

Several of the higher order effects have received recent attention. Markus (40) demonstrated that the effects of rotatory inertia, inertial and shear forces were inevitably to reduce the loss factor from that of the Ross, Ungar, and Kerwin theory. From the work of Mentel (41), however, it may be taken that the dilatational effect may be neglected in comparison to that of shear. Transverse compressional damping was found (42) to be significant only at the thickness-stretch resonance of the viscoelastic core. Thick and stiff cores, on the other hand, have been shown (43) to produce significant energy dissipation due to core bending, especially in the lower modes. In some recent work, D. K. Rao (44) has given a quantitative comparison of a number of the higher order effects on the loss factor and natural frequency of short, damped sandwich beams. Korites and Nelson found that the dissipative heating in the damping layer caused substantial reductions in loss factor (45). This introduces a serious nonlinearity into the analysis for the properties are then amplitude dependent. The influence of varying core geometry has been explored as well. Sandman (46) analyzed a segmented core tuned to the first resonant mode, while Rao and Stühler (47) allowed the core to have a linearly varying thickness. The nonlinear equations of motion for large amplitudes have been considered (48). The super-harmonic resonance was identified as a source of difficulty for the complex modulus method. It should also be noted that this last work contains a good review of the literature on linear sandwich plates with viscoelastic cores.

In summary, the theory for laminated beams with viscoelastic layers is well in hand. The Kerwin theory has been found over the past twenty years to have wide applicability and, when used properly, to be adequate for purposes of engineering design. The equations were considered in 1959 to be somewhat

cumbersome for design purposes, but the almost universal availability of computers has removed this obstacle.

## B. Layered Plates

It was noted in an earlier section that the loss factor of a plate with unconstrained layer is the same as that of the beam, but that that of the plate with constrained layer is not. In view of the greater complexity of plate vibrations, it should not be too surprising that the development and analysis of constrained layers for plates trailed the development of treatments for beams by about a decade.

There was, however, a closely related structural configuration which received considerable attention early in the sixties - the sandwich plate. In a series of papers, Y. Y. Yu developed a theory for such plates and in 1962 considered (49) the damping due to a viscoelastic core in a symmetrically constructed three layer plate of arbitrary thickness ratios and material properties, subject to a restriction of small values of material loss factors. Sandwich plates were not found to be efficient for damping low frequency motions.

Ditaranto and McGraw examined three layer plates with viscoelastic cores and unsymmetrical construction in 1969 (50). They noted that, as in the case of the beam, the relationship between loss factor and frequency was independent of boundary condition. Further, some results for plates could be deduced from results obtained for beams. Although a general theory was given, only results for simply-supported sandwich plates were presented. These results are less general than those of Yu due to the use of a simpler assumed deformation, analogous to those earlier applied to beams. Thus, the bending of the shear layer is ignored, as is the rotatory inertia. In 1974, Sadasiva Rao and Nakra returned to the more general displacements assumptions of Yu and considered the bending of unsymmetrical, simply-supported sandwich plates (51). The extensional effect was found to be of importance only in very stiff cores, but the rotatory inertia of the elastic elements proved to be significant.



Attempts to develop approximate analyses of multilayered plates have paralleled the efforts for beams. A treatment employing multiple viscoelastic layers separated by constraining layers cut in rectangular segments of optimal dimension was fabricated and tested (52). The design was found to be effective and well-modeled by an equivalent layer approach based on that given by Plunkett and Lee for beams (12). Jones (53) applied multiple layer treatments with continuous constraining layers to panels and successfully predicted their effectiveness.

Although analytic methods have been very successful in developing the differential equations for damped plates, solutions have been obtained only for the simply supported edge condition. The numerical methods have been applied to obtain results for other boundary conditions. Finite element results have been compared with experiment for a free-free square plate (54,55) and for a circular sandwich plate (56). In addition to simple plates, the damping of the vibrations of stiffened panels has also been considered (57).

To summarize, the constrained layer damping of plates has received much less attention over the past two decades than have damped beams. The governing equations have been developed, but the complexity of the boundary condition has precluded the obtaining of extensive results. It has been established that the loss factor vs frequency expression is independent of boundary condition, but the natural frequency for a given boundary condition must still be determined, and this requires the solution of the coupled problem. There has also been less success in developing innovative and effective geometrical arrangements for plate damping; increased effort in this area is indicated.

### C. Other Geometries

Some advances in the analysis of the damping of free layers have also occurred. The analysis of torsion of a metal strip coated on one side by a polymer was recently developed (58) and the coated torsional pendulum proposed as a means of experimentally determining the complex modulus.

Annular inserts of high damping materials placed in holes drilled in highly stressed regions of vibrating members have been suggested (59) as being a means of enhancing the damping. The wisdom of deliberately introducing a material discontinuity in a region of high stress appears dubious, however.

Tuned dampers have been developed and successfully applied. In some, the deformation of the viscoelastic layer is predominantly axial, in others the inertia of the mass is used to develop a shear deformation (60). In none, however, has anything more than a rudimentary analysis been necessary. The same geometrical arrangement as in the tuned damper has proven to be a popular and successful means of determining material properties.

Sandwich and coated shell structures have also been considered. For the most part, adaptations of the analytical methods developed for plates have been employed. Finite element methods will undoubtedly receive extensive application to the design and prediction of damping for shell structures.

## V. SUMMARY

Substantial effort has been expanded over the past two decades in advancing the understanding and effectiveness of constrained layer damping treatments. During that period, the characteristics of and procedures for finding optimal designs have come to be better understood, innovative geometrical arrangements for obtaining the necessary large shear strains have been developed, and the limitation of the theory to the simply supported end condition has been removed. The access to computers, and even pocket calculators, has proven to be an enormous advantage, for the design equations are burdensome with hand calculation, but inconsequential with computational assistance. Larger machines have been and will be applied to some of the remaining tasks, such as plates of general boundary condition and material properties varying with amplitude, temperature, and frequency. The well known propensity of the polymeric materials to have a significant temperature dependence has led designers to seek means of incorporating two materials of differing properties in the same damping addition. This approach appears to be a promising means of achieving near optimal effectiveness over a range of temperatures, or equivalently, frequencies.

Of no small significance is that new materials have been developed (61), and material properties have been determined with greater certainty than was the case two decades ago. This has permitted the designer to proceed with greater confidence. The confidence of the designer has also been enhanced by the growing number of cases in which constrained layer damping treatments have been proven to be successful as means of controlling the amplitude of resonant vibration (62,63,64). Certainly these successes will lead design engineers to increased efforts to use constrained layer damping technology.

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